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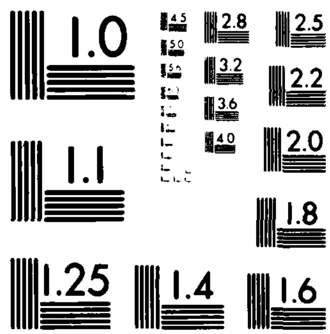

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EXTREME VALUES OF BIRTH AND DEATH PROCESSES AND QUEUES

by

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Abstract

We study the asymptotic behavior of maximum values of birth and death processes over large time intervals. In most cases, the distributions of these maxima, under standard linear normalizations, either do not converge or they converge to a degenerate distribution. However, by allowing the birth and death rates to vary in a certain manner as the time interval increases, we show that the maxima do indeed have three possible limit distributions. Two of these are classical extreme value distributions and the third one is a new distribution. This third distribution is the best one for practical applications. Our results are for transient as well as recurrent birth and death processes and related queues. For transient processes, the focus is on the maxima conditioned that they are finite.

Keywords: Extreme values, birth and death processes, M/M/s queues, limit theorems.

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1. Introduction

When modeling the dynamics of a parameter of a system by a stochastic process, the questions one addresses depend on the nature of the parameter. In some instances, the extreme values of the parameter rather than its usual values may be of paramount interest. In a manufacturing plant, for example, a typical parameter is the queue length of parts waiting to be processed at a work station. Small to moderate values of the queue may indicate that the system is operating successfully and the queue fluctuations are unimportant. On the other hand, large queues may call for extraordinary measures such as allocation of auxiliary storage space, employee overtime, or rescheduling of production. A natural question is: What is the probability that the queue will exceed a specific critical value in a certain time period? Extreme value questions like this are the topic of this paper. More specifically, our focus is on characterizing the asymptotic behavior of the maxima of birth and death processes and related queues.

The gist of our study is illustrated by the following results for the M/M/s queue. Consider such a queueing process in which customers arrive to s servers according to a Poisson process with rate λ, and the independent, exponentially distributed service times have mean μ^{-1}. Let M_n denote the maximum queue length in the time interval $[0,T_n]$, where T_n is the nth time the system becomes empty. Our interest is in finding norming constants a_n, $b_n > 0$ and a non-degenerate distribution G such that

(1.1) $$\lim_{n\to\infty} P((M_n - a_n)/b_n \leq x) = G(x),$$

for each continuity point of G. When such a_n, b_n, G exist, we say that

M_n has the limit distribution G. Otherwise, we say that M_n does not have a limit distribution. As usual, we consider only linear normalizations.

We can write $M_n = \max\{Y_1,\ldots,Y_n\}$, where Y_k is the maximum of the queue in the interval $[T_{k-1},T_k]$. Since the queuing process is Markovian, then $Y_1,Y_2,\ldots$ are independent identically distributed random variables. From the classical extreme value theory for independent identically distributed variables (see for instance Galambos (1978) or Leadbetter et al. (1983)), we know that the possible limit distributions for M_n are only $\exp(-x^{-\gamma})$, $x > 0$, or $\exp(-e^{-x})$, $x \in R$. One consequence is as follows; this is a special case of Theorem 2.5.

THEOREM 1.1. If the queueing process is null recurrent ($\lambda = s\mu$), then

$$\lim_{n\to\infty} P(M_n/(n(\lambda/\mu)^s s!) \le x) = e^{-x^{-1}}, \; x > 0.$$

Cohen (1969) proved an analogue of this for the M/G/1 and G/M/1 queues; related studies on extreme values of queue lengths and waiting times are Heyde (1971) and Iglehart (1972).

Another classical result for discrete random variables is that the convergence (1.1) can take place only if

$$\lim_{m\to\infty} P(Y_1 = m)/P(Y_1 > m-1) = 0.$$

Consequently, we have the following anomaly.

THEOREM 1.2. If the queueing process is positive recurrent ($\lambda < s\mu$), then M_n does not have a limit distribution.

