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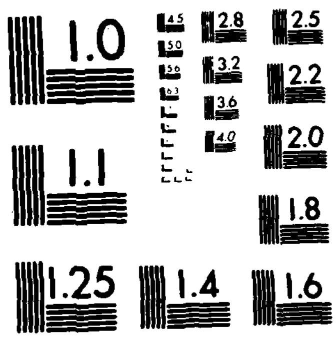
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ORDERING DISTRIBUTIONS BY SCALED ORDER STATISTICS

by

Marco Scarsini[(1),(2)]
Istituto di Matematica Finanziaria
Università di Parma

and

Moshe Shaked[(3)]
Department of Mathematics
University of Arizona

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Abstract

Motivated by applications in reliability theory, we define a preordering $(X_1, \ldots X_n) \overset{(k)}{\underset{\sim}{\le}} (Y_1, \ldots, Y_n)$ of nonnegative random vectors by requiring the k-th order statistic of $a_1X_1, \ldots, a_nX_n$ to be stochastically smaller than the k-th order statistic of $a_1Y_1, \ldots, a_nY_n$ for all choices of $a_i > 0$, $i = 1, 2, \ldots, n$. We identify a class of functions $M_{k,n}$ such that $\underset{\sim}{X} \overset{(k)}{\underset{\sim}{\le}} \underset{\sim}{Y}$ if and only if $E\phi(\underset{\sim}{X}) \le E\phi(\underset{\sim}{Y})$ for all $\phi \epsilon M_{k,n}$. Some preservation results related to the ordering $\overset{(k)}{\underset{\sim}{\le}}$ are obtained. Some examples and applications of the results are given.

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Key words and phrases: Stochastic orderings, order statistics.

1. Introduction.

Consider the set $\mathcal{X}_n$ of all n-dimensional random vectors which are nonnegative with probability one. For $\underline{z} = (z_1, \ldots, z_n) \in R_+^n \equiv \{\underline{z}:\underline{z} \ge \underline{0}\}$, denote by $\underline{z}_{(k)} = (z_1, \ldots, z_n)_{(k)}$ the k-th smallest z_i in $\{z_1, \ldots, z_n\}$. Thus for $\underline{Z} = (Z_1, \ldots, Z_n) \in \mathcal{X}_n$, the k-th order statistic of $Z_1, \ldots, Z_n$ is $\underline{Z}_{(k)} = (Z_1, \ldots, Z_n)_{(k)}$.

It is possible to introduce various orderings on $\mathcal{X}_n$. In this paper we consider the preordering '$\overset{(k)}{\underset{\sim}{\le}}$' defined for $\underline{X}, \underline{Y} \in \mathcal{X}_n$ by

$$\underline{X} \overset{(k)}{\underset{\sim}{\le}} \underline{Y}$$

if and only if for all $a_i > 0$, $i = 1, \ldots, n$,

$$(a_1X_1, \ldots, a_nX_n)_{(k)} \overset{st}{\le} (a_1Y_1, \ldots, a_nY_n)_{(k)}$$

where here '$\overset{st}{\le}$' denotes the usual (univariate) stochastic ordering. (For univariate random variables X and Y the notation $X \overset{st}{\le} Y$ means $Eg(X) \le Eg(Y)$ for all nondecreasing Borel measurable functions g for which the expectations exist.)

The following two results show the relationship between the preordering $\overset{(k)}{\underset{\sim}{\le}}$ and other well known orderings (see also Scarsini (1985)).

<u>Proposition 1.1</u>. Let $\underline{X}$ and $\underline{Y}$ be members of $\mathcal{X}_n$ with distributions F and G respectively. Then the following two conditions are equivalent:

(i) $\underline{X} \overset{(n)}{\underset{\sim}{\le}} \underline{Y}$,

(ii) $F(\underline{t}) \ge G(\underline{t})$ for all $\underline{t} \in R_+^n$.

<u>Proof</u>: Clearly (i) is equivalent to

$$\max(a_1X_1, \ldots, a_nX_n) \overset{st}{\le} \max(a_1Y_1, \ldots, a_nY_n)$$

whenever $a_i > 0$, $i = 1, \ldots, n$. The latter is the same as

$$F(\frac{t}{a_1}, \ldots, \frac{t}{a_n}) \ge G(\frac{t}{a_1}, \ldots, \frac{t}{a_n}) \text{ whenever } t \ge 0,\ a_i > 0,\ i = 1, \ldots, n,$$

which is the same as

(1.1) $F(t_1, \ldots, t_n) > G(t_1, \ldots, t_n)$ whenever $t_i > 0$, $i = 1, \ldots, n$.

