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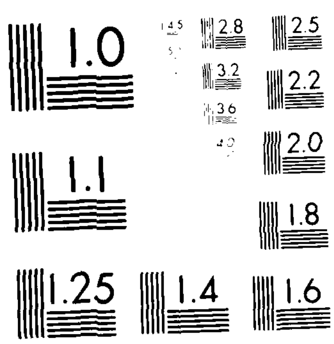

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MOVING AVERAGE MODELS WITH BIVARIATE EXPONENTIAL AND GEOMETRIC DISTRIBUTIONS

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Abstract

Two classes of finite and infinite moving average sequences of bivariate random vectors are considered. The first class has bivariate exponential marginals while the second class has bivariate geometric marginals. The theory of positive dependence is used to show that in various cases the two classes consist of associated random variables. Association is then applied to establish moment inequalities and to obtain approximations to some joint probabilities of the bivariate processes.

Key Words: bivariate exponential and geometric distributions, association, bivariate moving averages and bivariate point processes.

1. Introduction and Summary

A primary stationary model in time series analysis is the $p\times 1$ moving average (MA) model given by:

$$\underset{\sim}{X}(n) = \sum_{j=-\infty}^{\infty} A(j)\underset{\sim}{\varepsilon}(n-j), \quad n = 0,\pm 1,\pm 2,\ldots, \tag{1.1}$$

where $A(j)$, $j = 0,\pm 1,\pm 2,\ldots$, is a sequence of $p\times p$ parameter matrices such that $\sum_{j=-\infty}^{\infty} \|A(j)\| < \infty$, and $\underset{\sim}{\varepsilon}(n)$, $n = 0,\pm 1,\pm 2,\ldots$, is a sequence of uncorrelated $p\times 1$ random vectors with mean zero and common covariance matrix. It is well known that this model emerges from many physically realizable systems (see, for example, Hannan (1970), p. 9). However, in some physical situations where the random vectors $\underset{\sim}{X}(n)$ are either positive or discrete, the preceeding assumptions on the $\underset{\sim}{\varepsilon}(n)$ sequence are inappropriate (see Lewis (1980), p. 152).

Several researchers, addressing themselves to this problem, have been constructing univariate stationary MA models and univariate stationary autoregressive moving average (ARMA) models where the random variables $X(n)$ have exponential or gamma distributions, and discrete models where $X(n)$ assumes values in a common set. Lawrance and Lewis (1977, 1980) present stationary MA models where the random variables $X(n)$ have exponential distributions; Gaver and Lewis (1980) consider stationary ARMA type models where the random variables $X(n)$ have gamma distributions. Jacobs and Lewis (1978a,b, 1983) construct ARMA type models where the random variables $X(n)$ are discrete and assume values in a common finite set. The aforementioned models have been used in the various fields of applied probability and time series analysis, for example, these models have been used to model and analyze univariate point processes with correlated service and correlated interarrival times (see Jacobs (1978)). Details concerning univariate geometric MA processes and the corresponding point processes may be found in Langberg and Stoffer (1985).

In this paper we present two classes of finite and infinite MA sequences of bivariate random vectors. The first class has exponential marginals while the second class has geometric marginals. Within each class of models, the sequences are classified according to their order of dependence on the past. For the sake of clarity we restrict ourselves to bivariate MA sequences. However these models can be extended in a straight forward way. We use the theory of positive dependence to show that in a variety of cases the two classes of MA sequences are associated. We then apply the association to establish some moment and probability inequalities.

In Section 2 we define the bivariate exponential and geometric distributions which are the underlying distributions of our two classes, and present a variety of examples of such distributions. Further in Section 2 we define the concept of association and present a variety of bivariate exponential and geometric distributions that are associated. In Section 3 we construct the two classes of MA sequences proving that they have exponential or geometric marginals and showing that if the underlying distribution is associated, so is the related MA sequence. Finally in Section 3 we present the autocovariance matrices for both classes of sequences. In Section 4 we indicate how to relate bivariate point processes to the bivariate exponential or geometric MA processes discussed in Section 3. Also, in Section 4 we utilize positive dependence properties to obtain some probability bounds and moment inequalities for the bivariate processes

2. Preliminaries

In this section we present definitions and prove some basic results to be used in the sequel. First, we present a definition of a bivaraite geometric distribution.

Definition 2.1. Let M,N be random variables assuming values in the set $\{1,2,\ldots\}$.

We say that (M,N) has a bivariate geometric distribution if M and N have goemetric distributions.

<u>Examples 2.2</u>. (a) Let N be geometric. Then (N,N) is bivariate geometric. (b) Let M and N be independent geometric random variables, then (M,N) is bivariate geometric. (c) Let N_1,N_2,N_3 be independent geometric random variables, and put $M = \{\min(N_1,N_3)\}$, $N = \{\min(N_2,N_3)\}$. Then (M,N) has the Esary-Marshall (1974) bivariate geometric distribution. (d) Let $P_{00},P_{01},P_{10},P_{11}$ be in [0,1] such that (i) $P_{00}+P_{01}+P_{10}+P_{11} = 1$, (ii) $P_{01}+P_{11} < 1$ and $P_{10}+P_{11} < 1$, and let M, N be random variables assuming values in the set {1,2,...} determined by:

$$(2.3) \qquad P(M>a,\ N>b) = \begin{cases} P_{11}^{a}[P_{01}+P_{11}]^{b-a}, & b \geq a, \\ P_{11}^{b}[P_{10}+P_{11}]^{a-b}, & b \leq a, \quad a,b=1,2,\dots . \end{cases}$$

Then (M,N) has the Block (1977) fundamental bivariate geometric distribution (see also Block and Paulson (1984)). (e) Let (M_1,M_2) be bivariate geometric and let $(N_1(j),N_2(j))$, $j=1,2,\dots$, be an iid sequence of random vectors with bivariate geometric distributions which are independent of (M_1,M_2). Then $(\sum_{j=1}^{M_1}N_1(j),\sum_{j=1}^{M_2}N_2(j))$ has a bivariate geometric distribution.