This non-convergence theorem is surprising, especially in light of Theorem 1.1, since positive recurrent processes generally have nicer properties than null recurrent ones. Cohen (1969) and Anderson (1970) give insights on typical liminf's and limsup's for the distribution of $(M_n - a_n)/b_n$.

Our point of departure is to establish the convergence (1.1) for queues and birth and death processes in spite of the non-convergence described in Theorem 1.2 and its generalization, Theorem 2.3. Our approach is to allow the birth and death parameters (λ, μ, s for the M/M/s queue) to vary with n when considering the convergence of M_n. Such parameter variations in limit theorems are not uncommon. A classic example is that if S_n is a binomial random variable with parameters n,p and if $p = p_n$ varies with n such that $np_n \to \lambda > 0$ as $n \to \infty$, then the distribution of S_n converges weakly to a Poisson distribution with mean λ.

Here is an example of our major results. Suppose M_n is the maximum, as above, of an M/M/s queue, where the arrival rate $\lambda = \lambda(n)$ and service rate $\mu = \mu(n)$ depend on n. Let $\rho_n = \lambda(n)/(s\mu(n))$.

THEOREM 1.3. Suppose that $\rho_n < 1$ for each n and that $\rho_n \to 1$. The possible limit distributions for M_n are $G_0(x) = \exp(-x^{-1})$, $x > 0$; $G_\infty(x) = \exp(-e^{-x})$, $x \in R$; and

$$(1.2) \qquad G_0(x) = \exp(-c/(e^x - 1)), \ x > 0, \text{ for } 0 < c < \infty.$$

The M_n has the limit distribution G_c, where $0 < c < \infty$, if and only if $n(1-\rho_n)/s! \to c$. Appropriate norming constants a_n, b_n are as in Theorem 3.1.

Note that G_0 and G_∞ are classical extreme value distributions. The third distribution (1.2) has not appeared in the literature before. Numerical work has shown that this new distribution is the best one for practical approximations. Namely, for the standard M/M/s queue with traffic intensity $\rho = \lambda/s\mu$,

$$P(-M_n \log\rho < x) \cong \exp(c_n/(e^k-i)),\ x > 0,$$

where $c_n = n(1-\rho)$. This approximation is good for $n > 15$ and for all $\rho < 1$.

This completes our introduction. Here is what lies ahead: Section 2 consists of preliminaries, including classical convergence and non-convergence theorems for extreme values of birth and death processes; Section 3 contains our main results for recurrent birth and death processes; Section 4 contains analogous results for transient processes; and Section 5 gives applications to M/M/s and related queues.

2. Preliminaries

We shall consider a continuous-time birth and death process on the nonnegative integers with birth rates $\lambda_0, \lambda_1, \lambda_2, \ldots$ and death rates $\mu_0 = 0$, $\mu_1, \mu_2, \ldots$ This is a Markov process that evolves as follows: Upon entering state k, the process remains there for an exponentially distributed time with mean $(\lambda_k+\mu_k)^{-1}$, and then it moves to state k+1 or k-1 according to the respective probabilities $\lambda_k/(\lambda_k+\mu_k)$ and $\mu_k/(\lambda_k+\mu_k)$. We assume for now that the process is positive recurrent or null recurrent - transient processes are discussed in Section 4. We also assume, for convenience, that the process at time zero begins in state zero. Let M_n denote the maximum value of the process in the time interval $[0,T_n]$, where T_n is the time of the nth visit of the process to state zero. The M_n and T_n are finite valued since the process is recurrent. We shall study the asymptotic behavior of the distribution of M_n as $n \to 0$.