Finally, using standard limiting arguments, (1.1) is equivalent to (ii).||

Similarly one can prove:

Proposition 1.2. Let $\underline{X}$ and $\underline{Y}$ be members of $\mathcal{X}_n$ with distributions F and G respectively. Then the following two conditions are equivalent:

(iii) $\underline{X} \overset{(1)}{\underset{\sim}{<}} \underline{Y}$,

(iv) $\overline{F}(\underline{t}) < \overline{G}(\underline{t})$ for all $\underline{t} \in R_+^n$,

where $\overline{F}(t_1, \ldots, t_n) \equiv P(X_1 > t_1, \ldots, X_n > t_n)$ and $\overline{G}(t_1, \ldots, t_n) \equiv P(Y_1 > t_1, \ldots, Y_n > t_n)$ are the corresponding survival functions.

Thus the ordering $\overset{(1)}{\underset{\sim}{<}}$ is the same as the one discussed by Rüschendorf (1980) and Mosler (1984). Mosler (1984) also discusses the ordering described in (ii) which is equivalent to $\overset{(n)}{\underset{\sim}{<}}$ as is shown in Proposition 1.1.

The purpose of this paper is to study the preordering $\overset{(k)}{\underset{\sim}{<}}$ for $k = 1, 2, \ldots, n$. We state the main results in Section 2. Some applications and examples are given in Section 3.

2. The main results.

For $m \in \{1, \ldots, n\}$ let ψ_m be the set of all subsets of $\{1, \ldots, n\}$ of size m. For $I = \{i_1, \ldots, i_m\} \in \psi_m$ and $x_{i_1}, \ldots, x_{i_m} > 0$, let $\underline{x}_I$ denote $(x_{i_1}, \ldots, x_{i_m})$ and let $(\underline{x}_I, \infty)$ denote $\lim_{\underline{x}_{I^c} \to \infty} (x_1, \ldots, x_n)$ and $(\underline{x}_I, \underline{0})$ denote $\lim_{\underline{x}_{I^c} \to \underline{0}} (x_1, \ldots, x_n)$. In this paper I^c denotes the complement of I in $\{1, \ldots, n\}$. Also, ∞ denotes a single ∞ or a vector $(\infty, \ldots, \infty)$ of a proper size. Similarly, $\underline{0}$ denotes a vector of zeroes of a proper size.

Let M_1 denote the class of all bounded distribution functions on R_+^n

with no singularities at ∞, that is, $f \in M_1$, if and only if there exists a measure μ on R_+^n such that $f(\underline{t}) = \mu([\underline{0}, \underline{t}])$ and for some bound $L < \infty$ we have $f(\underline{t}) \to L$ as $\underline{t} \to \infty$. Note that in this paper, μ is allowed to have positive mass at the origin or along any of the axis or on sets of the form $\{(\underline{x}_I, \underline{0}): \underline{x} \in R_+^m\}$ for some $I \in \psi_m$. Clearly every $f \in M_1$ determines a unique measure μ as described above and vice versa. In the sequel, for

$\underline{x} \in R_+^n$, $I \in \psi_m$ and $f \in M_1$, we denote $\lim_{\underline{x}_{I^c} \to \infty} f(x_1, \ldots, x_n)$ by $f(\underline{x}_I, \infty)$ and $\lim_{\underline{x}_{I^c} \to \underline{0}} f(x_1, \ldots, x_n)$ by $f(\underline{x}_I, \underline{0})$.

For $k \in \{1, \ldots, n\}$ let $M_{k,n}$ be the class of functions $\phi: R_+^n \to R$ such that

$$\phi(x_1, \ldots, x_n) = \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I \in \psi_m} f(\underline{x}_I, \infty), \text{ for some } f \in M_1, \tag{2.1}$$

where $\sum_{I \in \psi_m}$ denotes the sum over all $\binom{n}{m}$ elements of ψ_m. We sometimes supress the subscript n of $M_{k,n}$ and just write M_k. Note that for $k = 1$, the above definition and the previous definition of M_1 coincide.

<u>Theorem 2.1.</u> Let $\underline{X}, \underline{Y} \in \mathcal{X}_n$. Then the following two conditions are equivalent:

(v) $\underline{X} \overset{(k)}{\underset{\sim}{\le}} \underline{Y}$.

(vi) $E\phi(\underline{X}) \le E\phi(\underline{Y})$ for all $\phi \in M_k$ such that the expectations exist.

