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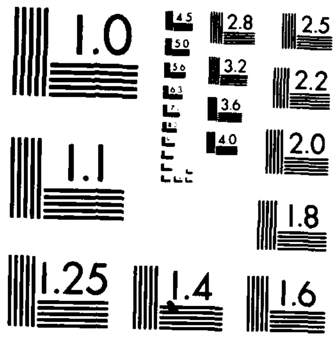

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INEQUALITIES FOR PROBABILITY CONTENTS OF CONVEX SETS VIA GEOMETRIC AVERAGE

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Abstract

It is shown that: If (X_1, X_2) is a permutation invariant central convex unimodal random vector and if A is a symmetric (about $\underline{0}$) permutation invariant convex set then $P\{(aX_1, \frac{X_2}{a}) \varepsilon A\}$ is nondecreasing as a varies from 0+ to 1 and is nonincreasing as a varies from 1 to ∞ (that is, $P\{(a_1X_1, a_2X_2) \varepsilon A\}$ is a Schur-concave function of $(\log a_1, \log a_2)$). Some extensions of this result for the n-dimensional case are discussed. Applications are given for elliptically contoured distribution and scale parameter families.

1. Introduction and motivation

Let $\underline{X} = (X_1,\ldots,X_n)$ have a density function f which is absolutely continuous with respect to the Lebesgue measure and denote, for $a_i > 0$, $i = 1,\ldots,n$,

$$D_\infty(\underline{a}) = \{\underline{x}: |x_i| < a_i,\ i = 1,\ldots,n\}, \tag{1.1}$$

$$D_2(\underline{a}) = \{\underline{x}: \sum(x_i/a_i)^2 \leq 1\} \tag{1.2}$$

the n-dimensional rectangle and ellipsoid (which depend on the vector $\underline{a} = (a_1,\ldots,a_n)$) respectively.

A function ψ is said to be Schur-concave in $\underline{a}$ (respectively $\underline{a}^2 \equiv (a_1^2,\ldots,a_n^2)$) if $\psi(\underline{a}) \leq \psi(\underline{b})$ whenever $\underline{a} \succ \underline{b}$ (respectively $\underline{a}^2 \succ \underline{b}^2$) where $\succ$ denotes the majorization relation; see, e.g., Marshall and Olkin (1979).

It is known (Tong (1982)) that if $f(\underline{x})$ is a Schur-concave function of $\underline{x}$ then $P\{\underline{X} \in D_\infty(\underline{a})\}$ (respectively, $P\{\underline{X} \in D_2(\underline{a})\}$) is a Schur-concave function of $\underline{a}$ (respectively, $\underline{a}^2$). Such a result depends on the diversity of the elements of $\underline{a}$ and $\underline{a}^2$ when the <u>arithmetic mean</u> is kept fixed.

Since the volumes (Vol) of $D_\infty(\underline{a})$ and $D_2(\underline{a})$ are multiples of $\prod_{i=1}^{n} a_i$, it follows that if $\underline{a} \succ \underline{b}$ $[\underline{a}^2 \succ \underline{b}^2]$ then $\mathrm{Vol}(D_\infty(\underline{a})) \leq \mathrm{Vol}(D_\infty(\underline{b}))$ $[\mathrm{Vol}(D_2(\underline{a})) \leq \mathrm{Vol}(D_2(\underline{b}))]$ with strict inequality if $\underline{b}[\underline{b}^2]$ is not a permutation of $\underline{a}[\underline{a}^2]$. Consequently, in the inequalities

$$P\{\underline{X} \in D_\infty(\underline{a})\} \leq P\{\underline{X} \in D_\infty(\underline{b})\},$$

$$P\{\underline{X} \in D_2(\underline{a})\} \leq P\{\underline{X} \in D_2(\underline{b})\},$$

the difference in probability contents could be partly due to the difference in the volumes of the two sets. In view of this fact Perlman (1982) suggested that a corresponding result will be of interest if the volumes of the sets are kept fixed. This can be accomplished by inequalities via the majorization

$$(\log a_1,\ldots,\log a_n) \succ (\log b_1,\ldots,\log b_n).$$

Such a majorization inequality depends on the diversity of the elements of $\underline{a}$ and $\underline{a}^2$ when the geometric mean is kept fixed.