We can write $M_n = \max\{Y_1,\ldots,Y_n\}$, where Y_k is the maximum of the birth and death process in the time interval $[T_{k-1},T_k]$, here $T_0 = 0$. Since the process is Markovian, the random variables $Y_1, Y_2, \ldots$ are independent and identically distributed. Consequently,

$$P(M_n \leq x) = P(Y_1 \leq x,\ldots,Y_n \leq x) = F(x)^n,$$

where F is the distribution of the Y_k's. To obtain an expression for F, note that the successive states of the birth and death process form a simple discrete-time random walk that moves from state k to state k+1 or k-1 according to the respective probabilities $\lambda_k/(\lambda_k+\mu_k)$ and $\mu_k/(\lambda_k+\mu_k)$. Then clearly F(x) is the probability that, starting from state 1, the random walk reaches state 0 before it exceeds x. Thus, from Section I.12 of Chung (1967), we know that

$$F(x) = 1 - \Big(\sum_{k=0}^{x} r_k\Big)^{-1} \tag{2.1}$$

where $r_0 = 1$ and $r_k = (\mu_1 \cdots \mu_k)/(\lambda_1 \cdots \lambda_k)$, $k \geq 1$. Furthermore, the birth and death process is recurrent when $\Sigma_{k=0}^{\infty} r_k = \infty$ and is transient when $\Sigma_{k=0}^{\infty} r_k < \infty$.

Here are some asymptotic properties of the ratio λ_k/μ_k, depending on whether the birth and death process is transient or recurrent; we let

$$\rho = \liminf_{k\to\infty} \lambda_k/\mu_k \quad \text{and} \quad \bar{\rho} = \limsup_{k\to\infty} \lambda_k/\mu_k. \tag{2.2}$$

LEMMA 2.1. If $\Sigma_{k=0}^{\infty} r_k < \infty$, then $\lambda_k/\mu_k > 1$ for an infinite number of k's, and $\rho \geq 1$. If $\Sigma_{k=0}^{\infty} r_k = \infty$, then $\bar{\rho} \leq 1$.

<u>Proof</u>. Suppose $\Sigma_{k=0}^{\infty} r_k < \infty$. For any $N \geq 1$, we have

$$r_{N-1} \sum_{k=N}^{\infty} (\mu_N \cdots \mu_k)/\lambda_N \cdots \lambda_k) = \sum_{k=N}^{\infty} r_k < \infty.$$

Thus λ_k/μ_k must be > 1 for an infinite number of k's; otherwise, the first sum would be infinite. To prove $\rho \geq 1$, fix $\varepsilon > 0$ and let N be such that $\lambda_k/\mu_k \geq \rho - \varepsilon$, $k \geq N$. Then

$$\infty > \sum_{k=0}^{\infty} r_k = \sum_{k=0}^{N-1} r_k + r_N \sum_{m=0}^{\infty} (\rho - \varepsilon)^{-m}.$$

Consequently, $(\rho-\varepsilon)^{-1} < 1$ for each $\varepsilon > 0$, and hence $\rho \geq 1$. The second assertion is proved similarly.

Keep in mind that, for now, we are considering only recurrent birth and death processes. Our interest is in the weak convergence of the distribution

(2.3) $P((M_n - a_n)/b_n \leq x) = F(a_n + b_n x)^n = [1 - (1 - F(a_n + b_n x))]^n$,

where a_n and $b_n > 0$ are constants. It is well known that, for any $\gamma_n \varepsilon R$ and $-\infty \leq \gamma \leq \infty$, the convergence $(1+\gamma_n)^n \to e^{\gamma}$ is equivalent to $n\gamma_n \to \gamma$. This property applied to (2.3) translates into the following known result (cf Corollary 1.3.1 of Galambos (1978) or Theorem 1.5.1 of Leadbetter et al. (1983)). Here $0 \leq g(x) < \infty$.

CONVERGENCE CRITERION 2.2. $P((M_n - a_n)/b_n \leq x) \to e^{-g(x)}$ as $n \to \infty$ if and only if $n(1 - F(a_n + b_n x)) \to g(x)$ as $n \to \infty$. Note that this criterion is also true when F varies with n.