<u>Proof</u>. Suppose $\underline{X} \overset{(k)}{\underset{\sim}{\le}} \underline{Y}$. Let $a_i > 0$, $i = 1, \ldots, n$. From David (1970), p. 75 it follows that for every $x > 0$,

$$P\{(a_1X_1, \ldots, a_nX_n)_{(k)} > x\}$$

$$= \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{\{i_1,\ldots,i_m\} \in \psi_m} P\{a_{i_1}X_{i_1} > x, \ldots, a_{i_m}X_{i_m} > x\}.$$

Thus

$$P\{(a_1X_1, \ldots, a_nX_n)_{(k)} > x\}$$

$$= E[\sum_{m=n-k+1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \chi_{[(\underline{t}_I,\underline{0}),\infty]}(\underline{X})]$$

where $t_i \equiv x/a_i$, $i = 1, \ldots, n$ and χ_A is the indicator function of the set A. Now, if $\underline{X} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y}$ and F and G are the distribution functions of $\underline{X}$ and $\underline{Y}$ respectively, then

$$(2.2) \quad \int_{R_+^n} [\sum_{m=n-k+1}^{n} (-1)^{n-m+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \chi_{[(\underline{t}_I,\underline{0}),\infty]}(\underline{x})] dF(\underline{x})$$

$$< \int_{R_+^n} [\sum_{m=n-k+1}^{n} (-1)^{n-m+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \chi_{[(\underline{t}_I,\underline{0}),\infty]}(\underline{x})] dG(\underline{x}).$$

Let $\phi \varepsilon M_k$ and let f be the corresponding member of M_1 as described in (2.1). Then

$$E\phi(\underline{X}) = \int_{R_+^n} \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} f(\underline{x}_I,\infty) dF(\underline{x})$$

$$= \int_{R_+^n} \sum_{m=n-k-1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} [\int_{R_+^n} \chi_{[\underline{0},(\underline{x}_I,\infty)]}(\underline{u}) df(\underline{u})] dF(\underline{x})$$

$$= \int_{R_+^n} [\int_{R_+^n} \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \chi_{[(\underline{u}_I,\underline{0}),\infty]}(\underline{x}) dF(\underline{x})] df(\underline{u})$$

$$\underset{(2.2)}{<} \int_{R_+^n} [\int_{R_+^n} \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1} \binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \chi_{[(\underline{u}_I,\underline{0}),\infty]}(\underline{x}) dG(\underline{x})] df(\underline{u})$$

$$= E\phi(\underline{Y})$$

and (vi) follows.

Clearly $\sum_{m=n-k+1}^{n} (-1)^{m-n+k-1}\binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \chi_{[(\underline{t}_I,\underline{0}),\infty]}(\cdot)$ is a function in M_k whenever $\underline{t} > \underline{0}$. It follows that (vi) implies (2.2) which is equivalent to (v).||

Note that with $k = 1$, Theorem 2.1 yields Theorem 3(a) of Rüschendorf (1980) for measures on R_+^n.

The following preservation results will be used in Section 3.

Theorem 2.2. Assume that $\underline{X} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y}$ for some $k \varepsilon \{1, \ldots, n\}$ and $\underline{X}$ and $\underline{Y}$ in $\mathfrak{X}_n$. Let $b_i:R_+ \to R_+$ be a right continuous nondecreasing function, $i = 1, \ldots, n$. Then

(2.3) $(b_1(X_1), \ldots, b_n(X_n)) \overset{(k)}{\underset{\sim}{\leq}} (b_1(Y_1), \ldots, b_n(Y_n))$.

Theorem 2.2 can be proved by an explicit computation of $P\{(a_1b_1(X_1), \ldots, a_nb_n(X_n))_{(k)} > t\}$ and $P\{a_1b_1(Y_1), \ldots, a_nb_n(Y_n))_{(k)} > t\}$. However we note that it is also an immediate consequence of Theorem 2.1 and the following Lemma 2.3. Lemma 2.3 is also used in the proof of Theorem 2.4. The proof of Lemma 2.3 is easy and is omitted.

Lemma 2.3. If $\phi(\cdot, \ldots, \cdot) \varepsilon M_{k,n}$ and if $b_i:R_+\to R_+$ is right continuous nondecreasing function, $i = 1, \ldots, n$, then the function defined by $\phi(b_1(\cdot), \ldots, b_n(\cdot))$ is also a member of M_k.

The following result is important because it yields Theorems 2.6 and 2.7 below as special cases.

Theorem 2.4. Let $\underline{X}$, $\underline{Y}$, $\underline{Z}$ and $\underline{W}$ be random vectors in $\mathfrak{X}_n$ such that

$$\underline{X} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y},$$
$$\underline{Z} \overset{(k)}{\underset{\sim}{\leq}} \underline{W},$$

and $\underline{X}$ and $\underline{Z}$ are independent and $\underline{Y}$ and $\underline{W}$ are independent. Then

$$\underline{S} \equiv (c_1(X_1,Z_1), \ldots, c_n(X_n,Z_n))$$
$$\overset{(k)}{\underset{\sim}{\leq}} (c_1(Y_1,W_1), \ldots, c_n(Y_n,Z_n)) \equiv \underline{T}$$

Whenever $c_i: R_+^2 \to R_+$ is right continuous nondecreasing function, $i = 1, \ldots, n$.