In this paper we derive such an inequality for a large class of density functions and a large class of convex sets. Our most general results are given for the bivariate case. An extension to the n-dimensional case appears to be difficult [for reasons to be discussed in Section 3] except for some special cases such as the case of independent identically distributed random variables or when the underlying joint density is spherically symmetric. The class of convex sets considered includes D_∞ and D_2 as special cases and special applications are given for elliptically contoured distributions and scale parameters families. In all these cases, universal upper bounds on the probability contents can be given by substituting the values of the a_i's by their geometric mean.

2. The inequalities.

Before proceeding we first show that the condition of Schur-concavity is no longer adequate for the problem under study.

Example 2.1. Let $\underline{X} = (X_1, X_2)$ have the uniform density over the region

$$\{(x_1, x_2): |x_1 - x_2| < 4,\ 2 < |x_1 + x_2| < 4\}$$

(which is a Schur-concave function of $\underline{x}$). Then the probability content of $D_\infty(\underline{a})$ is zero for $\underline{a} = (1,1)$ and is positive for all $\underline{a}$ satisfying $a_1 = a_2^{-1} \neq 1$.

In order to derive our inequalities we recall the following definitions. For two vectors $\underline{x}$ and $\underline{y}$ write $\underline{x} >^t \underline{y}$ if $\underline{x}$ and $\underline{y}$ agree in all but two coordinates, say i and j, $i < j$, $x_i < x_j$ and $y_i = x_j$ and $y_j = x_i$.

Definition 2.2. Let $\underline{a} = (a_1,\ldots,a_n)$ where $a_1 \le a_2 \le \ldots \le a_n$. We say that a function $\phi(\underline{a}, \underline{x})$ is decreasing in transposition (DT; see Hollander, Proschan and Sethuraman (1977, 1981)) or arrangement increasing (AI; see Marshall and Olkin (1979, Section 6.F)) if

(a) $f(\underline{a}\Pi, \underline{x}\Pi) = f(\underline{a},\underline{x})$ for all permutation matrices Π and vectors $\underline{x}$ and $\underline{a}$ as above,

and

(b) $f(\underline{a},\underline{x}) \ge f(\underline{a},\underline{y})$ whenever $\underline{x} >^t \underline{y}$.

The following result plays a key role in the subsequent theorems.

Theorem 2.3. Let $(X_1,\ldots,X_n)$ have a density f and let A be a subset of R^n. If f and I_A (the indicator function of A) are such that $f(\frac{x_1}{a_1},\ldots,\frac{x_n}{a_n})$ and $I_A(\frac{x_1}{a_1},\ldots,\frac{x_n}{a_n})$ are AI in $\underline{a} \in (0,\infty)^n$ and $\underline{x} \in R^n$, then $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ is Schur-concave in $(\log a_1,\ldots,\log a_n)$ [i.e., $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\} \le P\{(\frac{X_1}{b_1},\ldots,\frac{X_n}{b_n}) \in A\}$ whenever

$(\log a_1,\ldots,\log a_n) \succ (\log b_1,\ldots, \log b_n)$].

Proof. Let ϕ and ψ be two n-variate real functions such that

$$g_1(a_1,\ldots,a_n;\ x_1,\ldots,x_n) \equiv \phi\left(\frac{x_1}{a_1},\ldots,\frac{x_n}{a_n}\right) \quad \text{is AI on} \quad (0,\infty)^n \times R^n$$

and

$$g_2(x_1,\ldots,x_n;\ a_1,\ldots,a_n) \equiv \psi\left(\frac{x_1}{a_1},\ldots,\frac{x_n}{a_n}\right) \quad \text{is AI on} \quad R^n \times (0,\infty)^n.$$