In the following remark we show that Examples 2.2a, 2.2b, 2.2c, but not 2.2e, are particular cases of Example 2.2d.

<u>Remarks 2.4</u>. (a) Let $P_{10} = P_{01} = 0$ in equation (2.3). Then we obtain the distribution introduced in Example 2.2a. (b) Let $P_{11} = (P_{11}+P_{10})(P_{11}+P_{01})$ in (2.3). Then we obtain the bivariate geometric distribution introduced in Example 2.2b. (c) Let $P_{11} \geq (P_{11}+P_{10})(P_{11}+P_{01})$ in (2.3) and let N_1,N_2,N_3 be independent geometric random variables with parameters $P_{11}(P_{11}+P_{10})^{-1}$, $P_{11}(P_{11}+P_{01})^{-1}$, $P_{11}^{-1}(P_{11}+P_{10})(P_{11}+P_{01})$, respectively. Put $M = \{\min(N_1,N_3)\}$ and $N = \{\min(N_2,N_3)\}$. Then (M,N) is stochastically equal to the Esary-Marshall bivariate geometric distribution given in Example 2.2c.

Next, we present a definition of a bivariate exponential distribution.

Definition 2.5 Let E_1, E_2 be random variables assuming values in $(0,\infty)$. We say that (E_1,E_2) has a bivariate exponential distribution if E_1 and E_2 have exponential distributions.

Examples 2.6 (a) Let E be exponential. Then (E,E) is bivariate exponential. (b) Let E_1, E_2 be independent exponentials. Then (E_1,E_2) has a bivariate exponential distribution. (c) Let X_1, X_2, X_3 be independent exponentials and put $E_1 = \{\min(X_1,X_3)\}$, $E_2 = \{\min(X_2,X_3)\}$. Then (E_1,E_2) has the Marshall-Olkin (1967) bivariate exponential distribution. (d) Let (M,N) have a bivariate geometric distribution and let $(E_1(j),E_2(j))$, $j=1,2,\ldots$, be an iid sequence of random vectors with bivariate exponential distributions, independent of M and N. Then $(\sum_{j=1}^{M} E_1(j), \sum_{j=1}^{N} E_2(j))$ has a bivariate exponential distribution. (e) Let $0 \leq \alpha \leq 1$. Then (E_1,E_2) determined by $P\{E_1 > x, E_2 > y\} = \exp\{-x-y-\alpha xy\}$, $x,y > 0$, has a Gumbel (1960) bivariate exponential distribution. (f) Let $|\alpha| \leq 1$. Then (E_1,E_2) determined by $P\{E_1 \leq x, E_2 \leq y\} = (1-e^{-x})(1-e^{-y})(1+\alpha e^{-x-y})$, $x,y > 0$, has a bivariate Gumbel (1960) exponential distribution. (g) Let $\alpha \geq 1$. Then (E_1,E_2) determined by $P\{E_1 > x, E_2 > y\} = e^{-(x^\alpha+y^\alpha)^{1/\alpha}}$, $x,y > 0$, is bivariate exponential. (h) Let (X,Y) be a random vector with continuous marginal distributions F and G, respectively. Then the random vector $(-\ln[1-F(X)], -\ln[1-G(Y)])$ is bivariate exponential.

Example 2.6(d) has been used by several researchers to generate bivariate distributions (for example Arnold (1975), Downton (1970), and Hawkes (1972) to mention a few). In the following remarks we illustrate how some of the bivariate exponential distributions are obtained from Example 2.6(d).

Remarks 2.7. (a) M = N and let $E_1(j)$, $E_2(j)$ be independent exponentials, $j=1,2,\ldots$. Then we obtain the distribution introduced by Downton (1970). (b) Let (M,N) be as in Example 2.2(d) and let $E_1(j)$, $E_2(j)$ be independent exponentials, $j=1,2,\ldots$. Then we obtain the bivariate exponential distribution introduced by Hawkes (1972) and Paulson (1973). (c) Let (M,N) be as in Example 2.2(c) and let $E_1(j) = E_2(j)$, $j = 1,2,\ldots$, Then we obtain the Marshall-Olkin (1967) distribution given in Example 2.6(c) (for details see Marshall-Olkin (1967)).

Finally we present a concept of positive dependence.

Definition 2.8 Let $\underset{\sim}{T} = (T_1,\ldots,T_n)$, $n = 1,2,\ldots$, be a multivariate random vector. We say that the random variables $T_1,\ldots,T_n$ are associated if for all pairs of measurable bounded functions $f,g\colon R^n \to R$ both nondecreasing in each argument $\mathrm{cov}(f(\underset{\sim}{T}),g(\underset{\sim}{T})) \geq 0$.

Remarks 2.9. (a) Note that independent random variables are associated and that nondecreasing functions of associated random variables are associated (cf. Barlow and Proschan (1975) pp. 30-31). Thus the components of the vector given in Example 2.2(c) and the components of the vector given in Example 2.6(c) are associated. (b) Let (E_1,E_2) be as in Example 2.6(e) with $\alpha > 0$, or as in 2.6(f) with $-1 \leq \alpha < 0$. Since $P\{E_1 > x,\ E_2 > y\} < P\{E_1 > x\}P\{E_2 > y\}$ for $x,y > 0$, E_1 and E_2 are not associated. (c) Let (X,Y) be as in Example 2.6(h). Then $-\ln[1-F(X)]$ and $-\ln[1-G(Y)]$ are associated if and only if X and Y are associated (cf. Barlow and Proschan (1975), Proposition 3, p. 30).

