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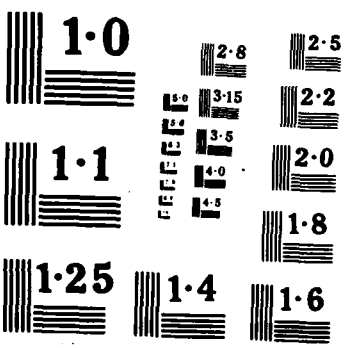
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ON STRONGLY CONSISTENT ESTIMATES OF REGRESSION COEFFICIENTS WHEN THE ERRORS ARE NOT INDEPENDENTLY AND IDENTICALLY DISTRIBUTED *

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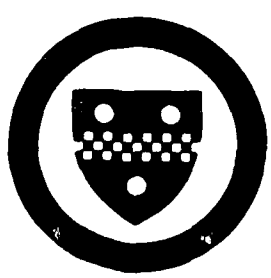

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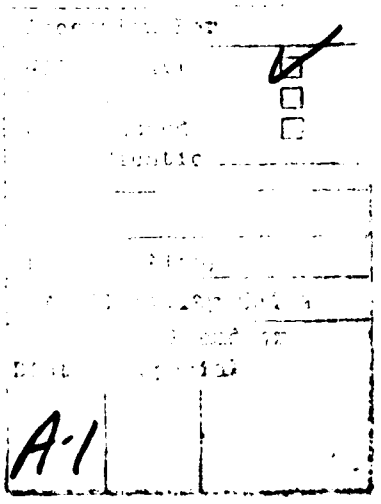
ABSTRACT

In this paper, the author proposes two methods of estimation of the regression coefficients when the errors are not distributed identically and independently and are of nonzero mean. The estimates proved in this paper are shown to be strongly consistent and mean square consistent.

Key Words and Phrases: Correlated errors, heterogeneous errors, regression coefficients, strongly consistent



1. INTRODUCTION

Suppose we have a system of linear equations

$$Y_i = X_{i_1}\beta_1 + \ldots + X_{i_p}\beta_p, \quad i = 1, 2, \ldots, m \tag{1}$$

where Y_i, X_{i_j}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p$ are known while $\beta_1, \ldots, \beta_p$ are the unknowns. There are many ways to define a solution for the determination of $\underset{\sim}{\beta}' = (\beta_1, \ldots, \beta_p)$. One well known way is the so-called Chebyshev approximation, which seeks to minimize

$$\overline{Q}_n(\underset{\sim}{\beta}) = \max_{1 \le i \le n} |Y_i - \underset{\sim}{X}_i'|$$

where $\underset{\sim}{X}_i = (X_{i_1}, \ldots, X_{i_p})'$. Such a solution, denoted by $\overline{\underset{\sim}{\beta}}_n$, can be computed by the method of linear programing (see [2] and [6]).

Now if $Y_1, Y_2, \ldots$ are observed with random errors, then, instead of (1), we have the linear regression model

$$Y_i = \underset{\sim}{X}_i'\beta + e_i, \quad i = 1, 2, \ldots, n \tag{2}$$

where $e_1, \ldots, e_n$ are random errors. Usually, in (2), it is assumed that the expectations of errors are zero and have finite second moment with orthogonality or asymptotic independence. To solve this problem, one can use the famous least square estimation (LSE) method. In the literature, there are a lot of papers concerning with LSE and many important results are obtained (see [1], [3], [5]). But the unbiasedness and consistency of LSE strongly depend on the assumption that the expectations of errors are zero, and this assumption is not realistic sometimes. The means of errors of measurements may be different. Similarly, it is not always realistic to assume that the errors are distributed independently. In such situations, it is of interest to obtain consistent estimates of $\underset{\sim}{\beta}$. In this paper, we propose two methods of obtaining consistent estimates of $\underset{\sim}{\beta}$.

Both of these methods are motivated by the so-called Chebshev approximation.

The first method is to use the measure

$$Q_n(\underset{\sim}{\beta}) = \max_{1\le i\le n} (Y_i - \underset{\sim}{X}_i'\underset{\sim}{\beta}) - \min_{1\le i\le n} (Y_i - \underset{\sim}{X}_i'\underset{\sim}{\beta})$$

which is never negative. So we can find the solution $\hat{\underset{\sim}{\beta}}_n$ which minimizes $Q_n(\underset{\sim}{\beta})$. We refer to $\hat{\underset{\sim}{\beta}}$ as MD estimate (that is, the estimate based on Maximum Difference between residuals).

The second method is to use the measure

$$\bar{Q}_n(\underset{\sim}{\beta}) = \max_{1\le i\le n} |Y_i - \underset{\sim}{X}_i'\underset{\sim}{\beta}|.$$

Denote by $\bar{\underset{\sim}{\beta}}_n$ the value of $\underset{\sim}{\beta}$ which minimizes $\bar{Q}_n(\underset{\sim}{\beta})$. We shall call $\bar{\underset{\sim}{\beta}}_n$ MA estimator (that is, the estimator based on the Maximum Absolute value of residuals).

In Section 2, we shall prove the strong consistency of $\hat{\underset{\sim}{\beta}}_n$ whereas in Section 3, we shall prove its mean square consistency. In Section 4, we will prove that estimates of endpoints of error support based on the residuals, with respect to $\hat{\underset{\sim}{\beta}}_n$, are strong by consistent, and establish the strong consistency of the estimate of second moment of errors, based on $\hat{\underset{\sim}{\beta}}_n$, when the error sequence is stationary and ergodic. In Section 5, we establish the strong consistency of $\bar{\underset{\sim}{\beta}}_n$ and its mean square consistency is given in Section 6. In Section 7, we shall prove the consistency of estimates of the largest value of endpoints and that of the second moment of errors when the error sequence is stationary and ergodic. Here we emphasize the fact that we do not use independence (even in the asymptotic case) of the error sequence in proving our results.