The following lemma will be used in the proof of Theorem 2.4.

Lemma 2.5. Let $\underline{X}$, $\underline{Y}$, $\underline{Z}$ and $\underline{W}$ be as in Theorem 2.4. Then

(2.4) $\quad E\psi(\underline{X},\underline{Z}) \leq E\psi(\underline{Y},\underline{W})$

for all ψ such that

(2.5) $\quad \phi_{\underline{z}}(\cdot) \equiv \psi(\cdot,\underline{z}) \in M_{k,n}$ for all $\underline{z} \geq \underline{0}$,

(2.6) $\quad \phi^{\underline{x}}(\cdot) \equiv \psi(\underline{x},\cdot) \in M_{k,n}$ for all $\underline{x} \geq 0$,

provided the expectations in (2.4) exist.

Proof. Denote by $F_{\underline{X}}$, $F_{\underline{Y}}$, $F_{\underline{Z}}$, $F_{\underline{W}}$ the distributions of $\underline{X}$, $\underline{Y}$, $\underline{Z}$, $\underline{W}$, respectively. Let ψ satisfy (2.5) and 2.6). Then

$$
\begin{aligned}
E\psi(\underline{X},\underline{Z}) &= \int[\int\psi(\underline{x},\underline{z})dF_{\underline{Z}}(\underline{z})]dF_{\underline{X}}(\underline{x}) \\
&\underset{(2.6)}{\leq} \int[\int\psi(\underline{x},\underline{w})dF_{\underline{W}}(\underline{w})]dF_{\underline{X}}(\underline{x}) \\
&= \int [\int\psi(\underline{x},\underline{w})dF_{\underline{X}}(\underline{x})]dF_{\underline{W}}(\underline{w}) \\
&\underset{(2.5)}{\leq} \int[\int\psi(\underline{y},\underline{w})dF_{\underline{Y}}(\underline{y})]dF_{\underline{W}}(\underline{w}) \\
&= E\psi(\underline{Y},\underline{W}),
\end{aligned}
$$

and (2.4) follows.||

Proof of Theorem 2.4. Let $\phi \in M_k$. Consider the function $\eta_{\underline{z}}$ defined, for a fixed $\underline{z} \geq \underline{0}$, by $\eta_{\underline{z}}(\cdot, \ldots, \cdot) = \phi(c_1(\cdot,z_1), \ldots, c_n(\cdot,z_n))$. By Lemma 2.3, $\eta_{\underline{z}} \in M_k$ for all $\underline{z} \geq 0$. Similarly the function $\eta^{\underline{x}}$, which is defined for each fixed $\underline{x} \geq \underline{0}$, by $\eta^{\underline{x}}(\cdot, \ldots, \cdot) = \phi(c_1(x_1,\cdot), \ldots, c_n(x_n,\cdot))$, is also a member of M_k for all $\underline{x} > \underline{0}$. Hence, by Lemma 2.5,

$$E\phi(\underline{S}) = E\phi(c_1(x_1,Z_1), \ldots, c_n(X_n,Z_n))$$

$$\leq E\phi(c_1(Y_1,W_1), \ldots, c_n(Y_n,W_n)) = E\phi(\underline{T}).$$

This is true for every $\phi \in M_k$. Hence $\underline{S} \overset{(k)}{\underset{\sim}{\leq}} \underline{T}$.||

Theorems 2.6 and 2.7 below are special cases of Theorem 2.4. Theorem 2.6 shows that the ordering $\overset{(k)}{\underset{\sim}{\leq}}$ is preserved under convolutions.

Theorem 2.6. Let $\underline{X}$, $\underline{Y}$, $\underline{Z}$ and $\underline{W}$ be as in Theorem 2.4. Then

$$\underline{X} + \underline{Z} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y} + \underline{W}.$$

One consequence of Theorem 2.6 is the following. If $\underline{X}$, $\underline{Y}$, and $\underline{Z}$ belongs to $\mathcal{X}_n$ such that $\underline{X} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y}$ and $\underline{Z}$ is independent of $\underline{X}$ and $\underline{Y}$ then

$$\underline{X} + \underline{Z} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y} + \underline{Z}.$$

Theorem 2.7. Let $\underline{X}$, $\underline{Y}$, $\underline{Z}$ and $\underline{W}$ be as in Theorem 2.4. Then

$$(\min(X_1,Z_1), \ldots, \min(X_n,Z_n)) \overset{(k)}{\underset{\sim}{\leq}} (\min(Y_1,W_1), \ldots, \min(Y_n,W_n)),$$

$$(\max(X_1,Z_1), \ldots, \max(X_n,Z_n)) \overset{(k)}{\underset{\sim}{\leq}} (\max(Y_1,W_1), \ldots, \max(Y_n,W_n)).$$

In the next theorem, $\underline{X}^{(i)}$ denotes $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ and $\underline{Y}^{(i)}$ denotes $(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$, $i = 1, \ldots, n$.