The following lemma provides sufficient conditions for some of the bivariate distributions presented in Examples 2.2 and 2.6 to be associated.

Lemma 2.10. Let $\underset{\sim}{Q} = (Q_1,Q_2)$ be a random vector with components assuming values in the set $\{1,2,\ldots\}$ and let $\underset{\sim}{R}(j) = (R_1(j),\ R_2(j))$, $j = 1,2,\ldots$, be an iid sequence of nonnegative random vectors independent of $\underset{\sim}{Q}$. If Q_1 and Q_2 are associated, and $R_1(1)$ and $R_2(1)$ are associated, then $\sum_{j=1}^{Q_1} R_1(j)$ and $\sum_{j=1}^{Q_2} R_2(j)$ are associated.

Proof: Let $f,g\colon R^2 \to R$ be measurable bounded functions nondecreasing in each argument and let $X = \sum_{j=1}^{Q_1} R_1(j)$ and $Y = \sum_{j=1}^{Q_2} R_1(j)$. First note that

$$\mathrm{cov}\{f(X,Y),g(X,Y)\} = E\{\mathrm{cov}(f(X,Y),g(X,Y))|\underset{\sim}{Q}\}$$
$$+ \mathrm{cov}\{Ef(X,Y)|\underset{\sim}{Q},\ Eg(X,Y)|\underset{\sim}{Q}\}.$$

Now, $Ef(X,Y)|\underset{\sim}{Q}$ and $Eg(X,Y)|\underset{\sim}{Q}$ are nondecreasing functions of Q_1 and Q_2. Since Q_1 and Q_2 are associated we have

$$\mathrm{cov}\{Ef(X,Y)|\underset{\sim}{Q},\ Eg(X,Y)|\underset{\sim}{Q}\} \geq 0.$$

Now, let $Q = \max(Q_1, Q_2)$. Since $f(X,Y)|\underset{\sim}{Q}$ and $g(X,Y)|\underset{\sim}{Q}$ are nondecreasing functions of $R_1(1),\ldots,R_1(Q),R_2(1),\ldots,R_2(Q)$, these random variables are associated (cf. Barlow and Proschan (1975), Theorem 2.2). Thus

$$\mathrm{cov}\{f(X,Y)|\underset{\sim}{Q},\ g(X,Y)|\underset{\sim}{Q}\} \geq 0.$$

Consequently, $\mathrm{cov}\{f(X,Y),g(X,Y)\} \geq 0$ and X and Y are associated. □

<u>Remarks 2.11</u> In particular, we conclude from Lemma 2.10 that: (a) The components of the bivariate geometric distribution given in Example 2.2(e) are associated provided that M_1 and M_2, and $N_1(1)$ and $N_2(1)$ are associated. (b) The components of the bivariate exponential distribution given in Example 2.6(d) are associated provided that M and N, and $E_1(1)$ and $E_2(1)$ are associated.

3. Model Constructions

In this section we construct two classes of finite and infinite MA sequences of bivariate random vectors. We denote the first class of sequences by $\{\underset{\sim}{X}(n,m) = (X_1(n,m),X_2(n,m)),\ n = 0,\pm1,\pm2,\ldots\}$ $m = 1,2,\ldots,\infty$, and the second class of sequences by $\{\underset{\sim}{G}(n,m) = (G_1(n,m),\ G_2(n,m)),\ n = 0,\pm1,\pm2,\ldots\}$ $m = 1,2,\ldots,\infty$. We show that each random vector $\underset{\sim}{X}(n,m)$ has a bivariate exponential distribution with a vector mean that does not depend on n or m and that each $\underset{\sim}{G}(n,m)$ has a bivariate geometric distribution with a vector mean independent of n or m. Within each class of sequences the order of dependence on the past is indicated by the parameter m. For each positive integer m, $\underset{\sim}{X}(n,m)$ and $\underset{\sim}{G}(n,m)$ depends only on the previous m variates $\{\underset{\sim}{X}(n-1,m),\ldots,\underset{\sim}{X}(n-m,m)\}$ and $\{\underset{\sim}{G}(n-1,m),\ldots,\underset{\sim}{G}(n-m,m)\}$, respectively, while $\underset{\sim}{X}(n,\infty)$ and $\underset{\sim}{G}(n,\infty)$ depends on all the preceeding random vectors $\{\underset{\sim}{X}(n,1,\infty),\underset{\sim}{X}(n-2,\infty),\ldots\}$ and $\{\underset{\sim}{G}(n-1,\infty),\underset{\sim}{G}(n-2,\infty),\ldots\}$, respectively. After constructing the various models we present sufficient conditions for the random variables $\{X_\ell(n_j,m)\}$ and $\{G_\ell(n_j,m)\}$, $\ell = 1,2$; $j = 1,2,\ldots,k$ to be associated, where $k = 1,2,\ldots$, and $n_1 < n_2 < \ldots < n_k \in \{0, \pm 1, \pm 2,\ldots\}$. We conclude this section by computing the autocovariance matrices for the two classes of sequences.

First, we construct the exponential class of sequences. Some notation is needed.