For example, let ϵ_n, n = 1, 2, ... be i.i.d. random variables with common uniform distribution over the interval [0,1]. Define the errors as

$$e_{2^k+i} = \epsilon_k,\ k = 1, 2, \ldots,\ i = 1, 2, \ldots, 2^k.$$

Such an error sequence satisfies all conditions in our theorems but it is not asymptotically independent.

Of course, if the errors have a common mean, our model will turn out to be the usual linear regression model. In this paper, we do not assume the errors have common mean. For example, let ϵ_n be independent r.v. distributed on (0,1) and with density $p_n(x) \geq \epsilon > 0$. Also, assume that $E(\epsilon_n)$ are not equal. Define

$$e_{2^k+i} = \frac{1}{k}\,\epsilon_i + \epsilon_k,\ i = 1, 2, \ldots, 2^k,\ k = 1, 2, \ldots .$$

In this case, LS method does not work but our method still works.

2. STRONG CONSISTENCY OF $\hat{\underset{\sim}{\beta}}_n$

In this section we shall prove the following theorem:

Theorem 1. Suppose that e_i has a support included in the bounded interval $[a_1, a_2]$, and for any subsequence $\{e_{i_j} : j = 1, 2, \ldots\}$ of $\{e_i\}$ we have

$$\overline{\lim_{j\to\infty}} e_{i_j} = a_2, \quad \underline{\lim_{j\to\infty}} e_{i_j} = a_1, \text{ a.s.} \tag{3}$$

Also, suppose that $\{\underset{\sim}{X}_i\}$ is bounded, and for any *nonzero* p-vector $\underset{\sim}{d}$, $\lim_{n\to\infty} \underset{\sim}{x}_n'\underset{\sim}{d}$ does not exist. Then

$$\lim_{n\to\infty} \hat{\underset{\sim}{\beta}}_n = \underset{\sim}{\beta}, \text{ a.s.} \tag{4}$$

Proof. Several preliminary facts are in order.

1. *Define a function*

$$g(\underset{\sim}{\alpha}) = \overline{\lim} \underset{\sim}{\alpha}'\underset{\sim}{x}_n - \underline{\lim} \underset{\sim}{\alpha}'\underset{\sim}{x}_n$$

on R^p, then $g(\underset{\sim}{\alpha})$ is continuous on R^p, and $g(\underset{\sim}{\alpha}) > 0$ when $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$.

The second conclusion follows directly from the assumption of the theorem, and the first is an easy consequence of the boundedness of $\{\underset{\sim}{x}_i\}$.

Denote by B the surface of the unit sphere in R^p i.e. $B = \{\underset{\sim}{\alpha}: \|\underset{\sim}{\alpha}\| = 1\}$. Because B is compact, g is continuous, and $g(\underset{\sim}{\alpha}) > 0$ for $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$, we have

$$c \hat{=} \inf_{\underset{\sim}{\alpha}\in B} g(\underset{\sim}{\alpha}) > 0. \tag{5}$$

2. Define B and c as above. Then we can find a positive integer m and a subset $D = \{\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_m\}$ of B such that for any $\underset{\sim}{\alpha} \in B$, there exists $\underset{\sim}{\alpha}_i \in D$ with the property

$$\sup \{|(\underset{\sim}{\alpha} - \underset{\sim}{\alpha}_i)' \underset{\sim}{x}_j| : j = 1, 2, \ldots\} < c/4 \tag{6}$$

The proof is obvious. If $\|\underset{\sim}{x}_j\| \leq M$, $j = 1, 2, \ldots$, then we need only to find m large enough so that there exists a $\frac{c}{8M}$ - net of B consisting of m points. This $\frac{c}{8M}$ - net is chosen as D.

For every $\underset{\sim}{\alpha}_i \in D$, we can find out two subsequences $\{\underset{\sim}{x}_{i1j}\}_{j=1}^{\infty}$ and $\{\underset{\sim}{x}_{i2j}\}_{j=1}^{\infty}$ of $\{\underset{\sim}{x}_n\}$ such that

$$\lim_{j\to\infty} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_{i1j} = \overline{\lim_{n\to\infty}}\ \underset{\sim}{\alpha}_i' \underset{\sim}{x}_n, \quad \lim_{j\to\infty} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_{i2j} = \underline{\lim_{n\to\infty}}\ \underset{\sim}{\alpha}_i' \underset{\sim}{x}_n.$$

Denote the values of y and e corresponding to $\underset{\sim}{x}_{irj}$ by y_{irj}, e_{irj}, $r = 1, 2$, i.e. if $\underset{\sim}{x}_{irj} = \underset{\sim}{x}_k$, then $y_{irj} = y_k$, $e_{irj} = e_k$. Put

$$E_{i1} = \{\overline{\lim_{j\to\infty}}\ e_{i1j} = a_2\}, \quad E_{i2} = \{\underline{\lim_{j\to\infty}}\ e_{i2j} = a_1\}$$

$$E = \bigcap_{i=1}^{m} (E_{i1} \cap E_{i2}). \tag{7}$$

3. $P(E) = 1$.

In fact, since $P(E_{i1}) = P(E_{i2}) = 1$, $i = 1, \ldots, m$ by (3), we get the result.

Employ the symbol ω to represent a real number sequence $(c_1, c_2, \ldots)$, $a_1 \leq c_1 \leq a_2$, $i = 1, 2, \ldots$. The event E determined by (7) can be viewed as a subset $\tilde{E}$ of the ω-space i.e. $E = \{(e_1, e_2, \ldots) \in \tilde{E}\}$.