We begin by characterizing when the maximum M_n does not have a limit distribution and when it might have one. Here we use the notation (2.2) and $\alpha_{km} = \prod_{\ell=k+1}^{m} \lambda_\ell/\mu_\ell$.

THEOREM 2.3. (i) The maximum M_n does not have a limit distribution if and only if

(2.4) $$\liminf_{n\to\infty} \sum_{k=0}^{m-1} \alpha_{km} < \infty.$$

(ii) Inequality (2.4) holds when $\Sigma_{k=0}^{\infty} \limsup_{m\to\infty} \alpha_{km} < \infty$ or when $\bar{\rho} < 1$.

(iii) Inequality (2.4) does not hold when $\Sigma_{k=0}^{\infty} \liminf_{m\to\infty} \alpha_{km} = \infty$ or when $\rho > 1$. The condition $\rho > 1$ is equivalent to $\lim_{k\to\infty} \lambda_{\mu}/\mu_k = 1$.

(iv) If $\alpha_k = \lim_{m\to\infty} \alpha_{km}$ exists for each k, then inequality (2.4) holds if and only if $\Sigma_{k=0}^{\infty} \alpha_k < \infty$.

Proof. (i) We know, from Theorem 1.7.13 of Leadbetter *et al.* (1983), that there exist x_n and $0 < \tau < \infty$ such that $n(1 - F(x_n)) \to \tau$ as $n \to \infty$ if and only if

$$(F(m) - F(m-1))/(1 - F(m-1)) \to 0 \quad \text{as } m \to \infty.$$

The latter condition, in light of (2.1), is

$$r_m / \Sigma_{k=0}^{m-1} r_k = \Big(\sum_{k=0}^{m-1} \alpha_{km}\Big)^{-1} \to 0 \quad \text{as } m \to \infty.$$

Then, by Criterion 2.2, the M_n does not have a limit distribution if and only if the last convergence does not hold, which is equivalent to (2.4).

(ii) If the sum $\Sigma_{k=0}^{\infty} \limsup_{m\to\infty} \alpha_{km}$ is finite, then by Fatou's lemma this finite sum is an upper bound for $\limsup_{m\to\infty} \Sigma_{k=0}^{m-1} \alpha_{km}$ and hence (2.4) holds. Now suppose $\bar{\rho} < 1$. Then for any c in the interval $(\bar{\rho}, 1)$, there is a number N such that $\lambda_k/\mu_k \le c$ for $k > N$. It follows that, for $m > N$,

$$\text{(2.5)} \qquad \begin{aligned} \sum_{k=0}^{m} \alpha_{km} &= r_m^{-1} \sum_{k=0}^{N} r_k + \sum_{k=N+1}^{m} \alpha_{km} \\ &\le r_N^{-1} c^{m-N} \sum_{k=0}^{N} r_k + \sum_{\ell=0}^{m-N} c^{\ell} \\ &\to 1/(1-c) \quad \text{as } m \to \infty. \end{aligned}$$

Thus (2.4) holds.

(iii) By Fatou's lemma, the sum $\Sigma_{k=0}^{\infty} \liminf_{m\to\infty} \alpha_{km}$ is a lower bound for the left side of (2.4). Thus, if this sum is infinite, then (2.4) does not hold. Now suppose $\rho > 1$. Then for any $c < 1$, there is an N such that $\lambda_k/\mu_k > c$ for $k > N$. The display (2.5) clearly holds with the inequality reversed and so

$$\liminf_{m\to\infty} \sum_{k=0}^{m} \alpha_{km} > 1/(1-c).$$

Letting $c \to 1$ implies that (2.4) does not hold. To prove that $\rho > 1$ if and only if $\lim_{k\to\infty} \lambda_k/\mu_k = 1$, we need only show that $\rho > 1$ implies $\rho = \bar{\rho} = 1$. But this is true because, by Lemma 2.1, we know that $\rho \leq \bar{\rho} \leq 1$.

(iv) This part is a consequence of parts (ii) and (iii).