<u>Notation 3.1</u>. Throughout, n ranges over the integers and m,j over the positive integers. Let $\underset{\sim}{E}(n) = (E_1(n),E_2(n))$ be iid bivariate exponential random vectors with mean vector $(\lambda_1^{-1},\lambda_2^{-1})$; $\lambda_1,\lambda_2 > 0$. Let $\beta_1(n,j)$ and $\beta_2(n,j)$ be parameters taking values in $[0,1]$ and let $B(n,j)$ be a 2×2 diagonal matrix with $B(n,j) = \mathrm{diag}\{\beta_1(n,j),\ \beta_2(n,j)\}$. Further let $(I_1(n,j),I_2(n,j))$ be independent bivariate random vectors independent of all the $\underset{\sim}{E}(n)$ such that $I_1(n,j)$ and $I_2(n,j)$ are Bernoulli with parameters $1-\beta_1(n,j)$ and $1-\beta_2(n,j)$, respectively. Let $V_q(n,j)$ be a 2×2 random diagonal matrix defined by $V_q(n,j) = \mathrm{diag}\{\prod_{k=q}^{j} I_1(n,k),\ \prod_{k=q}^{j} I_2(n,k)\}$, $q \in \{1,2,\ldots,j\}$, and for ease of notation we put $V_1(n,j) \equiv V(n,j)$. Finally <u>let a sum (product) over an empty set of indices be equal to zero (one)</u>.

We now present the class of exponential sequences. For $m = 1,2,\ldots,$ and $n = 0, \pm 1, \pm 2,\ldots$, let

$$\underset{\sim}{X}(n,m) = \sum_{r=0}^{m-1} V(n,r)B(n,r+1)\underset{\sim}{E}(n-r) + V(n,m)\underset{\sim}{E}(n-m) \tag{3.2}$$

and

$$\underset{\sim}{X}(n,\infty) = \sum_{r=0}^{\infty} V(n,r)B(n,r+1)\underset{\sim}{E}(n-r). \tag{3.3}$$

We show in Corollary 3.8 and Lemma 3.9 that for all n,m, $\underset{\sim}{X}(n,m)$ and $\underset{\sim}{X}(n,\infty)$ have bivariate exponential distributions. Next, we construct the geometric class. Some notation is needed.

Notation 3.4. Let p_1, p_2 be real numbers in (0,1] and let $\alpha_1(n)$ and $\alpha_2(n)$ be a sequence of parameters such that $p_j \le \alpha_j(n) \le 1$, $j = 1,2$. Further, let $\underset{\sim}{N}(n) = (N_1(n), N_2(n))$ be independent bivariate geometric vectors with mean vector $(p_1^{-1}\,\alpha_1(n),\ p_2^{-1}\alpha_2(n))$ and let $\underset{\sim}{M}(n) = (M_1(n),\ M_2(n))$ be iid bivariate geoemtrics, independent of all $\underset{\sim}{N}(n)$, with mean vector (p_1^{-1}, p_2^{-1}). Finally, let $(J_1(n,j), J_2(n,j))$ be independent random vectors, independent of all $\underset{\sim}{M}(n)$ and $\underset{\sim}{N}(n)$, such that $J_i(n,j)$ is Bernoulli with parameter $(1-\alpha_i(n))$, $i = 1,2$, and let $U_q(n,j)$ be a 2×2 random diagonal matrix $U_q(n,j) = \text{diag}\{\prod_{k=q}^{j} J_1(n,k),\ \prod_{k=q}^{j} J_2(n,k)\}$, $q \in \{1,2,\ldots,j\}$. To ease the notation we put $U_1(n,j) \equiv U(n,j)$.

We now present the class of geometric sequences. For $m = 1,2,\ldots,$ and $n = 0, \pm 1, \pm 2, \ldots,$ let

$$\underset{\sim}{G}(n,m) = \sum_{r=0}^{m} U(n,r)\underset{\sim}{N}(n-r) + U(n,m+1)\underset{\sim}{M}(n-m) \tag{3.5}$$

and

$$\underset{\sim}{G}(n,\infty) = \sum_{r=0}^{\infty} U(n,r)\underset{\sim}{N}(n-r). \tag{3.6}$$

Next, we show that $\underset{\sim}{X}(n,m)$ and $\underset{\sim}{G}(n,m)$ have bivariate exponential and geometric distributions, respectively. The following lemma is needed.

Lemma 3.7 For $n = 0, \pm 1, \pm 2, \ldots,$ and $m,q = 1,2,\ldots,$ let

$$\underset{\sim}{Y}_q(n,m) = \sum_{r=0}^{m-1} V_q(n,r+q-1)B(n,r+q)\underset{\sim}{E}(n-r-q+1) + V_q(n,m+q-1)\underset{\sim}{E}(n-m-q+1)$$

and

$$\underset{\sim}{H}_q(n,m) = \sum_{r=0}^{m} U_q(n,r+q-1)\underset{\sim}{N}(n-r-q+1) + U_q(n,m+q)\underset{\sim}{M}(n-m-q+1).$$

Then for all n,m, and q, $\underset{\sim}{Y}_q(n,m)$ has a bivariate exponential distribution with mean vector $(\lambda_1^{-1}, \lambda_2^{-1})$ and $\underset{\sim}{H}_q(n,m)$ has a bivariate geometric distribution with mean vector (p_1^{-1}, p_2^{-1}).

Proof: We prove the result of the lemma by an induction argument on m. For $m = 1$,

$$\underset{\sim}{Y}_q(n,1) = B(n,q)\underset{\sim}{E}(n-q+1) + V_q(n,q)\underset{\sim}{E}(n-q)$$

and for $m = 0$,

$$\underset{\sim}{H}_q(n,0) = \underset{\sim}{N}(n-q+1) + U_q(n,q)\underset{\sim}{M}(n-q+1).$$

