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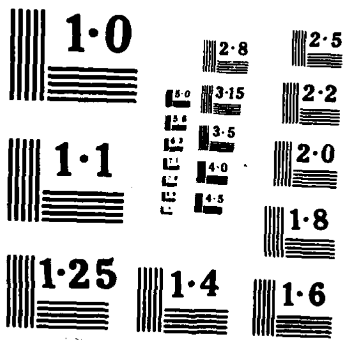
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A STRUCTURE THEOREM ON BIVARIATE POSITIVE QUADRANT DEPENDENT DISTRIBUTIONS AND TESTS FOR INDEPENDENCE IN TWO-WAY CONTINGENCY TABLES*

by

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ABSTRACT

In this paper, the set of all bivariate positive quadrant dependent distributions with fixed marginals is shown to be compact and convex. Extreme points of this convex set are enumerated in some specific examples. Applications are given in testing the hypothesis of independence against strict positive quadrant dependence in the context of ordinal contingency tables. Various procedures based upon certain functions of the eigenvalues of a random matrix are also proposed for testing for independence in two-way contingency table. The performance of some tests one of which is based on eigenvalues of a random matrix is compared.

AMS 1980 Subject Classifications: Primary 62H05; Secondary 62H17, 62H15

Key words and phrases: Asymptotic distributions, compact set, contingency tables, convex set, eigenvalues, extreme points, gamma ratio, hypothesis of independence, positive quadrant dependent distributions, power function.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the set of all bivariate positive quadrant dependent distributions with fixed marginals is shown to be compact and convex. Extreme points of this convex set are enumerated in some specific examples. Applications are given in testing the hypothesis of independence against strict positive quadrant dependence in the context of ordinal contingency tables. Various procedures based upon certain functions of the eigenvalues of a random matrix are also proposed for testing for independence in two-way contingency table. The

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) performance of some tests one of which is based on eigenvalues of a random matrix is compared.

1. INTRODUCTION

Cross-classified data having ordered categories arise in many investigations conducted by medical, physical, natural and social scientists. Statistical methods have been developed and continue to evolve to analyze such data. Many of these methods are tailored to answer specific questions and issues raised. For reviews of the literature in this area, the reader is referred to Agresti [1] and Goodman [6].

We begin with a general description of a problem tackled in this paper. Let $\mathcal{B}$ be the Borel σ - field on the real line, R and $\mathcal{B}\times\mathcal{B}$ the product σ - field on R×R. Let μ be a probability measure on $\mathcal{B}\times\mathcal{B}$ and μ_1 and μ_2 the corresponding marginal probability measures on $\mathcal{B}$, i.e., $\mu_1(B) = \mu(B\times R)$ and $\mu_2(B) = \mu(R\times B)$ for every B in $\mathcal{B}$. Following Lehmann [14], μ is said to be a <u>positive quadrant</u>

<u>dependent</u> if

$$\mu\{[c,\infty)\times[d,\infty)\} \geq \mu_1\{[c,\infty)\}\mu_2\{[d,\infty)\}$$

for every c,d in R. In the jargon of random vairables, the above notion can be rephrased as follows. Let X and Y be two random variables with some joint probability distribution function F. X and Y are said to be positive quadrant dependent if

$$P\{X \geq c,\ Y \geq d\} \geq P\{X \geq c\}\ P\{Y \geq d\}$$

for all c,d in R. For various properties of positive quadrant dependence, see Lehmann [14] or Eaton [3]. In this paper, we look at the notion of positive quadrant dependence from a global point of view. Let M_{PQD} denote the set of all positive quadrant dependent probability measures μ on $\mathcal{B}\times\mathcal{B}$. It is natural to think along the following lines. If M_{PQD} is a convex set and compact in some decent topology, then the set of extreme points of M_{PQD} will be non-empty. See Phelps [20]. Moreover, every member of M_{PQD} can be expressed as a mixture (in some sense) of extreme points of M_{PQD}. There are certain properties of distributions which are preserved under mixtures. Under these circumstances, it suffices to examine extensively the extreme points so as to make comments on the members of M_{PQD}. But this line of reasoning fails since M_{PQD} is not a convex set as the the following example demonstrates.

Let μ be a probability measure on $\mathcal{B} \times \mathcal{B}$ with support contained in $\{(1,1), (1,2), (2,1), (2,2)\}$. Such a probability measure can be written as

μ =

	1	2	
1	p_{11}	p_{i2}	p_1
2	p_{21}	p_{22}	p_2
	q_1	q_2	1 ,

where $p_{ij} = \mu\{(i,j)\}$, $i = 1,2$; $j = 1,2$; $p_i = \mu_1(\{i\})$, $i = 1,2$ and $q_j = \mu_2(\{j\})$, $j = 1,2$. Then $\mu \in M_{PQD}$ if and only if $p_2 q_2 \leq p_{22} \leq p_2 \wedge q_2$, where $p_2 \wedge q_2$ denotes the minimum of p_2 and q_2. For the desired example, let μ and η be the probability measures with the same support $\{(1,1), (1,2), (2,1), (2,2)\}$ given by

μ =

	1	2	
1	¼	¼	½
2	¼	¼	½
	½	½	1

η =

	1	2	
1	2/9	1/9	1/3
2	4/9	2/9	2/3
	2/3	1/3	1 .

μ and η are positive quadrant dependent but $\frac{1}{2}\mu + \frac{1}{2}\eta$ is not.

We can identify some natural subsets of M_{PQD} as convex sets. Let λ and ν be two probability measures on $\mathcal{B}$. Let $M_{PQD}(\lambda,\nu)$ be the collection of all probability measures μ in M_{PQD} such that $\mu_1 = \lambda$ and $\mu_2 = \nu$, i.e.,

$$M_{PQD}(\lambda,\nu) = \{\mu \in M_{PQD};\ \mu_1 = \lambda \text{ and } \mu_2 = \nu\}.$$

In Section 2, we show that $M_{PQD}(\lambda,\nu)$ is a compact convex set in the weak topology on the space of all probability measures M on $\mathcal{B} \times \mathcal{B}$. Using this result, one obtains a decomposition of M_{PQD} as

$$M_{PQD} = \bigcup_{\lambda} \bigcup_{\nu} M_{PQD}(\lambda,\nu),$$

where the union is taken over all probability measures λ, ν on $\mathcal{B}$.

In Section 3, we concentrate on the case when both λ and ν have finite support. We describe a method of enumerating all extreme points of $M_{PQD}(\lambda,\nu)$ with the help of some examples. In Section 4, using the structure of $M_{PQD}(\lambda,\nu)$, we compare the performance of some tests for testing independence against strict positive quadrant dependence.

2. Main Results

In this section, we show that for any two probability measures λ and ν on $\mathcal{B}$, $M_{PQD}(\lambda,\nu)$ is compact and convex.

We need the following definitions and results in this connection.

Let (X,d) be a Polish space, i.e., a complete separable metric space. Let $\mathcal{B}_X$ be the Borel σ-field on X and M_X the space of all probability measures on $\mathcal{B}_X$. M_X is equipped with weak topology.

Definition 1. A subset S of M_X is said to be uniformly tight if for every $\varepsilon > 0$ there exists a compact subset C of X such that

$$\mu(C) > 1-\varepsilon \quad \text{for every } \mu \text{ in } S.$$

The following is known as Prohorov's theorem.

Proposition 2 A subset S of M_X is relatively compact if and only if S is uniformly tight. S is compact if and only if S is closed and uniformly tight.

Proof. See Billingsley [, Theorems 6.1 and 6.2, p.37].

Theorem 3 Let $M(\lambda,\nu)$ be the collection of all probability measures μ on $\mathcal{B} \times \mathcal{B}$ such that $\mu_1 = \lambda$ and $\mu_2 = \nu$. Then $M(\lambda,\nu)$ is compact.

Proof. It is obvious that $M(\lambda,\nu)$ is a closed subset of M, the space of all probability measures on $\mathcal{B} \times \mathcal{B}$. We show that $M(\lambda,\nu)$ is uniformly tight. Let $\varepsilon > 0$. There exist compact subsets C_1 and C_2 of R such that $\lambda(C_1^c) < \varepsilon/2$ and $\nu(C_2^c) < \varepsilon/2$. $C_1 \times C_2$ is a compact subset of $R \times R$. Let $\mu \in M(\lambda,\nu)$. Then

$$\mu[(C_1 x C_2)^c] \le \mu(C_1^c x R \cup R x C_2^c)$$

$$\le \mu(C_1^c x R) + \mu(R x C_2^c)$$

$$= \mu_1(C_1^c) + \mu_2(C_2^c)$$

$$= \lambda(C_1^c) + \nu(C_2^c)$$

$$< \varepsilon .$$

This completes the proof in view of Proposition 2.

The following result is the main result of this section.

Theorem 4 For any given probability measures λ and ν on $\mathcal{B}$, $M_{PQD}(\lambda,\nu)$ is compact and convex.

Proof. $M_{PQD}(\lambda,\nu)$ is a closed subset of $M(\lambda,\nu)$ follows from the following observation. Let μ^n, $n \ge 1$ be a sequence in $M_{PQD}(\lambda,\nu)$ converging weakly to a μ in $M(\lambda,\nu)$. Then for any c,d in R, (See Billingsley [2, p. 11]),

$$\mu\{[c,\infty)x[d,\infty)\} \ge \limsup_{n\to\infty} \mu^n\{[c,\infty)x[d,\infty)\}$$

$$\ge \lambda\{[c,\infty)\}\nu\{[d,\infty)\}.$$

Hence $\mu \in M_{PQD}(\lambda,\nu)$. This implies that $M_{PQD}(\lambda,\nu)$ is compact. We now show that $M_{PQD}(\lambda,\nu)$ is convex. Let $\mu,\eta \in M_{PQD}(\lambda,\nu)$ and $0 \le \alpha \le 1$. Then for any c,d in R,

$$(\alpha\mu + (1-\alpha)\eta)\{[c,\infty)\times[d,\infty)\} - \lambda\{[c,\infty)\}\nu\{[d,\infty)\}$$

$$= \alpha\mu\{[c,\infty)\times[d,\infty)\} + (1-\alpha)\eta\{[c,\infty)\times[d,\infty)\} -$$

$$\lambda\{[c,\infty)\}\nu\{[d,\infty)\}$$

$$\geq \alpha\lambda\{[c,\infty)\} \ \nu\{[d,\infty)\} + (1-\alpha)\lambda\{[c,\infty)\}\nu\{[d,\infty)\}$$

$$- \lambda\{[c,\infty)\}\nu\{[d,\infty)\} = 0.$$

Consequently, $\alpha\mu + (1-\alpha)\eta \in M_{PQD}(\lambda,\nu)$. This completes the proof.

3. Extreme Points

In this section, we assume that the support of λ is $\{1,2,\cdots,m\}$ and that of ν is $\{1,2,\cdots,n\}$. Let $p_i = \mu(\{i\})$, $i = 1,2,\cdots,m$ and $q_j = \nu(\{j\})$, $j = 1,2,\cdots,n$. In this case, we use the suggestive notation $M_{PQD}(p_1,p_2,\cdots,p_m;\ q_1,q_2,\cdots,q_n)$ for $M_{PQD}(\lambda,\nu)$. Any μ in $M_{PQD}(\lambda,\nu)$ can be written in the following form

$$\begin{array}{cccc|c}
\hline
p_{11} & p_{12} & \cdots & p_{1n} & p_1 \\
p_{21} & p_{22} & \cdots & p_{2n} & p_2 \\
\cdots & \cdots & \cdots & & \vdots \\
p_{m1} & p_{m2} & \cdots & p_{mn} & p_m \\
\hline
q_1 & q_2 & \cdots & q_n & 1
\end{array},$$

where $p_{ij} = \mu\{(i,j)\}$, $i = 1,2,\cdots,m$; $j = 1,2,\cdots,n$. In other words, $M_{PQD}(p_1,p_2,\cdots,p_m;\ q_1,q_2,\cdots,q_n)$ is the collection of all matrices (p_{ij}) of order $m \times n$ such that each $p_{ij} \geq 0$, row sums $p_1,p_2,\cdots,p_m$, column sums $q_1,q_2,\cdots,q_n$ and the joint distribution is positive quadrant dependent. The compact convex set $M_{PQD}(p_1,p_2,\cdots,p_m;\ q_1,q_2,\cdots,q_n)$ has a finite number of extreme points. We now describe a method of enumerating the extreme points in some special cases for illustration from which the general technique can easily be perceived. As we shall see shortly in Section 4, the knowledge of extreme points has considerable bearing on the power of tests of independence against strict positive quadrant dependence.

Example 1. $m = 2$ and $n = 2$.

Let $p_1,p_2;\ q_1,q_2$ be specified such that each of p_1,p_2,q_1,q_2 is positive and $p_1 + p_2 = 1 = q_1 + q_2$. It can be easily verified that if a matrix

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

with non-negative entries, row sums p_1,p_2 and column sums q_1,q_2 belongs to $M_{PQD}(p_1,p_2;\ q_1,q_2)$, then

$$p_2q_2 \le p_{22} \le p_2 \wedge q_2.$$

Conversely, if the number p_{22} satisfies the above inequality, then the matrix

$$\begin{bmatrix} p_1-q_2+p_{22} & q_2-p_{22} \\ p_2-p_{22} & p_{22} \end{bmatrix}$$

belongs to $M_{PQD}(p_1,p_2;\ q_1,q_2)$. There are only two extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2)$. These are given by

$$\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \text{ and } \begin{bmatrix} q_1 & q_2-p_2 \\ 0 & p_2 \end{bmatrix} \quad \text{if} \quad p_2 \wedge q_2 = p_2,$$

$$\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \text{ and } \begin{bmatrix} p_1 & 0 \\ p_2-q_2 & q_2 \end{bmatrix} \quad \text{if} \quad p_2 \wedge q_2 = q_2.$$

Every member of $M_{PQD}(p_1,p_2;\ q_1,q_2)$ is a convex combination of these two extreme points.

<u>Example 2</u> $m = 2$ and $n = 3$.

Let p_1, p_2, q_1, q_2, q_3 be five positive numbers satisfying $p_1 + p_2 = 1 = q_1 + q_2 + q_3$. If the matrix (p_{ij}) belongs to $M_{PQD}(p_1, p_2; q_1, q_2, q_3)$, then

$$p_2 q_3 \leq p_{23} \leq p_2 \wedge q_3 \quad \text{and}$$

$$(p_2 q_2 + p_2 q_3) \vee p_{23} \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23}),$$

where $a \vee b$ indicates the maximum of the numbers a and b. Conversely, if p_{22} and p_{23} are two numbers satisfying the above inequalities, then the matrix

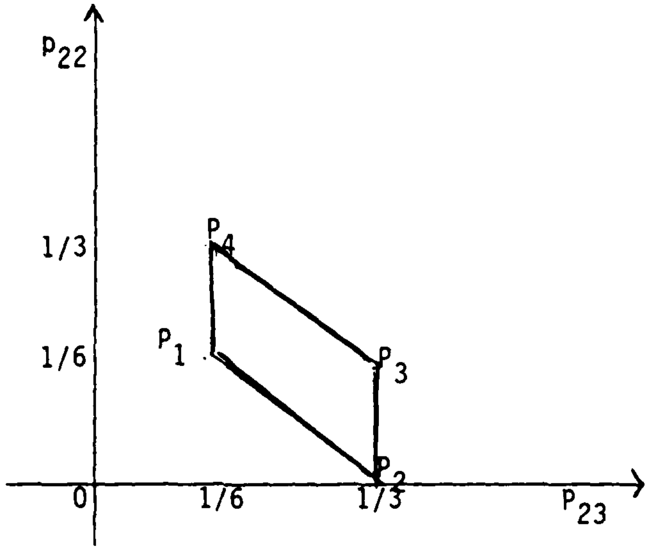
$$\begin{bmatrix} q_1 - p_2 + p_{22} + p_{23} & q_2 - p_{22} & q_3 - p_{23} \\ p_2 - p_{22} - p_{23} & p_{22} & p_{23} \end{bmatrix}$$

belongs to $M_{PQD}(p_1, p_2; q_1, q_2, q_3)$. The impact of this observation is that the numbers p_{22} and p_{23} in the matrix (p_{ij}) determine whether the matrix (p_{ij}) belongs to $M_{PQD}(p_1, p_2; q_1, q_2, q_3)$ or not. These two inequalities determine a simplex in the $p_{22} - p_{23}$ plane. As a simple illustration, take $p_1 = p_2 = \frac{1}{2}$; $q_1 = q_2 = q_3 = 1/3$. The determining inequalities are

$$1/6 \leq p_{23} \leq 1/3 \quad \text{and}$$

$$1/3 = p_{23} \vee 1/3 \leq p_{22} + p_{23} \leq \tfrac{1}{2} \wedge (1/3 + p_{23}) = \tfrac{1}{2}.$$

These inequalities determine the following simplex in the $p_{22} - p_{23}$ plane.



There are four extreme points of the set $M_{PQD}(1/2,1/2;\ 1/3,1/3,1/3)$ given by

$$\begin{bmatrix} 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 \end{bmatrix}, \quad \begin{bmatrix} 1/6 & 1/3 & 0 \\ 1/6 & 0 & 1/3 \end{bmatrix},$$

$$\begin{bmatrix} 1/3 & 1/6 & 0 \\ 0 & 1/6 & 1/3 \end{bmatrix}, \quad \begin{bmatrix} 1/3 & 0 & 1/6 \\ 0 & 1/3 & 1/6 \end{bmatrix}$$

corresponding to the four points P_1, P_2, P_3, P_4 respectively. Every member of $M_{PQD}(1/2,1/2;\ 1/3,1/3,1/3)$ is a convex combination of these four extreme points.

<u>Example 3</u> $m = 3$ and $n = 4$.