4. For any $\epsilon > 0$, $t \geq \epsilon$, $\alpha_i \in D$, put $\underset{\sim}{d}_t = \underset{\sim}{\beta} + t\underset{\sim}{\alpha}_i$, then for any $\omega \in \tilde{E}$, there exists $N_i(\omega)$ such that

$$\max_{1\leq k\leq n} (y_k(\omega) - \underset{\sim}{x}_k' \underset{\sim}{d}_t) - \min_{1\leq k\leq n} (y_k(\omega) - \underset{\sim}{x}_k' \underset{\sim}{d}_t) \geq a_2 - a_1 + \frac{2}{3}tc \tag{8}$$

when $n \geq N_i(\omega)$.

The proof is as follows: For simplicity of notation we shall use y_k, e_k to express $y_k(\omega)$ and $e_k(\omega)$. Then

$$y_k - \underset{\sim}{x}_k' \underset{\sim}{d}_t = \underset{\sim}{x}_k' \underset{\sim}{\beta} + e_k - \underset{\sim}{x}_k' (\underset{\sim}{\beta} + t\underset{\sim}{\alpha}_i) = e_k - t\underset{\sim}{x}_k' \underset{\sim}{\alpha}_i . \tag{9}$$

Take $\eta > 0$ whose value will be given later. Since $\overline{\lim_{j\to\infty}} e_{ilj} = a_2$ and $\lim_{j\to\infty} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_{ilj} = \overline{\lim_{n\to\infty}} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_n$, we can find $j_1 = j_1(\eta,\omega)$ so that

$$e_{ilj_1} \geq a_2 - \eta \tag{10}$$

$$\underset{\sim}{\alpha}_i' \underset{\sim}{x}_{ilj_1} < \overline{\lim_{n\to\infty}} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_n + \eta. \tag{11}$$

Take $N_{i1} = N_{i1}(\eta,\omega)$ large enough so that when $n \geq N_{i1}$ we have

$$\underset{\sim}{x}_{ilj_1} \in \{\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n\}.$$

Then by (9)-(11), when $n \geq N_{i1}$ we have

$$\max_{1\leq k\leq n} (y_k - \underset{\sim}{x}_k' \underset{\sim}{d}_t) \geq e_{ilj_1} - t\underset{\sim}{x}_{ilj_1}' \underset{\sim}{\alpha}_i \geq a_2 - \eta - t(\overline{\lim_{n\to\infty}} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_n + \eta). \tag{12}$$

Similarly, we can prove that there exists $N_{i2} = N_{i2}(\eta,\omega)$ such that when $n \geq N_{i2}$ we have

$$\min_{1\leq k\leq n} (y_k - \underset{\sim}{x}_k' \underset{\sim}{d}_t) \leq a_1 + \eta - t(\overline{\lim_{n\to\infty}} \underset{\sim}{\alpha}_i' \underset{\sim}{x}_n - \eta). \tag{13}$$

Put $N_i = N_i(\eta,\omega) = \max\{N_{i1}(\eta,\omega), N_{i2}(\eta,\omega)\}$. Then using (12), (13) and noticing the definition of c in (5), we get

$$\max_{1\le k\le n}(y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t) - \min_{1\le k\le n}(y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t)$$

$$\ge a_2 - \eta - t(\varlimsup_{n\to\infty} \underset{\sim}{\alpha}_i'\underset{\sim}{x}_n + \eta) - \{a_1 + \eta - t(\varliminf_{n\to\infty} \underset{\sim}{\alpha}_i'\underset{\sim}{x}_n - \eta)$$

$$\ge a_2 - a_1 - 2(t+1)\eta + tc. \tag{14}$$

Now specify $\eta = \frac{c\epsilon}{6(\epsilon+1)}$. Since $\frac{ct}{6(t+1)} \ge \frac{c\epsilon}{6(\epsilon+1)}$ for $t \ge \epsilon$, it follows from (14) that for $t \ge \epsilon$

$$\max_{1\le k\le n}(y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t) - \min_{1\le k\le n}(y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t)$$

$$\ge a_2 - a_1 - 2(t+1)\frac{ct}{6(t+1)} + tc = a_2 - a_1 + \frac{2}{3}tc.$$

This proves the assertion.

Now turn to the proof of theorem 1. Fix $\omega \in \tilde{E}$ and $\epsilon > 0$, and again use y_i, e_i, N_i etc. to express $y_i(\omega)$, $e_i(\omega)$, $N_i(\omega)$. Put $N = N(\omega) = \max_{1\le i\le m} N_i(\omega)$. Suppose that $\underset{\sim}{\ell} \in R^p$ satisfies $\|\underset{\sim}{\ell} - \underset{\sim}{\beta}\| \ge \epsilon$, then $\underset{\sim}{\ell}$ can be written as $\underset{\sim}{\ell} = \underset{\sim}{\beta} + t\underset{\sim}{\alpha}$, $t \ge \epsilon$, $\underset{\sim}{\alpha} \in B$. By 2, we can choose $\underset{\sim}{\alpha}_i \in D$ so that (6) is true. Remember that $d_t = \underset{\sim}{\beta} + t\underset{\sim}{\alpha}_i$, we have

$$y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell} = y_k - \underset{\sim}{x}_k'(\underset{\sim}{\beta} + t\underset{\sim}{\alpha}) = y_k - \underset{\sim}{x}_k'(\underset{\sim}{d}_t + t(\underset{\sim}{\alpha} - \underset{\sim}{\alpha}_i))$$

$$= y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t - t\underset{\sim}{x}_k'(\underset{\sim}{\alpha} - \underset{\sim}{\alpha}_i) \ge y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t - tc/4$$

and

$$y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell} \le y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t + tc/4.$$

