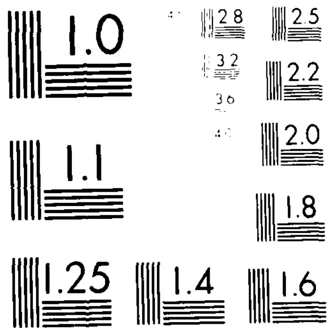
AD-A169 970 A NEGATIVE RESULT ABOUT SOME CONCEPTS OF NEGATIVE DEPENDENCE(U) FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS K JOAG-DEV ET AL JUL 85 AFOSR-TR-86-0454 1/1

UNCLASSIFIED F49620-85-C-0007 F/G 12/1 NL



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T DOCUMENTATION PAGE

SSION NO. AFOSR-TR- 86-0454	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and subtitle) A NEGATIVE RESULT ABOUT SOME CONCEPTS OF NEGATIVE DEPENDENCE	5. TYPE OF REPORT & PERIOD COVERED Technical 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Kumar Joag-Dev and Frank Proschan	8. CONTRACT OR GRANT NUMBER(s) AFOSR F49620-85-C-0007
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Florida State University Tallahassee, FL 32306-3033	10. PROGRAM ELEMENT, PROJECT, TASK AREA, & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS The U.S. Air Force Air Force Office of Scientific Research Bolling Air Force Base, DC 20332	12. REPORT DATE July 1985 13. NUMBER OF PAGES 6
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)

unlimited distribution

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS

Qualitative dependence, negative dependence, PF_2 functions, pairwise dependence, conditional negative dependence.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The key result is:

Theorem. Let X_1, X_2 be independent random variables and suppose there exist real numbers c, t_1, t_2 such that $t_2 > t_1$ and

$$P[X_1 \geq c | X_1 + X_2 = t_1] > P[X_1 \geq c | X_1 + X_2 = t_2],$$

where conditioning events have positive probability. Then there exists a random variable X_3 independent of X_1, X_2 such that the conditional distribution of (X_1, X_2, X_3) given the sum $\sum_1^3 X_i$ is not pairwise NQD.

Other negative results concerning negative dependence are presented.

AFOSR-TR- 86-0454

A NEGATIVE RESULT ABOUT SOME CONCEPTS OF NEGATIVE DEPENDENCE

by

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FSU Statistics Report M703

AFOSR Technical Report No. 85-179

July, 1985

The Florida State University
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Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF under Grant ~~AFOSR 84-0208~~[1] and ~~AFOSR~~ 85-C-0007[2].
F49620

Key words: Qualitative dependence, negative dependence, PF_2 functions, pairwise dependence, conditional negative dependence.

AMS Classification Nos.: Primary 62H05; Secondary 62H10.

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Abstract

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Theorem. Let X_1, X_2 be independent random variables and suppose there exist real numbers c, t_1, t_2 such that $t_2 > t_1$ and

$$P[X_1 \geq c \mid X_1 + X_2 = t_1] > P[X_1 \geq c \mid X_1 + X_2 = t_2],$$

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Other negative results concerning negative dependence are presented.

1. Introduction.

Suppose Peter and Paul inherit a fortune. If X_1, X_2 are their shares, then it is clear that (X_1, X_2) should be 'negatively dependent' according to every reasonable notion of negative dependence. We want to consider the question of a multivariate analog of this simple bivariate fact. For example, suppose X_1, X_2, X_3 are independent random variables representing the shares of Peter, Paul and Mary. Suppose that the sum $X_1 + X_2 + X_3$ is t. To what extent does the conditional distribution of (X_1, X_2, X_3) exhibit negative dependence? If X_i possess log concave density (also known as PF_2 condition) then utilizing a monotonicity result of Efron (1965), Joag-Dev and Proschan (1983) show that the above distribution does satisfy a strong negative dependence relation called 'negative association' (NA). Other concepts of negative dependence such as 'negative orthant dependence', 'reverse rule', etc. are established for a long list of multivariate distributions, such as multinomial, Dirichlet, hypergeometric, etc., by Block, Savits and Shaked (1982), Ebrahimi and Ghosh (1981), and others. Most of these examples can be perceived as the conditional distributions, obtained by fixing the sum of the independent PF_2 random variables. Note that assuming only finite variance, if the random variables are independent and identically distributed, then clearly the conditional covariance of every pair is negative when the total sum is fixed. A natural question would be: does the conditional joint distribution exhibit a condition such as pairwise negative quadrant dependence? Note that although this condition is stronger than negative covariance, it is weaker than 'negative upper orthant dependence' (NUOD) or 'negative lower orthant dependence' (NLOD) which in turn are weaker than NA (see the next section for the definitions). We show that without the monotonicity resulting from PF_2, such a negative dependence condition does not hold.

2. Results.

Next we define some of the standard notions of negative dependence.