Let $p_1, p_2, p_3, q_1, q_2, q_3, q_4$ be seven positive numbers satisfying $p_1 + p_2 + p_3 = 1 = q_1 + q_2 + q_3 + q_4$. If $(p_{ij}) \in M_{PQD}(p_1, p_2, p_3; q_1, q_2, q_3, q_4)$, then

(1) $p_3 q_4 \leq p_{34} \leq p_3 \wedge q_4$,

(2) $p_{34} \vee (p_3 q_3 + p_3 q_4) \leq p_{33} + p_{34} \leq p_3 \wedge (q_3 + p_{34})$,

(3) $p_{34} \vee (p_2 q_4 + p_3 q_4) \leq p_{24} + p_{34} \leq q_4 \wedge (p_2 + p_{34})$,

(4) $(p_{33} + p_{34}) \vee (p_3 q_2 + p_3 q_3 + p_3 q_4) \leq p_{32} + p_{33} + p_{34} \leq p_3 \wedge (q_2 + p_{33} + p_{34})$,

(5) $(p_{33} + p_{24} + p_{34}) \vee (p_2 q_3 + p_2 q_4 + p_3 q_3 + p_3 q_4) \leq p_{23} + p_{24} + p_{33} + p_{34} \leq (p_2 + p_{33} + p_{34}) \wedge (q_3 + p_{24} + p_{34})$,

and

(6) $(p_{23} + p_{24} + p_{34} + p_{32} + p_{33}) \vee (p_2 q_2 + p_2 q_3 + p_2 q_4 + p_3 q_2 + p_3 q_3 + p_3 q_4) \leq$
$p_{22} + p_{23} + p_{24} + p_{32} + p_{33} + p_{34} \leq$
$(p_2 + p_{32} + p_{33} + p_{34}) \wedge (q_2 + p_{23} + p_{24} + p_{33} + p_{34})$.

Conversely, if $p_{34}, p_{33}, p_{24}, p_{32}, p_{23}, p_{22}$ are six numbers satisfying the above inequalities, then these six numbers determine uniquely a member of $M_{PQD}(p_1, p_2, p_3; q_1, q_2, q_3, q_4)$. We explain how to enumerate all extreme points of the convex set in the simple example $p_1 = p_2 = p_3 = 1/3$ and $q_1 = q_2 = q_3 = q_4 = 1/4$. (The technique in the general case is similar to this special example.) The six inequalities above now become

(1) $1/12 \leq p_{34} \leq 3/12$,

(2) $p_{34} \vee 2/12 \leq p_{33} + p_{34} \leq 4/12 \wedge (3/12 + p_{34}) = 4/12$,

(3) $p_{34} \vee 2/12 \leq p_{24} + p_{34} \leq 3/12 \wedge (4/12 + p_{34}) = 3/12$,

(4) $(p_{33} + p_{34}) \vee 3/12 \leq p_{32} + p_{33} + p_{34} \leq 4/12 \wedge (3/12 + p_{33} + p_{34}) = 4/12$,

(5) $(p_{33} + p_{24} + p_{34}) \vee 4/12 \leq p_{23} + p_{24} + p_{33} + p_{34} \leq (4/12 + p_{33} + p_{34}) \wedge (3/12 + p_{24} + p_{34}) = 3/12 + p_{24} + p_{34}$,

(6) $(p_{23} + p_{32} + p_{33} + p_{24} + p_{34}) \vee 6/12 \leq p_{22} + p_{23} + p_{24} + p_{32} + p_{33} + p_{34} \leq (4/12 + p_{32} + p_{33} + p_{34}) \wedge (3/12 + p_{23} + p_{24} + p_{33} + p_{34})$.

The first step in the determination of extreme points is to get rid of the maximum and minimum symbols by splitting some or all inequalities above. For example, inequality (1) can be written as $1/12 \leq p_{34} \leq 2/12$ and $2/12 \leq p_{34} \leq 3/12$. Inequality (3) can be written as $p_{34} \vee 2/12 \leq p_{33} + p_{34} \leq 3/12$ and $3/12 \leq p_{33} + p_{34} \leq 4/12$. The above set of inequalities are equivalent to the following four sets of inequalities.

I (1) $1/12 \leq p_{34} \leq 2/12$

(2) $2/12 \leq p_{33} + p_{34} \leq 3/12$

(3) $2/12 \leq p_{24} + p_{34} \leq 3/12$

(4) $3/12 \leq p_{32} + p_{33} + p_{34} \leq 4/12$

(5) Same as above

(6) Same as above

II (1) $1/12 \leq p_{34} \leq 2/12$

(2) $3/12 \leq p_{33} + p_{34} \leq 4/12$

(3) $2/12 \leq p_{24} + p_{34} \leq 3/12$

(4) $p_{33} + p_{34} \leq p_{32} + p_{33} + p_{34} \leq 4/12$

(5) Same as above

(6) Same as above

III (1) $2/12 \leq p_{34} \leq 3/12$

(2) $p_{34} \leq p_{33} + p_{34} \leq 3/12$

(3) $p_{34} \leq p_{24} + p_{34} \leq 3/12$

(4) $3/12 \leq p_{32} + p_{33} + p_{34} \leq 4/12$

(5) Same as above

(6) Same as above

IV (1) $2/12 \leq p_{34} \leq 3/12$

(2) $3/12 \leq p_{33} + p_{34} \leq 4/12$

(3) $p_{34} \leq p_{24} + p_{34} \leq 3/12$

(4) $p_{33} + p_{34} \leq p_{32} + p_{33} + p_{34} \leq 4/12$

(5) Same as above

(6) Same as above

The maximum and minimum symbols in inequalities (5) and (6) stay put in spite of splitting the inequalities (1) and (2). In order to neutralize these symbols, we introduce the following auxiliary inequalities.

(2,3) $p_{33} + p_{24} + p_{34} \leq 4/12$

or

$4/12 \leq p_{33} + p_{24} + p_{34}$

(2,3,4,5) $p_{23} + p_{32} + p_{33} + p_{24} + p_{34} \leq 6/12$

or

$6/12 \leq p_{23} + p_{32} + p_{33} + p_{24} + p_{34}$

(4,5) $4/12 + p_{32} + p_{33} + p_{34} \leq 3/12 + p_{23} + p_{24} + p_{33} + p_{34}$

or

$3/12 + p_{23} + p_{24} + p_{33} + p_{34} \leq 4/12 + p_{32} + p_{33} + p_{34}.$

Now, a choice of each of the auxiliary inequalities (2,3), (2,3,4,5) and (4,5) is appended to each set of the inequalities I, II, III and IV. This would generate 32 sets of inequalities equivalent to the four sets I, II, III and IV of inequalities. To save space, we will not reproduce these 32 sets of inequalities. A sample set of inequalities is produced below for further discussion.

<u>A sample set of inequalities chosen from 32 sets of inequlities described above</u>

(1) $2/12 \le p_{34} \le 3/12$

(2) $3/12 \le p_{33} + p_{34} \le 4/12$

(3) $p_{34} \le p_{24} + p_{34} \le 3/12$

(2,3) $p_{33} + p_{24} + p_{34} \le 4/12$

(4) $p_{33} + p_{34} \le p_{32} + p_{33} + p_{34} \le 4/12$

(2,3,4,5) $6/12 \le p_{23} + p_{32} + p_{33} + p_{24} + p_{34}$

(5) $4/12 \le p_{23} + p_{24} + p_{33} + p_{34} \le 3/12 + p_{24} + p_{34}$

(4,5) $4/12 + p_{32} + p_{33} + p_{34} \le 3/12 + p_{23} + p_{24} + p_{33} + p_{34}$

(6) $p_{23} + p_{32} + p_{33} + p_{24} + p_{34} \le p_{22} + p_{23} + p_{24} + p_{32} + p_{33} + p_{34}$

$$\le 4/12 + p_{32} + p_{33} + p_{34}.$$

The above set of inequalities is obtained from the set IV of inequalities by appending the first choice of (2,3), the second choice of (2,3,4,5) and the first choice of (4,5).

In order to obtain a member of $M_{PQD}(1/3,1/3,1/3;\ 1/4,1/4,1/4,1/4)$ we proceed as follows. Set the central expression in each of the main six inequalities equal to the quantity either on the left or the right of the inequalities and then solve the system of equations thus arize in the unknowns $p_{34}, p_{33}, p_{24}, p_{32}, p_{23}, p_{22}$ making sure that the constraints imposed by the auxiliary inequalities are satisfied.These system of equations are easy to solve. The solution will give a member of $M_{PQD}(1/3,1/3,1/3;\ 1/4,1/4,1/4,1/4)$.

Generate members of $M_{PQD}(1/3,1/3,1/3;\ 1/4,1/4,1/4,1/4)$ by following the above procedure for each set of the 32 sets of inequalities. The set of extreme points of the convex set of interest is a subset of these solutions. There will be a large amount of duplicates and some of the solutions obtained are already convex combinations of other solutions. After considerable amount of weeding, we got the following matrices as the entire collection of extreme points of the convex set $M_{PQD}(1/3,1/3,1/3;\ 1/4,1/4,1/4,1/4)$.

1. $\begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 \\ 1/12 & 1/12 & 1/12 & 1/12 \\ 1/12 & 1/12 & 1/12 & 1/12 \end{bmatrix}$

2. $\begin{bmatrix} 2/12 & 0 & 1/12 & 1/12 \\ 0 & 2/12 & 1/12 & 1/12 \\ 1/12 & 1/12 & 1/12 & 1/12 \end{bmatrix}$

3. $\begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 \\ 2/12 & 0 & 1/12 & 1/12 \\ 0 & 2/12 & 1/12 & 1/12 \end{bmatrix}$

4. $\begin{bmatrix} 2/12 & 0 & 1/12 & 1/12 \\ 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 2/12 & 1/12 & 1/12 \end{bmatrix}$

5. $\begin{bmatrix} 3/12 & 0 & 0 & 1/12 \\ 0 & 1/12 & 2/12 & 1/12 \\ 0 & 2/12 & 1/12 & 1/12 \end{bmatrix}$

6. $\begin{bmatrix} 1/12 & 2/12 & 0 & 1/12 \\ 1/12 & 0 & 2/12 & 1/12 \\ 1/12 & 1/12 & 1/12 & 1/12 \end{bmatrix}$

7. $\begin{bmatrix} 2/12 & 1/12 & 0 & 1/12 \\ 0 & 1/12 & 2/12 & 1/12 \\ 1/12 & 1/12 & 1/12 & 1/12 \end{bmatrix}$

8. $\begin{bmatrix} 3/12 & 1/12 & 0 & 0 \\ 0 & 0 & 2/12 & 2/12 \\ 0 & 2/12 & 1/12 & 1/12 \end{bmatrix}$

9. $\begin{bmatrix} 2/12 & 0 & 2/12 & 0 \\ 1/12 & 1/12 & 0 & 2/12 \\ 0 & 2/12 & 1/12 & 1/12 \end{bmatrix}$

10. $\begin{bmatrix} 1/12 & 1/12 & 2/12 & 0 \\ 2/12 & 0 & 0 & 2/12 \\ 0 & 2/12 & 1/12 & 1/12 \end{bmatrix}$

11. $$\begin{bmatrix} 1/12 & 2/12 & 0 & 1/12 \\ 2/12 & 1/12 & 0 & 1/12 \\ 0 & 0 & 3/12 & 1/12 \end{bmatrix}$$

12. $$\begin{bmatrix} 3/12 & 0 & 0 & 1/12 \\ 0 & 3/12 & 0 & 1/12 \\ 0 & 0 & 3/12 & 1/12 \end{bmatrix}$$

13. $$\begin{bmatrix} 1/12 & 0 & 2/12 & 1/12 \\ 1/12 & 2/12 & 1/12 & 0 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

14. $$\begin{bmatrix} 1/12 & 2/12 & 0 & 1/12 \\ 1/12 & 0 & 3/12 & 0 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

15. $$\begin{bmatrix} 2/12 & 1/12 & 0 & 1/12 \\ 0 & 1/12 & 3/12 & 0 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

16. $$\begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 \\ 2/12 & 0 & 2/12 & 0 \\ 0 & 2/12 & 0 & 2/12 \end{bmatrix}$$

17. $$\begin{bmatrix} 2/12 & 0 & 1/12 & 1/12 \\ 1/12 & 1/12 & 2/12 & 0 \\ 0 & 2/12 & 0 & 2/12 \end{bmatrix}$$

18. $$\begin{bmatrix} 2/12 & 1/12 & 0 & 1/12 \\ 1/12 & 0 & 3/12 & 0 \\ 0 & 2/12 & 0 & 2/12 \end{bmatrix}$$

19. $$\begin{bmatrix} 1/12 & 2/12 & 1/12 & 0 \\ 1/12 & 0 & 2/12 & 1/12 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

20. $$\begin{bmatrix} 2/12 & 1/12 & 1/12 & 0 \\ 0 & 1/12 & 2/12 & 1/12 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

21. $$\begin{bmatrix} 2/12 & 2/12 & 0 & 0 \\ 0 & 0 & 3/12 & 1/12 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

22. $$\begin{bmatrix} 2/12 & 0 & 2/12 & 0 \\ 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 2/12 & 0 & 2/12 \end{bmatrix}$$

23. $$\begin{bmatrix} 1/12 & 1/12 & 2/12 & 0 \\ 2/12 & 0 & 1/12 & 1/12 \\ 0 & 2/12 & 0 & 2/12 \end{bmatrix}$$

24. $$\begin{bmatrix} 1/12 & 1/12 & 1/12 & 1/12 \\ 1/12 & 1/12 & 2/12 & 0 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$$

25. $\begin{bmatrix} 2/12 & 0 & 1/12 & 1/12 \\ 0 & 2/12 & 2/12 & 0 \\ 1/12 & 1/12 & 0 & 2/12 \end{bmatrix}$

26. $\begin{bmatrix} 2/12 & 0 & 1/12 & 1/12 \\ 1/12 & 1/12 & 2/12 & 0 \\ 0 & 2/12 & 0 & 2/12 \end{bmatrix}$

27. $\begin{bmatrix} 1/12 & 1/12 & 2/12 & 0 \\ 0 & 2/12 & 1/12 & 1/12 \\ 2/12 & 0 & 0 & 2/12 \end{bmatrix}$

28. $\begin{bmatrix} 1/12 & 3/12 & 0 & 0 \\ 0 & 0 & 3/12 & 1/12 \\ 2/12 & 0 & 0 & 2/12 \end{bmatrix}$

29. $\begin{bmatrix} 2/12 & 0 & 1/12 & 1/12 \\ 1/12 & 3/12 & 0 & 0 \\ 0 & 0 & 2/12 & 2/12 \end{bmatrix}$

30. $\begin{bmatrix} 1/12 & 2/12 & 0 & 1/12 \\ 2/12 & 1/12 & 1/12 & 0 \\ 0 & 0 & 2/12 & 2/12 \end{bmatrix}$

31. $\begin{bmatrix} 0 & 2/12 & 1/12 & 1/12 \\ 3/12 & 1/12 & 0 & 0 \\ 0 & 0 & 2/12 & 2/12 \end{bmatrix}$

32. $\begin{bmatrix} 1/12 & 1/12 & 2/12 & 0 \\ 1/12 & 2/12 & 1/12 & 0 \\ 1/12 & 0 & 0 & 3/12 \end{bmatrix}$

33. $\begin{bmatrix} 2/12 & 0 & 2/12 & 0 \\ 0 & 3/12 & 1/12 & 0 \\ 1/12 & 0 & 0 & 3/12 \end{bmatrix}$

34. $\begin{bmatrix} 1/12 & 3/12 & 0 & 0 \\ 1/12 & 0 & 3/12 & 0 \\ 1/12 & 0 & 0 & 3/12 \end{bmatrix}$

35. $\begin{bmatrix} 1/12 & 0 & 3/12 & 0 \\ 2/12 & 2/12 & 0 & 0 \\ 0 & 1/12 & 0 & 3/12 \end{bmatrix}$

36. $\begin{bmatrix} 1/12 & 2/12 & 1/12 & 0 \\ 2/12 & 0 & 2/12 & 0 \\ 0 & 1/12 & 0 & 3/12 \end{bmatrix}$

37. $\begin{bmatrix} 3/12 & 0 & 1/12 & 0 \\ 0 & 2/12 & 2/12 & 0 \\ 0 & 1/12 & 0 & 3/12 \end{bmatrix}$

38. $\begin{bmatrix} 1/12 & 3/12 & 0 & 0 \\ 2/12 & 0 & 2/12 & 0 \\ 0 & 0 & 1/12 & 3/12 \end{bmatrix}$

39. $$\begin{bmatrix} 2/12 & 2/12 & 0 & 0 \\ 1/12 & 1/12 & 2/12 & 0 \\ 0 & 0 & 1/12 & 3/12 \end{bmatrix} .$$

4. Testing independence against strict positive quadrant dependence

Let X and Y be two random variables with known marginal distributions and unknown joint distribution. We want to test the hypothesis that X and Y are independent against the hypothesis that they are strictly positive quadrant dependent. By strict positive quadrant dependence we mean positive quadrant dependence but not independence. The data consist of N independent realizations of the vector (X,Y). Let τ be a test proposed for testing the hyposthesis of independence based on the given data. Let λ be the distribution of X and ν that of Y. Let $M_{PQD}(\lambda,\nu)$ be the collection of all bivariate distributions with fixed marginals λ and ν which are positive quadrant dependent. The power function of the test τ can be defined formally as follows.

$$\beta_\tau(\mu) = \Pr\{ \tau \text{ rejects the null hypothesis } / \mu\}$$

for μ in $M_{PQD}(\lambda,\nu)$. The above probability is computed when the joint distribution of X and Y is μ. The calculation of the power function of the test τ whose domain of definition is $M_{PQD}(\lambda,\nu)$ is very tedious. Moreover, if we wish to compare the performance of two tests to discriminate the null hypothesis of independence against the alternative hypothesis of strict positive quadrant dependence, we need to compare their power functions. This comparison then becomes doubly more difficult to achieve. But the following theorem asserts that it suffices to compare the powers at the extreme points of $M_{PQD}(\lambda,\nu)$ only.