From these inequalities and (8), we have

$$\max_{1\le k\le n}(y_k - x_k'\ell) - \min_{1\le k\le n}(y_k - x_k'\ell)$$

$$\ge \max_{1\le k\le n}(y_k - x_k'd_t) - \min_{1\le k\le n}(y_k - x_k'd_t) - tc/2$$

$$\ge a_2 - a_1 + tc/6$$

for $n \ge N$. On the other hand, we know that

$$\max_{1\le k\le n}(y_k - x_k'\beta) - \min_{1\le k\le n}(y_k - x_k'\beta)$$

$$= \max_{1\le k\le n} e_k - \min_{1\le k\le n} e_k \le a_2 - a_1.$$

So we have $Q_n(\ell) > Q_n(\beta)$ for $n \ge N$ and $\|\ell - \beta\| \ge \epsilon$. According to the definition of $\hat{\beta}_n$, we have $\|\hat{\beta}_n - \beta\| < \epsilon$ when $n \ge N$. Since $\epsilon > 0$ is arbitrary, this proves that $\lim_{n\to\infty} \hat{\beta}_n = \beta$ when $\omega \in E$. The proof of Theorem 1 is concluded in view of 3.

Let us mention two important examples of this theorem.

1. $\{e_i\}$ is a m-dependent and identically distributed sequence and the support of e_i is bounded, where m is a positive integer.

Let a_1, a_2 express the infimum and supremum of the support of e_1 respectively.

(I) $m = 1$.

It is obvious that $\overline{\lim_{j\to\infty}} e_{i_j} = a_2$, $\underline{\lim_{j\to\infty}} e_{i_j} = a_1$ for any subsequence $\{e_{i_j}\}$ of $\{e_i\}$.

(II) $m > 1$.

Take arbitrarily a subsequence $\{e_{i_j}\}$ of $\{e_i\}$. We can further choose an iid. subsequence $\{\bar{e}_i\}$ from $\{e_{i_j}\}$. By (I) we have $\overline{\lim_{i\to\infty}} \bar{e}_i = a_2$, a.s. So $a_2 \ge \overline{\lim_{j\to\infty}} e_{i_j} \ge \overline{\lim_{i\to\infty}} \bar{e}_i = a_2$, a.s. In a similar way we prove the other assertion of (3).

2. Let $\{e_i\}$ be a stationary sequence, the support of e_1 being bounded (a_1, a_2 denote the left and right endpoint of the support) and suppose that the following condition (weak dependency) holds:

$$\lim_{n\to\infty}[\sup\{|p(A)p(B_n)-p(AB_n)|: A\in\sigma(e_i), B_n\in\sigma(e_{i+n},e_{i+n+1},\ldots)\}] = 0 \qquad (15)$$

for any fixed i, where $\sigma(e_i, e_j, \ldots)$ express the σ-field generated by $e_i, e_j, \ldots$.

For this case we again take an arbitrary subsequence $\{e_{i_j}\}$ of $\{e_i\}$, for fixed $\epsilon > 0$, put $E_j = \{e_{i_j} \leq a_2 - \epsilon\}$. It needs only to verify that

$$P(E_v E_{v+1} E_{v+2} \ldots) = 0, \text{ for any } v. \qquad (16)$$

Use C_{in} to denote the quantity under the limit sign of (15). Fix $r > v$ and take n_1 large enough to make $C_{i_r n_1} < 1/2^r$. Notice that $i_r + n_1 \leq i_{r+n_1}$. Further, choose n_2 large enough to satisfy $C_{i_{r+n_1} n_2} < 1/2^{r+1}$. In general, after determining n_m, we choose n_{m+1} large enough so that $C_{i_{r+n_1+\ldots+n_m} n_{m+1}} < 1/2^{r+m}$. Then

$$\begin{aligned}
&P(E_v E_{v+1} E_{v+2} \ldots) \\
&\leq P(E_r E_{r+n_1} E_{r+n_1+n_2+n_3} \ldots) \\
&\leq 1/2^r + P(E_{r+n_1} E_{r+n_1+n_2} E_{r+n_1+n_2+n_3} \ldots)P(E_r) \\
&\leq 1/2^r + 1/2^{r+1} + P(E_{r+n_1+n_2} E_{r+n_1+n_2+n_3} \ldots)P(E_r)P(E_{r+n_1}) \\
&\leq \ldots \\
&\leq (1/2^r + 1/2^{r+1} + \ldots) + P(E_r)P(E_{r+n_1})\ldots P(E_{r+n_1+\ldots+n_m})
\end{aligned}$$

for any nature number m. Since $P(E_r) = P(E_{r+n_1}) = \ldots < 1$, letting $m \to \infty$ in the above expression we get

$$P(E_v\ E_{v+1}\ E_{v+2}\ \ldots) \leq 1/2^{r-1}.$$

Thus (16) is true. (16) shows that

$$\overline{\lim_{j\to\infty}}\ e_{i_j} = a_2, \text{ a.s.}$$

Similarly, we can prove

$$\underline{\lim_{j\to\infty}}\ e_{i_j} = a_1, \text{ a.s.}$$

□

3. MEAN SQUARE CONSISTENCY OF $\hat{\beta}_n$

In this section we shall prove the following theorem:

Theorem 2. Under the assumption of theorem 1 we have

$$\lim_{n\to\infty} E(\|\hat{\beta}_n - \beta\|^2) = 0.$$

Proof. Define $h(\alpha) = \lim_{n\to\infty} [\max_{1\le i\le n} \alpha' x_i - \min_{1\le i\le n} \alpha' x_i]$. Because $\{x_i\}$ is bounded, $h(\alpha)$ is finite everywhere and further, it is easily seen from the assumption of theorem 1 that $h(\alpha) > 0$ when $\alpha \neq 0$, and $h(\alpha)$ is continuous everywhere on R^p. So the infremum of $h(\alpha)$ on the surface of unit sphere B is greater than zero : $\tilde{c} = \inf\{h(\alpha): \alpha \in B\} > 0$. Find on B a finite subset $\tilde{D} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q\}$ with the following property. For any $\alpha \in B$, there exists $i \in \{1, \ldots, q\}$ such that

$$\sup\{|(\alpha - \tilde{\alpha}_i)' x_j|: j=1, 2, \ldots\} < \tilde{c}/4. \tag{17}$$