Then

$$(2.1) \qquad g(\underline{a};\underline{b}) \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_1(\underline{a};\underline{x})\ g_2(\underline{x};\underline{b})d\underline{x}$$

is AI on $(0,\infty)^n \times (0,\infty)^n$. The proof of this statement is the same as the proof of 6.F.12 of Marshall and Olkin (1979) except that two of the R^n's there are replaced by $(0,\infty)^n$. Substitute $y_i = x_i/b_i$ in the integral in (2.1) to see that the function g in (2.1) is of the form

$$(2.2) \qquad g(\underline{a};\underline{b}) = \left(\prod_{i=1}^{n} b_i\right) h\left(\frac{b_1}{a_1},\ldots,\frac{b_n}{a_n}\right)$$

for some function h on $(0,\infty)^n$. The function $\hat{h}$ defined by $\hat{h}(\underline{a};\underline{b}) \equiv h\left(\frac{b_1}{a_1},\ldots,\frac{b_n}{a_n}\right)$ is AI on $(0,\infty)^n \times (0,\infty)^n$. To see it write $\hat{h}(\underline{a};\underline{b}) = \left(\prod_{i=1}^{n} b_i\right)^{-1} g(\underline{a};\underline{b})$. Since $g(\underline{a};\underline{b})$ is AI it follows from Lemma 3.1 of Hollander, Proschan and Sethuraman (1977) that $\hat{h}(\underline{a};\underline{b})$ is AI on $(0,\infty)^n \times (0,\infty)^n$.

Since $h\left(\frac{b_1}{a_1},\ldots,\frac{b_n}{a_n}\right)$ is AI on $(0,\infty)^n \times (0,\infty)^n$ it follows from 6.F.6 of Marshall and Olkin (1979) [replacing one of the R's there by $(0,\infty)$] that

$h(e^{b_1}/e^{a_1},\ldots,e^{b_n}/e^{a_n})$ is AI on R^n. Thus, from 6.F.3.a of Marshall and Olkin (1979) it follows that $h(e^{c_1},\ldots,e^{c_n})$ is Schur-concave in $\underline{c} \in R^n$. That is, the function $h(c_1,\ldots,c_n)$ of (2.2) is Schur-concave in $(\log c_1,\ldots,\log c_n)$.

Denote

$$\chi(\underline{a}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\frac{x_1}{a_1},\ldots,\frac{x_n}{a_n})\ \psi(x_1,\ldots,x_n)d\underline{x}$$

Put $b_1 = \ldots = b_n = 1$ in (2.2) to obtain $\chi(\underline{a}) = h(a_1^{-1},\ldots,a_n^{-1})$. Since $h(\underline{a})$ is Schur-concave in $(\log a_1,\ldots,\log a_n)$ it follows that also $\chi(\underline{a})$ is Schur-concave in $(\log a_1,\ldots, \log a_n)$. Theorem 2.3 now follows by setting $\phi(\underline{x}) \equiv f(\underline{x})$ and $\psi(\underline{x}) \equiv I_A(\underline{x})$. ||

A natural question to ask is how the AI property of Theorem 2.3 is realted to more familiar and easily checked conditions such as unimodality and Schur-concavity. To answer this we recall some definitions from Dharmadhikari and Jogdeo (1976).

Definition 2.4. A random vector $\underline{X}$ (or its distribution) is called central convex unimodal if its distribution is in the closed convex hull of the set of all uniform distributions on symmetric compact convex bodies in R^n.

Definition 2.5. A random vector $\underline{X}$ (or its distribution) is called monotone unimodal if for every symmetric convex set $C \subset R^n$ and every $\underline{x} \neq \underline{0}$, the quantity $P\{\underline{X} \in C + k\underline{x}\}$ is nonincreasing in $k \geq 0$.