Theorem 2.8. Let $\underline{X}, \underline{Y} \in \mathcal{X}_n$. Suppose $\underline{X} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y}$. (a) If $1 \leq k \leq n$ then

(2.7) $\underline{X}^{(i)} \overset{(k-1)}{\underset{\sim}{\leq}} \underline{Y}^{(i)}$.

(b) If $\underline{X}$ and $\underline{Y}$ have all their mass on $\{\underline{x}: x_1 > 0, \ldots, x_n > 0\}$, [i.e., if none of the X_i's or Y_i's is zero with positive probability] and if $1 \leq k \leq n - 1$, then

(2.8) $\underline{X}^{(i)} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y}^{(i)}$.

Proof of (a). By definition, $\underline{X} \overset{(k)}{\underset{\sim}{\leq}} \underline{Y}$ if and only if $(a_1 X_1, \ldots,$

$a_nX_n)_{(k)} \overset{st}{\le} (a_1Y_1, \ldots, a_nY_n)_{(k)}$ for all $a_i > 0$, $i = 1, \ldots, n$. Hence

$$(2.9) \quad \lim_{a_i \downarrow 0} (a_1X_1, \ldots, a_nX_n)_{(k)} \overset{st}{\le} \lim_{a_i \downarrow 0} (a_1Y_1, \ldots, a_nY_n)_{(k)}$$

for all $a_j > 0$, $j \neq i$. But (2.9) is the same as

$(a_1X_1, \ldots, a_{i-1}X_{i-1}, a_{i+1}X_{i+1}, \ldots, a_nX_n)_{(k-1)}$

$\overset{st}{\le}(a_1Y_1, \ldots, a_{i-1}Y_{i-1}, a_{i+1}Y_{i+1}, \ldots, a_nY_n)_{(k-1)}$ for all $a_j > 0$, $j \neq i$,

and (2.7) follows.||

<u>Proof of (b)</u>. Again $\underline{X} \overset{(k)}{\underset{\sim}{\le}} \underline{Y}$ if and only if $(a_1X_1, \ldots,$

$a_nY_n)_{(k)} \overset{st}{\le} (a_1Y_1, \ldots, a_nY_n)_{(k)}$ for all $a_i > 0$, $i = 1, \ldots, n$. Hence

$$(2.10) \quad \lim_{a_i \to \infty} (a_1X_1, \ldots, a_nX_n)_{(k)} \overset{st}{\le} \lim_{a_i \to \infty} (a_1Y_1, \ldots, a_nY_n)_{(k)}$$

for all $a_j > 0$, $j \neq i$. Since $\underline{X}$ and $\underline{Y}$ are positive with probability one, it follows that (2.10) is the same as

$(a_1X_1, \ldots, a_{i-1}X_{i-1}, a_{i+1}X_{i+1}, \ldots, a_nX_n)_{(k)}$

$\overset{st}{\le}(a_1Y_1, \ldots, a_{i-1}Y_{i-1}, a_{i+1}Y_{i+1}, \ldots, a_nY_n)_{(k)}$ for all $a_j > 0$, $j \neq i$, and

(2.8) follows.||

Theorem 2.8 says that if the n-dimensional vectors $\underline{X}$ and $\underline{Y}$ satisfy $\underline{X} \overset{(k)}{\underset{\sim}{\le}} \underline{Y}$ then (under the proper conditions) the (n-1)-dimensional marginals satisfy the orderings $\overset{(k-1)}{\underset{\sim}{\le}}$ and $\overset{(k)}{\underset{\sim}{\le}}$. By induction, for proper choices of k, the (n-2)-dimensional marginals satisfy the orderings $\overset{(k-2)}{\underset{\sim}{\le}}$, $\overset{(k-1)}{\underset{\sim}{\le}}$ and $\overset{(k)}{\underset{\sim}{\le}}$, and so on. The reader may wish to list all the orderings that follow, for example, for $n = 6$, from $\overset{(4)}{\underset{\sim}{\le}}$ for the m-dimensional marginals, $m = 5, 4, 3, 2$. In general, if $\underline{X}$ and $\underline{Y}$ satisfy the conditions of Theorem 2.8 (a) and (b) then, for $1 \le m \le n$, any m-dimensional marginal of $\underline{X}$ is $\overset{(\ell)}{\underset{\sim}{\le}}$ than the respective m-dimensional marginal of $\underline{Y}$ whenever $\max(1,m+k-n) \leq \ell \leq \min(k,m)$.