Now we prove the existence of N such that

$$\max_{1\le i\le n} x_i'\alpha - \min_{1\le i\le n} x_i'\alpha > \tilde{c}/4, \text{ for every } \alpha \in B \text{ when } n \ge N.$$

In fact, by the definition of $\tilde{c}$, we can find out N such that

$$\max_{1\le k\le n} x_k'\alpha_i - \min_{1\le k\le n} x_k'\alpha_i > 3\tilde{c}/4, \quad i = 1, \ldots, q$$

when $n \ge N$. For any $\alpha \in B$ choose α_i such that (17) holds, then we have

$$\max_{1\le k\le n} x_k'\alpha - \min_{1\le k\le n} x_k'\alpha \ge (\max_{1\le k\le n} x_k'\alpha_i - \tilde{c}/4) - (\min_{1\le k\le n} x_k'\alpha_i + \tilde{c}/4)$$

$$= \max_{1\le k\le n} x_k'\alpha_i - \min_{1\le k\le n} x_k'\alpha_i - \tilde{c}/2 > 3\tilde{c}/4 - \tilde{c}/2 = \tilde{c}/4$$

when $n \ge N$. Now put $t = 2(a_2 - a_1)/\tilde{c}$. If $\ell \in R^p$ and $\|\ell - \beta\| > t$, then ℓ can be written as $\ell = \beta + \tilde{t}\alpha$, $\alpha \in B$, $\tilde{t} > t$. Thus for $n \ge N$, we get

$$Q_n(\underset{\sim}{\ell}) = \max_{1\le k\le n}(y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell}) - \min_{1\le k\le n}(y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell})$$

$$= \max_{1\le k\le n}(e_k - \tilde{t}\underset{\sim}{x}_k'\underset{\sim}{\alpha}) - \min_{1\le k\le n}(e_k - \tilde{t}\underset{\sim}{x}_k'\underset{\sim}{\alpha})$$

$$\ge \tilde{t}(\max_{1\le k\le n}\underset{\sim}{x}_k'\underset{\sim}{\alpha} - \min_{1\le k\le n}\underset{\sim}{x}_k'\underset{\sim}{\alpha}) - (\max_{1\le k\le n} e_k - \min_{1\le k\le n} e_k)$$

$$\ge \tilde{\tilde{t}}c/4 - (a_2 - a_1) \ge 2(a_2 - a_1) > Q_n(\underset{\sim}{\beta}).$$

This shows that $\{\hat{\underset{\sim}{\beta}}_N, \hat{\underset{\sim}{\beta}}_{N+1}, \ldots\}$ are uniformly bounded. Since $P(\lim_{n\to\infty} \hat{\underset{\sim}{\beta}}_n = \underset{\sim}{\beta}) = 1$ by theorem 1, the assertion of theorem 2 follows from the dominant convergent theorem.

4. ESTIMATION OF THE ENDPOINTS OF THE SUPPORT AND THE SECOND MOMENT

Define

$$\hat{a}_{1n} = \min_{1\le i\le n} (y_i - \underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}}_n), \quad \hat{a}_{2n} = \max_{1\le i\le n} (y_i - \underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}}_n).$$

Theorem 3. Under the assumption of theorem 1, $\hat{a}_{1n}$, $\hat{a}_{2n}$ are strong consistent estimates of a_1, a_2 respectively. (So $\hat{a}_{2n} - \hat{a}_{1n}$ is a strong consistent estimate of $a_2 - a_1$.)

Proof. Take $\hat{a}_{2n}$ for instance. Because $\lim_{n\to\infty} \max_{1\le i\le n} e_i \ge \overline{\lim}_{n\to\infty} e_n$, by (3), we have $P(\lim_{n\to\infty} \max_{1\le i\le n} e_i = a_2) = 1$. Now

$$\max_{1\le i\le n} (y_i - \underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}}_n) = \max_{1\le i\le n} (y_i - \underset{\sim}{x}_i'\underset{\sim}{\beta} - \underset{\sim}{x}_i'(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}))$$

$$= \max_{1\le i\le n} (y_i - \underset{\sim}{x}_i'\underset{\sim}{\beta}) + J_n = \max_{1\le i\le n} e_i + J_n$$

where $|J_n| \le \sup \|\underset{\sim}{x}_i\| \, \|\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}\|$. Because $\lim_{n\to\infty} \hat{\underset{\sim}{\beta}}_n = \underset{\sim}{\beta}$, a.s. in view of theorem 1 and $\{\underset{\sim}{x}_i\}$ is bounded, we derive that $\lim_{n\to\infty} J_n = 0$, a.s. Combining this with $P(\lim_{n\to\infty} \max_{k\le i\le n} e_i = a_2) = 1$, we get $\lim \hat{a}_{2n} = a_2$, a.s.

Now suppose that $e_1, e_2, \ldots$ is a strictly stationary ergodic sequence. As an estimate of Ee_1^2, we use $\hat{S}_n = 1/n \sum_{i=1}^{n} (y_i - \underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}}_n)^2$.