If $\underline{X}$ has a density f and is central convex unimodal then the set $\{\underline{x}:f(\underline{x}) \geq u\}$ is convex and symmetric, that is, $\underline{X}$ is unimodal according to Anderson (1955). It is well known (see, e.g., Dharmadhikari and Jogdeo

(1976)) that every central convex unimodal random vector is monotone unimodal. Wells (1978) showed that there exist (in R^2) monotone unimodal random vectors which are not central convex unimodal.

Theorem 2.6. If (X_1, X_2) with a Schur-concave density $f(x_1,x_2)$ is monotone unimodal, if $f(x_1, -x_2)$ is Schur-concave and if $A \subset R^2$ is measurable symmetric (about $\underline{0}$) permutation invariant and convex, then $P\{(\frac{X_1}{a_1}, \frac{X_2}{a_2}) \in A\}$ is Schur-concave in $(\log a_1, \log a_2)$.

Remark 2.7. Dharmadhikari and Jogdeo (1976) showed that every monotone unimodal random vector is symmetric (about $\underline{0}$). From this it follows that the Schur-concave density f in Theorem 2.6 is symmetric not only about $\{(x_1,x_2):x_1 = x_2\}$ but also about $\{(x_1,x_2): x_1 + x_2 = 0\}$. Thus $f(\frac{x_1}{a_1}, \frac{x_2}{a_2})$ cannot be AI in $\underline{a} \in (0,\infty)^2$ and $\underline{x} \in R^2$. However the restriction of f to $\{(x_1,x_2): x_1 + x_2 > 0\}$, or equivalently the conditional density of (X_1,X_2) given that $X_1 + X_2 > 0$, can be AI (see proof of Theorem 2.6 below) and this suffices to yield the conclusion of Theorem 2.6.

Proof of Theorem 2.6. Let $\hat{A} = A \cap \{(x_1,x_2): x_1 + x_2 > 0\}$. It will be shown that $P\{(\frac{1}{a_1} X_1, \frac{1}{a_2} X_2) \in A | X_1 + X_2 > 0\}$ which is equal to

$$P\{(\frac{1}{a_1} X_1, \frac{1}{a_2} X_2) \in \hat{A} | X_1 + X_2 > 0\} \tag{2.3}$$

is Schur-concave in $(\log a_1, \log a_2)$. Theorem 2.6 then follows from the symmetry of $f(x_1,x_2)$ about $\{(x_1,x_2): x_1 + x_2 = 0\}$; see Remark 2.7.

Let $B = \{(x_1,x_2): x_1 + x_2 > 0\}$. To prove that (2.3) is Schur-concave in $(\log a_1, \log a_2)$ it suffices, by Theorem 2.3, to show that

$g(x_1,x_2) \equiv 2f(x_1,x_2)\, I_B(x_1,x_2)$ and $I_{\hat{A}}$ (here $I_{\hat{A}}$ and I_B are the indicator functions of $\hat{A}$ and B) satisfy

(2.4) $g(\frac{x_1}{a_1}, \frac{x_2}{a_2})$ is AI in $\underline{a} \in (0,\infty)^2$ and $\underline{x} \in R^2$

and

(2.5) $I_{\hat{A}}(\frac{x_1}{a_1}, \frac{x_2}{a_2})$ is AI in $\underline{a} \in (0,\infty)^2$ and $\underline{x} \in R^2$.

Only the proof of (2.4) will be given. The proof of (2.5) is similar. To prove (2.4) it suffices to show that

(2.6) $f(c_1x_1, c_2x_2) \le f(c_2x_1, c_1x_2)$ whenever $c_1 > c_2 > 0$ and $x_1 + x_2 \ge 0$.

Fix $c_1 > c_2 > 0$. First we prove (2.6) when $x_1 > x_2 > 0$. Denote $(y_1,y_2) = \frac{c_1x_1+c_2x_2}{c_2x_1+c_1x_2}(c_2x_1, c_1x_2)$ and note that $(c_1x_1,c_2x_2) \succ (y_1,y_2)$ and that $c_1x_1 + c_2x_2 \ge c_2x_1 + c_1x_2$. Thus

$$
\begin{aligned}
& f(c_1x_1, c_2x_2) \\
& \le f(y_1,y_2) \qquad \text{(by Schur-concavity)} \\
& \le f(c_2x_1, c_1x_2) \qquad \text{(by monotone unimodality)}
\end{aligned}
$$

as was to be shown.