By computing the characteristic functions of the components of $\underset{\sim}{Y}_q(n,1)$ and $\underset{\sim}{H}_q(n,0)$ one can verify that the results of the lemma hold for all n,q. Assume now that the results of the lemma hold for m, and all n,q, Noting that

$$\underset{\sim}{Y}_q(n,m+1) = B(n,q)\underset{\sim}{E}(n-q+1) + V_q(n,q)\left[\sum_{r=0}^{m-1} V_{q+1}(n,r+q)B(n,r+q+1)\underset{\sim}{E}(n-q-r)\right.$$
$$\left. + V_{q+1}(n,m+q)\underset{\sim}{E}(n-m-q)\right]$$

and

$$\underset{\sim}{H}_q(n,m+1) = \underset{\sim}{N}(n-q+1) + U_q(n,q)\left[\sum_{r=0}^{m} U_{q+1}(n,r+q)\underset{\sim}{N}(n-q-r)\right.$$
$$\left. + U_{q+1}(n,m+q+1)\underset{\sim}{M}(n-m-q)\right]$$

we see that, by induction, the terms in the brackets are bivariate exponential with mean $(\lambda_1^{-1},\lambda_2^{-1})$ and bivariate geometric with mean (p_1^{-1},p_2^{-1}), respectively. Since these terms are independent of $\underset{\sim}{E}(n-q+1)$ and $\underset{\sim}{N}(n-q+1)$, respectively, it follows as in the case $m = 1$ and $m = 0$, respectively, that $\underset{\sim}{Y}_q(n,m+1)$ and $\underset{\sim}{H}_q(n,m+1)$ have the appropriate distributions for all n and q. □

Note that $\underset{\sim}{X}(n,m)$ and $\underset{\sim}{G}(n,m)$ given by (3.2) and (3.5), respectively, are equal to $\underset{\sim}{Y}_1(n,m)$ and $\underset{\sim}{H}_1(n,m)$, respectively. Thus, we conclude from Lemma 3.7 that:

Corollary 3.8 For all n and m, $\underset{\sim}{X}(n,m)$ has a bivariate exponential distribution with mean vector $(\lambda_1^{-1},\lambda_2^{-1})$, and $\underset{\sim}{G}(n,m)$ has a bivariate geometric distribution with mean vector (p_1^{-1},p_2^{-1}).

Next, we show that $\underset{\sim}{G}(n,\infty)$ given by (3.6) is bivariate geometric, and if for all n and i = 1,2

$$\lim_{m\to\infty} \prod_{j=1}^{m} [1-\beta_i(n,j)] = 0 \tag{3.9}$$

then $\underset{\sim}{X}(n,\infty)$ given by (3.3), is bivariate exponential.

Lemma 3.10. (a) For all n, $\underset{\sim}{G}(n,\infty)$ has a bivariate geometric distribution with mean vector (p_1^{-1},p_2^{-1}). (b) If condition (3.9) holds, then for all n, $\underset{\sim}{X}(n,\infty)$ has a bivariate exponential distribution with mean vector $(\lambda_1^{-1},\lambda_2^{-1})$.

Proof: Let m be a positive integer. Since $\lim_{m\to\infty} [1-\alpha_j(n)]^m \le \lim_{m\to\infty} (1-p_j)^m = 0$, $j = 1,2$, $\underset{\sim}{G}(n,m) \xrightarrow{P} \underset{\sim}{G}(n,\infty)$ as $m\to\infty$. By (3.9), $\underset{\sim}{X}(n,m) \xrightarrow{P} \underset{\sim}{X}(n,\infty)$ as $m\to\infty$. Thus in particular $\underset{\sim}{G}(n,m) \xrightarrow{D} \underset{\sim}{G}(n,\infty)$ and $\underset{\sim}{X}(n,m) \xrightarrow{D} \underset{\sim}{X}(n,\infty)$ as $m\to\infty$ and the results of the lemma follow from Corollary 3.8. □

Note that for (3.9) to hold, it suffices that for all n and i = 1,2, $\inf\{\beta_i(n,j),\ j = 1,2,\ldots\} > 0$. Next, we investigate some of the dependency aspects of both classes.

Remarks 3.11. (a) For fixed m, the sequences $\{\underset{\sim}{X}(n,m),\ n = 0,\pm1,\pm2,\ldots\}$ and $\{\underset{\sim}{G}(n,m),\ n = 0,\pm1,\pm2,\ldots\}$ are m-dependent (that is, if n_1 and n_2 are integers such that $|n_1-n_2| > m$, then $\underset{\sim}{X}(n_1,m)$ and $\underset{\sim}{X}(n_2,m)$ are independent as are $\underset{\sim}{G}(n_1,m)$ and $\underset{\sim}{G}(n_2,m)$). (b) Clearly if we choose m to be a function, say ψ, of n with $\psi(n) \in \{1,2,\ldots\}$ for all n, then the dependency of $\underset{\sim}{X}(n,\psi(n))$ and $\underset{\sim}{G}(n,\psi(n))$ on the past varies with n. (c) It is easy to see that for all n, $\underset{\sim}{X}(n,\infty)$ and $\underset{\sim}{G}(n,\infty)$ depends on all preceeding random vectors $\{\underset{\sim}{X}(q,\infty),\ -\infty < q < n\}$ and $\{\underset{\sim}{G}(q,\infty),\ -\infty < q < n\}$, respectively.

We now investigate a positive dependence aspect of both classes.

Lemma 3.12. Suppose that $E_1(1)$ and $E_2(1)$ are associated. Then for all positive integers m,k and all integers $n_1 < n_2 < \dots < n_k$, the random variables $\{X_i(n_j,m),\ i = 1,2;\ j = 1,\dots,k\}$ are associated.

Proof: By Barlow and Proschan (1975, Theorem 2.2, p. 31 and Proposition 4, p. 30) the random variables $E_i(n_j)$, $I_i(n_j,q)$, $i = 1,2;\ j = 1,\dots,k$, and $q = 1,\dots,m$ are associated. Since the $X_i(n_j,m)$, $i = 1,2;\ j = 1,\dots,k$ are nondecreasing functions of the previous collection of associated random variables the result of the lemma follows by Barlow and Proschan (1975, Proposition 3, p. 30). $\square$

In a similar way one can prove the following lemma.