3. Examples and applications.

In this section some examples of functions ϕ in M_k will be given and then an application in reliability theory will be discussed.

In general, every distribution function in M_1 determines a function ϕ in $M_{k,n}$ as described in (2.1). In the following examples we describe in some detail some functions ϕ in $M_{2,3}$.

Example 3.1. Let F_i, $i = 1, 2, 3$ and G be univariate probability distributions on R_+ and let U_1, U_2, U_3 and W be independent random variables with distributions F_1, F_2, F_3 and G respectively. If $V_i \equiv \max(U_i,W)$, $i = 1, 2, 3$, then the joint distribution of V_1, V_2 and V_3 is given by

$$f(v_1,v_2,v_3) = F_1(v_1)F_2(v_2)F_3(v_3)G(\min(v_1,v_2,v_3)), \quad v_i \geq 0, \ i = 1, 2, 3.$$

Thus the function ϕ defined by

$$\begin{aligned}\phi(v_1,v_2,v_3) &= f(v_1,v_2,\infty) + f(v_1,\infty,v_3) + f(\infty, v_2,v_3) - 2f(v_1,v_2,v_3)\\ &= F_1(v_1)F_2(v_2)G(\min(v_1,v_2))\\ &\quad + F_1(v_1)F_3(v_3)G(\min(v_1,v_3))\\ &\quad + F_2(v_2)F_3(v_3)G(\min(v_2,v_3))\\ &\quad -2\, F_1(v_1)F_2(v_2)F_3(v_3)G(\min(v_1,v_2,v_3)), \quad v_i \geq 0, \ i = 1,2,3,\end{aligned}$$

is a member of $M_{2,3}$.

For example, if F_1, F_2, F_3 and G are the uniform distributions on the interval $[0,1]$ then for $0 \leq v_i \leq 1$, $i = 1,2,3$,

$$\begin{aligned}\phi(v_1,v_2,v_3) &= v_1v_2\min(v_1,v_2) + v_1v_3\min(v_1,v_3)\\ &\quad + v_2v_3\min(v_2,v_3) - 2v_1v_2v_3\min(v_1,v_2,v_3).\end{aligned}$$

Example 3.2. Let U_1, U_2, U_3, W and F_1, F_2, F_3, G be as in Example 3.1.

Denote $\overline{F}_i = 1 - F_i$, $\overline{G} = 1 - G$. If $V_i = \min(U_i, W)$, $i = 1, 2, 3$, then the joint survival function of V_1, V_2 and V_3 is given by

$$(3.1)\quad \overline{f}(v_1,v_2,v_3) = P\{V_1 > v_1, V_2 > v_2, V_3 > v_3\}$$
$$= \overline{F}_1(v_1)\overline{F}_2(v_2)\overline{F}_3(v_3)\overline{G}(\max(v_1,v_2,v_3)),\ v_i > 0,\ i = 1, 2, 3.$$

The joint distribution f of V_1, V_2 and V_3 can be obtained from $\overline{f}$ using the formula

$$(3.2)\quad f(v_1,v_2,v_3) = 1 - \overline{f}(v_1,0,0) - \overline{f}(0,v_2,0) - \overline{f}(0,0,v_3)$$
$$+ \overline{f}(v_1,v_2,0) + \overline{f}(v_1,0,v_3) + \overline{f}(0, v_2,v_3)$$
$$- \overline{f}(v_1,v_2,v_3),\ v_i > 0,\ i = 1, 2, 3.$$

The function ϕ defined by

$$(3.3)\quad \phi(v_1,v_2,v_3) = f(v_1,v_2,\infty) + f(v_1,\infty,v_3) + f(\infty, v_2,v_3)$$
$$- 2f(v_1,v_2,v_3),\ v_i > 0,\ i = 1, 2, 3,$$

belongs to $M_{2,3}$.

Plugging (3.2) into (3.3) one obtains

$$(3.4)\quad \mu(v_1,v_2,v_3) = 1 - \overline{f}(v_1,v_2,0) - \overline{f}(v_1,0,v_3) - \overline{f}(0,v_2,v_3)$$
$$+ 2\overline{f}(v_1,v_2,v_3),\ v_i > 0,\ i = 1, 2, 3.$$

Choosing various univariate distribution functions F_i, $i = 1, 2, 3$, and G in (3.1) and plugging the resulting $\overline{f}$ in (3.4), one can obtain explicit expressions for various members of $M_{2,3}$.