Theorem 5 Let μ, μ^1, $\mu^2, \cdots$, μ^k be members of $M_{PQD}(\lambda,\nu)$ such that $\mu = \alpha_1\mu^1 + \alpha_2\mu^2 + \cdots + \alpha_k\mu^k$ for some $\alpha_1, \alpha_2, \cdots, \alpha_k \geq 0$ with $\sum_{i=1}^{k} \alpha_i = 1$. Then

$$\beta_\tau(\mu) = \sum_{i=1}^{k} \alpha_i \beta_\tau(\mu^i)$$

The above result can be used as follows. Suppose each of X and Y takes finitely many values. Then the convex set $M_{PQD}(\lambda,\nu)$ has finitely many extreme points and every member of $M_{PQD}(\lambda,\nu)$ can be written as a convex combination of these extreme points. Then the power of the test τ

evaluated at any given μ in $M_{PQD}(\lambda,\nu)$ is precisely the same convex combination of the powers of the test evaluated at each of the extreme points. This result also points out that in order to compare the performance of two given tests, it suffices to compare the powers evaluated at the extreme points. As an illustration, we consider the case when X takes values 1 and 2, and Y takes values 1, 2 and 3. Let n_{ij} = total number of (X,Y)'s with $X = i$ and $Y = j$, $i = 1,2$ and $j = 1,2,3$. The data can be arranged in the form of a contingency table as follows.

$$\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{bmatrix}$$

In this section, we compare the performance of two tests for testing the null hypothesis of independence against the alternative of strict positive quadrant dependence in the context of 2 x 3 contingency tables above.

$\underline{T_1}$: <u>Test based-on gamma ratio</u>

Let the bivariate distribution of X and Y be given by

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} .$$

The Gamma Ratio (see Goodman and Kruskal [25]) of X and Y is defined by

$$\gamma = \frac{\Pi_c - \Pi_d}{\Pi_c + \Pi_d} ,$$

where $\Pi_c = 2p_{11}(p_{22} + p_{23}) + 2p_{12}p_{13}$ and $\Pi_d = 2p_{13}(p_{21} + p_{22}) + 2p_{12}p_{21}$.

One can show that $\gamma = 0$ if X and Y are independent. See

Agresti (1984, p.160). One can also show that $\gamma \geq 0$ if X and Y are positive quadrant dependent. An estimate of γ based on the sample given above is given by

$$\hat{\gamma} = \frac{C - D}{C + D},$$

where C = The total number of concordant pairs $= n_{11}(n_{22} + n_{23}) + n_{12}n_{23}$ and D = The total number of discordant pairs $= n_{13}(n_{21} + n_{22}) + n_{12}n_{21}$.

The following is a natural test based on $\hat{\gamma}$ for testing the above null hypothesis against the specific alternative mentioned thereby.

Test T_1 : Reject the null hypothesis if $\hat{\gamma} \geq a$.

T_2 : Test based on eigen values

Let the marginal distribution of X be given by p_1 and p_2, and that of Y by q_1, q_2 and q_3. Let

$$Q = \begin{bmatrix} \frac{p_{11}}{\sqrt{p_1 q_1}} & -\frac{p_{12}}{\sqrt{p_1 q_2}} & \frac{p_{13}}{\sqrt{p_1 q_3}} \\ \frac{p_{21}}{\sqrt{p_2 q_1}} & -\frac{p_{22}}{\sqrt{p_2 q_2}} & -\frac{p_{23}}{\sqrt{p_2 q_3}} \end{bmatrix}.$$

Let κ_1 and κ_2 be the eigen values of QQ^T, where T denotes operation transpose on matrices. We give some properties of these eigen values below. For further details, see Lancaster [11] and [12], O'Neill ([17], [18], and [19]).

Properties

1. One of the eigen values is always equals unity. Let us use κ_1 for this eigen value.

2. If X and Y are independent, $\kappa_2 = 0$.

3. If X and Y are strictly positive quadrant dependent, then $\kappa_2 > 0$.

We estimate $\kappa_1 + \kappa_2$ based on the data given above as follows.

Let

$$B = \begin{bmatrix} \frac{n_{11}}{N\sqrt{p_1 q_1}} & \frac{n_{12}}{N\sqrt{p_1 q_2}} & \frac{n_{13}}{N\sqrt{p_1 q_3}} \\ \frac{n_{21}}{N\sqrt{p_2 q_1}} & \frac{n_{22}}{N\sqrt{p_2 q_2}} & \frac{n_{23}}{N\sqrt{p_2 q_3}} \end{bmatrix} .$$

Let $\hat{\kappa}_1^2$ and $\hat{\kappa}_2^2$ be the eigen values of BB^T. Then we propose $\hat{\kappa} = \hat{\kappa}_1 + \hat{\kappa}_2$ as an estimator of $\kappa_1 + \kappa_2$.

Test T_2 : Reject the null hypothesis if $\hat{\kappa} \geq a$.

We discuss the performance of these two tests in the case of two specific examples given below.

Example 1 $p_1 = p_2 = 1/2$ and $q_1 = q_2 = q_3 = 1/3$.

Example 2 $p_1 = 1/4$, $p_2 = 3/4$ and $q_1 = 1/2$, $q_2 = 1/4$, $q_3 = 1/4$.

Let us now elaborate on some of the properties of the eigen values of QQ^T. The second eigen value of QQ^T, κ_2, can be worked out explicitly.

$$\kappa_2 = \frac{(p_{11}p_{22} - p_{12}p_{21})^2}{p_1 p_2 q_1 q_2} + \frac{(p_{11}p_{23} - p_{13}p_{21})^2}{p_1 p_2 q_1 q_3} + \frac{(p_{12}p_{23} - p_{13}p_{22})^2}{p_1 p_2 q_2 q_3} .$$

From this it follows that $\kappa_2 = 0$ if and only if X and Y are independent.

At this juncture, we want to make some remarks on the definition of the matrix B above. In order to develop an estimator of κ_2, it is natural to divide each frequency n_{ij} in B by the square root of the product of the corresponding marginal totals $n_{i.}$ and $n_{.j}$. See O'Neill ([17], [18], [19]). If we had proceeded as outlined by O'Neill, one of the eigen values of BB^T would always be equal to unity. In our definition of B, it is not true that one of the eigen values of BB^T would always be equal to unity. Since we know the marginal probability laws of X and Y, we need to estimate only p_{ij}'s by n_{ij}/N's. This what motivated us to define the matrix B the way it was defined above. However, in view of the above formula for κ_2, one could estimate κ_2 directly without having to define the matrix B. Accordingly, let

$$\tilde{\kappa}_2 = \frac{(n_{11}n_{22} - n_{12}n_{21})^2}{N^4 p_1 p_2 q_1 q_2} + \frac{(n_{11}n_{23} - n_{12}n_{21})^2}{N^4 p_1 p_2 q_1 q_3} + \frac{(n_{12}n_{23} - n_{13}n_{22})^2}{N^4 p_1 p_2 q_2 q_3} .$$

We can build a test based on the statistic $\tilde{\kappa}_2$.

<u>Test T_3:</u> Reject the null hypothesis if $\tilde{\kappa}_2 \geq a$.

We discuss the performance of these three tests in the case of two specific examples given below.

<u>Example 1</u> $p_1 = p_2 = 1/2$ and $q_1 = q_2 = q_3 = 1/3$.

<u>Example 2</u> $p_1 = 1/4$, $p_2 = 3/4$ and $q_1 = 1/2$, $q_2 = 1/4$, $q_3 = 1/4$.

The performance of T_1 and T_3 was compared in detail in Subramanyam and Bhaskara Rao [12]. We now compare the performance of the tests T_1, T_2 and T_3 together under the level of significance $\alpha = 0.01, 0.025, 0.05$ and $N = 15, 20$ and 25. The exact distribution of $\hat{\gamma}$, $\hat{\kappa}$ and $\hat{\kappa}_2$ is evaluated for each of the sample sizes $N = 15, 20$ and 25 and the power of the tests T_1, T_2 and T_3 is evaluated at each of the extreme point distributions of the above examples using these exact distributions. The graphs*of these distributions are given at the end of this section. Nguyen and Sampson [8] evaluated the powers of tests based on some other statistics at some specific alternative distributions by simulating these distributions.

*The authors wish to thank Ron Chao for his valuable help in the computations.

Example No.	Extreme Points	γ	κ_2
1.	$P_1 = \begin{bmatrix} 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 \end{bmatrix}$	0	0
	$P_2 = \begin{bmatrix} 1/6 & 1/3 & 0 \\ 1/6 & 0 & 1/3 \end{bmatrix}$	1/2	2/3
	$P_3 = \begin{bmatrix} 1/3 & 1/6 & 0 \\ 0 & 1/6 & 1/3 \end{bmatrix}$	1	2/3
	$P_4 = \begin{bmatrix} 1/3 & 0 & 1/6 \\ 0 & 1/3 & 1/6 \end{bmatrix}$	1/2	2/3
2.	$P_5 = \begin{bmatrix} 2/16 & 1/16 & 1/16 \\ 6/16 & 3/16 & 3/16 \end{bmatrix}$	0	0
	$P_6 = \begin{bmatrix} 2/16 & 2/16 & 0 \\ 6/16 & 2/16 & 4/16 \end{bmatrix}$	1/4	1/6
	$P_7 = \begin{bmatrix} 4/16 & 0 & 0 \\ 4/16 & 4/16 & 4/16 \end{bmatrix}$	1	1/3
	$P_8 = \begin{bmatrix} 3/16 & 0 & 1/16 \\ 5/16 & 4/16 & 3/16 \end{bmatrix}$	4/10	1/8

Table 1: Power functions of the tests T_1, T_2, and T_3.

Sample size: N	15			20			25		
Distribution	$\beta_{T_1}(\cdot)$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$
Example 1									
P_1 (size)	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
P_2	0.0768	0.3944	0.5021	0.1389	0.6477	0.6892	0.2080	0.8817	0.8339
P_3	0.9999	0.3944	0.5360	1.0000	0.6477	0.8276	1.0000	0.8817	0.9569
P_4	0.0768	0.3944	0.5360	0.1389	0.6477	0.8276	0.2080	0.8817	0.9569
Critical value:a	0.90	1.99	0.66	0.78	1.73	0.49	0.71	1.60	0.39
Example 2									
P_5 (size)	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
P_6	0.1224	0.0540	0.1037	0.0661	0.0821	0.1360	0.0348	0.1144	0.1670
P_7	0.9866	0.1110	0.0923	0.9968	0.1824	0.1855	0.9992	0.2501	0.3131
P_8	0.3664	0.0403	0.0194	0.2719	0.0548	0.0348	0.1984	0.0654	0.0595
Critical value:a	1.00	2.05	0.77	1.00	1.78	0.56	1.00	1.62	0.44

Table 2: Power functions of the tests T_1, T_2, and T_3.

Sample size: N	15			20			25		
Distribution	$\beta_{T_1}(\cdot)$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$
Example 1									
P_1 (size)	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025
P_2	0.1733	0.6165	0.6581	0.2602	0.8693	0.8241	0.3334	1.0000	0.9365
P_3	0.9999	0.6165	0.7817	1.0000	0.8693	0.9467	1.0000	1.0000	0.9925
P_4	0.1733	0.6165	0.7817	0.2602	0.8693	0.9467	0.3334	1.0000	0.9925
Critical value: a	0.80	1.83	0.51	0.69	1.61	0.38	0.62	1.50	0.30
Example 2									
P_5 (size)	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025
P_6	0.1224	0.1181	0.1747	0.0661	0.1492	0.2139	0.0348	0.1884	0.2569
P_7	0.9866	0.2152	0.1993	0.9968	0.2802	0.3508	0.9993	0.4071	0.5017
P_8	0.3664	0.0965	0.0571	0.2719	0.1047	0.3508	0.1984	0.1352	0.1500
Critical value: a	1.00	1.87	0.56	1.00	1.66	0.42	1.00	1.52	0.33

Table 3: Power functions of the tests T_1, T_2 and T_3

Sample size: N	15			20			25		
Distribution	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$	$\beta_{T_1}(\ )$	$\beta_{T_2}(\ )$	$\beta_{T_3}(\ )$
P_1 (size)	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
P_2	0.3229	0.7602	0.7771	0.3927	0.9964	0.9209	0.4761	0.9999	0.9768
P_3	0.9999	0.7602	0.8927	1.0000	0.9964	0.9822	1.0000	0.9999	0.9976
P_4	0.2936	0.7602	0.8927	0.4142	0.9964	0.9822	0.4761	0.9999	0.9976
Critical value: a	0.69	1.73	0.40	0.60	1.52	0.30	0.53	1.43	0.24
P_5 (size)	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
P_6	0.1224	0.1870	0.2511	0.0661	0.2312	0.3144	0.0780	0.3030	0.3524
P_7	0.9866	0.3286	0.3526	0.9968	0.4447	0.5357	0.9992	0.5727	0.6650
P_8	0.3667	0.1615	0.1279	0.2719	0.1969	0.2143	0.2399	0.2366	0.2847
Critical value: a	1.00	1.74	0.43	1.00	1.55	0.31	0.73	1.44	0.25

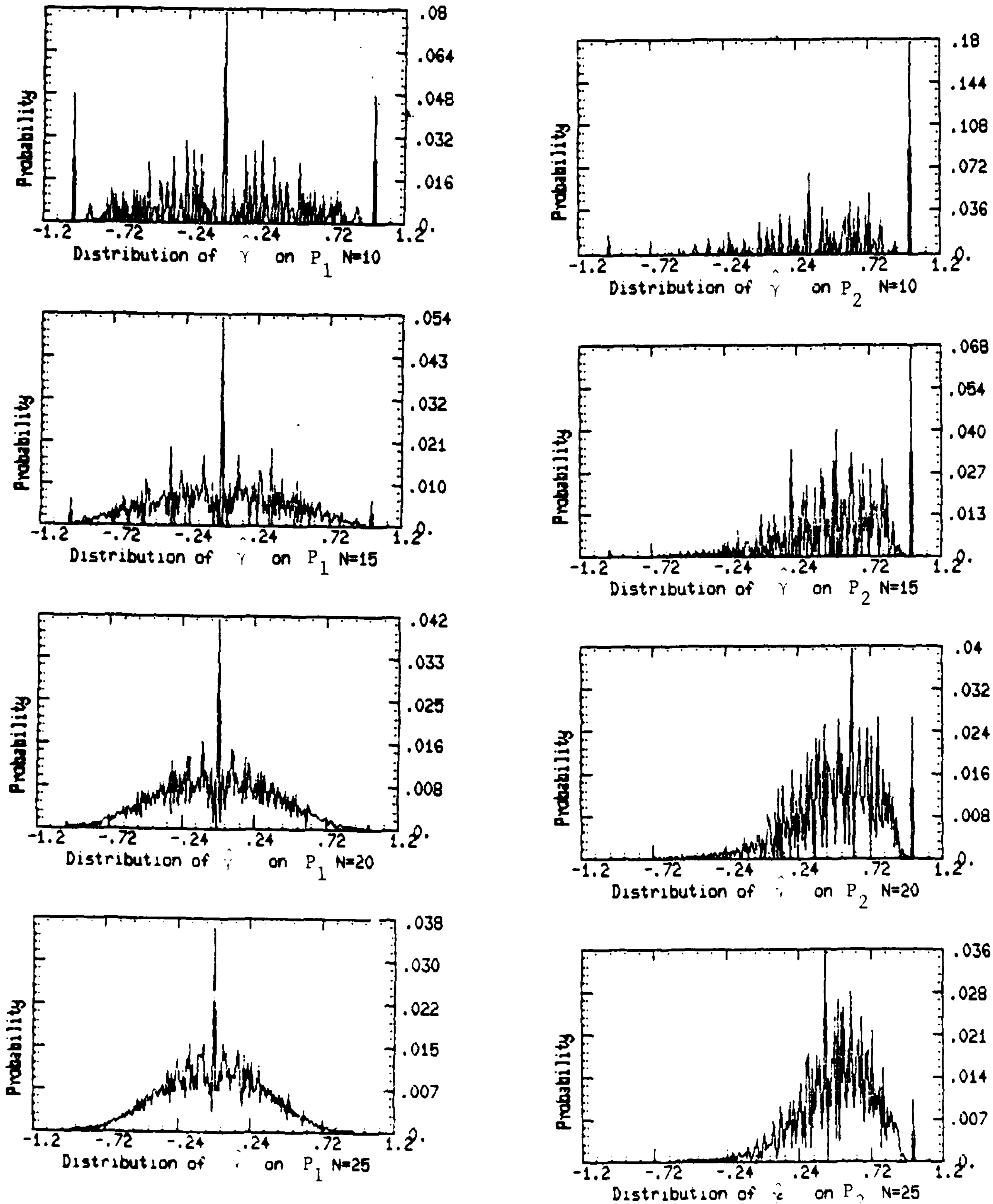
Conclusions

1. The power of the test T_1 dominates the power of the other two tests at the extreme point distribution P_3 in Example 1 and P_7 in Example 2. This is not surprising as the Gamma Ratio achieves the perfect value unity under P_3 in Example 1 and P_7 in Example 2.