Theorem 4. Suppose that $\{e_i\}$ is a strictly stationary ergodic sequence. Assume that the conditions of theorem 1 are true. Then

$$\lim_{n\to\infty} \hat{S}_n = Ee_1^2, \text{ a.s.} \tag{18}$$

Proof. We have

$$\hat{S}_n = \frac{1}{n}\sum_{i=1}^{n}(y_i - \underset{\sim}{x}_i'\underset{\sim}{\beta} - \underset{\sim}{x}_i'(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}))^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}[\underset{\sim}{x}_i'(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})]^2 - \frac{2}{n}\sum_{i=1}^{n} e_i\underset{\sim}{x}_i'(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}) + \frac{1}{n}\sum_{i=1}^{n} e_i^2$$

$$= J_1 + J_2 + J_3. \tag{19}$$

In view of ergodicity of the sequence [4], we gain

$$\lim_{n\to\infty} J_3 = Ee_1^2, \text{ a.s.} \tag{20}$$

Write $M = \sup_i \|\underset{\sim}{x}_i\|$, then $M < \infty$. Hence

$$J_2 \le M^2\|\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}\| \to 0, \text{ a.s.} \tag{21}$$

Finally, $\|e_i\underset{\sim}{x}_i'(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})\| \le (|a_2| + |a_1|)M\|\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}\|$, so we obtain

$$J_1 \to 0, \text{ a.s.} \tag{22}$$

By (19)-(22), (18) follows. □

5. STRONG CONSISTENCY OF $\bar{\beta}_n$

In this section we shall prove the following theorem:

Theorem 5. Suppose that each e_i has a support included in the bounded intervals $[a_1, a_2]$ and for any subsequence $\{e_{i_j}: j = 1, 2, \ldots\}$ of $\{e_i\}$ we have

$$\overline{\lim_{j\to\infty}} e_{i_j} = a_2, \quad \underline{\lim_{j\to\infty}} e_{i_j} = a_1, \quad \text{a.s.} \tag{23}$$

Also suppose that $\{\underset{\sim}{x}_i\}$ is bounded, and for any nonzero p-vector $\underset{\sim}{d}$

$$\overline{\lim} \; \underset{\sim}{x}_n' \underset{\sim}{d} > 0.$$

Then

$$\lim \bar{\beta}_n = \beta, \quad \text{a.s.} \tag{24}$$

Proof. Without losing generality, we suppose that $|a_1| < a_2$, othersise consider the model $-y_n = \underset{\sim}{x}'_n(-\beta) - e_n$.

Several preliminary facts are in order.

(I) Define a function

$$\bar{g}(\underset{\sim}{\alpha}) = \overline{\lim} \; \underset{\sim}{x}_n'\underset{\sim}{\alpha}$$

on R^p. Then $\bar{g}(\underset{\sim}{\alpha})$ is continuous on R^p, and $\bar{g}(\underset{\sim}{\alpha}) > 0$ when $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$.

The second conclusion follows directly from the assumption of the theorem, and the first is an easy consequence of the boundedness of $\{\underset{\sim}{x}_i\}$.

Denote by B the surface of the unit sphere in R^p, i.e. $B = \{\underset{\sim}{\alpha}: \|\underset{\sim}{\alpha}\| = 1\}$. Because B is compact, $\bar{g}$ is continuous, and $\bar{g}(\underset{\sim}{\alpha}) > 0$ for $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$, we have

$$\bar{c} \triangleq \inf_{\underset{\sim}{\alpha}\in B} \bar{g}(\underset{\sim}{\alpha}) > 0. \tag{25}$$

(II) Define B and $\bar{c}$ as above, then we can find a positive integer $\bar{m}$ and a subset $\bar{D} = \{\underset{\sim}{\bar{\alpha}}_1, \ldots, \underset{\sim}{\bar{\alpha}}_{\bar{m}}\}$ of B such that for any $\underset{\sim}{\alpha} \in B$, there exists $\underset{\sim}{\bar{\alpha}}_i \in \bar{D}$ with the property

$$\sup\{|(\underset{\sim}{\alpha} - \underset{\sim}{\bar{\alpha}}_i)'\underset{\sim}{x}_j| : j = 1, 2, \ldots\} < \bar{c}/3. \tag{26}$$

The proof has been given in §2.

For every $\underset{\sim}{\bar{\alpha}}_i \in \bar{D}$, we can find out a subsequence $\{\underset{\sim}{x}_{ij}\}_{j=1}^{\infty}$ of $\{\underset{\sim}{x}_n\}$ such that

$$\lim_{j\to\infty} \underset{\sim}{\bar{\alpha}}_i'\underset{\sim}{x}_{ij} = \overline{\lim} \, \underset{\sim}{\bar{\alpha}}_i'\underset{\sim}{x}_n, \quad i = 1, 2, \ldots, \bar{m}.$$

Denote the values of y and e corresponding to $\underset{\sim}{x}_{ij}$ by y_{ij}, e_{ij}, i.e. if $\underset{\sim}{x}_{ij} = \underset{\sim}{x}_k$, then $y_{ij} = y_k$, $e_{ij} = e_k$. Put

$$F_i = \{\overline{\lim_{j\to\infty}} \, e_{ij} = a_2\}$$
$$F = \bigcap_{i=1}^{\bar{m}} F_i. \tag{27}$$

(III) $P(F) = 1$.

In fact, since $P(F_i) = 1$, $i = 1, \ldots, \bar{m}$ by (23), we get the result.

Employ the symbol $\bar{\omega}$ to represent a real number sequence $(\bar{c}_1, \bar{c}_2, \ldots)$, $a_1 \le \bar{c}_i \le a_2$, $i = 1, 2, \ldots$. The event F determined by (27) can be viewed as a subset $\tilde{F}$ of the $\bar{\omega}$-space, i.e. $F = \{(e_1, e_2, \ldots) \in \tilde{F}\}$.

(IV) For any $\varepsilon > 0$, $t \ge \varepsilon$, $\underset{\sim}{\bar{\alpha}}_i \cdot D$, put $\underset{\sim}{d}_t = \underset{\sim}{\beta} - t\underset{\sim}{\bar{\alpha}}_i$. Then for any $\omega \in \tilde{F}$, there exists $N_i(\bar{\omega})$ such that

$$\max_{1\le k\le n} |y_k(\bar{\omega}) - \underset{\sim}{x}_k'\underset{\sim}{d}_t| \ge a + \frac{2}{3}t\bar{c} \tag{28}$$

when $n \ge N_i(\bar{\omega})$.

