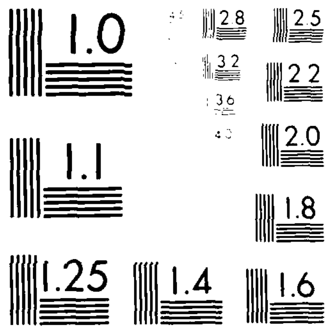
AD-A170 081 OPTIMAL ALLOCATION OF MULTISTATE COMPONENTS(U) ILLINOIS UNIV AT CHICAGO CIRCLE DEPT OF MATHEMATICS STATISTIC.. E EL-NEWEIHI ET AL SEP 85 TR-85-4 AFOSR-TR-86-0424 1/1

UNCLASSIFIED AFOSR-80-0170 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART

AFOSR-TR- 86-0424

(2)

AD-A170 081

THE UNIVERSITY OF ILLINOIS AT CHICAGO

OPTIMAL ALLOCATION OF MULTISTATE COMPONENTS

by

EMAD EL-NEWEIHI[1]
FRANK PROSCHAN[1]
and
JAYARAM SETHURAMAN[2]

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE

OPTIMAL ALLOCATION OF MULTISTATE COMPONENTS

by

EMAD EL-NEWEIHI[1]
FRANK PROSCHAN[1]
and
JAYARAM SETHURAMAN[2]

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 86-0414	2. GOVT ACCESSION NO. AD-A170081	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Optimal Allocation of Multistate Components		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Emad El-Neweihi, Frank Proschan, and Jayaram Sethuraman		8. CONTRACT OR GRANT NUMBER(s) AFOSR 80-0170
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics, Statistics, and University of Illinois at Chicago Chicago, IL 60680		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS [illegible]
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Air Force Air Force Office of Scientific Research Bolling Air Force Base, Washington, D.C. 20332		12. REPORT DATE September 1985
		13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution unlimited

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Multistate components and systems, optimal allocation, majorization, Schur functions, reliability, series systems, parallel-series systems, spares.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we present some results in the optimal allocation of multistate components to k series systems so that some performance characteristic like expected number of systems functioning at level α or higher, the probability that at least one of the systems functions at level α or higher, etc. is maximized. Our basic mathematical tools are majorization and Schur functions; the methods used and some of the theorems obtained are those of "Optimal Allocation of Components in Parallel-series and Series-Parallel Systems" El-Neweihi E., Proschan, F., and Sethuraman, J., Report (1984). In addition, we show

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) how these results may be used to obtain fruitful applications in reliability theory.

Optimal Allocation of Multistate Components

by

Emad El-Neweihi[1], Frank Proschan[1], and Jayaram Sethuraman[2]

September, 1985

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
Chicago, Illinois 60607

and

Florida State University
Department of Statistics
Tallahassee, Florida 32306

University of Illinois at Chicago
Technical Report No. 85-4
Statistical Laboratory

[1]Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant Numbers AFOSR 80-0170 and AFOSR 85-C-007. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon.

[2]Research supported by the U.S. Army Research Office under Grant DAAG 29-82-K-0168.

AMS (1981) Subject Classification: Primary 62B10; secondary 62B15

Key words: Multistate components and systems, optimal allocation, majorization, Schur functions, reliability, series systems, parallel-series systems, spares.

Optimal Allocation of Multistate Components

by

Emad El-Neweihi, Frank Proschan and Jayaram Sethuraman

ABSTRACT

In this paper we present some results in the optimal allocation of multistate components to k series systems so that some performance characteristic like expected number of systems functioning at level α or higher, the probability that at least one of the systems functions at level α or higher, etc. is maximized. Our basic mathematical tools are majorization and Schur functions; the methods used and some of the theorems obtained are those of "Optimal Allocation of Components in Parallel-Series and Series-Parallel Systems" El-Neweihi, E., Proschan, F., and Sethuraman, J., Report (1984). In addition, we show how these results may be used to obtain fruitful applications in reliability theory.

A-1

1. Introduction.

The theory of multistate systems has been developed to describe more adequately the performance of systems and their components each of which operates at more than the usually assumed two levels of performance: "functioning" or "failed". (For a recent survey of the subject see [3].)