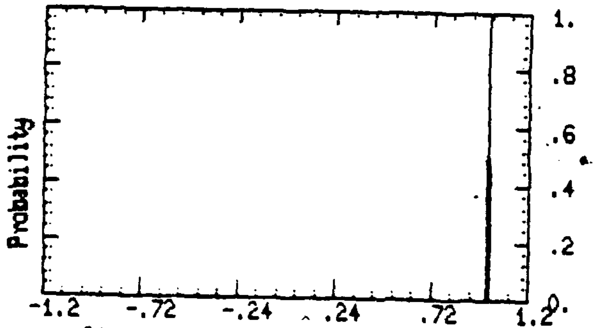
2. On the whole T_2 seems to perform well in comparison with the other two tests. Even under the distribution P_3 in Example 1 and P_7 in Example 2, T_2 is not overpowered by T_1.

3. Some extensive studies are needed to be carried out to see whether T_2 is preferrable to the other two under different sets of marginal distributions.

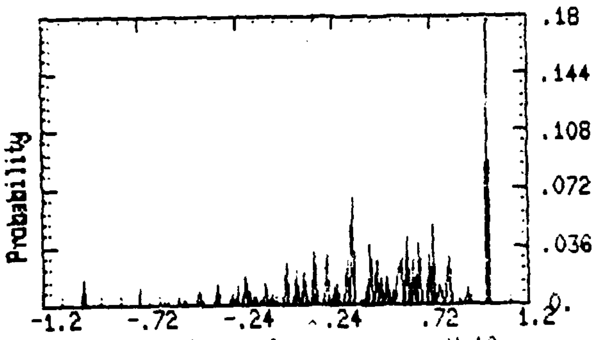
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\gamma}$ FOR N= 10,15,20,25 UNDER P_1 AND P_2:



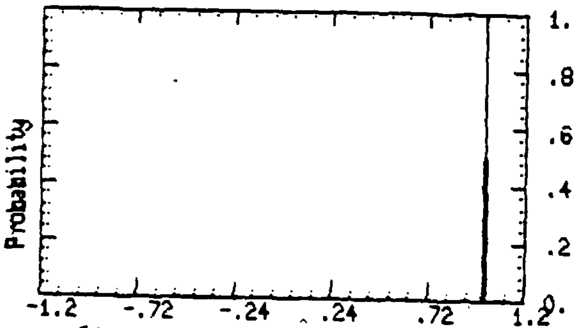
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\gamma}$ FOR N= 10,15,20,25 UNDER P_3 AND P_4.



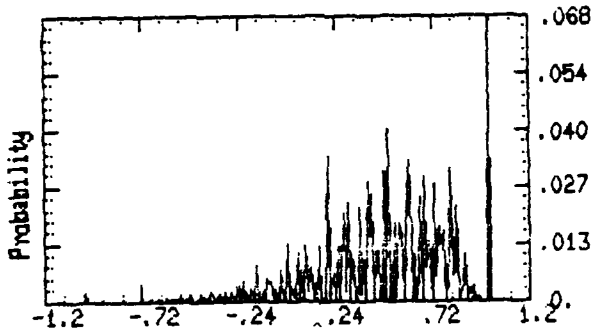
Distribution of $\hat{\gamma}$ on P_3 N=10



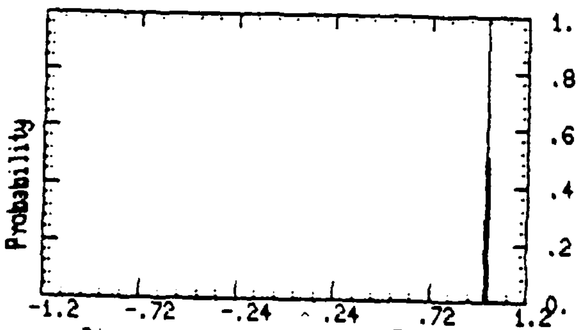
Distribution of $\hat{\gamma}$ on P_4 N=10



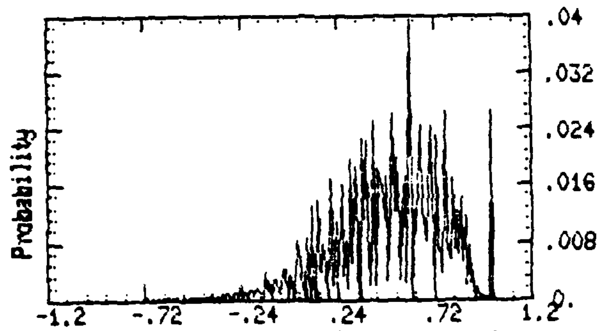
Distribution of $\hat{\gamma}$ on P_3 N=15



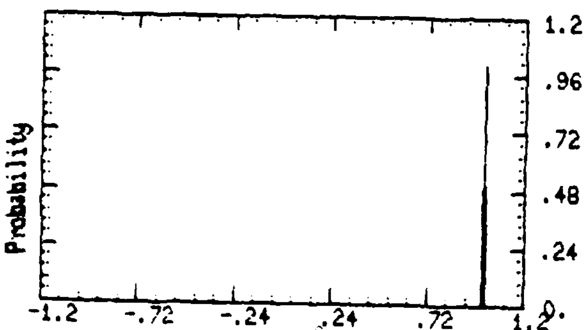
Distribution of $\hat{\gamma}$ on P_4 N=15



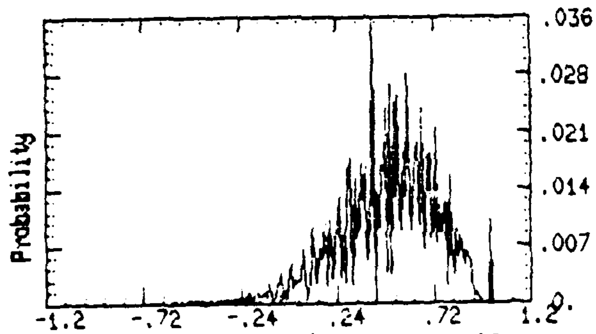
Distribution of $\hat{\gamma}$ on P_3 N=20



Distribution of $\hat{\gamma}$ on P_4 N=20

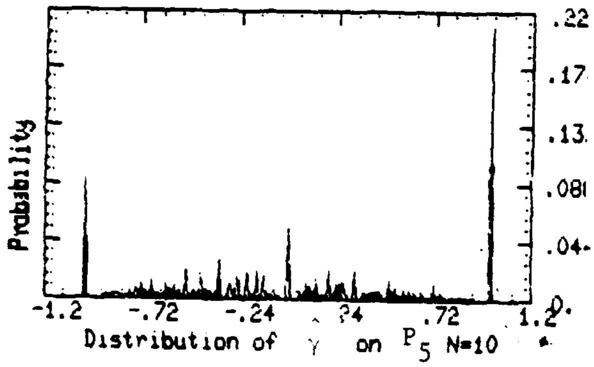


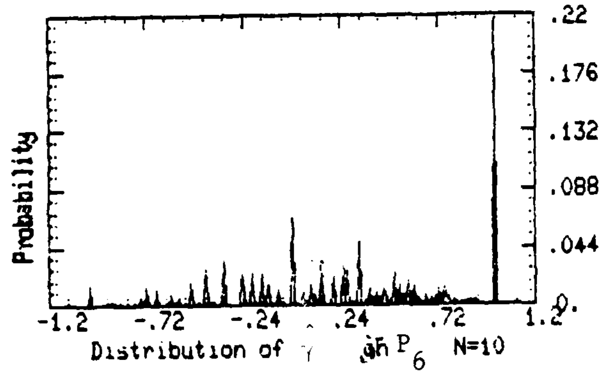
Distribution of $\hat{\gamma}$ on P_3 N=25

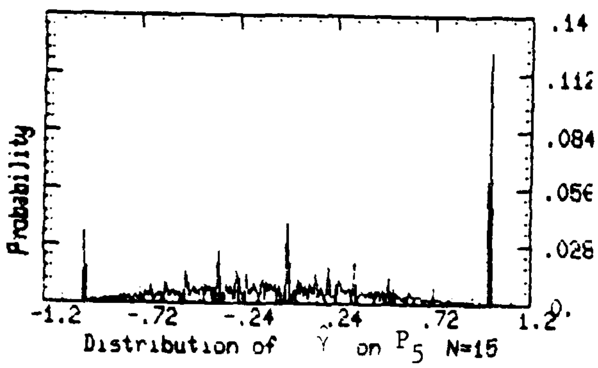


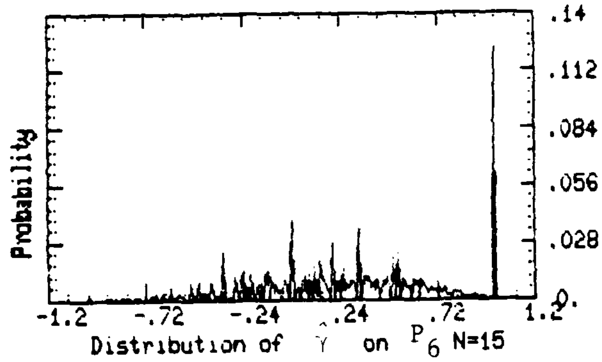
Distribution of $\hat{\gamma}$ on P_4 N=25

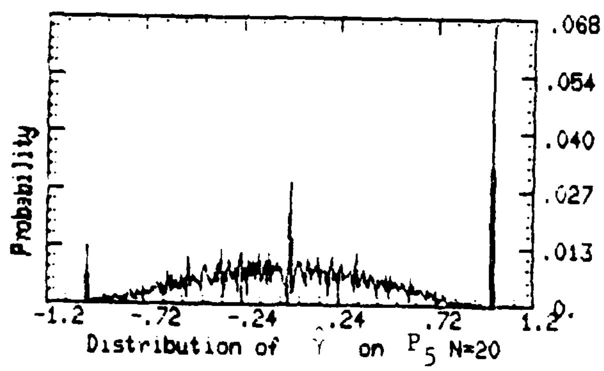
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\gamma}$ FOR N= 10,15,20,25 UNDER P_5 AND P_6.

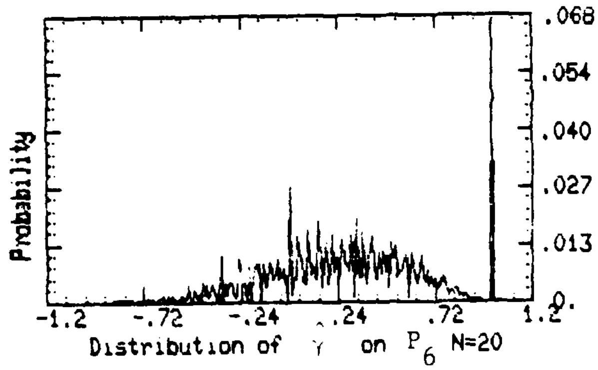


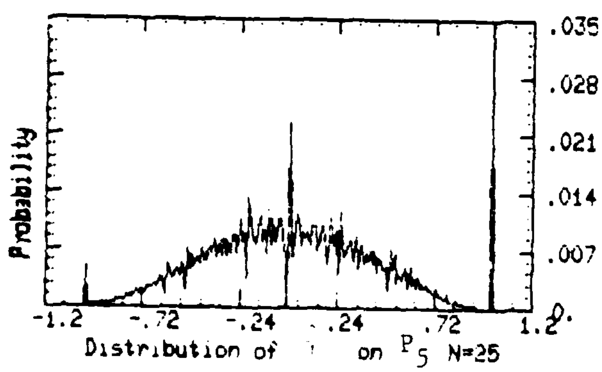


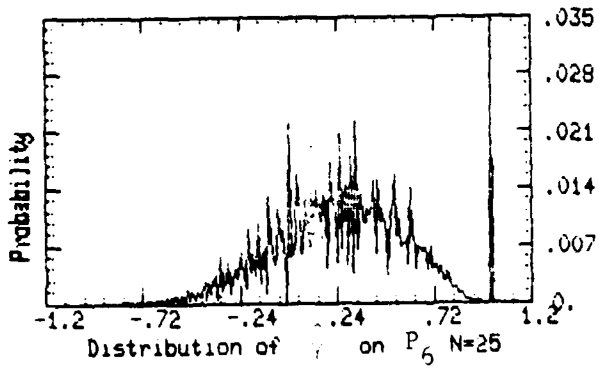




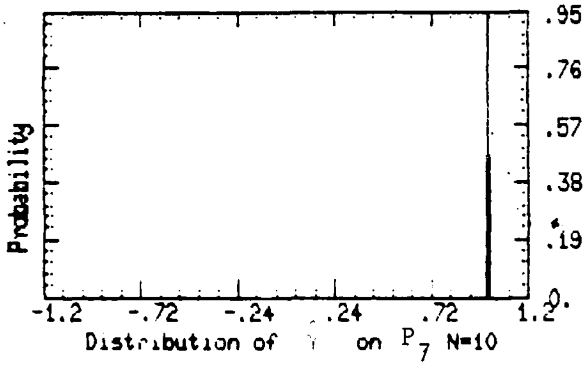


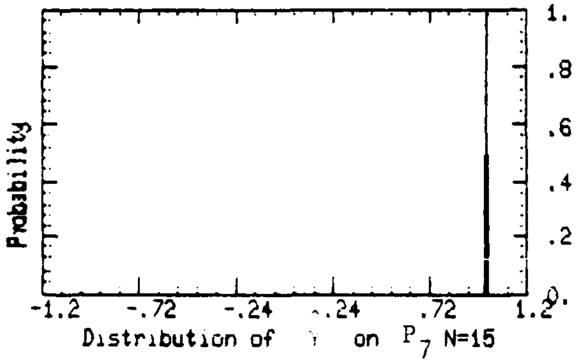


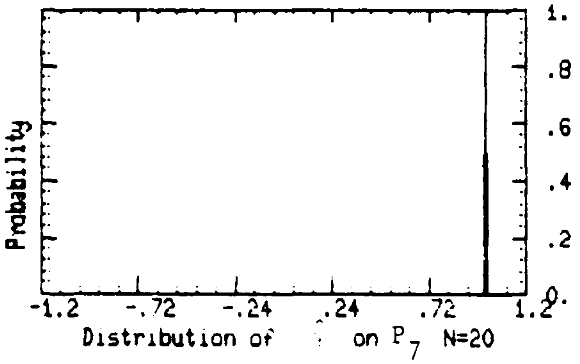


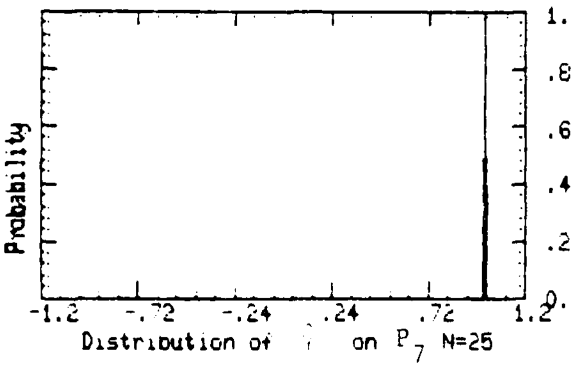


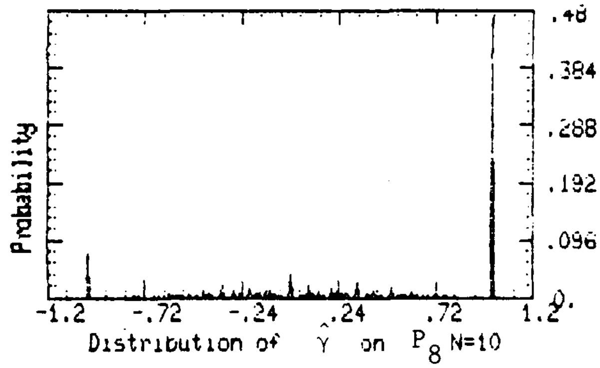
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\gamma}$ FOR N= 10,15,20,25 UNDER P_7 AND P_8.