The preceding theorem yields the negative conclusion that the existence of a limit distribution for M_n is the exception rather than the rule: There is no limit distribution for a typical positive recurrent process with $\limsup_{k\to\infty} \lambda_k/\mu_k < 1$, but one might exist for an atypical process with $\lim_{k\to\infty} \lambda_k/\mu_k = 1$ (such as a null recurrent process). For those instances when there might be a limit distribution for M_n, we have the following properties from the classical extreme value theory.

PROPERTIES OF M_n 2.4. (a) The possible limit distributions for M_n are only $\exp(-x^{-\gamma})$, $x > 0$, or $\exp(-e^{-x})$, $x \in R$.

(b) The first of these distributions is the limit if and only if

$$(2.6) \qquad (1 - F(tx))/(1 - F(t)) = \sum_{k=0}^{t} r_k / \sum_{k=0}^{tx} r_k \to x^{\gamma} \quad \text{as } x\to\infty.$$

Appropriate norming constants are $a_n = 0$ and $b_n = \min\{m: \sum_{k=0}^{m} r_k > n\}$.

(c) The second distribution in (a) is the limit distribution for M_n if and only if there is a positive function $g(t)$ such that

$$(2.7) \qquad \left[1 - F(t + xg(t))\right]/(1 - F(t)) = \sum_{k=0}^{t} r_k / \sum_{k=0}^{t+xg(t)} r_k \to c^{-x} \quad \text{as } t \to \infty.$$

In this case, one can choose

$$g(t) = \sum_{k=0}^{t} r_k \sum_{k=t}^{\infty} \left(\sum_{m=0}^{k} r_m \right)^{-1},$$

and $a_n = \min\{m: \Sigma_{k=0}^{m} r_k > n\}$ and $b_n = g(a_n)$.

A special case of (b) is as follows. This applies, for instance, to a null recurrent process with $\lambda_k = \mu_k$, $k > s$, for some s (such as the M/M/s queue in Theorem 1.1).

THEOREM 2.5. If $\Sigma_{k=0}^{\infty} |1 - \lambda_k/\mu_k| < \infty$, then

$$(2.8) \qquad \lim_{n\to\infty} P(M_n/nb \le x) = e^{-x^{-1}}, \qquad x > 0$$

where $b = \prod_{k=1}^{\infty} \lambda_k/\mu_k$.

<u>Proof</u>. A basic property of infinite products of real numbers is that $\prod_{k=1}^{\infty} (1-a_k)$ exists when $\Sigma_{k=1}^{\infty} |a_k| < \infty$. In light of this, the hypothesis implies the existence of the limit b. Now

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \prod_{\ell=1}^{k} \mu_\ell/\lambda_\ell = b^{-1}.$$

Consequently, $n^{-1} \Sigma_{k=0}^{n} r_k \to b^{-1}$, and so

$$n\left(1 - F(nbx)\right) = \left[bx(nbx)^{-1} \sum_{k=0}^{nbx} r_k\right]^{-1} \to x^{-1} \quad \text{as } n \to \infty.$$

This convergence and Criterion 2.2 yield (2.8).

3. Main Results

We saw above that the maximum M_n does not have a limit distribution for a wide class of birth and death processes, including those in which

$\limsup_{k\to\infty} \lambda_k/\mu_k < 1$. However, M_n might have a limit distribution when $\lim_{k\to\infty} \lambda_k/\mu_k = 1$. These negative and slightly positive findings prompted us to explore the convergence of M_n when the parameters $\lambda_k = \lambda_{nk}$ and $\mu_k = \mu_{nk}$ vary with n such that λ_{nk}/μ_{nk} is nearly 1 for large n and k. This is the basis of our following results.