Proof. For simplicity of notation, we shall use y_k, e_k to express $y_k(\bar{\omega})$ and $e_k(\bar{\omega})$, when

$$y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t = \underset{\sim}{x}_k'\underset{\sim}{\beta} + e_k - \underset{\sim}{x}_k'(\underset{\sim}{\beta} - t\underset{\sim}{\bar{\alpha}}_i) = e_k + t\underset{\sim}{x}_k'\underset{\sim}{\bar{\alpha}}_i. \tag{29}$$

Take $\bar{\eta} = \min\{a_2, \frac{\bar{c}\epsilon}{3(\epsilon+1)}\}$. Since $\overline{\lim}_{j\to\infty} e_{ij} = a_2$ and $\lim_{j\to\infty} \underset{\sim}{\alpha}_i'\underset{\sim}{x}_{ij} = \overline{\lim}_{n\to\infty} \underset{\sim}{\alpha}_i'\underset{\sim}{x}_n$, we can find $k = k(\bar{\eta},\bar{\omega})$ such that

$$e_{ik} > a_2 - \bar{\eta} \tag{30}$$

$$\underset{\sim}{\alpha}_i'\underset{\sim}{x}_{ik} > \bar{c} - \bar{\eta}. \tag{31}$$

Take $N_i = N_i(\bar{\eta},\bar{\omega})$ large enough so that when $n \geq N_i$ we have

$$\underset{\sim}{x}_{ik} \in \{\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n\}.$$

Then by (29)-(31), when $n \geq N_i$ we have

$$\begin{aligned}
&\max_{1\leq k\leq n} |y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t| \geq |e_{ik} + t\underset{\sim}{x}_{ik}'\underset{\sim}{\bar{\alpha}}_i| \\
&> a_2 - \bar{\eta} + t(\bar{c} - \bar{\eta}) = a_2 + t\bar{c} - (1 + t)\bar{\eta} \\
&\geq a_2 + \tfrac{2}{3}t\bar{c}
\end{aligned}$$

since $\frac{\bar{c}t}{3(t+1)} \geq \frac{\bar{c}\epsilon}{3(\epsilon+1)}$ for $t \geq \epsilon$.

This proves the assertion.

Now turn to the proof of Theorem 5. Fix $\bar{\omega} \in \tilde{F}$ and $\epsilon > 0$, and again use y_i, e_i, N_i, etc. to express $y_i(\bar{\omega})$, $e_i(\bar{\omega})$, $N_i(\bar{\omega})$. Put $N = N(\bar{\omega}) = \max_{1\leq i\leq n} N_i(\bar{\omega})$. Suppose that $\underset{\sim}{\ell} \in R^p$ satisfies $\|\underset{\sim}{\ell} - \underset{\sim}{\beta}\| \geq \epsilon$, then $\underset{\sim}{\ell}$ can be written as $\underset{\sim}{\ell} = \underset{\sim}{\beta} - t\underset{\sim}{\alpha}$, $t \geq \epsilon$, $\underset{\sim}{\alpha} \in B$. By (II) we can choose $\underset{\sim}{\bar{\alpha}}_i \in \bar{D}$ so that (26) is true. Remember that $\underset{\sim}{d}_t = \underset{\sim}{\beta} - t\underset{\sim}{\bar{\alpha}}_i$, we have

$$y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell} = y_k - \underset{\sim}{x}_k'(\underset{\sim}{\beta} - t\underset{\sim}{\alpha}) = y_k - \underset{\sim}{x}_k'(\underset{\sim}{d}_t + t(\overline{\underset{\sim}{\alpha}}_i - \underset{\sim}{\alpha}))$$

$$= y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t - t\underset{\sim}{x}_k'(\underset{\sim}{\alpha} - \overline{\underset{\sim}{\alpha}}_i).$$

Then

$$|y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell}| \geq |y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t| - t\overline{c}/3.$$

From this inequality and (28), we have

$$\max_{1\leq k\leq n} |y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell}| \geq \max_{1\leq k\leq n} |y_k - \underset{\sim}{x}_k'\underset{\sim}{d}_t| - t\overline{c}/3$$

$$> a_2 + \frac{2}{3}t\overline{c} - t\overline{c}/3 = a_2 + \frac{1}{3}t\overline{c}$$

for $n \geq N$. On the other hand, we know that

$$\max_{1\leq k\leq n} |y_k - \underset{\sim}{x}_k'\underset{\sim}{\beta}| = \max_{1\leq k\leq n} |e_k| \leq a_2.$$

So we have $\overline{Q}_n(\underset{\sim}{\ell}) > \overline{Q}_n(\underset{\sim}{\beta})$ for $n \geq N$ and $\|\underset{\sim}{\ell} - \underset{\sim}{\beta}\| \geq \epsilon$. According to the definitions of $\overline{\underset{\sim}{\beta}}_n$, we have $\|\overline{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}\| < \epsilon$ when $n \geq N$. Since $\epsilon > 0$ is arbitrary, this proves that $\lim \overline{\underset{\sim}{\beta}}_n = \underset{\sim}{\beta}$ when $\overline{\omega} \in \tilde{F}$. The proof of Theorem 5 is completed in view of (III). □

6. MEAN SQUARE CONSISTENCY OF $\overline{\beta}_n$

In this section we shall prove the following theorem:

Theorem 6. Under the assumption of theorem 5, we have

$$\lim_{n\to\infty} E(\|\overline{\beta}_n - \beta\|^2) = 0.$$

Proof. Without losing generality, we suppose that $|a_1| < a_2$.