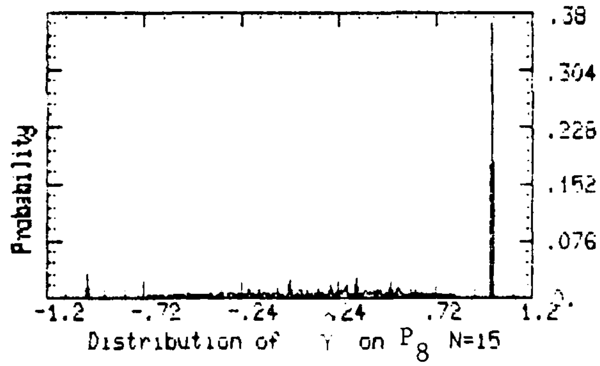


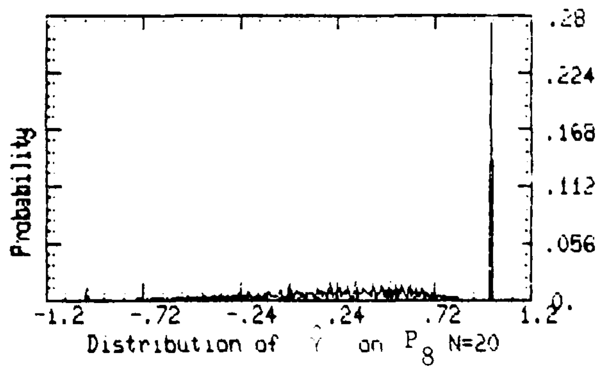


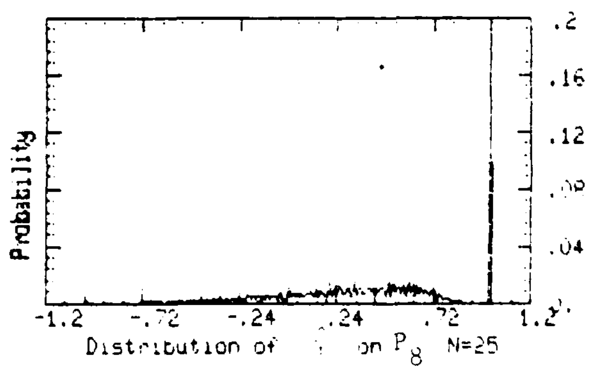




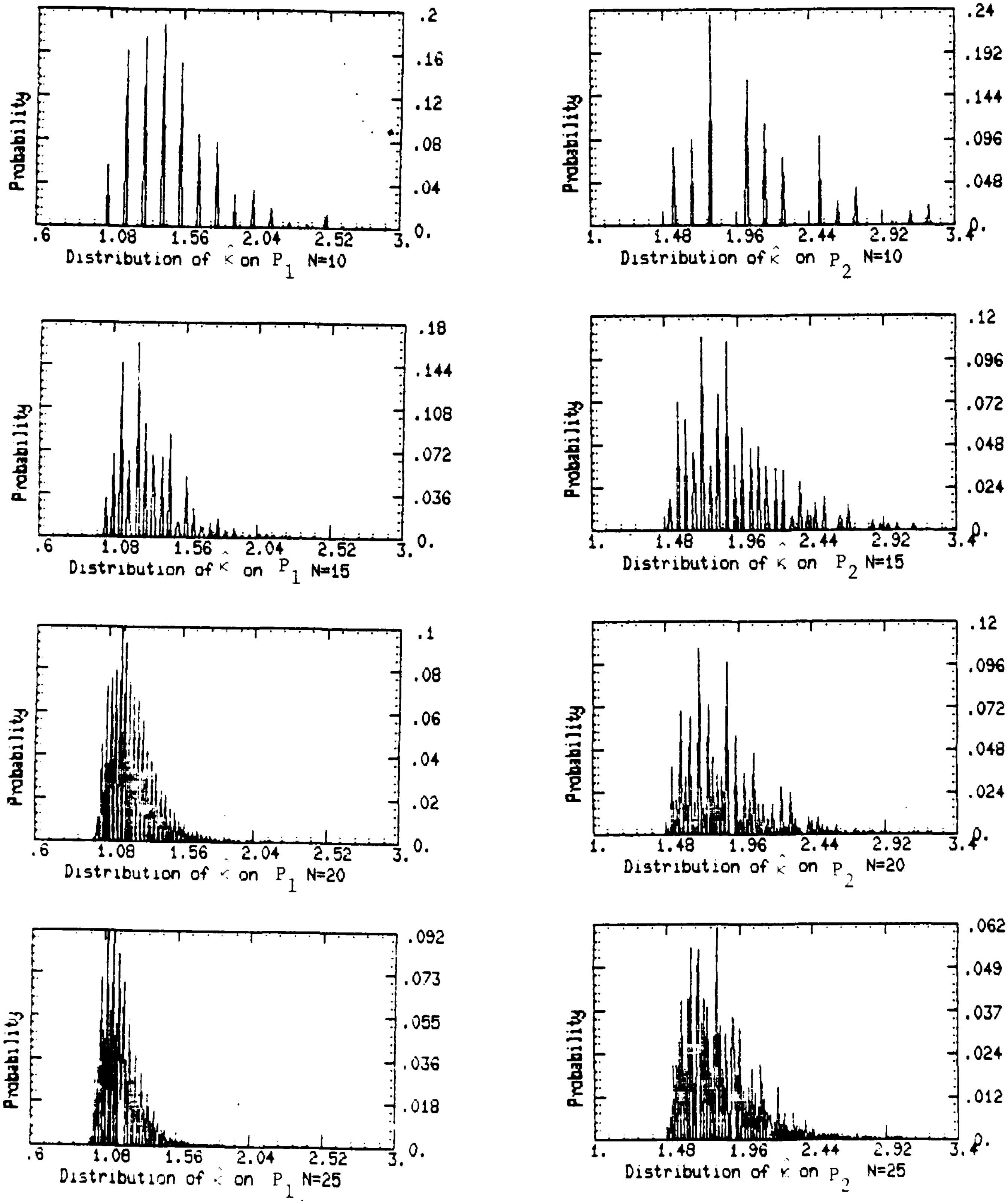




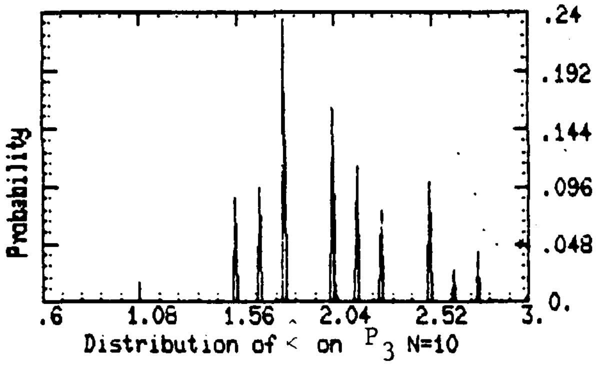


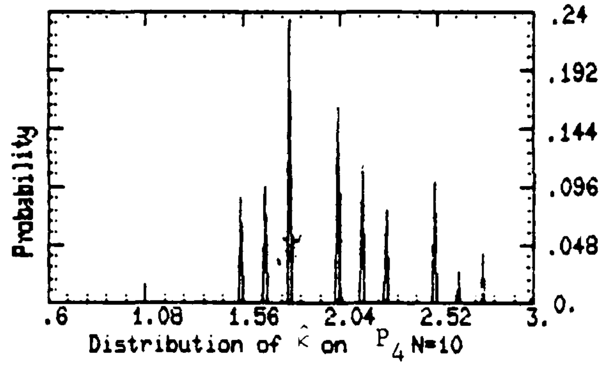


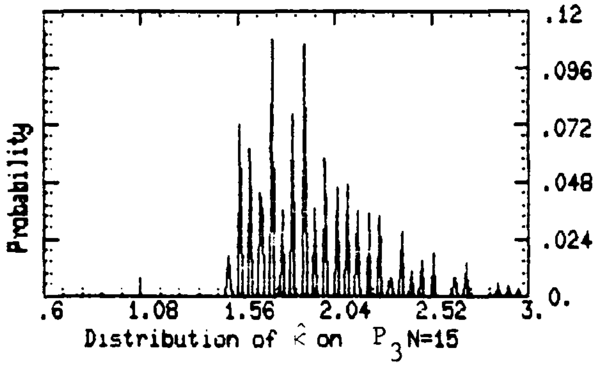
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa} = \hat{\kappa}_1 + \hat{\kappa}_2$ FOR N = 10,15,20,25 UNDER P_1 AND P_2.

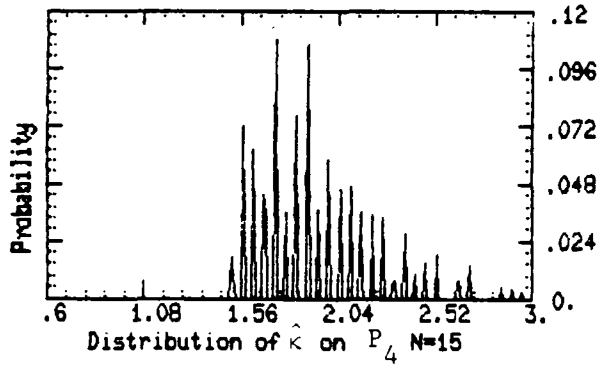


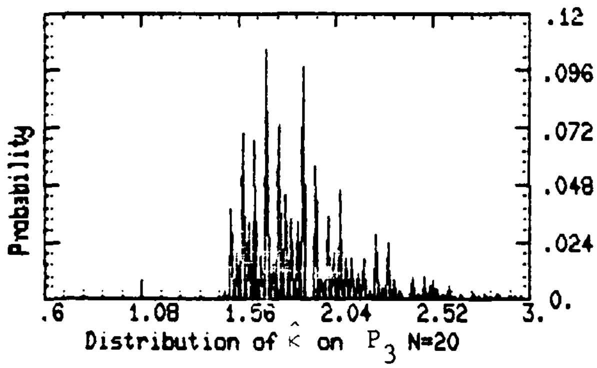
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa} = \hat{\kappa}_1 + \hat{\kappa}_2$ FOR N= 10,15,20,25 UNDER P_3 AND P_4.

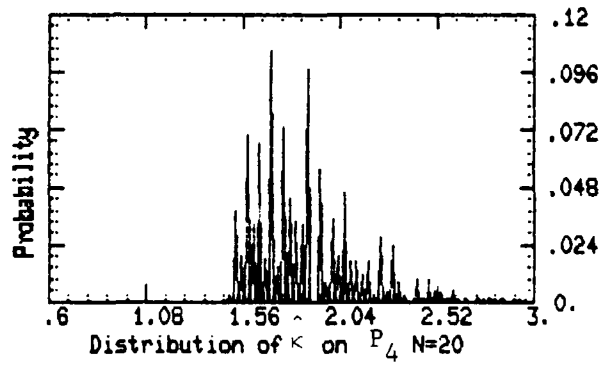


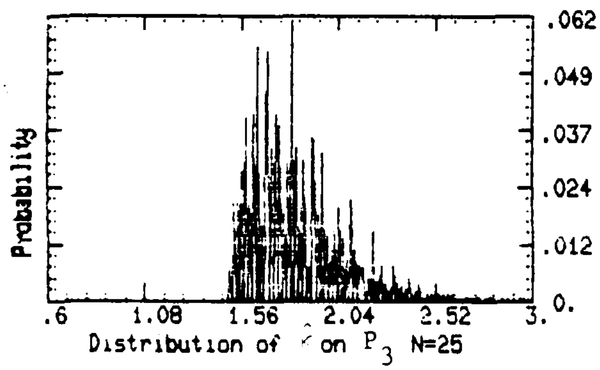


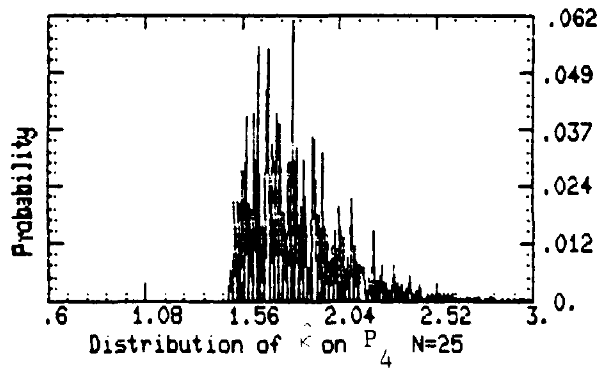




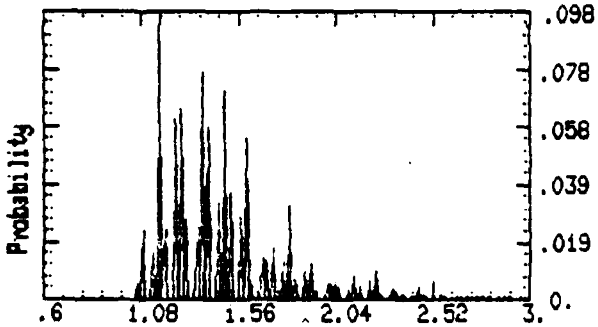




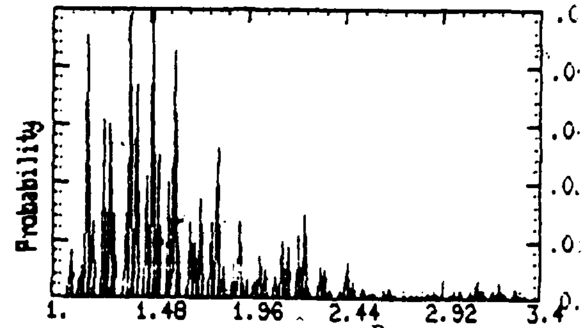




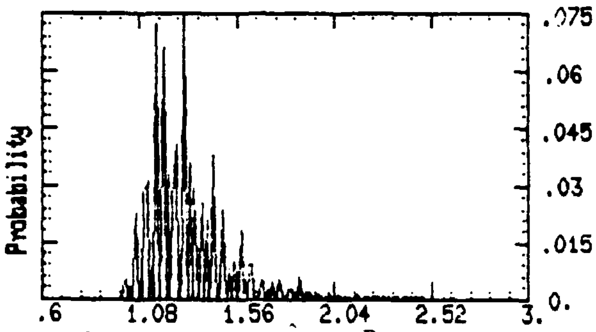
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa}+\hat{\kappa}_1+\hat{\kappa}_2$ FOR N= 10,15,20,25 UNDER P_5 AND P_6.



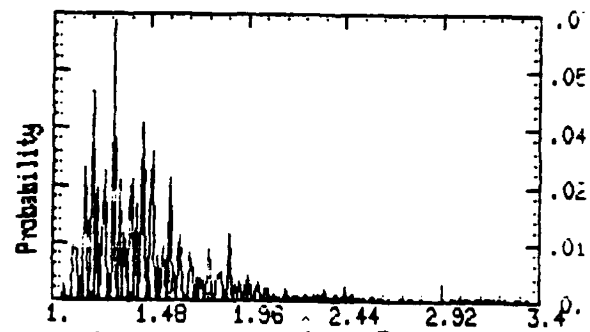
Distribution of $\hat{\kappa}$ on P_5 N=10



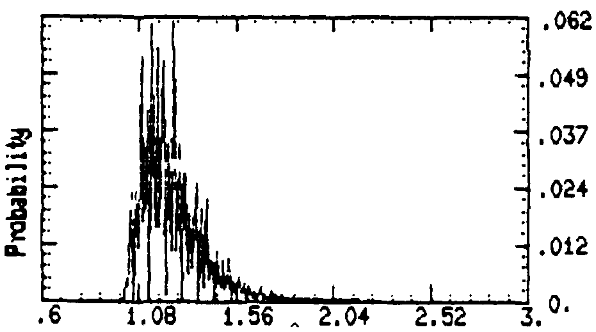
Distribution of $\hat{\kappa}$ on P_6 N=10



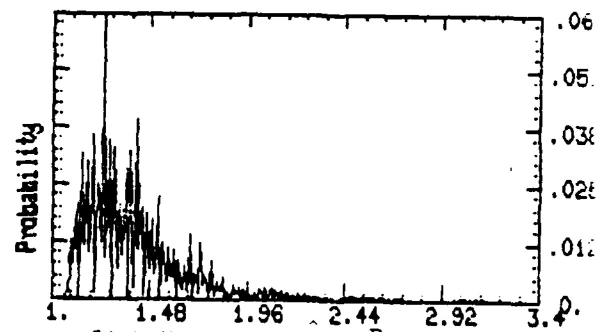
Distribution of $\hat{\kappa}$ on P_5 N=15



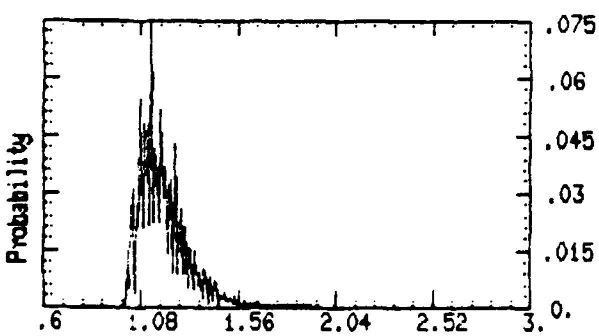
Distribution of $\hat{\kappa}$ on P_6 N=15



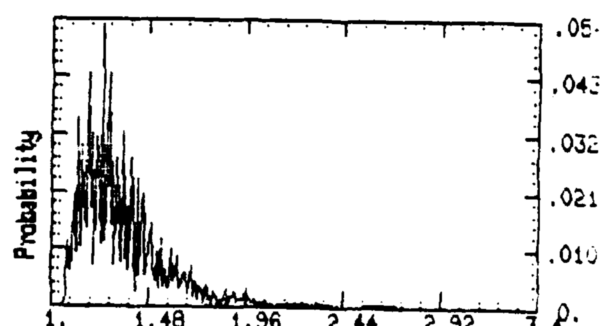
Distribution of $\hat{\kappa}$ on P_5 N=20



Distribution of $\hat{\kappa}$ on P_6 N=20

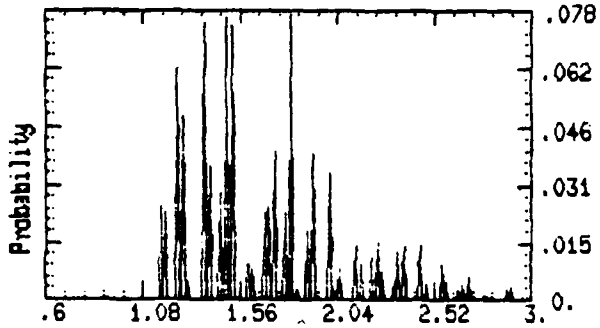


Distribution of $\hat{\kappa}$ on P_5 N=25

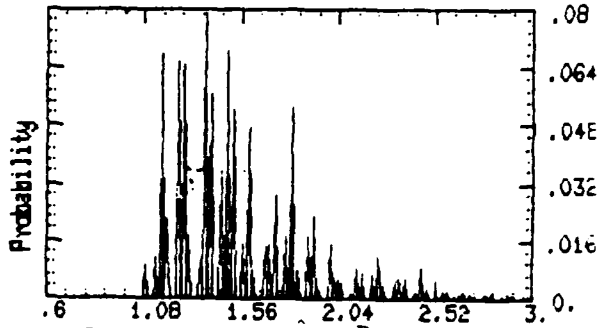


Distribution of $\hat{\kappa}$ on P_6 N=25

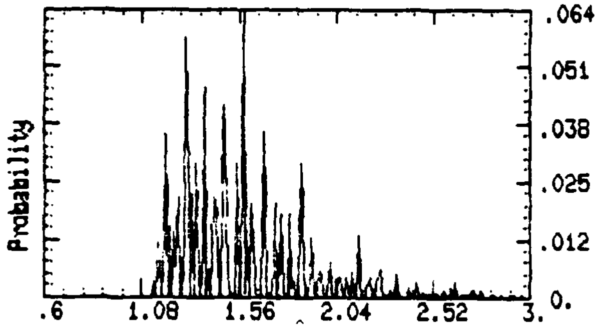
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa} = \hat{\kappa}_1 + \hat{\kappa}_2$ FOR N= 10,15,20,25 UNDER P_7 and P_8.