Now assume $x_1 \ge 0 > x_2$ (and $x_1 + x_2 \ge 0$). Since (X_1,X_2) is monotone unimodal it follows that $(X_1,-X_2)$ is monotone unimodal. Its density h is given by $h(x_1,x_2) = f(x_1,-x_2)$. By assumption, the density of $(X_1,-X_2)$ is Schur-concave. Hence, it follows from the preceding argument that when

$x_1 \geq -x_2 > 0$,

$$h(c_1x_1, -c_2x_2) \leq h(c_2x_1, -c_1x_2) ,$$

that is,

$$f(c_1x_1, c_2x_2) \leq f(c_2x_1, c_1x_2) ,$$

as was to be shown.

When $x_2 \geq x_1 > 0$ (and $x_1 + x_2 \geq 0$) it can be shown as above that (2.6) holds.||

It is known that a permutation invariant, central convex unimodal density is Schur-concave (see, e.g., Marshall and Olkin (1974) or Tong (1980), p. 108). Also it is clear (using Remark 2.7) that (X_1, X_2) has a permutation invariant central convex unimodal density if and only if $(X_1, -X_2)$ has. Thus we obtain the following result as a corollary of Theorem 2.6.

Theorem 2.8. If (X_1, X_2) has a permutation invariant central convex unimodal density and if $A \subset R^2$ is measurable symmetric (about $\underline{0}$) permutation invariant and convex, then $P\{(\frac{X_1}{a_1}, \frac{X_2}{a_2}) \in A\}$ is Schur-concave in $(\log a_1, \log a_2)$.

Remark 2.9. Note that the class of density functions (and subsets) in Theorem 2.8 is a proper subclass of Schur-concave function (and subsets). The additional condition there seems to be symmetry (about $\underline{0}$) and unimodality, and the latter is not met for the density in Example 2.1.

If $\underline{X} = (X_1, \ldots, X_n)$ has a density of the form $f(x_1, \ldots, x_n) = g(\sum_{i=1}^{n} x_i^2)$

[that is, $\underline{X}$ is spherically symmetric] and if g is nonincreasing then $\underline{X}$ is central convex unimodal. In this case, for a particular kind of sets A, we can extend Theorem 2.8 to the n-dimensional case $(n \geq 2)$.

Theorem 2.10. If $\underline{X} = (X_1,\ldots,X_n)$ has a density f of the form $f(\underline{x}) = g(\sum_{i=1}^{n} \varphi(x_i))$ for some nonincreasing g and if A is of the form $A = \{\underline{x}: \sum_{i=1}^{n} \psi(x_i) \leq \lambda\}$ for some $\lambda > 0$ where φ and ψ are non-negative, symmetric about 0, convex and nondecreasing on $[0,\infty)$, then $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ is Schur-concave in $(\log a_1,\ldots,\log a_n)$.

Proof. It is possible to prove this result by showing directly (as in the proof of Theorem 2.6) that the conditional density of $\underline{X}$ given $\sum_{i=1}^{n} x_i > 0$ and the indicator function of $A \cap \{\underline{x}: \sum_{i=1}^{n} x_i > 0\}$ satisfy the condition of Theorem 2.3. However here we use a conditioning argument and derive the desired result from Theorem 2.8.

First notice that it suffices to prove the Schur-concavity of $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ in $(\log a_1,\ldots,\log a_n)$ by fixing $a_3,\ldots,a_n$ and showing that this quantity is Schur-concave in $(\log a_1, \log a_2)$.