Consider a sequence of recurrent birth and death processes indexed by n = 1,2,..., where the nth process has the respective birth and death rates λ_{nk} and μ_{nk} when in state k = 0,1,... For the nth process, let M_n denote its maximum up to the time of its nth return to state zero. This M_n has the same meaning as the one in Section 2, but here its defining parameters λ_{nk}, μ_{nk} vary with n as well as with k. That is, M_n is the maximum of n independent random variables with the common distribution

$$F_n(x) = 1 - \left(\sum_{k=0}^{x} r_{nk}\right)^{-1}$$

where $r_{n0} = 0$ and $r_{nk} = (\mu_{n1}\cdots\mu_{nk})/(\lambda_{n1}\cdots\lambda_{nk})$, $k \geq 1$.

We shall assume that, for each n, there is a positive number $\rho_n < 1$ and a positive integer s_n such that

$$\lambda_{nk}/\mu_{nk} = \rho_n \quad \text{for } k \geq s_n, \tag{3.1}$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{s_n} r_{nk} = 0. \tag{3.2}$$

The parameter s_n may be bounded, unbounded, or independent of n, such as $s_n = 1$. Assumption (3.1), with $\rho_n < 1$, implies that $\Sigma_{k=0}^{\infty} r_{nk} \geq \Sigma_{k=s_n}^{\infty} \rho_n^{-k}$

$= \infty$, which ensures that the nth birth and death process is recurrent. Assumption (3.1) is satisfied automatically for M/M/s queues: the s_n is the number of servers. Assumption (3.2) holds when s_n and r_{nk} are bounded or when $r_{nk} \leq B_n$ for $k \leq s_n$ and $s_n B_n/n \to 0$.

We shall show that when $\rho_n \to 1$, the possible limit distributions for $(M_n - a_n)/b_n$ are as follows. Here $c = \lim_{n\to\infty} c_n$ and

$$(3.3) \qquad c_n = n(1 - \rho_n)r_{ns_n}^{-1} = n(1 - \rho_n)\prod_{k=1}^{s_n} \lambda_{nk}/\mu_{nk}.$$

Case	Limit Distribution	Norming Constants
$c = 0$	$G_0(x) = \exp(-x^{-1}),\ x>0,$	$a_n = s_n - 1 \quad b_n = n\prod_{k=1}^{s_n} \lambda_{nk}/\mu_{nk}$
$0 < c < \infty$	$G_c(x) = \exp(-c/(e^x-1))$	$a_n = s_n - 1 \quad b_n = -1/\log\rho_n$
$c=\infty$	$G_\infty(x) = \exp(-e^{-x}),\ x\varepsilon R,$	$a_n = s_n - 1 - \log c_n/\log\rho_n$ $b_n = -/\log\rho_n.$

An easy check shows that the distributions G_c $(0 \leq c \leq \infty)$ are of distinct type: G_c and $G_{c'}$ are of the same type ($G_c(x) = G_{c'}(a + bx)$ for some a,b) if and only if $c = c'$.

The following result gives sufficient conditions for the existence of limit distributions for M_n.

THEOREM 3.1 Suppose (3.1) and (3.2) hold and $\rho_n \to 1$. If $c_n \to c$ as $n \to \infty$, where $0 \leq c \leq \infty$, then

$$(3.4) \qquad \lim_{n\to\infty} P\big((M_n - a_n)/b_n \leq x\big) = G_c(x), \quad x\varepsilon R,$$

where the G_c, a_n, b_n corresponding to the limit c are displayed above.

<u>Proof</u>. By the Convergence Criterion 2.2, it suffices to show that $c_n \to c$ implies

$$(3.5) \qquad \lim_{n\to\infty} n(1-F_n(a_n + b_nx)) = x^{-1} \qquad \text{when } c = 0$$

$$= c/(e^x-1) \qquad \text{when } 0 < c < \infty$$

$$= e^{-x} \qquad \text{when } c = \infty.$$

To this end, let $m_n(x)$ denote the integer part of $a_n + b_n x$. Using (3.2), we have

$$(3.6) \qquad n(1 - F_n(a_n + b_n x)) = (n^{-1} \sum_{k=0}^{m_n(x)} r_{nk})^{-1}$$