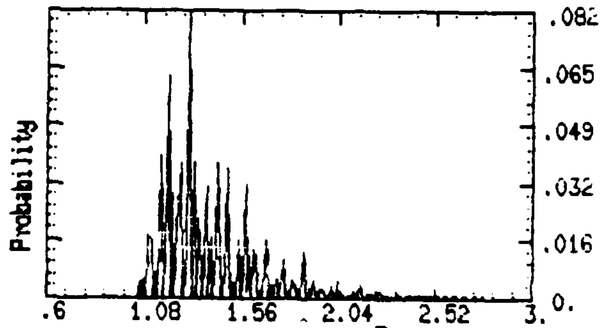
Distribution of $\hat{\kappa}$ on P_7 N=10



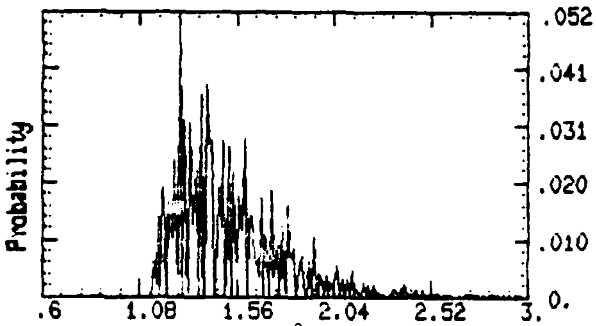
Distribution of $\hat{\kappa}$ on P_8 N=10



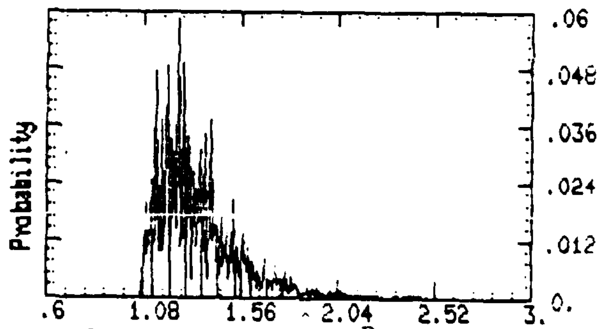
Distribution of $\hat{\kappa}$ on P_7 N=15



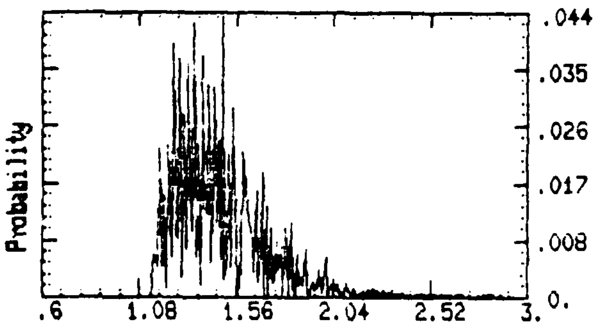
Distribution of $\hat{\kappa}$ on P_8 N=15



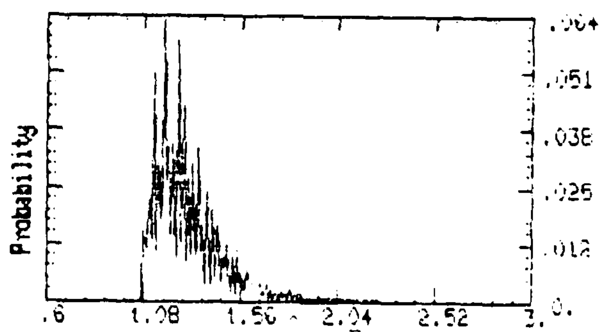
Distribution of $\hat{\kappa}$ on P_7 N=20



Distribution of $\hat{\kappa}$ on P_8 N=20

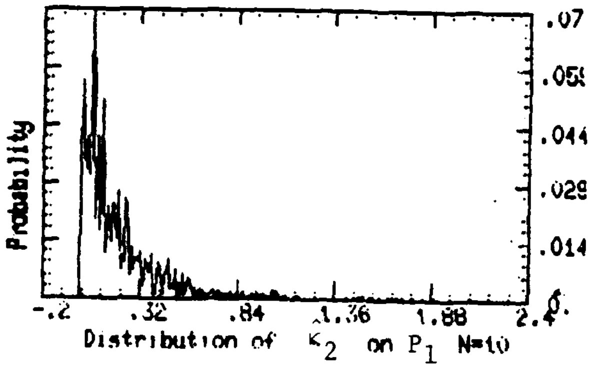


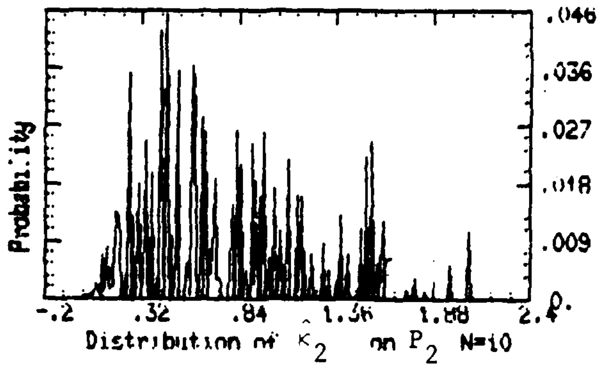
Distribution of $\hat{\kappa}$ on P_7 N=25

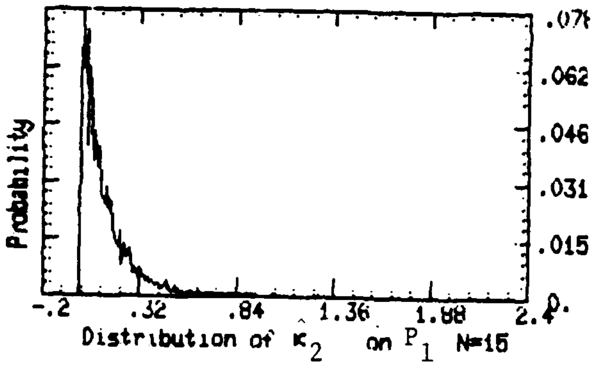


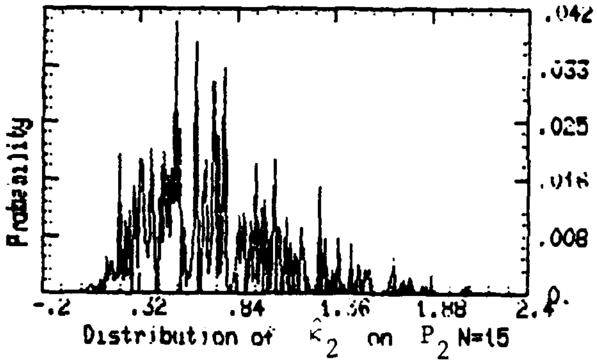
Distribution of $\hat{\kappa}$ on P_8 N=25

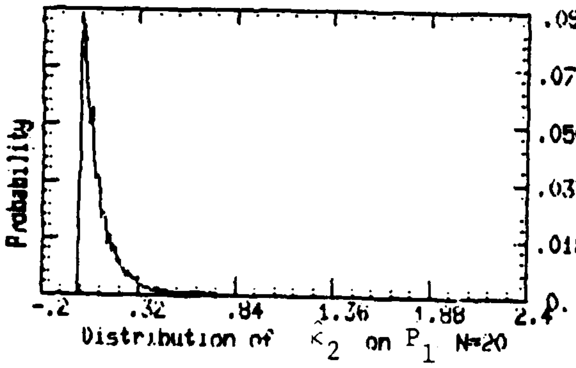
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa}_2$ FOR N = 10,15,20,25 UNDER P_1 AND P_2.

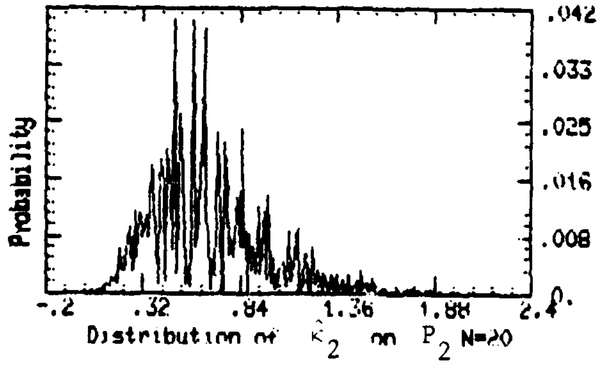


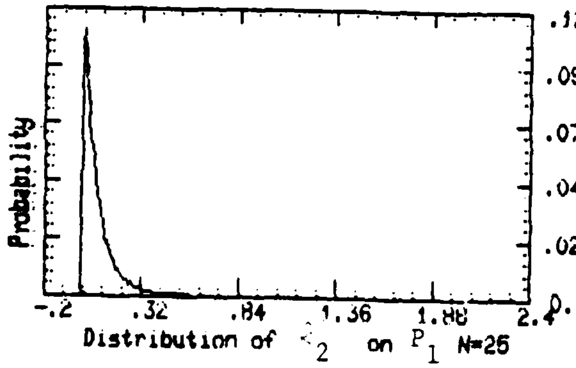


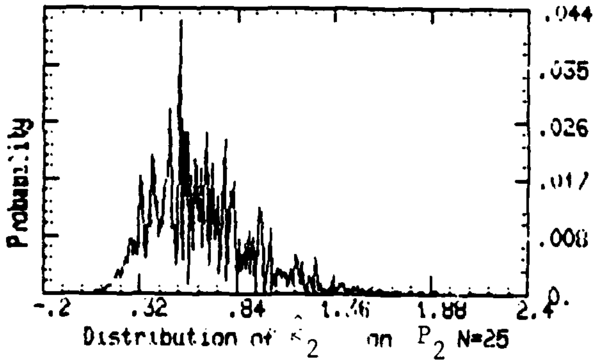




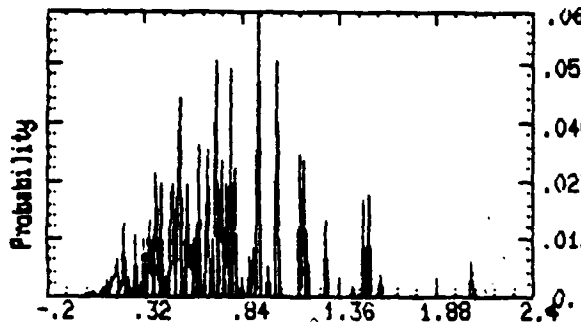




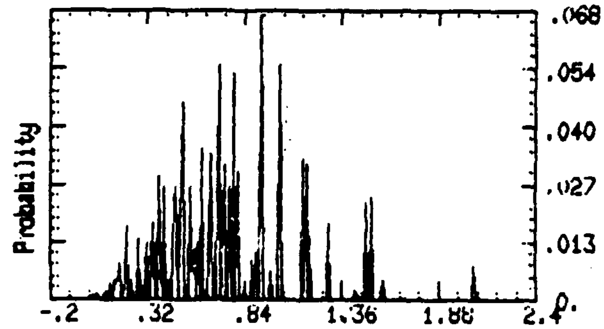




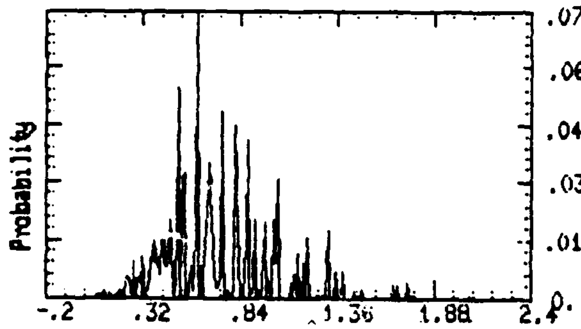
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa}_2$ FOR N= 10,15,20,25 UNDER P_3 AND P_4.



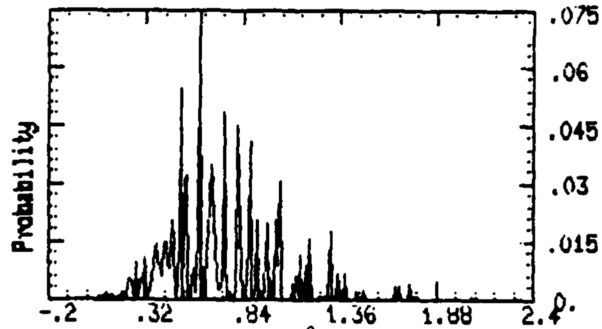
Distribution of $\hat{\kappa}_2$ on P_3 N=10



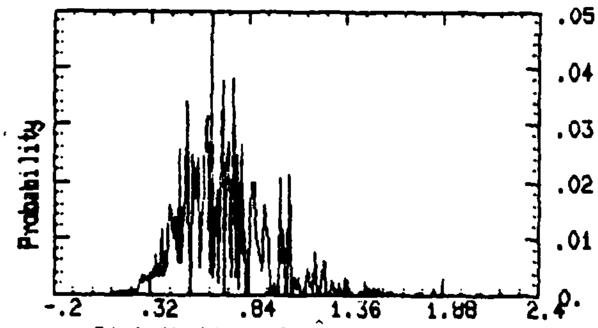
Distribution of $\hat{\kappa}_2$ on P_4 N=10



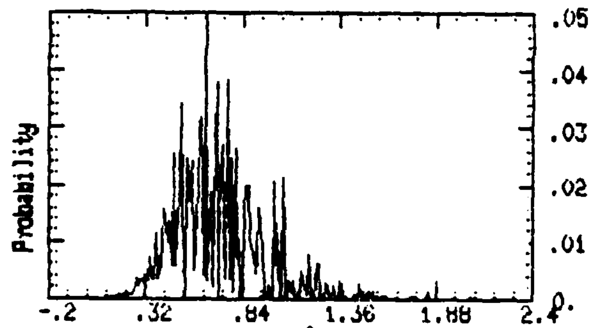
Distribution of $\hat{\kappa}_2$ on P_3 N=15



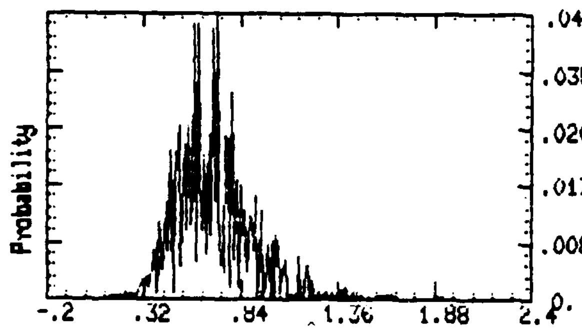
Distribution of $\hat{\kappa}_2$ on P_4 N=15



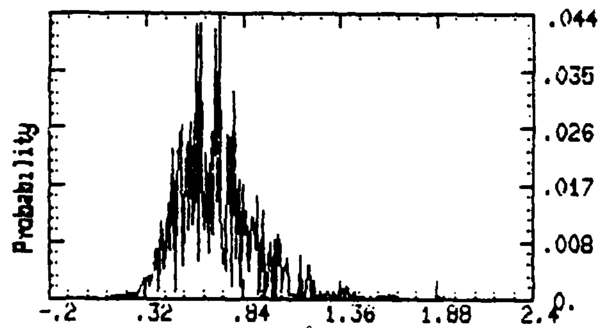
Distribution of $\hat{\kappa}_2$ on P_3 N=20



Distribution of $\hat{\kappa}_2$ on P_4 N=20

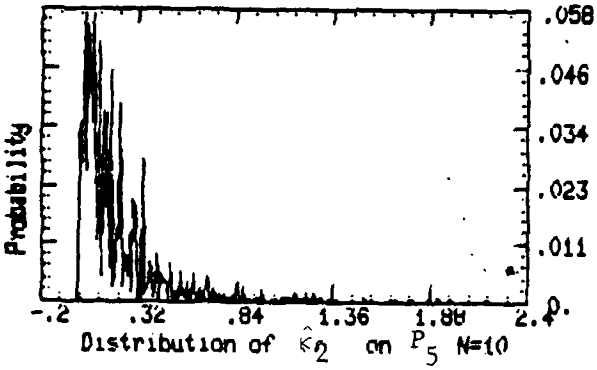


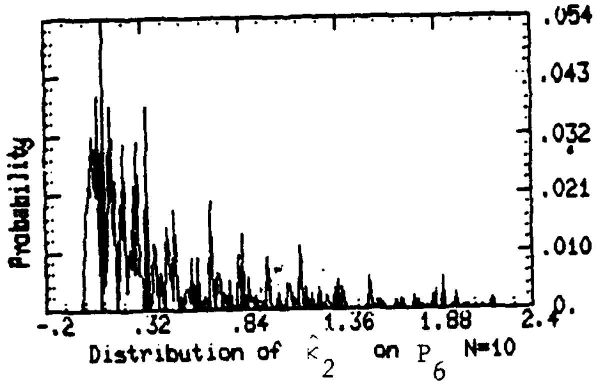
Distribution of $\hat{\kappa}_2$ on P_3 N=25

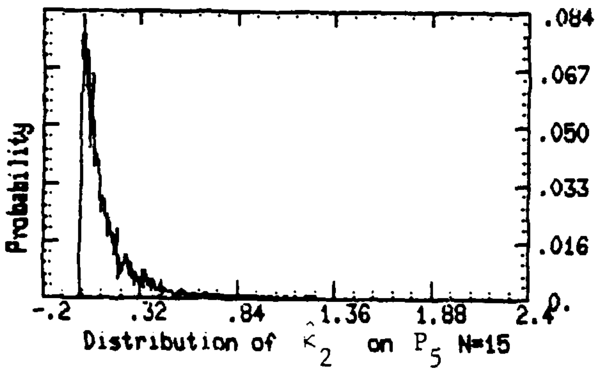


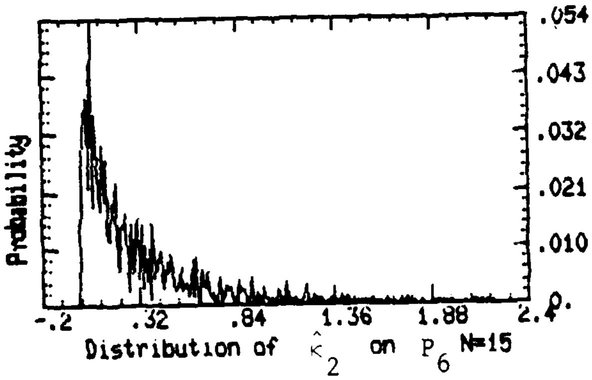
Distribution of $\hat{\kappa}_2$ on P_4 N=25

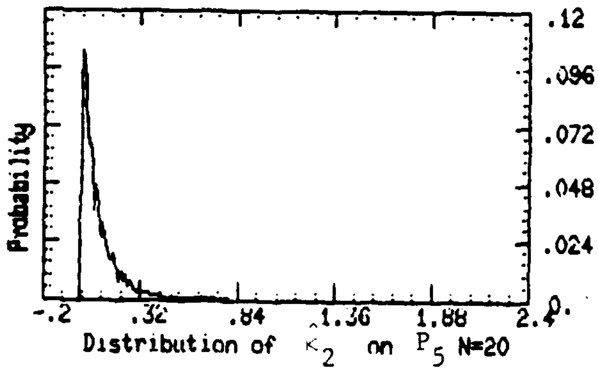
GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa}_2$ FOR N= 10,15,20,25 UNDER P_5 and P_6.