For fixed $x_3,\ldots,x_n$ and $a_3,\ldots,a_n$, consider

$$P\{(\frac{X_1}{a_1}, \frac{X_2}{a_2}, \ldots, \frac{X_n}{a_n}) \in A | X_3 = x_3, \ldots, X_n = x_n\} \tag{2.7}$$

which can be written as

$$P\{(\frac{X_1}{a_1}, \frac{X_2}{a_2}) \in \tilde{A}(\frac{x_3}{a_3},\ldots,\frac{x_n}{a_n})\},$$

where

$$\tilde{A}(y_3,\ldots,y_n) = \{(x_1,x_2): (x_1,x_2,y_3,\ldots,y_n) \in A\}$$
$$= \{(x_1,x_2): \psi(x_1) + \psi(x_2) \leq \lambda - \sum_{i=3}^{n} \psi(y_i)\}.$$

The conditional density of (X_1,X_2) given $X_i = x_i$, $i=3,\ldots,n$, is of the form $h(\phi(x_1) + \phi(x_2))$ where h is nonincreasing. Hence it is permutation invariant central convex unimodal. Also, $\tilde{A}(\frac{x_3}{a_3},\ldots,\frac{x_n}{a_n})$ is measurable symmetric permutation invariant and convex. Hence by Theorem 2.8, the quantity in (2.7) is Schur-concave in $(\log a_1, \log a_2)$.

The family of functions which are Schur-concave in $(\log a_1, \log a_2)$ is a convex cone, hence the unconditional probability $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ is Schur-concave in $(\log a_1, \log a_2)$. ||

Remark 2.11. The assumption that $\underline{X}$ is absolutely continuous in Theorems 2.6, 2.8 and 2.10 is not essential. If $\underline{X}$ does not have a density then it can be approximated by a sequence of absolutely continuous random vectors which have the required properties (such as symmetry about $\underline{0}$, permutation invariance and unimodality) and the conclusions of Theorems 2.6, 2.8 and 2.10 will apply to $\underline{X}$ by weak convergence.

Remark 2.12. Since $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ is a Schur-concave function of $(\log a_1,\ldots,\log a_n)$ if and only if $P\{(a_1X_1,\ldots,a_nX_n) \in A\}$, $P\{(X_1,\ldots,X_n) \in \hat{A}(a_1,\ldots,a_n)\}$ and $P\{X_1,\ldots,X_n) \in \hat{A}(\frac{1}{a_1},\ldots,\frac{1}{a_n})\}$ are (where $\hat{A}(a_1,\ldots,a_n) = \{(a_1x_1,\ldots,a_nx_n): (x_1,\ldots,x_n) \in A\}$), it is seen that, under the conditions of Theorems 2.3, 2.6, 2.8 and 2.10 (see also Remark 2.11), all these probability contents are Schur-concave functions of $(\log a_1,\ldots,\log a_n)$.

3. Some applications.

In this section we study some applications of Theorem 2.8 and 2.10 and discuss the possibility of n-dimensional generalizations (apart from Theorem 2.10).

Application 3.1. A class of convex sets which is of special interest is of the form $A = \{(x_1,x_2): \sum_{i=1}^{2} \phi(x_i) \leq \lambda\}$ for some $\lambda > 0$ and a function ϕ which is nonnegative, symmetric about 0, convex and nondecreasing on $(0,\infty)$. In particular,

$$D_k = \{(x_1,x_2): \sum_{i=1}^{2} x_i^{\,k} \leq 1\},\ k = 2,4,6,8,\ldots,\infty,$$

are in this class.

Karlin and Rinott (1983, Theorem 24) showed, among other things, that if the nonnegative random vector $\underline{X}$ has a Schur-concave density f then

$$(3.1)\qquad P\{(\frac{X_1}{a_1^{(k-1)/k}}, \frac{X_2}{a_2^{(k-1)/k}}) \in D_k\}$$

is a Schur-concave function of (a_1,a_2) $[a_i>0, i=1,2]$ for $k \geq 1$. A related question is whether or not

$$(3.2)\qquad P\{(\frac{X_1}{a_1},\frac{X_2}{a_2}) \in D_k\},\ k = 2,4,6,8,\ \ldots,\ \infty,$$