In this paper we present some results in the optimal allocation of multistate components to k series systems so that some performance characteristic like expected number of systems functioning at level α or higher, the probability that at least one of the systems functions at level α or higher, etc., is maximized. Our basic mathematical tools are majorization and Schur functions; the methods used and some of the theorems obtained are those of [2]. In addition, we show how these results may be used to obtain useful applications in reliability theory. The present paper and reference [2] are the only papers exploiting the elegance and power of majorization and Schur functions to solve optimal allocation problems in reliability as far as we know.

Preliminaries.

For vector $\underline{x} = (x_1, \ldots, x_n)$, let $x_{[1]} \geq \ldots \geq x_{[n]}$ denote the *decreasing rearrangement* of the coordinates of $\underline{x}$. We say vector $\underline{x}$ *majorizes* vector $\underline{y}$ if

$$\sum_1^k x_{[i]} \geq \sum_1^k y_{[i]} \text{ for } k = 1, \ldots, n - 1$$

and

$$\sum_1^n x_{[i]} = \sum_1^n y_{[i]}.$$

We say a function f: $R^n \to R$ is *Schur-convex (Schur-concave)* if $f(\underline{x}) \geq f(\underline{y})$ $(f(\underline{x}) \leq f(\underline{y}))$ whenever $\underline{x} \overset{m}{>} \underline{y}$.

Examples of Schur-convex (Schur-concave) functions are $\sum_{i=1}^{n} f_1(x_i)$ and $\prod_{i=1}^{n} f_2(x_i)$, where $f_1: R \to R$ is convex (concave) and $f_2: R \to R$ is log-convex (log-concave).

A random variable X is said to be *stochastically larger* than a random variable Y if $P(X > x) \geq P(Y > x)$ for all real x; we write $X \overset{st}{>} Y$. For vectors, $\underline{X} \overset{st}{>} \underline{Y}$ if $f(\underline{X}) \overset{st}{>} f(\underline{Y})$ for every increasing function f.

A multistate series system consisting of n multistate components is a system whose structure function is given by $\phi(\underline{x}) = \min_{1 \leq i \leq n} x_i$ where x_i represents the state of component i and takes its value in a common state space $S \subseteq [0,\infty)$, $i = 1, \ldots, n$. A multistate parallel system can be similarly defined.

2. Optimum Allocation of Multistate Components.

We review a general optimal allocation result for multistate systems recently obtained in [2]. We also present additional related results. We then describe in some detail reliability models in which these results can be used.

The following theorem is basic in [2]; the proof depends on the useful tools of majorization and Schur functions. We present an outline of this proof; the interested reader may consult [2] for further details.

2.1. Theorem. Let $P_1, \ldots, P_k$ be the disjoint min path sets of a parallel-series system having path lengths $n_1, \ldots, n_k$. Without loss of generality, assume that $n_1 \leq \ldots \leq n_k$. Suppose that there are $n = n_1 + n_2 + \ldots + n_k$ independent components with reliabilities $p_1, \ldots, p_n$ (at time t_o, say) to be allocated among the path sets. Then the reliability of the system (at time t_o) is maximized when the n_1 most reliable components are allocated to P_1, the n_2 next most reliable components are allocated to $P_2, \ldots,$ and finally, the n_k least reliable components are allocated to P_k.

Proof. Let $x_i = \log \prod_{j \in P_i} p_j$, $i = 1, \ldots, k$. The allocation described in the hypothesis maximizes $\underline{x} = (x_1, \ldots, x_k)$ in the sense of majorization. The reliability function of the system, $1 - \prod_{i=1}^{k} (1 - e^{-x_i})$ is a Schur-convex function of $\underline{x}$, and hence the reliability is maximized by the allocation described. ||

The following theorem shows that the allocation described above also maximizes the expected number of working min path sets which are now viewed as k separate series systems.