For example, if U_i, $i = 1, 2, 3$, and W are uniformly distributed on $[0,1]$ then for $0 < v_i < 1$, $i = 1, 2, 3$,

$$(3.5)\quad \begin{aligned} \phi(v_1,v_2,v_3) = 1 &- (1-v_1)(1-v_2)(1-\max(v_1,v_2)) \\ &- (1-v_1)(1-v_3)(1-\max(v_1,v_3)) \\ &+ 2(1-v_1)(1-v_2)(1-v_3)(1-\max(v_1,v_2,v_3)). \end{aligned}$$

If V_i, $i = 1, 2, 3$, and W are standard (mean one) exponential random variables [then (V_1,V_2,V_3) has a Marshall-Olkin (1967) multivariate exponential distribution] then $G(t) = F_i(t) = 1 - e^{-t}$, $t > 0$, $i = 1, 2, 3$. Plugging these in (3.1) and (3.4) one obtains the following member of $M_{2,3}$:

$$(3.6)\quad \phi(v_1,v_2,v_3) = 1 - e^{-v_1-v_2-\max(v_1,v_2)} - e^{-v_1-v_3-\max(v_1,v_3)}$$
$$- e^{-v_2-v_3-\max(v_2,v_3)} + 2e^{-v_1-v_2-v_3-\max(v_1,v_2,v_3)}, \quad v_i > 0,\ i = 1, 2, 3.$$

Of course, that ϕ of (3.6) is a member of $M_{2,3}$ can also follow at once from Theorems 2.1, 2.2 and the fact that ϕ of (3.5) is in $M_{2,3}$.

Example 3.3. The function $\overline{F}$ defined by

$$\overline{F}(v_1,v_2,v_3) = (1+v_1+v_2+v_3)^{-1}, \quad v_i > 0,\ i = 1, 2, 3$$

is a survival function (see, e.g., Takahashi (1965)). Substituting it in (3.2) one obtains a distribution function F. Substituting this F in (3.4) one obtains ϕ, a member of $M_{2,3}$, defined by

$$\phi(v_1,v_2,v_3) = 1 - (1+v_1+v_2)^{-1} - (1+v_1+v_3)^{-1} - (1+v_2+v_3)^{-1}$$
$$+ 2(1+v_1+v_2+v_3)^{-1}, \quad v_i > 0,\ i = 1, 2, 3.$$

Application 3.4 (reliability theory). Every collection $X_1, \ldots, X_n$ of nonnegative random variables can be thought of as a collection of lifelengths of devices. For $a_i > 0$, $i = 1, \ldots, n$, the scaled lifelengths $a_1X_1, \ldots, a_nX_n$ have been of interest in many studies (see, e.g., El-Neweihi (1984), Marshall and Shaked (1985a,b) and references therein). Note that $(a_1X_1, \ldots, a_nX_n)_{(k)}$ is the lifelength of an $(n-k+1)$-out-of-n system with component lifetimes $a_1X_1, \ldots, a_nX_n$.

Let $\underline{X} = (X_1, \ldots, X_n)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$ be two vectors of random lifelengths with distributions F and G respectively. In some applications the condition $\underline{X} \overset{(k)}{\underset{\sim}{<}} \underline{Y}$ naturally holds or is not very hard to prove (see below for more details). Then Theorem 2.1 yeidls a host of useful inequalities.

In order to verify $\underline{X} \overset{(1)}{\underset{\sim}{<}} \underline{Y}$ [respectively, $\underline{X} \overset{(n)}{\underset{\sim}{<}} \underline{Y}$], it follows from Propositions 1.1 and 1.2 that all that one has to do is to show $\overline{F}(\underline{t}) < \overline{G}(\underline{t})$ [respectively $F(\underline{t}) > G(\underline{t})$], $\underline{t} > \underline{0}$. In order to verify $\overset{(k)}{\underset{\sim}{<}}$ for $1 < k < n$, one just has to show (see the proof of Theorem 2.1)

$$(3.7) \qquad \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1}\binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \overline{F}(\underline{t}_I, 0),\ \underline{t} > 0,$$

$$< \sum_{m=n-k+1}^{n} (-1)^{m-n+k-1}\binom{m-1}{n-k} \sum_{I\varepsilon\psi_m} \overline{G}(\underline{t}_I, \underline{0}),\ \underline{t} > 0,$$

or, equivalently,

$$(3.8) \qquad \sum_{m=k}^{n} (-1)^{m-k}\binom{m-1}{k-1} \sum_{I\varepsilon\psi_m} F(\underline{t}_I, \infty) > \sum_{m=k}^{n} (-1)^{m-k}\binom{m-1}{k-1} \sum_{I\varepsilon\psi_m} G(\underline{t}_I, \infty),\ \underline{t} > 0.$$