Lemma 3.13. Suppose that $M_1(1)$ and $M_2(1)$ are associated and that for all n, $N_1(n)$ and $N_2(n)$ are associated. Then for all positive integers m,k and all integers $n_1 < n_2 < \dots < n_k$, the random variables $\{G_i(n_j,m),\ i = 1,2;\ j = 1,\dots,k\}$ are associated.

Now, we prove similar results for the infinite dependence sequences $\{X(n,\infty),\ n = 0,\pm 1,\pm 2,\dots\}$ and $\{G(n,\infty),\ n = 0,\pm 1,\pm 2,\dots\}$.

Lemma 3.14.(a) Suppose that $M_1(1)$ and $M_2(1)$ are associated, and that for all n, $N_1(n)$ and $N_2(n)$ are associated. Then for all positive integers k and all integers $n_1 < n_2 < \dots < n_k$, the random variables $\{G_i(n_j,\infty),\ i = 1,2;\ j = 1,\dots,k\}$ are associated. (b) If $E_1(1)$ and $E_2(1)$ are associated and condition (3.9) holds then for all positive integers k and all integers $n_1 < n_2 < \dots < n_k$, the random variables $\{X_i(n_j,\infty),\ i = 1,2 \text{ and } j = 1,\dots,k\}$ are associated.

Proof: By similar arguments to the ones given in the proof of Lemma 3.10 we conclude that the two sequences $\{G_1(n_1,m),G_2(n_1,m),\dots,G_1(n_k,m),G_2(n_k,m)\}$ and $\{X_1(n_1,m),X_2(n_1,m),\dots,X_1(n_k,m),X_2(n_k,m)\}$ converge in distribution as $m \to \infty$ to $\{G_1(n_1,\infty),G_2(n_1,\infty),\dots,G_1(n_k,\infty),G_2(n_k,\infty)\}$ and $\{X_1(n_1,\infty),X_2(n_1,\infty),\dots,X_1(n_k,\infty),X_2(n_k,\infty)$ respectively. By Lemma 3.12 the random variables $\{X_i(n_j,m),\ i = 1,2;\ j = 1,\dots,k\}$ are associated for all m and by Lemma 3.13 the $\{G_i(n_j,m),\ i=1,2;\ j=1,\ldots,k\}$ are associated for all m. Consequently, the results of the lemma follow by Esary, Proschan and Walkup (1967, Proposition 4). □

Next, we compute the autocovariance matrices for both classes of sequences. Some notation is needed.

<u>Notation 3.15</u>. Let Ξ_E, Ξ_M, and $\Xi_N(n)$ be the covariance matrices of $\underset{\sim}{E}(1)$, $\underset{\sim}{M}(1)$, and $\underset{\sim}{N}(n)$, respectively. For $h=0,1,2,\ldots$, let $\Gamma_X^m(n,h) = \mathrm{cov}(\underset{\sim}{X}(n,m),\underset{\sim}{X}(n+h,m))$ and $\Gamma_G^m(n,h) = \mathrm{cov}(\underset{\sim}{G}(n,m),\ \underset{\sim}{G}(n+h,m))$, $n = 0,\pm1,\pm2,\ldots$, and $m = 1,2,\ldots,\infty$. Further, let $A(n,j)$ be a 2×2 diagonal matrix, $A(n,j) = \mathrm{diag}\{[1-\alpha_1(n)]^j,\ [1-\alpha_2(n)]^j\}$, I be the 2×2 identity matrix, and χ the indicator function.

By some simple calculations we obtain for $n=0,\pm1,\pm2,\ldots;\ m=1,2,\ldots,\infty$, and $h = 1,2,\ldots$ (but not zero),

$$(3.16)\quad \Gamma_X^m(n,h) = \sum_{r=0}^{m-h-1} B(n,r+1)\{\prod_{j=1}^{r}[I-B(n,j)]\}\ \Xi_E\{\prod_{j=1}^{r+h}[I-B(n+h,j)]\}B(n+h,r+h-1)$$

$$+\ B(n,m-h+1)\{\prod_{j=1}^{m-h}[I-B(n,j)]\}\ \Xi_E\{\prod_{j=1}^{m}[I-B(n+h,j)]\}.$$

We may obtain the off-diagonal elements of $\Gamma_X^m(n,0)$ from (3.16) by setting $B(n,m+1)=I$ and $h = 0$. The diagonal elements of $\Gamma_X^m(n,0)$ are the variances of $X_1(n,m)$ and $X_2(n,m)$, namely, λ_1^{-2} and λ_2^{-2}, respectively. In a similar way we obtain for $n = 0,\ \pm1,\pm2,\ldots,\ m = 1,2,\ldots,\infty$, and $h = 1,2,\ldots$ (but not zero),

$$(3.17)\qquad \Gamma_G^m(n,h) = \sum_{r=0}^{m-h} A(n,r)\ \Xi_N(n-r)A(n+h,r+h)$$

$$+\chi_{\{0\}}(h)A(n,m+1)\Xi_M A(n,m+1).$$

We may obtain the off-diagonal elements of $\Gamma_G^m(n,0)$ from (3.17) by setting $h=0$; the diagonal elements are the variances of $G_1(n,m)$ and $G_2(n,m)$, namely, $(1-p_1)p_1^{-2}$ and $(1-p_2)p_2^{-2}$, respectively.

4. Inequalities

Throughout this section we fix m, $m = 1,2,\ldots,\infty$, and hence suppress it from our notation, that is, $\underset{\sim}{X}(n,m)$ and $\underset{\sim}{G}(n,m)$ are represented by $\underset{\sim}{X}(n)$ and $\underset{\sim}{G}(n)$, respectively.