2.2. Theorem. Let $S_1, \ldots, S_k$ be k series systems of sizes $n_1 \le \ldots \le n_k$, respectively. Suppose that there are $n = n_1 + \ldots + n_k$ independent components with reliabilities $p_1, \ldots, p_n$ (at time t_o, say) to be allocated among the series systems. Then the expected number of working subsystems (at time t_o) is maximized when the n_1 most reliable components are allocated to S_1, the n_2 next most reliable components are allocated to S_2, ..., and finally, the n_k least reliable components are allocated to S_k.

Proof. Let $\underline{x} = (x_1, \ldots, x_k)$ be as defined in the proof of Theorem 2.1. The expected number of working systems is $\sum_{i=1}^{k} e^{x_i}$ which is a Schur-convex function of $\underline{x}$. The result thus follows. ||

Note that the optimal allocation in Theorem 2.1 and Theorem 2.2 does not depend on the actual values of the reliabilities $p_1, \ldots, p_n$, but only on their ordering.

The above results can be extended to cover the case of multistate components and systems. Consider a parallel-series system as described in Theorem 2.1, except that now each component has a common state space $S \subseteq [0,\infty)$; common choices for S are $\{0, 1, \ldots, M\}$ and the unit interval $[0,1]$. Let $X_1(t), \ldots, X_n(t)$ denote the states of the components $1, \ldots, n$ and $X(t)$ denote the state of the system at time t, $t \ge 0$. Suppose that

$$(2.1) \qquad X_1(t) \overset{st}{<} X_2(t) \overset{st}{<} \cdots \overset{st}{<} X_n(t)$$

for each $t \geq 0$. Theorem 2.3 just below describes the best allocation of components to path sets to maximize the system state $X(t)$ in the stochastic sense uniformly in t.

2.3. Theorem. Consider a parallel-series system as described above with component states satisfying (2.1). To maximize the system state stochastically, uniformly in t, allocate components as follows: The best n_1 components (those with stochastically largest states) to P_1, the next best n_2 components to $P_2, \ldots,$ and finally the n_k worst components to P_k.

Proof. Fix a time $t > 0$ and state $\alpha \geq 0$. Let $P_i^\alpha(t) = P(X_i(t) \geq \alpha)$, $1 \leq i \leq n$. These probabilities are in increasing order in i for any fixed t and α in view of condition (2.1). Since the optimal allocation in Theorem 2.1 depends only on the ordering among such probabilities, the same allocation maximizes the system state stochastically, uniformly in t. ||

Now let us view the above min path sets as k separate multistate series systems and call them $S_1, \ldots, S_k$. For $\alpha \geq 0$, let $N^\alpha(t)$ be the number of systems functioning at a level α or higher. We have the following theorem.

2.4. Theorem. Let $S_1, \ldots, S_k$ and $N^\alpha(t)$ be as described above. Then the allocation described in Theorem 2.3 maximizes $EN^\alpha(t)$ uniformly in α and t.

Proof. Fix a time $t \geq 0$ and a state $\alpha \geq 0$. Then $E(N^\alpha(t)) = \sum_{i=1}^{k} \prod_{j \in S_i} P_j^\alpha(t)$, where $P_j^\alpha(t) = P(X_j(t) \geq \alpha)$, $j = 1, \ldots, n$. The result now follows from Theorem 2.2 and Condition (2.1). ||

In the results we have considered so far, all the components are tacitly assumed to be of the same type and thus can be interchanged freely. We now consider a situation in which we have multistate components of m different types. Let $X_1^j(t) \overset{st}{\leq} X_2^j(t) \overset{st}{\leq} \dots \overset{st}{\leq} X_k^j(t)$ be the states of k components of type j at time t, $j = 1, \dots, m$, $t \geq 0$. We wish to build k multistate series systems each containing one component of each type. Let $N^\alpha(t)$ be the number of systems that function at level α or higher, where $\alpha \geq 0$ corresponds to an arbitrary but fixed state. We wish to maximize either $E(N^\alpha(t))$ or $P(N^\alpha(t) \geq 1)$. The following theorem gives the optimal allocation.