Clearly the ordering '$\overset{st}{<}$' implies '$\overset{(k)}{\underset{\sim}{<}}$' (here, for every two random vectors $\underline{X}$ and $\underline{Y}$, the notaton $\underline{X} \overset{st}{<} \underline{Y}$ means

$$(3.9) \qquad Eg(\underline{X}) < Eg(\underline{Y})$$

for all measurable nondecreasing functions g for which the expectations exist). Note however that (3.9) is a much stronger requirement than (3.7) or (3.8). Hence (3.9) may not hold when (3.7) or (3.8) hold. Even if (unknown to the researcher) (3.9) holds it still may be possible only to show (3.7) or (3.8). Thus the advantage of '$\overset{(k)}{\underset{\sim}{<}}$' over '$\overset{st}{<}$' is in the relative simplicity of its verification and the fact that it still implies many useful inequalities.

Since the ordering '$\overset{st}{<}$' implies '$\overset{(k)}{<}$' it follows that the inequalities of Theorem 2.1 apply in many applications in reliability theory and elsewhere. For example, Block, Savits and Shaked (1985) give conditions under which a nonnegative random vector $\underline{T} = (T_1, \ldots, T_n)$ satisfies

(3.10) $$[(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n) | T_i = t'] \overset{st}{\le} [(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n) | T_i = t]$$

whenever $0 < t < t'$, $i = 1, \ldots, n$. Denoting the left hand side of (3.10) by $\underline{X}$ and the right hand side of (3.10) by $\underline{Y}$ (here $\underline{X}$ and $\underline{Y}$ are $(n-1)$-dimensional) it follows that $\underline{X} \underset{\sim}{\overset{(k)}{\le}} \underline{Y}$ and the inequalities described in Theorem 2.1 apply.

<u>Application 3.5</u> (systems with spare parts). Consider two systems of n components. Denote the lifetimes of the components of the first system by $X_1, \ldots, X_n$ and of the second system by $Y_1, \ldots, Y_n$. Suppose each component in each system has a cold spare which starts to live upon the failure of the component. Denote the lifetimes of the spares by $Z_1, \ldots, Z_n$ and $W_1, \ldots, W_n$ where Z_i [respectively, W_i] is the lifetimes of the spare which replaces the component with lifetime X_i [respectively, Y_i], $i = 1, \ldots, n$. The X_i's [respectively, Y_i's, Z_i's, W_i's] among themselves may be dependent, but we assume that $\underline{X}$ is independent of $\underline{Z}$ and $\underline{Y}$ is independent of $\underline{W}$.

The lifetime T_1 of the first system then is determined by $\underline{X} + \underline{Z}$, say

(3.11) $$T_1 = \tau(\underline{X}+\underline{Z})$$

where τ is a coherent life function in the sense of Esary and Marshall (1970). Suppose that the lifetime T_2 of the second system is

(3.12) $$T_2 = \tau(\underline{Y}+\underline{W})$$

where the τ in (3.12) is the same as the τ in (3.11).

If $\underline{X} \underset{\sim}{\overset{(k)}{\le}} \underline{Y}$ and $\underline{Z} \underset{\sim}{\overset{(k)}{\le}} \underline{W}$ then, by Theorem 2.6, $\underline{X} + \underline{Z} \underset{\sim}{\overset{(k)}{\le}} \underline{Y} + \underline{W}$. Thus various inequalities regarding T_1 and T_2 can be obtained from Theorem 2.1. For example, if τ is the coherent life function which corresponds to an $(n-k+1)$-out-of-n system then we have

$$T_1 \overset{st}{\le} T_2. \tag{3.13}$$

If the spares are warm standbys then, using the above notation, the lifetime $\tilde{T}_1$ of the first system is determined by $(\max(X_1,Z_1), \ldots, \max(X_n,Z_n))$, say

$$\tilde{T}_1 = \tau(\max(X_1,Z_1), \ldots, \max(X_n,Z_n))$$

and the lifetime $\tilde{T}_2$ of the second system is

$$\tilde{T}_2 = \tau(\max(Y_1,W_1), \ldots, \max(Y_n,W_n)).$$

Using Theorem 2.7 we get

$$\tilde{T}_1 \overset{st}{\le} \tilde{T}_2$$

when τ is as described before (3.13).

As another example, suppose $n-1$ of the components with lifetimes X_i [respectively Y_i] are used, with the correspnding spare parts, for an $(n-k)$-out-of-$(n-1)$ system. Denote the lifetime of the resulting system by S_1 [respectively, S_2]. Then, from Theorem 2.8 it follows that

$$S_1 \overset{st}{\le} S_2.$$

Similar stochastic ordering applies to systems which use "second-hand" components as described in Block, Bueno, Savits and Shaked (1984). We omit the details.

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