In the point process theory of the models, the behavior of the vector of sums $S_{\underset{\sim}{X}}(\underset{\sim}{r}) = (S_{X_1}(r_1), S_{X_2}(r_2))$ where $S_{X_i}(r_i) = \sum_{n=1}^{r_i} X_i(n)$, $i = 1,2$ and $T_{\underset{\sim}{G}}(\underset{\sim}{r}) = (T_{G_1}(r_1), T_{G_2}(r_2))$ where $T_{G_i}(r_i) = \sum_{n=1}^{r_i} G_i(n)$, are of interest, $r_1, r_2 = 1,2,\ldots$. For example, if $\underset{\sim}{X}(n)$ is a vector of bivariate exponential interarrival times of a point process $M_{\underset{\sim}{X}}(\underset{\sim}{t}) = (M_{X_1}(t_1), M_{X_2}(t_2))$ which are the number of arrivals by times $t_1, t_2 > 0$, then

$$P\{M_{X_1}(t_1) \le r_1,\ M_{X_2}(t_2) \le r_2\} = P\{S_{X_1}(r_1) > t_1,\ S_{X_2}(r_2) > t_2\}.$$

Similarly, if $\underset{\sim}{G}(n)$ is a vector of bivariate geometric waiting times of a count process $N_{\underset{\sim}{G}}(\underset{\sim}{r}) = (N_{G_1}(r_1), N_{G_2}(r_2))$ which are the number of occurrences by trials $r_1, r_2 = 1,2,\ldots$, then $N_{G_i}(r_i) = T_{G_i}(r_i)$, $i = 1,2$.

We now utilize positive dependence properties to obtain probability bounds for the sums $S_{\underset{\sim}{X}}(\underset{\sim}{r})$ and $T_{\underset{\sim}{G}}(\underset{\sim}{r})$ and moment inequalities for the processes $\underset{\sim}{X}(n)$ and $\underset{\sim}{G}(n)$. First, we define two concepts of positive dependence.

Definition 4.1 Let $q = 2,3,\ldots$, and let $\underset{\sim}{X} = (X_1,\ldots,X_q)$ be a random vector. We say that $\underset{\sim}{X}$ is positively upper orthant dependent (PUOD) [positively lower orthant dependent (PLOD)] if for all real numbers $t_1,\ldots,t_q$

$$P\{X_j > t_j,\ j = 1,\dots,q\} \geq \prod_{j=1}^{q} P\{X_j > t_j\}$$

$$[P\{X_j \leq t_j,\ j = 1,\dots,q\} \geq \prod_{j=1}^{q} P\{X_j \leq t_j\}].$$

Remarks 4.2. (a) In the bivariate case ($q=2$) $\underset{\sim}{X}$ is PUOD iff $\underset{\sim}{X}$ is PLOD. (b) For $q > 2$ the two concepts of positive dependence are not equivalent. (c) If $X_1,\dots,X_q$ are associated then clearly $\underset{\sim}{X}$ is PUOD and PLOD. (d) Let $f_1,\dots,f_q$: $(-\infty,\infty) \to [0,\infty)$ be measurable nondecreasing (nonincreasing) functions and let $\underset{\sim}{X}$ be PUOD (PLOD). Then

$$E \prod_{j=1}^{q} f_j(X_j) \geq \prod_{j=1}^{q} Ef_j(X_j) \tag{4.3}$$

(see Lehman (1966)). For the sake of completeness we present the following definition.

Definition 4.4 Let X,Y be random variables. We say that X is stochastically less than or equal to Y, and write $X \overset{s}{\leq} Y$ is for every real number, $t, P(X>t) \leq P(Y>t)$.

Remark 4.5. Let $f: (-\infty,\infty) \to [0,\infty)$ be a measurable nondecreasing function and let $X \overset{s}{\leq} Y$. Then $Ef(X) \leq Ef(Y)$ (see Lehmann (1966)).

Next, we discuss the inheritance of positive dependence properties.

Lemma 4.6. Suppose that for $q = 1,2,\dots,$ the random variables $\{X_i(n),\ i = 1,2;\ n = 1,2,\dots,q\}$ are associated and the random variables $\{G_i(n),\ i = 1,2;\ n = 1,\dots,q\}$ are associated. Then for $r_1, r_2 = 1,2,\dots,$ and $t_1, t_2 > 0$ we have that (i) $\{S_{X_i}(r_i), i=1,2\}$ are associated, (ii) $\{T_{G_i}(r_i),\ i = 1,2\}$ or equivalently $\{N_{G_i}(r_i),\ i = 1,2\}$ are associated and (iii) $\{M_{X_i}(t_i),\ i = 1,2\}$ are PUOD and PLOD.

Proof: Parts (i) and (ii) follow from the facts that $S_{X_i}(r_i)$ and $T_{G_i}(r_i)$, $i = 1,2$, are nondecreasing functions of associated random variables (cf. Remark 2.9a). To show (iii), let $f_i = \chi\{S_{X_i}(r_i) > t_i\}$ and $g_i = \chi\{S_{X_i}(r_i) \leq t_i\}$, $i = 1,2$, where χ is the indicator function. By Barlow and Proschan (1975, Proposition 3, p. 30), f_i and g_i, $i = 1,2$ are associated since $S_{X_i}(r_i)$, $i = 1,2$, are associated. Hence by (4.3),

$$P\{M_{X_i}(t_i) > r_i,\ i=1,2\} = E \prod_{i=1}^{2} f_i \geq \prod_{i=1}^{2} Ef_i = \prod_{i=1}^{2} P\{M_{X_i}(t_i) > r_i\}$$

and

$$P\{M_{X_i}(t_i) \leq r_i,\ i=1,2\} = E \prod_{i=1}^{2} g_i \geq \prod_{i=1}^{2} Eg_i = \prod_{i=1}^{2} P\{M_{X_i}(t_i) \leq r_i\}. \qquad \square$$

For the stationary models, we may obtain bounds for sums $S_{\underset{\sim}{X}}(\underset{\sim}{r})$ and $T_{\underset{\sim}{G}}(\underset{\sim}{r})$ using gamma and negative binomial distributions, respectively. First, we concentrate on $S_{\underset{\sim}{X}}(\underset{\sim}{r})$.