<u>2.5. Theorem.</u> Let $S_1, \dots, S_k$ denote the k series systems. To maximize $E(N^\alpha(t))$ or $P(N^\alpha(t) \geq 1)$ uniformly in α and t, assemble the systems as follows:

The best components are assembled together, the next best components are assembled together, ..., and finally the worst components are assembled together. Thus system S_i is built with components whose states are $X_i^1(t)$, ..., $X_i^m(t)$, $i = 1, \dots, k$.

<u>Proof.</u> Let $P_i^{j,\alpha}(t) = P(X_i^j(t) \geq \alpha)$, $j = 1, \dots, m$ and $i = 1, \dots, k$. Let $x_i^\alpha = \log \prod_{j=1}^m P^{j,\alpha}_{\psi^j(i)}$, $i = 1, \dots, k$, where $\psi^1, \dots, \psi^m$ are m permutations of $\{1, \dots, k\}$. Let $\underline{x}^\alpha = (x_1^\alpha, \dots, x_k^\alpha)$. The assembly described in the hypothesis of the theorem maximizes $\underline{x}^\alpha$ in the sense of majorization. Since both $E(N^\alpha(t))$ and $P(N^\alpha(t) \geq 1)$ can be expressed as Schur-convex functions of $\underline{x}^\alpha$, the result now follows. ||

<u>2.6. Remark.</u> The model discussed just above is related to a binary model given in [1] .

3. <u>Applications in Reliability.</u>

In this section we describe models to which theorems of section 2 can be applied. In all models the state space is $\{0, 1, \dots, M\}$.

3.1. Models. Consider n independent binary components. Suppose that each component is supported by $M-1$ functioning spares that do not deteriorate until put into use. We say that "position" or "socket" i is in state M when the original component of type i is still functioning and none of the $M-1$ spares have been used; the position is in state $M-1$ when the original component has failed and has been replaced by a spare, leaving $M-2$ spares available for replacement; ...; and finally, the position is in state 0 when no spares are available and the component in use fails.

Next let T_j^i be the life length of the j^{th} spare for component type i, $1 \le i \le n$, $1 \le j \le m$. Note that each original component in operation at time 0 is viewed as a member of its spares kit. Assume that for each i, $T_1^i, \ldots, T_M^i$ are independently and exponentially distributed with parameters $\lambda_1^i, \ldots, \lambda_M^i$ respectively. Let $\lambda_{(1)}^i \le \lambda_{(2)}^i \le \ldots \le \lambda_{(M)}^i$ be a rearrangement of $\lambda_1^i, \ldots, \lambda_M^i$, $i = 1, \ldots, n$. Assume that $\lambda_{(\ell)}^1 \ge \lambda_{(\ell)}^2 \ge \ldots \ge \lambda_{(\ell)}^n$ for $\ell = 1, \ldots, M$.

Note that for each pair (i,t), the distribution of $X_i(t)$ depends on the order in which we have been using the spares. However, a little reflection shows that the following order maximizes $X_i(t)$ stochastically, uniformly in t for each i: Start with the spare whose parameter is $\lambda_{(1)}^i$. Upon its failure, replace it with the spare whose parameter is $\lambda_{(2)}^i$, ..., and finally use the spare whose parameter is $\lambda_M^{(i)}$. Let $X_i^*(t)$ be the random state of component i corresponding to the above order. Then clearly $X_1^*(t) \overset{st}{<} X_2^*(t) \overset{st}{<} \ldots \overset{st}{<} X_n^*(t)$ for each t. The allocation described in Theorem 2.3 maximizes stochastically the system state uniformly in t.

Thus, employing the order described above for using the spares corresponding to each socket, together with the optimal allocation of Theorem 2.3 yields the best stochastic performance of a parallel-series system formed from the n multistate components. ||

Similarly Theorem 2.4 gives us the allocation that maximizes the expected number of $N^{\alpha}(t)$ (uniformly in α and t), where $N^{\alpha}(t)$ is the number of systems functioning at a level α or higher among a total of k series systems built from the above multistate components.

3.2. Remark. In the example given in [2], $\lambda_1^i = \lambda_2^i = \ldots = \lambda_M^i = \lambda_i$, $i = 1, \ldots, n$. Thus, in this case, the order in which the spares are used is immaterial.