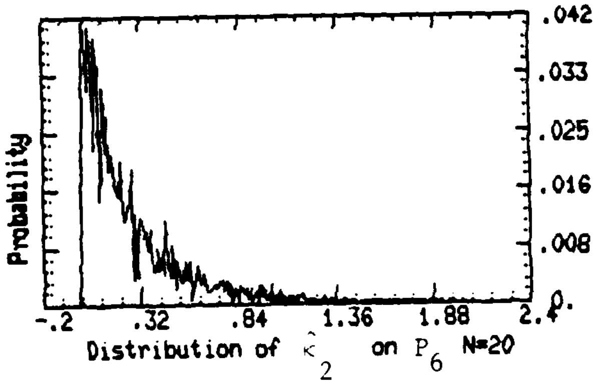


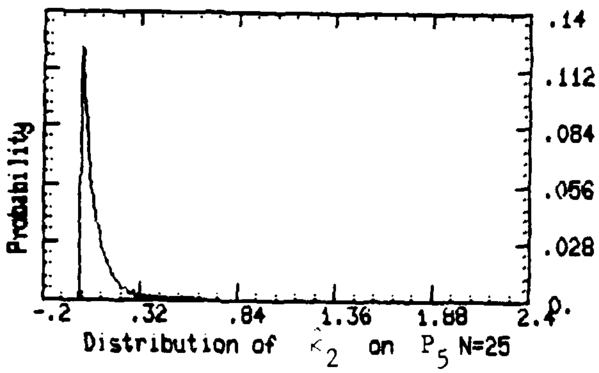


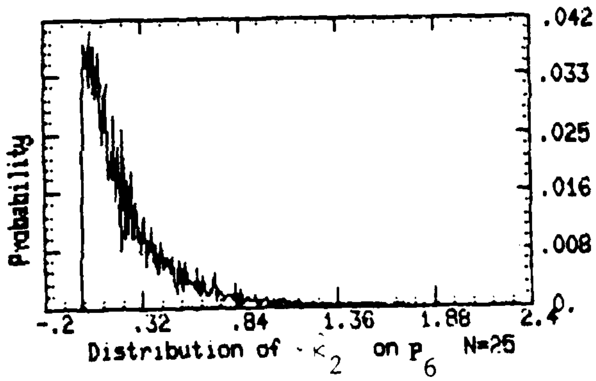




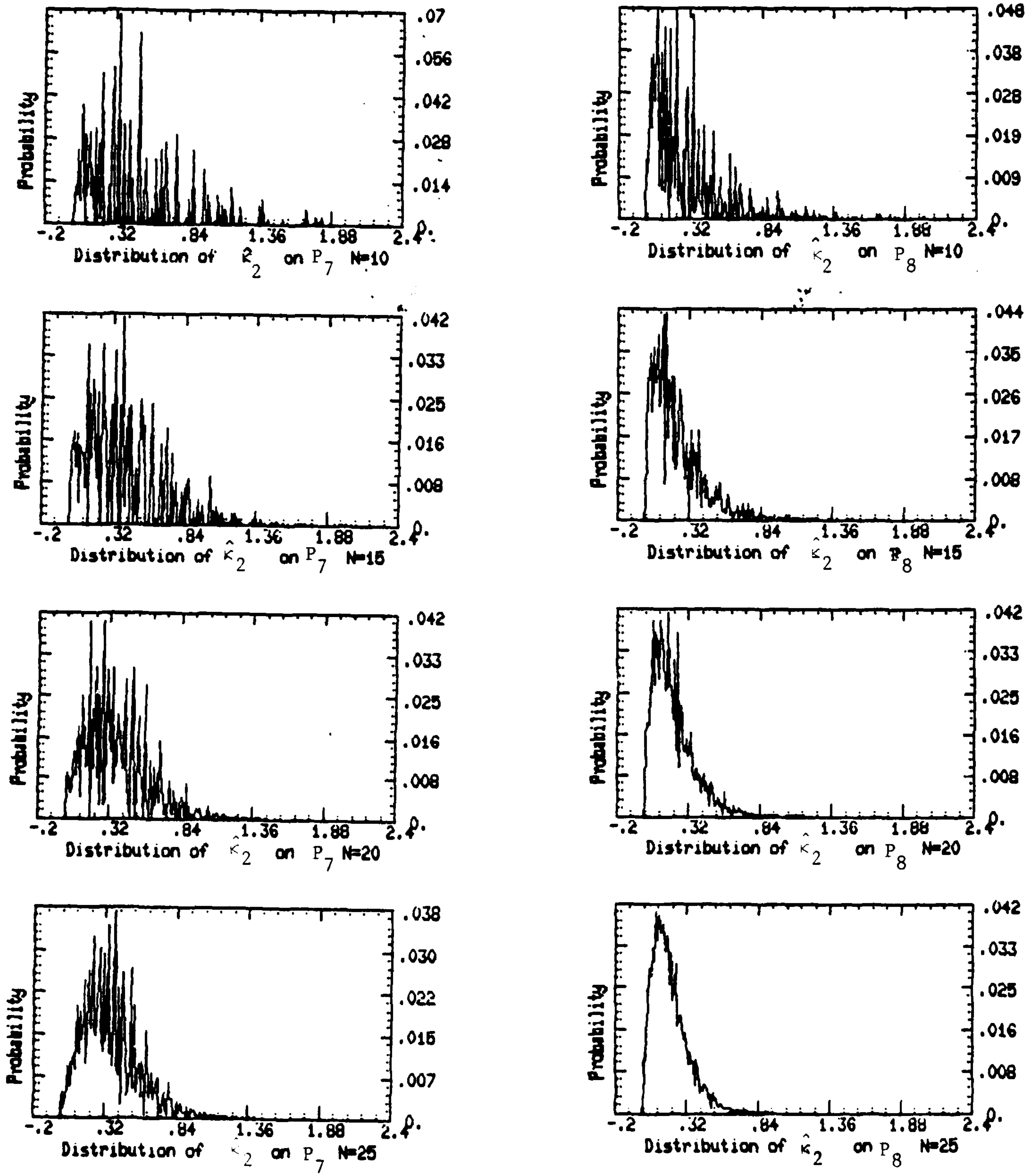








GRAPHS OF THE DISTRIBUTIONS OF $\hat{\kappa}_2$ FOR N = 10,15,20,25 UNDER P_7 and P_8.



5. Inference on the structure of dependence

In two-way contingency tables, the χ^2 test for independence has been widely used. When the test for independence is rejected, it is of interest to study the structure of dependence between the $a+1$ rows and $b+1$ columns. In this section, we write the matrices $F = (f_{ij})$ and $B = (b_{ij})$ in terms of their eigenvalues and eigenvectors by singular value decomposition; here $f_{ij} = p_{ij}/\sqrt{p_i q_j}$, $b_{ij} = n_{ij}/\sqrt{n_i n_j}$, $p_i = p_{i1} + \dots + p_{i,b+1}$, $q_j = p_{1j} + \dots + p_{a+1,j}$, $n_{i\cdot} = n_{i1} + \dots + n_{i,b+1}$ and $n_{\cdot j} = n_{1j} + \dots + n_{a+1,j}$ Taking advantage of the above decomposition, we propose procedures to find out as to which of the last t eigenvalues of FF' are zero. The distribution theory associated with the above procedures are also investigated. Some aspects of the above problem were discussed by O'Neill ([17], [18], [19]). The problem of determination of the rank of F is discussed in a forthcoming paper of Z.D. Bai, P.R. Krishnaiah and L.C. Zhao from the point of view of model selection using an information theoretic criterion. The above authors also established the strong consistency of their procedure. In this section, we use the notation $n = \sum_{i=1}^{a+1} \sum_{j=1}^{b+1} n_{ij}$ is fixed and the marginal totals $n_{i\cdot}$ and $n_{\cdot j}$ are random.

Consider the model

$$p_{ij} = p_i q_j \zeta_{ij} \tag{5.1}$$

$i = 1, 2, \dots, a+1$, $j = 1, 2, \dots, b+1$. Without losing generality we assume that $a \le b$. Under the above model, it is of interest to test for the structure of ζ_{ij}. From singular value decomposition of a matrix, it is known (e.g., see Lancaster [12]) that

$$F = \xi_o^* \eta_o^{*\prime} \delta_o + \sum_{u=1}^{a} \delta_u \xi_u^* \eta_u^{*\prime} \tag{5.2}$$

where $\delta_0 \geq \delta_1 \geq \ldots \geq \delta_a$ are the eigenvalues of F, $\delta_0 = 1$, ξ_u^* is the eigenvector of FF' corresponding to δ_u^2 and η_u^* is the eigenvector of F'F corresponding to δ_u^2. In (5.2)

$$\begin{aligned} \xi_u^* &= (\sqrt{p_1}\xi_{u1}, \ldots, \sqrt{p_{a+1}}\xi_{u,a+1})' \\ \eta_u^* &= (\sqrt{q_1}\eta_{u1}, \ldots, \sqrt{q_{b+1}}\eta_{u,b+1})' \qquad (5.3) \\ \xi_{o1} &= \ldots = \xi_{o,a+1} = 1, \ \eta_{o1} = \ldots = \eta_{o,b+1} = 1. \end{aligned}$$

We are interested in finding out as to how many δ_u's are equal to zero. This problem is analogous to the problem of studying the structure of interaction term in two-way classification model with one observation per cell. So, we will briefly discuss the above model.

Let

$$E(x_{ij}) = \mu + \alpha_i + \beta_j + \Delta_{ij} \qquad (5.4)$$

for i = 1, 2, ..., (a + 1), j = 1, 2, ..., (b + 1),

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_i \Delta_{ij} = \sum_j \Delta_{ij} = 0.$$

Also μ, α_i, β_j and Δ_{ij} respectively denote the general mean, i-th row effect, j-th column effect, and interaction in i-th row and j-th column. Also, let $\Delta = (\Delta_{ij})$ and $a < b$. We assume that x_{ij}'s are distributed independentally and with variance σ^2. The problem of testing the hypothesis $\Delta = 0$ was first considered by Fisher and MacKenzie [4], and later by Williams [23], Tukey [22] and others when the underlying distribution is normal. Fisher and MacKenzie [4] considered this problem using eigenvalues of certain matrix. For a review of the literature, the reader is referred to Krishnaiah and Yochmowitz [10]. Now, let $E(x_{ij}) = \log p_{ij}$, $\mu = \log c$, $\alpha_i = \log p_i$, $\beta_j = \log p_j$ and $\Delta_{ij} = \log \delta_{ij}$. Then the model (5.4) can be written as

$$p_{ij} = cp_i q_j \zeta_{ij}. \qquad (5.5)$$

But, here we do not assume that the conditions (5.5) are satisfied. But,

$$\log (\sum_i p_i) = \log (\sum_j q_j) = 0. \qquad (5.6)$$

We may assume that $c = 1$ (i.e., $\mu = 0$). We can write (5.6) as

$$p_{ij} = cp_i q_j \exp(\eta_{ij}) \qquad (5.7)$$

and express $\eta = (\eta_{ij})$ in terms of its eigenvalues and eigenvectors using spectral decomposition of a matrix. Then, we can draw inference on the rank of η. This problem is different from the problem of drawing inference on the rank of ζ except for the special case when the rank of η is zero. This special case is equivalent to the statement that the rank of ζ is one. In studying the interaction term in two-way classification model, Tukey [22] and Mandel [15] assumed certain structures on interaction term. We can assume similar structures on the models (5.6) and (5.7). As far as the models (5.6) and (5.7) are concerned, they are analogous to the well known two-way classification model with interaction and one observation per cell. But, the problems of estimation and distributions of test statistics are of different nature. In general we may also consider models of the form

$$p_{ij} = f(\alpha_i, \beta_j, \eta_{ij})$$

where $f(\cdot)$ is a suitably chosen function of α_1, β_j, and η_{ij}. For example $f(x,y,z) = f_1(x)f_2(y)f_3(z)$. As another possibility, we may also consider the model

$$p_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} \tag{5.8}$$

as in two-way classification with interaction.

Goodman [6] discussed the model (5.7) when η is written in terms of its eigenvalues and eigenvectors. O'Neil ([17], [18], [19]) discussed some aspects of the asymptotic distribution theory associated with finding the rank of the matrix ζ. In this section, we propose various test procedures for determination of the rank of ζ and investigate some problems on the asymptotic distributions of the test statistics.

We now discuss the problem of testing for the rank of the matrix ζ. If we know in advance the rank of ζ, we can use that knowledge in estimating the unknown parameters more accurately. For example, the maximum likelihood estimates of p_{ij}'s when the rank of F is one are not the same as when the rank of F is greater than one.

Now, let $B = (b_{ij})$ where

$$b_{ij} = n_{ij}/\sqrt{n_{i\cdot}n_{\cdot j}} \ . \tag{5.9}$$

Then, using the spectral decomposition of B, we have

$$B = \hat{\underset{\sim}{\xi}}{}_o^{*}\hat{\underset{\sim}{\eta}}{}_o^{*\prime}\hat{\delta}_o + \sum_{u=1}^{a} \hat{\delta}_u \hat{\underset{\sim}{\xi}}{}_u^{*}\hat{\underset{\sim}{\eta}}{}_u^{*\prime} \tag{5.10}$$

where $\hat{\delta}_o = 1$,

$$\hat{\underset{\sim}{\xi}}{}_u^{*} = ((n_{1\cdot}/n)^{1/2}\hat{\underset{\sim}{\xi}}_{u1},\ldots,(n_{a+1\cdot}/n)^{1/2}\hat{\underset{\sim}{\xi}}_{u,a+1}),$$

$$\hat{\underset{\sim}{\eta}}^*_u = ((n_{\cdot 1}/n)^{1/2}\hat{\underset{\sim}{\eta}}_{u1},\ldots,(n_{\cdot b+1}/n)^{1/2}\hat{\underset{\sim}{\eta}}_{u,b+1})'$$

$$\hat{\underset{\sim}{\xi}}_{o1} = \ldots = \hat{\underset{\sim}{\xi}}_{o,a+1} = 1, \ \hat{\eta}_{o1} = \ldots = \hat{\eta}_{o,b+1} = 1, \ \hat{\delta}_o = 1$$

Also, $\hat{\delta}_0 \geq \hat{\delta}_1 \geq \ldots \geq \hat{\delta}_a$ are the eigenvalues of B, $\hat{\underset{\sim}{\xi}}^*_{u1}$ is the eigenvector of BB' corresponding to $\hat{\underset{\sim}{\delta}}^2_u$ and $\hat{\underset{\sim}{\eta}}^*_u$ is the eigenvector of B'B corresponding to $\hat{\underset{\sim}{\delta}}^2_u$. Now, let $H_i:\delta_i^2 = 0$ $(i = 1, 2, \ldots, a)$ and $H = \bigcap_{i=1}^{a} H_i$. We can use $\psi(\hat{\underset{\sim}{\delta}}^2_1, \ldots, \hat{\delta}^2_a)$ to test H where $\psi(\cdot)$ is a suitable function of $\hat{\delta}^2_1, \ldots, \hat{\delta}^2_a$. For example, we can use $\hat{\delta}^2_1 + \ldots + \hat{\delta}^2_a$ or $\hat{\delta}^2_1$ as test statistics. Here we note that $n(\hat{\delta}^2_1 + \ldots + \hat{\delta}^2_a)$ is equivalent to χ^2_o where

$$\chi^2_o = \sum_i\sum_j\{n_{ij} - (n_{i\cdot}n_{\cdot j}/n)\}^2/n_{i\cdot}n_{\cdot j}. \tag{5.12}$$

When H_1 is true, O'Neil [17] showed that the joint distribution of $n\hat{\delta}^2_1, \ldots, n\hat{\delta}^2_a$ is the same as the joint distribution of the eigenvalues of the central Wishart matrix with *ab degrees of freedom*. Percentage points of the largest eigenvalue of the central Wishart matrix are given in Krishnaiah [8].