<u>Lemma 4.7</u>. Assume that $\beta_1(n,1)$ and $\beta_2(n,1)$ are equal to β_1 and β_2, respectively, for all n. Let $Y_i(r_i,\theta_i)$, $i=1,2$, be gamma random variables with parameters (r_i,θ_i). Then $S_{X_i}(r_i) \overset{s}{\geq} Y_i(r_i,\lambda_i\beta_i^{-1})$, $i=1,2$. If in addition, the random variables $\{X_i(n),\ i=1,2;\ n=1,\dots,q\}$, $q=1,2,\dots$, are associated, then for $x_1,x_2>0$,

(4.8) $P\{S_{X_1}(r_1) \geq x_1, S_{X_2}(r_2) \geq x_2\} \geq P\{Y_1(r_1,\lambda_1\beta_1^{-1}) \geq x_1\} P\{Y_2(r_2,\lambda_2\beta_2^{-1}) \geq x_2\}$.

<u>Proof</u>: From equation (3.2) and (3.3) we see that $X_i(n) \geq \beta_i E_i(n)$, $i=1,2$. Hence

$$\begin{aligned} P\{S_{X_i}(r_i) \geq x_i\} &= P\{\textstyle\sum_{n=1}^{r_i} X_i(n) \geq x_i\} \\ &\geq P\{\textstyle\sum_{n=1}^{r_i} \beta_i E_i(n) \geq x_i\} \\ &= P\{Y_i(r_i,\lambda_i\beta_i^{-1}) \geq x_i\}, \quad i=1,2, \end{aligned}$$

and the first assertion is proved. Equation (4.8) now follows from the first assertion and the fact that since $S_{X_1}(r_1)$ and $S_{X_2}(r_2)$ are associated (Lemma 4.6) they are also PUOD. $\square$

In a similar manner we obtain the following lemma for the sums $T_{\underset{\sim}{G}}(\underset{\sim}{r})$.

<u>Lemma 4.9</u>. Assume that $\alpha_1(n)$ and $\alpha_2(n)$ are equal to α_1 and α_2, respectively, for all n. Let $NB_i(r_i, \alpha_i)$, $i=1,2$, be negative binomial random variables with

parameters (r_i, θ_i). Then $T_{G_i}(r_i) \overset{s}{\geq} NB_i(r_i, p_i\alpha_i^{-1})$, $i = 1,2$. If in addition, the random variables $\{G_i(n),\ i = 1,2;\ n = 1,\ldots,q\}$, $q = 1,2,\ldots$, are associated, then for $a_1 \geq r_1$ and $a_2 \geq r_2$,

$$(4.10)\quad P\{T_{G_1}(r_1) \geq a_1, T_{G_2}(r_2) \geq a_2\} \geq P\{NB_1(r_1, p_1\alpha_1^{-1}) \geq a_1\}P\{NB_2(r_2, p_2\alpha_2^{-1}) \geq a_2\}.$$

Proof: From equation (3.5) or (3.6) we have that $G_i(n) \geq N_i(n)$, $i = 1,2$; $n = 1,2,\ldots$. Hence

$$\begin{aligned} P\{T_{G_i}(r_i) \geq a_i\} &= P\{\textstyle\sum_{n=1}^{r_i} G_i(n) \geq a_i\} \\ &\geq P\{\textstyle\sum_{n=1}^{r_i} N_i(n) \geq a_i\} \\ &= P\{NB_i(r_i, p_i\alpha_i^{-1}) \geq a_i\}, \quad i = 1,2, \end{aligned}$$

and the first assertion is proved. Equation (4.10) now follows from the association of $T_{G_1}(r_1)$ and $T_{G_2}(r_2)$. $\square$

Finally, we address ourselves to some moment inequalities.

Lemma 4.11. Let us assume that the random variables $\{X_i(n),\ i = 1,2;\ n = 1,\ldots,h\}$, $h = 1,2,\ldots$, are associated. Let $k_1,\ldots,k_q$ be positive integers and let $\ell_1, \ell_2, \ldots, \ell_q = 1,2$; $q = 1,2,\ldots$. Then

$$E \prod_{j=1}^{q} \{X_{\ell_j}(j)\}^{k_j} \geq \prod_{j=1}^{q} \{k_j! [\lambda_{\ell_j}]^{-k_j}\}.$$

Proof: The result follows by Corollary 3.8 and the association of the random vairables $\{X_{\ell_j}(n),\ j = 1,\ldots,q\}$. $\square$

Lemma 4.12. Let us assume that the random variables $\{G_i(n),\ i = 1,2;\ n = 1,\ldots,h\}$, $h = 1,2,\ldots$, are associated. Let $k_1,\ldots,k_q$ be positive integers and let $\ell_1, \ell_2, \ldots, \ell_q = 1,2$; $q = 1,2,\ldots$. Then

$$E \prod_{j=1}^{q} \{G_{\ell_j}(j)\}^{k_j} \geq \prod_{j=1}^{q} E\{G_{\ell_j}(1)\}^{k_j}.$$

Note that $G_{\ell_j}(1)$, is, by Corollary 3.8, a geometric random variable with mean $p_{\ell_j}^{-1}$, $j = 1,\ldots,q$.

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