3.3. Remark. In Model 3.1, it is assumed that for each i, $T_1^i, \ldots, T_M^i$ have exponential distributions with parameters satisfying certain conditions. Other distributions can be used provided the following conditions hold: 1) For each i, the random variables $T_1^i, \ldots, T_M^i$ are stochastically ordered. 2) For $i < j$, the smallest stochastically among $T_1^i, \ldots, T_M^i$ is stochastically less than the smallest among $T_1^j, \ldots, T_M^j$, the second smallest among $T_1^i, \ldots, T_M^i$ is stochastically less than the second smallest among $T_1^j, \ldots, T_M^j$, ..., and finally the largest stochastically among $T_1^i, \ldots, T_M^i$ is stochastically less than the largest stochastically among $T_1^j, \ldots, T_M^j$.

3.4. Model. Consider $n \cdot M$ binary components forming n parallel systems $S_1, \ldots, S_n$ of size M each. We now view each S_i as a single multistate system of components with $M+1$ states $0, 1, \ldots, M$, as follows: When all the M binary components in system S_i are functioning, then the socket corresponding to multistate component i is in state M. When the first binary component in system S_i fails, socket i is now in state $M-1$, and so on, until the last binary component in system S_i fails, and socket i is now in state 0, $i = 1, \ldots, n$. Let $\underline{T}^i = (T_1^i, \ldots, T_M^i)$ be the random vector representing the joint lifelengths of the M binary components in system S_i, $i = 1, \ldots, n$. Suppose that $\underline{T}^1 \overset{st}{<} \underline{T}^2 \overset{st}{<} \ldots \overset{st}{<} \underline{T}^n$. Let $X_1(t), \ldots, X_n(t)$ be the states of components $1, \ldots, n$ at time t, $t \geq 0$. Then $P(X_i(t) > j) = P(T^i_{(M-j)} > t)$ for each $j = 0$,

$\ldots, M-1$; $i = 1, \ldots, n$, and $t \geq 0$ where $T^i_{(\ell)}$ is the ℓ-th order statistic among $T^i_1, \ldots, T^i_M$. Clearly $X_1(t) \overset{st}{<} X_2(t) \overset{st}{<} \ldots \overset{st}{<} X_n(t)$ for each $t \geq 0$. Theorem 2.3 can now be used to provide us with the optimal allocation of the n multistate components to the min path sets of a parallel-series system of n components. ||

3.5. Model. Consider k binary components of type j, $j = 1, \ldots, m$. The i^{th} component of type j has a spares kit with $M-1$ functioning spares that do not deteriorate until put into use, $i = 1, \ldots, k$, $j = 1, \ldots, m$. We view each binary component together with its spares as a multistate component in the same fashion as described in Model 3.1. The life length of the i^{th} component of the j^{th} type as well as the life lengths of all of its spares have exponential distributions with the same parameter λ^j_i, $i = 1, \ldots, k$; $j = 1, \ldots, m$. All the random variables considered are independent. Assume $\lambda^j_1 \geq \lambda^j_2 \geq \ldots \geq \lambda^j_k$, $j = 1, \ldots, m$. Now let $X^j_i(t)$ be the state of the i^{th} component of type i at time t, where $i = 1, \ldots, k$, $j = 1, \ldots, m$, $t \geq 0$. Clearly $X^j_1(t) \overset{st}{<} X^j_2(t) \overset{st}{<} \ldots \overset{st}{<} X^j_k(t)$. Theorem 2.5 can now be used in the obvious fashion to obtain the optimal assembly of k series system. ||

References.

[1] On Optimal Assembly of Systems. Derman, C., Lieberman, G.J. and Ross, S.M. Nav. Res. Log. Quart. 19, 569-574 (1972).

[2] Optimal Allocation of Components in Parallel-Series and Series-Parallel Systems. El-Neweihi, E., Proschan, F., and Sethuraman, J. Report (1984). Submitted for publication.

[3] Degradable Systems: A Survey of Multistate System Theory. El-Neweihi, E. and Proschan, F. Commun. in Statist. 13(4), 405-432 (1984).

END

DTIC

8 — 86