are Schur-concave function of $(\log a_1, \log a_2)$. A simple modification of Example 2.1 [considering the conditional distribution of (X_1,X_2) there, given $X_1 \geq 0$, $X_2 \geq 0$] shows that the answer is no under the condition of Schur-concavity of f alone. However, Theorems 2.8 and 2.10 say that under the (stronger) conditions of symmetry and unimodality of f, the answer to this question is positive. Notice that, by Marshall and Olkin (1979, page 63, Table 2, (vi)), it follows that Schur-concavity of (3.2) in $(\log a_1, \log a_2)$ implies Schur-concavity of (3.2) in (a_1, a_2) $[a_i > 0,\ i = 1,2]$. The Schur-concavity of (3.2) in (a_1, a_2) is a stronger property than the Schur-concavity of (3.1) in (a_1, a_2) for $k = 2,4,6,8, \ldots, \infty$ (again apply (vi), Table 2, page 63 in Marshall and Olkin (1979)).

Application 3.2. (Elliptically contoured distributions). If f is of the form $f(x_1,x_2) = g((x_1,x_2)\Sigma^{-1}(x_1,x_2)')$ where g is nonincreasing and $\Sigma = (\sigma_{ij})$ has equal diagonal elements and is positive definite, then the conditions on f in Theorem 2.8 are satisfied. Thus, inequalities can be obtained through the Schur-concavity property in $(\log a_1, \log a_2)$. In particular, when combining with Application 3.1, one has the following result: If (X_1, X_2) is elliptically contoured distributed then, for D_∞ and D_2 defined in (1.1) and (1.2),

$$P\{(X_1,X_2) \in D_\infty(a, \tfrac{1}{a})\} \quad \text{and} \quad P\{(X_1,X_2) \in D_2(a, \tfrac{1}{a})\}$$

are decreasing as a varies away from 1. Consequently, when the area of such a rectangle (or ellipse) is fixed, then the maximum probability content is obtained when the rectangle becomes a square (the ellipse becomes a circle). For D_∞ this result has been obtained by Kunte and Rattihalli (1984).

Application 3.3. (Scale parameter families). Let $\underline{\theta} = (\theta_1, \theta_2)$, $\theta_i > 0$ (i=1,2), be a parameter vector and let (Y_1, Y_2) have density $g_{\underline{\theta}}(y_1,y_2) = (\theta_1\theta_2)^{-1} f(y_1/\theta_1, y_2/\theta_2)$. If f and A satisfy the conditions in Theorem 2.8 then $P\{(Y_1,Y_2) \in A\}$ is a Schur-concave function of

$(\log \theta_1, \log \theta_2)$.

Application 3.4. (Peakedness in bivariate distributions). If $f(x_1, x_2)$ satisfies the conditions in Theorem 2.8 then, for $a_i > 0$ $(i=1,2)$, $P\{|\sum_{i=1}^{2} a_i X_i| < \lambda\}$ is a Schur-concave function of $(\log a_1, \log a_2)$ for all $\lambda > 0$; that is, if $(\log a_1, \log a_2) \succ (\log b_1, \log b_2)$ then $\sum_{i=1}^{2} a_i X_i$ is more peaked than $\sum_{i=1}^{2} b_i X_i$. This result is to be compared with a result of Proschan (1965) who showed that $P\{|\sum_{i=1}^{n} a_i X_i| < \lambda\}$ is Schur-concave in $\underline{a}$ whenever $X_1, \ldots, X_n$ are independent with a common symmetric (about 0) log-concave density.