Define $\overline{h}(\alpha) = \lim_{n\to\infty} \max_{1\le i\le n} \alpha' x_i$. Because $\{x_i\}$ is bounded, $\overline{h}(\alpha)$ is finite everywhere and further it is easily seen from the assumption of theorem 5 that $\overline{h}(\alpha) > 0$ when $\alpha \neq 0$, and $\overline{h}(\alpha)$ is continuous everywhere on R^p. So the infimum of $\overline{h}(\alpha)$ on the surface of unit sphere B is greater than zero: $s = \inf\{\overline{h}(\alpha) : \alpha \in B\} > 0$. Find on B a finite subset $G = \{\beta_1, \ldots, \beta_r\}$ with the following property: For any $\alpha \in B$, there exists $i \in \{1, \ldots, r\}$ such that

$$\sup\{|(\alpha - \beta_i)' x_j| : j = 1, 2, \ldots\} < s/4. \tag{32}$$

Now we prove the existence of N such that

$$\max_{1\le i\le n} x_i'\alpha > s/4$$

for every $\alpha \in B$ when $n \ge N$.

In fact, by the definition of s, we can find out N such that

$$\max_{1\le k\le n} x_k'\alpha_i > 3s/4, \quad i = 1, \ldots, r$$

when $n \ge N$. For any $\alpha \in B$ choose α_i such that (32) holds. Then we have

$$\max_{1\le k\le n} x_k'\alpha = \max_{1\le k\le n} (x_k'\alpha_i + x_k'(\alpha - \alpha_i))$$

$$\ge \max_{1\le k\le n} x_k'\alpha_i - s/4 > s/2$$

when $n \ge N$. Now put $t = 4a_2/s$. If $\ell \in R^p$ and $\|\ell - \beta\| > t$, then ℓ can

be written as $\underset{\sim}{\ell} = \underset{\sim}{\beta} - \tilde{t}\underset{\sim}{\alpha}$, $\underset{\sim}{\alpha} \in B$, $\tilde{t} > t$. So that when $n \geq N$, we get

$$\overline{Q}_n(\underset{\sim}{\ell}) = \max_{1 \leq k \leq n} |y_k - \underset{\sim}{x}_k'\underset{\sim}{\ell}| = \max_{1 \leq k \leq n} |e_k + \tilde{t}\underset{\sim}{x}_k'\underset{\sim}{\alpha}|$$

$$\geq \tilde{t} \max_{1 \leq k \leq n} |\underset{\sim}{x}_k'\underset{\sim}{\alpha}| - \max_{1 \leq k \leq n} |e_k|$$

$$\geq \tilde{t}s/2 - a_2 > a_2 \geq \overline{Q}_n(\underset{\sim}{\beta}).$$

This shows that $\{\overline{\underset{\sim}{\beta}}_N, \overline{\underset{\sim}{\beta}}_{N+1}, \ldots\}$ are uniformly bounded. Since $P(\lim_{n\to\infty} \overline{\underset{\sim}{\beta}}_n = \underset{\sim}{\beta}) = 1$ by theorem 5, the assertion of theorem 6 follows from the dominant convergent theorem. □

7. ESTIMATION OF THE ENDPOINT OF THE SUPPORT WHICH HAS THE MAXIMUM ABSOLUTE VALUE AND SECOND MOMENT

Define $\hat{a}_n = \max_{1\le i\le n} |y_i - \underset{\sim}{x}_i'\overline{\underset{\sim}{\beta}}_n|$.

Theorem 7. Under the assumption of theorem 5, $\hat{a}_n$ is a strong consistent estimate of $\max\{|a_1|, |a_2|\}$.

Proof. Without losing generality, we suppose that $|a_1| < a_2$.

Because $\lim_{n\to\infty} \max_{1\le i\le n} e_i \ge \overline{\lim}\, e_n$, by (23) we have $P(\lim_{n\to\infty} \max_{1\le i\le n} e_i = a_2) = 1$.

Now

$$\max_{1\le i\le n} |y_i - \underset{\sim}{x}_i'\overline{\underset{\sim}{\beta}}_n| = \max_{1\le i\le n} |y_i - \underset{\sim}{x}_i'\underset{\sim}{\beta} + \underset{\sim}{x}_i'(\underset{\sim}{\beta} - \overline{\underset{\sim}{\beta}}_n)|$$

$$= \max_{1\le i\le n} |y_i - \underset{\sim}{x}_i'\underset{\sim}{\beta}| + J_n = \max_{1\le i\le n} e_i + J_n$$

where $|J_n| \le \sup\|\underset{\sim}{x}_i\|\, \|\overline{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}\|$. Because $\lim_{n\to\infty} \overline{\underset{\sim}{\beta}}_n = \underset{\sim}{\beta}$, a.s. and $\{\underset{\sim}{x}_i\}$ is bounded, we derive that $\lim_{n\to\infty} J_n = 0$, a.s. Combining this with $P(\lim_{n\to\infty} \max_{1\le i\le n} e_i = a_2) = 1$, we get $\lim_{n\to\infty} \hat{a}_n = a_2$, a.s.

Now suppose that $e_1, e_2, \ldots$ is a strictly stationary ergodic sequence. As an estimate of Ee_1^2, we use $\overline{s}_n = \frac{1}{n}\sum_{i=1}^{n} (y_i - \underset{\sim}{x}_i'\overline{\underset{\sim}{\beta}}_n)^2$.

Theorem 8. Suppose that $\{e_i\}$ is a strictly stationary ergodic sequence. Assume that the conditions of Theorem 5 are true. Then

$$\lim \overline{s}_n = Ee_1^2, \text{ a.s.}$$

If we substitute $\hat{\beta}_n$ by $\overline{\beta}_n$, the proof is the same as the proof of Theorem 4.

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