$$= \left(n^{-1} \sum_{k=0}^{s_n-1} r_{nk} + n^{-1} r_{ns_n} \sum_{k=s_n}^{m_n(x)} \rho_n^{-(k-s_n)}\right)^{-1}$$

$$= (o(1) + z_n(x))^{-1},$$

where

$$z_n(x) = (\rho_n^{s_n - m_n(x)} - \rho_n)/c_n \qquad \text{when } \rho_n < 1$$

$$= (m_n(x) - s_n)/(n r_{ns_n}^{-1}) \quad \text{when } \rho_n = 1.$$

Then to establish (3.5), it suffices to show that $z_n(x)^{-1}$ converges to the values on the right side of (3.5). Three cases present themselves.

Case 1: $c_n \to c = 0$. Here

$$m_n(x) = a_n + b_n x + O(1) = s_n - 1 + x n r_{ns_n}^{-1} + O(1).$$

First consider the special situation in which $\rho_n = 1$ for each n. Using $n^{-1} r_{ns_n} \to 0$, we have

$$z_n(x)^{-1} = n r_{ns_n}^{-1} / (m_n(x) - s_n)$$

$$= (x + o(1))^{-1} \to x^{-1} \text{ as } n \to \infty.$$

Next consider the general situation in which $\rho_n \to 1$. Because of the preceding, we may assume that $\rho_n < 1$ for each n.

Using

$$\log \rho_n = -(1 - \rho_n) + o(1 - \rho_n) \text{ and } e^u = 1 + u + o(u)$$

as $u \to 0$, it follows that

$$\rho_n^{s_n - m_n(x)} - \rho_n = \exp[(s_n - m_n(x))\log\rho_n] - \rho_n$$
$$= \exp[xc_n + 0(1-\rho_n)] - \rho_n$$
$$= 1 + xc_n + 0(1-\rho_n) + o(c_n) - \rho_n$$
$$= xc_n + 0(1-\rho_n) + o(c_n).$$

Substituting this in the expression for $z_n(x)$, and using $(1-\rho_n)/c_n = n^{-1}r_{ns_n} \to 0$, we have the desired convergence

$$z_n(x)^{-1} = [x + 0(1-\rho_n)/c_n + o(1)]^{-1} \to x^{-1} \quad \text{as } n \to \infty.$$

Case 2: $c_n \to c$ and $0 < c < \infty$. Here

$$m_n(x) = s_n - 1 - x/\log\rho_n + 0(1).$$

Using $\log\rho_n = o(1)$, we have

$$z_n(x)^{-1} = c_n/(\exp[(s_n - m_n(x))\log\rho_n] - \rho_n)$$
$$= c_n/(\exp[x + o(1)] - \rho_n)$$
$$\to c/(e^x - 1) \quad \text{as } n \to \infty.$$

Case 3: $c_n \to c = \infty$. Here

$$m_n(x) = s_n - 1 - x/\log\rho_n + \log c_n/\log\rho_n + 0(1).$$

Using $\log\rho_n = o(1)$, we have

$$\rho_n^{s_n - m_n(x)} = \exp[(s_n - m_n(x))\log\rho_n] = c_n e^{x+o(1)}.$$

Thus

$$z_n(x)^{-1} = (e^{x+o(1)} - \rho_n/c_n)^{-1} \to e^{-x} \quad \text{as } n \to \infty.$$

This completes the proof.

Theorem 3.1 says that the convergence of c_n is sufficient for M_n to have a limit distribution when $\rho_n \to 1$. The next result says that the convergence of c_n is necessary as well, and that M_n has no limit distribution other than those above.

THEOREM 3.2. Suppose (3.1) and (3.2) hold and $\rho_n \to 1$. The possible limit distributions for M_n are G_c, $0 \leq c \leq \infty$; and M_n has the limit distribution G_c if and only if $c_n \to c$ as $n \to \infty$.