Application 3.5. (Multivariate normal distributions). Let $\underline{X}$ be an $n \times 1$ random vector distributed as $N_n(\underline{\mu}, \Sigma)$. Das Gupta and Rattihalli (1984) considered the problem of selecting the region of largest confidence level for $\underline{\mu}$ from all regions of fixed Lebesgue measure, based on a single observation $\underline{X}$, Σ being a known positive-definite matrix. If one restricts attention to the class of translation-invariant regions then, it follows from Neyman-Pearson lemma, that such an optimal region is given by the correspoding concentration ellipsoid. Das Gupta and Rattihalli (1984), however, focused their attention only to a class of rectangular regions of fixed volume. In particular, they showed that if $\Sigma = \sigma^2 I$ then, subject to $\prod_{i=1}^{n} a_i = c\sigma^n$ (c is a constant), the probability $P\{\underline{X} \in D_\infty(\underline{a})\}$ is maximized when $a_1 = \ldots = a_n = c^{1/n}\sigma$. This fact follows also from Theorem 2.10 and not just for the rectangular region $D_\infty(\underline{a})$ but also for the elliptical region $D_2(\underline{a})$.

When $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$ then, subject to $\prod_{i=1}^{n} a_i = c$, the probabilities $P\{\underline{X} \in D_k(\underline{a})\}$, $k = 2, 4, \ldots, \infty$, are maximized when $a_i = \sigma_i a^*$ where $a^* = (c(\prod_{i=1}^{n} \sigma_i)^{-1})^{1/n}$. This result for the case $k = \infty$ has also been obtained by Das Gupta and Rattihalli (1984).

It is interesting to observe a difficulty for generalizing Theorem 2.8 to the n-dimensional case. For proving results concerning Schur-concave density functions (or random variables) one property is that: If the density $f(x_1,\ldots,x_n)$ of $(X_1,\ldots,X_n)$ is a Schur-concave function of $(x_1,\ldots,x_n)$ then the conditional density of (X_1,X_2) given $X_i = x_i$, $i = 3,\ldots,n$, is a Schur-concave function of (x_1,x_2) for every fixed $(x_3,\ldots,x_n)$; consequently the proof can be given for $n = 2$ first and then unconditioning as in the proof of Theorem 2.10. But in the current problem the symmetry condition (about $\underline{0}$) of $f(x_1,\ldots,x_n)$ does not yield the same property (hence we cannot justify (2.6)) for the conditional density of (X_1,X_2) given $(X_3,\ldots,X_n)$. Thus we do not yet know whether or not the following conjecture is true:

Conjecture. For $n > 2$, if $f(x_1,\ldots,x_n)$ [the density of $(X_1,\ldots,X_n)$] is permutation invariant and central convex unimodal, and if $A \subset R^n$ is measurable, symmetric about $\underline{0}$, permutation invariant and convex, then $P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ is a Schur-concave function of $(\log a_1,\ldots, \log a_n)$.

To remove the difficulty mentioned above one can, of course, consider the case in which $X_1,\ldots,X_n$ are independent and in which (as in the proof of Theorem 2.10) $\tilde{A}(x_3,\ldots,x_n)$ is symmetric about $(0,0)$ and convex (such as the class of convex sets $\{\underline{x}: \sum_{i=1}^{n} x_i^k \leq 1\}$ for $k = 2, 4, 6, 8,\ldots,\infty$). In particular, if $X_1,\ldots,X_n$ are independent identically distributed random variables whose common density is symmetric about 0 and log-concave, then the conditional density of (X_1,X_2) given $(X_3,\ldots,X_n)$ satisfies the conditions in Theorem 2.8. In this special case, if the set A depends on $(x_1,\ldots,x_n)$ only through $(|x_1|,\ldots,|x_n|)$, then inequalities for

$P\{(\frac{X_1}{a_1},\ldots,\frac{X_n}{a_n}) \in A\}$ can be derived by applying either Theorem 2.8 or Proposition 11.E.5.e of Marshall and Olkin (1979). In particular, if $X_1,\ldots,X_n$ are independent, identically distributed normal variables with mean 0, then the probability contents of $D_\infty(\underline{a})$ and $D_2(\underline{a})$, defined in (1.1) and (1.2) are Schur-concave functions of $(\log a_1,\ldots,\log a_n)$. The latter is the Okamato-Marshall-Olkin inequality (Marshall and Olkin (1979, P. 303)).

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