We have discussed before some procedures to test for the overall hypothesis $\delta^2_1 = \ldots = \delta^2_a = 0$. We will now discuss procedures for testing the subhypotheses H_t when H_1 is rejected. The hypothesis H_t is the same as the hypothesis that the rank of F is t. We will first consider the test procedure based upon T_1 where $T_q = \hat{\delta}^2_q + \ldots + \hat{\delta}^2_a$. In this procedure, we accept or reject H_1 according as

$$T_1 \lessgtr c_{1\alpha} \tag{5.13}$$

where

$$P[T_1 \leq c_{1\alpha}|H_1] = (1 - \alpha). \tag{5.14}$$

If H_1 is rejected, the hypothesis H_q is accepted or rejected according as

$$T_q \lesseqgtr c_{1\alpha}. \tag{5.15}$$

Starting with the test based upon T_1 we can also draw inference on testing the hypothesis H_{ij} as described below; here H_{ij} denotes the hypothesis $p_{ij} = p_i q_j$ for given values i,j. The χ_0^2 test statistic can be written as

$$\chi_0^2 = \underset{\sim}{z}'\underset{\sim}{z} \tag{5.16}$$

where

$$\underset{\sim}{z}' = (z_{11},\ldots,z_{1,b+1},\ldots,z_{a+1,1},\ldots,z_{a+1,b+1})$$

and

$$z_{ij} = (n_{ij} - (n_{i.}n_{.j}/n))/\sqrt{n_{i.}n_{.j}}.$$

But $\chi_0^2 = \max(\underset{\sim}{c}'\underset{\sim}{z})^2$ when the maximum is taken over all non-null $\underset{\sim}{c}$ subject to the restriction that $\underset{\sim}{c}'\underset{\sim}{c} = 1$. So, when H_1 is rejected, we can test the subhypotheses H_{ij} as follows. We accept or reject H_{ij} against two-sided alternatives according as

$$z_{ij}^2 \lesseqgtr c_{1\alpha}. \tag{5.17}$$

We can test the hypothesis $\bigcap_{i=1}^{t} \bigcap_{j=1}^{u} H_{ij}$ as follows. We accept or reject the above hypothesis against two-sided alternatives according as

$$\sum_{i=1}^{t} \sum_{j=1}^{u} z_{ij}^2 \lesseqgtr c_{1\alpha}$$

The hypothesis $\bigcap_{j=1}^{t} \bigcap_{j=1}^{u} H_{ij}$ implies the hypothesis

$$\sum_{i=1}^{t} \sum_{j=1}^{u} p_{ij} = (\sum_{i=1}^{t} p_i)(\sum^{u} q_j).$$

We will now discuss the problem of testing the hypotheses H_{ij} against the alternatives A_{ij} simultaneously where $A_{ij}: p_{ij} > p_i q_j$. We accept or reject H_{ij} against A_{ij} according as

$$z_{ij} \lessgtr c_{2\alpha} \tag{5.18}$$

where

$$P[\max z_{ij} \le c_{2\alpha} | H_1] = (1-\alpha) \tag{5.19}$$

and max z_{ij} denotes the maximum of the elements of $\underset{\sim}{z}$. The asymptotic joint distribution of the lelments of $\underset{\sim}{z}$ is a singular multivariate normal distribution. But, bounds on the critical values $c_{2\alpha}$ can be obtained by using Poincare's formula. Similarily, we can propose procedures to test hypotheses H_{ij} against A^*_{ij} where $A^*_{ij}: p_{ij} < p_i q_j$.

We now discuss the test based upon $\hat{\rho}_1^2$. We accept or reject H_1 according as

$$\hat{\rho}_1^2 \lessgtr c_{2\alpha} \tag{5.20}$$

where

$$P[\hat{\rho}_1^2 \lessgtr c_{2\alpha} | H_1] = (1-\alpha). \tag{5.21}$$

If H_1 is rejected, we accept or reject H_i according as

$$\hat{\rho}_i^2 \lessgtr c_{2\alpha}. \tag{5.22}$$

We will now derive the asymptotic nonnull distributions of certain functions of $n\hat{\rho}_1^2, \ldots, n\hat{\rho}_a^2$. The following lemmas are needed in the sequel.

Lemma 5.1 Let U: p x p be a symmetric matrix which can be expressed as

$$U = \Lambda + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \ldots \tag{5.23}$$

when $\Lambda = \text{diag}(\lambda_1,\ldots,\lambda_p)$. Also, let

$$\lambda_{q_1+\cdots+q_{\alpha-1}+1} = \cdots = \lambda_{q_1+\cdots+q_\alpha} = \theta_\alpha \qquad (5.24)$$

for $\alpha = 1,2,\ldots,r$, $q_0 = 0$ and $q_r = a$. In addition, let $\ell_1 \geq \cdots \geq \ell_p$ denote the eigenvalues of U. Then

$$\bar{\ell}_\alpha = \theta_\alpha + \varepsilon\bar{\ell}_\alpha^{(1)} + \varepsilon^2\bar{\ell}_\alpha^{(2)} + \cdots \qquad (5.25)$$

where

$$\bar{\ell}^{(1)} = \frac{1}{q_\alpha}\text{tr}\; U_{\alpha\alpha}^{(1)}$$

$$\bar{\ell}^{(2)} = \frac{1}{q_\alpha}\text{tr}\left[U_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha}\theta_{\alpha\beta}^{-1}\,U_{\alpha\beta}^{(1)}\,U_{\beta\alpha}^{(1)}\right]$$

$$\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$$

$$U = \begin{bmatrix} U_{11}^{(i)} & U_{12}^{(i)} & \cdots & U_{1r}^{(i)} \\ U_{21}^{(i)} & U_{22}^{(i)} & \cdots & U_{2r}^{(i)} \\ \vdots & \vdots & & \vdots \\ U_{r1}^{(i)} & U_{r2}^{(i)} & \cdots & U_{rr}^{(i)} \end{bmatrix}.$$

The above lemma is implicit in Kato [24]. It is also proved in Fujikoshi [5] by following the same lines as in Lawley [13].

<u>Lemma 5.2</u> Let $\psi_i(\ell_1,\ldots,\ell_a)$, $(i=1,2,\ldots,k)$, be analytic functions of $\ell_1,\ldots,\ell_a$ around $\lambda_1,\ldots,\lambda_a$ and let λ_i's have multiplicities as in (5.24). We assume that

$$\left.\frac{\partial T_i(\ell_1,\ldots,\ell_a)}{\partial \ell_{j_1}}\right|_{\underset{\sim}{\ell}=\underset{\sim}{\lambda}} = c_{ij_1} = a_{i\alpha}$$

$$\left.\frac{\partial^2 T_i(\ell_1,\ldots,\ell_a)}{\partial \ell_{j_1}\,\partial \ell_{j_2}}\right|_{\underset{\sim}{\ell}=\underset{\sim}{\lambda}} = c_{ij_1j_2} = a_{i\alpha\beta} \tag{5.26}$$

$$\left.\frac{\partial^3 T_i(\ell_1,\ldots,\ell_a)}{\partial \ell_{j_1}\,\partial \ell_{j_2}\,\partial \ell_{j_3}}\right|_{\underset{\sim}{\ell}=\underset{\sim}{\lambda}} = c_{ij_1j_2j_3} = a_{i\alpha\beta\gamma}$$

for $j_1 \in J_\alpha$, $j_2 \in J_\beta$, $j_3 \in J_\gamma$, $\underset{\sim}{\ell}' = (\ell_1,\ldots,\ell_a)$, $\underset{\sim}{\lambda}' = (\lambda_1,\ldots,\lambda_a)$ and J_α denotes the set of integers $q_1 +\ldots+ q_{\alpha-1} + 1,\ldots,q_1 +\ldots+ q_\alpha$. Then

$$L_i = \sqrt{n}\,\{T_i(\ell_1,\ldots,\ell_a) - T_i(\lambda_1,\ldots,\lambda_a)\}$$

$$= \sum_{\alpha=1}^{r} a_{i\alpha}\,\mathrm{tr}\,U_{\alpha\alpha}^{(1)} + \frac{1}{\sqrt{n}}\,[\sum_{\alpha=1}^{r} a_{i\alpha}\,\mathrm{tr}(U_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha} \theta_{\alpha\beta}^{-1}\,U_{\alpha\beta}^{(1)}\,U_{\beta\alpha}^{(1)}$$

$$+ \frac{1}{2} \sum_{\alpha}\sum_{\beta} a_{i\alpha\beta}\,\mathrm{tr}\,U_{\alpha\alpha}^{(1)}\,\mathrm{tr}\,U_{\beta\beta}^{(1)} +\ldots] \tag{5.27}$$

for $i = 1,2,\ldots,k$. Let H and K be orthogonal matrices and let

$$R_1 = H'BK.$$

If we choose the first columns of H and K such that

$$h_{i0} = (n_{i.}/n)^{\frac{1}{2}} \qquad i = 1,2,\ldots,(a+1)$$

$$k_{j0} = (n_{.j}/n)^{\frac{1}{2}} \qquad j = 1,2,\ldots,(b+1),$$

then

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}.$$

Similarly, let $\Omega_1 = H^{*\prime}FK^*$ where the first columns of H^* and K^* are given by

$$h^*_{i0} = p_i^{\frac{1}{2}}$$

$$k^*_{j0} = q_j^{\frac{1}{2}}$$

Then

$$\Omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & \Omega \end{pmatrix}.$$

So,

$$R_1R_1' = \begin{pmatrix} 1 & 0 \\ 0 & RR' \end{pmatrix}, \quad \Omega_1\Omega_1' = \begin{pmatrix} 1 & 0 \\ 0 & \Omega\Omega' \end{pmatrix}.$$

It is known (see O'Neill [17]) that $\hat{\rho}_1^2 \geq \ldots \geq \hat{\rho}_a^2$ are the eigenvalues of RR' whereas $\rho_1^2 \geq \ldots \geq \rho_a^2$ are the eigenvalues of $\Omega\Omega'$. Next, let

$$X = \sqrt{n}\,(R-\Omega) = (x_{ij}) \tag{5.28}$$

where $R = (r_{ij})$ and $\Omega = (\omega_{ij})$. Then x_{ij}'s are known (see O'Neill [17]) to be asymptotically distributed as multivariate normal with mean vector $\underset{\sim}{0}$ and the elements of the covariance matrix are given by

$$\mathrm{Cov}(x_{ij}, x_{k\ell}) = \sigma_{ij.k\ell} \text{ (say)}. \tag{5.29}$$

Using (5.28), we obtain

$$RR' = \Omega\Omega' + n^{-\frac{1}{2}}(\Omega X' + X\Omega') + n^{-1}XX'. \tag{5.30}$$

Now, let M: a×a and L: b×b be such that

$$M'\Omega L = (\mathrm{diag}(\rho_1,\dots,\rho_a)|0) = D_\rho.$$

Then,

$$S = M'RR'M = V + \frac{1}{\sqrt{n}} V^{(1)} + \frac{1}{n} V^{(2)} \tag{5.31}$$

where $V = D_\rho D_\rho'$, $V^{(1)} = (V^{(1)}_{\alpha\beta}) = D_\rho Z' + ZD_\rho'$, $Z = (z_{ij}) = M'XL$, and $V^{(2)} = (V^{(2)}_{\alpha\beta})$ $= M'XX'M$. Here $V^{(1)}_{\alpha\beta}$ and $V^{(2)}_{\alpha\beta}$ are of order $q_\alpha \times q_\beta$. Now, let $\ell_i = \hat{\rho}_i^2$, $\lambda_i = \rho_i^2$ and λ_i's have multiplicities as in (5.24). Then, using (5.26), we obtain

$$\begin{aligned} L_i &= \sqrt{n}\,\{T_i(\ell_1,\dots,\ell_p) - T_i(\lambda_1,\dots,\lambda_p)\} \\ &= \sum_{\alpha=1}^{r} a_{i\alpha}\,\mathrm{tr}\,V^{(1)}_{\alpha\alpha} + \frac{1}{\sqrt{n}}\Big[\sum_{\alpha=1}^{r} a_{i\alpha}\mathrm{tr}\{V^{(2)}_{\alpha\alpha} + \sum_{\beta\neq\alpha}\theta^{-1}_{\alpha\beta}V^{(1)}_{\alpha\beta}V^{(1)}_{\beta\alpha}\} \\ &\qquad + \tfrac{1}{2}\sum\sum a_{i\alpha\beta}\,\mathrm{tr}\,V^{(1)}_{\alpha\alpha}\,\mathrm{tr}\,V^{(1)}_{\beta\beta}\Big] \\ &\qquad + \text{terms of higher order.} \end{aligned} \tag{5.32}$$

But,

$$V^{(1)} = \begin{pmatrix} 2\rho_1 z_{11} & (\rho_2 z_{12} + \rho_1 z_{21}) & \cdots & (\rho_a z_{1a} + \rho_1 z_{a1}) \\ (\rho_1 z_{21} + \rho_2 z_{12}) & 2\rho_2 z_{22} & \cdots & (\rho_a z_{2a} + \rho_2 z_{a2}) \\ \vdots & \vdots & & \vdots \\ (\rho_1 z_{a1} + \rho_a z_{1a}) & (\rho_2 z_{a2} + \rho_a z_{2a}) & \cdots & 2\rho_a z_{aa} \end{pmatrix} \tag{5.33}$$

$$V^{(2)} = \begin{pmatrix} M_1'XX'M_1 & M_1'XX'M_2 & \cdots & M_1'XX'M_r \\ M_2'XX'M_1 & M_2'XX'M_2 & \cdots & M_2'XX'M_r \\ \vdots & \vdots & & \vdots \\ M_r'XX'M_1 & M_r'XX'M_2 & \cdots & M_r'XX'M_r \end{pmatrix}$$

where $M = (M_1,\ldots,M_r)$ and M_i is of order $a \times q_i$. So,

$$L_i = 2\sum_{\alpha=1}^{r} a_{i\alpha}\theta_\alpha(z_{q_1+\ldots+q_{\alpha-1}+1,q_1+\ldots+q_{\alpha-1}+1} +\ldots+ z_{q_1+\ldots+q_\alpha,q_1+\ldots+q_\alpha})$$

$$= \underset{\sim}{b}_i'\underset{\sim}{z}_0 \tag{5.35}$$

where

$$\underset{\sim}{b}_i' = (2a_{i1}\theta_1\underset{\sim}{1}_{q_1}',\ldots,2a_{ir}\theta_r\underset{\sim}{1}_{q_r}') \tag{5.36}$$

and $\underset{\sim}{z}_0' = (z_{11},\ldots,z_{aa})$. The asymptotic distribution of $B'\underset{\sim}{z}$ is multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix $B'\Sigma^*B$ where Σ^* is the covariance matrix of $\underset{\sim}{z}$ and $B = (\underset{\sim}{b}_1,\ldots,\underset{\sim}{b}_k)$. We can summarize the above results as follows:

Theorem 5.1 We assume that ρ_i's have multiplicities as given below:

$$\rho_{q_1+\ldots+q_{\alpha-1}} =\ldots= \rho_{q_1+\ldots+q_\alpha} = \theta_\alpha \tag{5.37}$$

for $\alpha = 1,2,\ldots,r$ where $q_0 = 0$, $q_1+\ldots+q_r = a$. Also, let $L_1,\ldots,L_k$ be functions of $\hat{\rho}_1^2,\ldots,\hat{\rho}_a^2$ satisfying the assumptions (5.25). Then, as $n \to \infty$, the joint distribution of $L_1,\ldots,L_k$ is multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix $B'\Sigma^*B$ where Σ^* is the covariance matrix of $\underset{\sim}{z}$, $B = (\underset{\sim}{b}_1,\ldots,\underset{\sim}{b}_k)$ and $\underset{\sim}{b}_i$'s are defined by (5.36).

Now, let $\sum_{\alpha=1}^{r} a_{i\alpha} trV_{\alpha\alpha}^{(1)} = 0$ for each i. Then,

$$L_i^* = n\{T_i(\ell_1,\ldots,\ell_p) - T_i(\lambda_1,\ldots,\lambda_p)\}$$

$$= \tfrac{1}{2}(v_1,\ldots,v_r)\, A_i \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix},$$

where

$$A_i = (a_{i\alpha\beta})$$
$$v_1 = 2(\rho_1 z_{11} + \ldots + \rho_{q_1} z_{q_1 q_1})$$

$$v_2 = 2(\rho_{q_1+1} z_{q_1+1,q_1+1} + \ldots + \rho_{q_1+q_2} z_{q_1+q_2,q_1+q_2})$$
$$\vdots$$
$$v_r = 2(\rho_{q_1+\ldots+q_{r-1}+1} z_{q_1+\ldots+q_{r-1}+1,q_1+\ldots+q_{r-1}+1} + \ldots$$

$$+ \rho_{q_1+\ldots+q_r} z_{q_1+\ldots+q_r,q_1+\ldots+q_r}).$$

Since ρ_i^2's have multiplicities, we can write $\underset{\sim}{v} = (v_1,\ldots,v_r)'$ as $\underset{\sim}{v} = E\underset{\sim}{z}_0$ where

$$E = (\underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_r),$$

$$\underset{\sim}{e}_1' = 2\theta_1(\underset{\sim}{1}_{q_1}', 0\ldots\ldots 0)$$

$$\underset{\sim}{e}_2' = 2\theta_2(0\ldots,0,\underset{\sim}{1}_{q_2}',0,\ldots,0)$$
$$\vdots$$
$$\underset{\sim}{e}_r' = 2\theta_r(0,\ldots\ldots 0,\underset{\sim}{1}_q')$$

As $n \to \infty$, $\underset{\sim}{v}$ is distributed as a multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix $E\Sigma^* E'$. So, the joint distribution of $L_1^*,\ldots,L_k^*$ is the same as that of correlated quadratic forms discussed in Khatri, Krishnaiah and Sen [7].

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8-86