Let $\underset{\sim}{Y} = (Y_1, \ldots, Y_k)$ be a k vector with real valued component random variables. The vector $\underset{\sim}{Y}$ is said to possess 'negative association' (NA) if for every partition of $\{1, 2, \ldots, n\}$ into A, $\bar{A}$ and every pair of co-ordinatewise nondecreasing functions f, g,

$$\text{(2.1)} \qquad \mathrm{COV}[f(Y_i,\ i \in A),\ g(Y_j,\ j \in \bar{A})] \le 0.$$

Condition (2.1) is stronger than 'negative upper orthant dependence' (NUOD) which requires

$$\text{(2.2)} \qquad P[Y_i \ge c_i,\ i = 1, \ldots, k] \le \prod_{i=1}^{k} P[Y_i \ge c_i],$$

for every set of constants $c_1, \ldots, c_k$. By reversing inequalities in the square brackets on both sides of (2.2) one obtains NLOD. It is easy to check that NA implies NLOD; however, between NUOD and NLOD neither implies the other for $k \ge 3$. For $k = 2$, the bivariate case, these two are equivalent and the condition is referred to as 'negative quadrant dependence' (NQD). It also follows that pairwise NQD condition is weaker than all the above.

Let X_i, $i = 1, \ldots, n$, be independent random variables from log concave densities. Efron (1965) shows that the conditional expectation, $E[g(X_1, \ldots, X_n) \mid \sum X_i = s]$ is nondecreasing in s, where g is an arbitrary co-ordinatewise nondecreasing function. This is the key tool in the proof given in Joag-Dev and Proschan (1983), to show NA for the conditional distribution of $\{X_i\}$ given $\sum X_i$.

<u>Theorem 2.1</u>. <u>Let</u> X_1, X_2 <u>be independent random variables and suppose there exist real numbers</u> c, t_1, t_2 <u>such that</u> $t_2 > t_1$ <u>and</u>

$$P[X_1 \geq c \mid X_1 + X_2 = t_1] > P[X_1 \geq c \mid X_1 + X_2 = t_2], \tag{2.3}$$

<u>where conditioning events have positive probability</u>. <u>Then there exists a random variable</u> X_3, <u>independent of</u> (X_1, X_2), <u>such that the conditional distribution of</u> (X_1, X_2, X_3) <u>given the sum</u> $\sum_{i=1}^{3} X_i$, <u>is not pairwise</u> NQD.

<u>Proof</u>. Define X_3 to be a binary random variable such that

$$P[X_3 = 0] = P[X_3 = t_2 - t_1] = \frac{1}{2}.$$

Let X_3 be independent of (X_1, X_2). Let the event $\sum_{i=1}^{3} X_i = t_2$ be denoted by A. The event A may be written as a disjoint union of A_1 and A_2 denoting the events $[X_1 + X_2 = t_1, X_3 = t_2 - t_1]$ and $[X_1 + X_2 = t_2, X_3 = 0]$ respectively. Using this notation, we have

$$P[X_1 \geq c \mid A] = \frac{P[A_1]}{P[A]} P[X_1 \geq c \mid A_1] + \frac{P[A_2]}{P[A]} P[X_1 \geq c \mid A_2]. \tag{2.4}$$

Using the independence of X_3 and (X_1, X_2) in (2.4), we get

$$P[X_1 \geq c \mid A] = \alpha P[X_1 \geq c \mid X_1 + X_2 = t_1] + (1 - \alpha) P[X_1 \geq c \mid X_1 + X_2 = t_2],$$

where $\alpha = P[A_1]/P[A]$, $0 < \alpha < 1$. Due to assumption (2.3) and the independence of X_3, it follows that

$$P[X_1 \geq c \mid X_3 = t_2 - t_1, X_1 + X_2 = t_1] > P[X_1 \geq c \mid A]. \tag{2.5}$$

However after some manipulation, we see that (2.5) is equivalent to

$$P[X_1 \geq c,\ X_3 \geq t_2 - t_1 | A] > P[X_1 \geq c | A] P[X_3 \geq t_2 - t_1 | A]. \tag{2.6}$$

Inequality (2.6) shows that conditionally, (X_1, X_3) is not NQD.

One may ask whether the assumption of a common distribution function will create NLOD. The following example shows that it does not. Let X_1, X_2, X_3 be independent random variables having a common discrete distribution on 0, 2, 3, with corresponding probabilities p_0, p_2, p_3 respectively. Let T denote the sum, $\sum_{i=1}^{3} X_i$. Now

$$P[X_i \leq 2,\ i = 1, 2, 3 | T = 6] = p_2^3/(3p_0p_3^2 + p_2^3),$$

while

$$P[X_1 \leq 2 | T = 6] = \frac{p_2^3 + p_3^2 p_0}{3p_0p_3^2 + p_2^3}.$$

Thus the NLOD condition would be violated if

$$\left(\frac{p_2^3}{3p_0p_3^2 + p_2^3}\right) > \left(\frac{p_2^3 + p_0p_3^2}{3p_0p_3^2 + p_2^3}\right)^3. \tag{2.7}$$

Put $a = p_2^3$, $b = p_0p_3^2$. Then condition (2.7) is equivalent to

$$\left(\frac{a}{a + 3b}\right) > \left(\frac{a + b}{a + 3b}\right)^3 \iff a(a + 3b)^2 > (a + b)^3 \iff 3a(a + 2b) > b^2. \tag{2.8}$$

The last inequality in (2.8) can easily be met when b is small or equivalently, p_2 is large. For example if $p_2 \geq \frac{1}{2}$ then it certainly holds.

3. Final Remark.

It seems that either a very strong negative dependence holds with the monotonicity condition while without it, even a somewhat weak condition does not hold. This brings out the crucial role played by the PF_2 property in conditional negative dependence.

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