<u>Proof</u>. Suppose M_n has the limit distribution H. Since $[0,\infty]$ is a closed set in the extended real line, there are positive integers $n_k \to \infty$ and c in $[0,\infty]$ such that $c_{n_k} \to c$ as $k \to \infty$. Then by Theorem 3.1, we know that M_{n_k} has the limit distribution G_c. Moreover, M_{n_k} also has the limit distribution H. From Khintchine's theorem on convergence to types of distributions (see for instance Theorem 1.2.3 of Leadbetter et al. (1983)), it follows that H and G are of the same type. This proves that any limit distribution of M_n must be one of the distributions G_c, $0 \leq c \leq \infty$.

We now prove that $c_n \to c$ is necessary and sufficient for M_n to have a limit distribution. The sufficiency follows from Theorem 3.1. To prove the necessity, suppose that M_n has the limit distribution G_c. Let c_{n_k} be any convergent subsequence of c_n and let $c' = \lim_{k\to\infty} c_{n_k}$. Arguing as in the last paragraph, it follows that $G_{c'}$ as well as G_c is a limit distribution of M_n and that $G_{c'}$ and G_c are of the same type. Consequently, $c' = c$. Thus, we have shown that any convergent subsequence of c_n must converge to c and hence $c_n \to c$.

APPROXIMATION OF $P(M_n \leq x)$ FOR PRACTICAL APPLICATIONS 3.3. Consider the maximum M_n of a birth and death process with rates λ_k, μ_k (without the artificial dependence on n) that satisfy $\lambda_k/\mu_k = \rho$, $k \geq s$, for some $0 < \rho < 1$ and $s \geq 1$. Theorems 3.1 and 3.2 yield the approximation

$$P((M_n - s + 1)(-\log\rho) \leq x) \cong \exp(c_n/(e^x - 1)), \quad x \geq 0, \tag{3.7}$$

where $c_n = n(1-\rho) \prod_{k=1}^{s} \lambda_k/\mu_k$ and n is large.

There are analogous approximations for $P(M_n < x)$ by G_0 when c_n is small and by G_∞ when c_n is large. However, these are not as good as (3.7), which is superior for any c_n. This is because G_0 and G_∞ are only theoretical limits for the two "unobtainable" values of c_n in $[0,\infty]$; they are not functions of the actual c_n as the right side of (3.7) is. For the case when the process is null recurrent (ρ=1), Theorem 2.4 yields the classical approximation $P(M_n/nb < x) \cong G_0(x)$, where $b = \prod_{k=1}^{s} \lambda_k/\mu_k$. We were pleasantly surprised that the approximation (3.7) is accurate even when ρ is not near one. This is apparently because ρ appears on the right as well as left side of (3.7). For the M/M/1 queue, we found from numerical computations that the difference between the two sides of (3.7) is below 0.018 when n = 15 and below 0.01 when n = 20, for any ρ in (0,1).

PROPERTIES OF THE LIMIT DISTRIBUTIONS 3.4. Let X_c denote a random variable with distribution G_c, $0 < c < \infty$, and let Y denote an exponentially distributed random variable with unit mean. Standard change-of-variable computations show that

$$X_0 \overset{\mathcal{D}}{=} 1/Y, \quad X_\infty \overset{\mathcal{D}}{=} -\log Y, \quad X_c \overset{\mathcal{D}}{=} \log(Y + c) - \log Y, \quad 0 < c < \infty,$$

where these are equalities in distribution. Solving the equations for Y and using obvious substitutions, we also have

$$X_0 \overset{\mathcal{D}}{=} = \exp(-X_\infty) \overset{\mathcal{D}}{=} c/(\exp(X_c) - 1)$$

$$X_\infty \overset{\mathcal{D}}{=} \log X_0 \overset{\mathcal{D}}{=} \log((\exp(X_c) - 1)/c)$$

$$X_c \overset{\mathcal{D}}{=} \log(1 + cX_0) \overset{\mathcal{D}}{=} \log(1 + c\exp(-X_\infty)).$$

The G_∞ can be viewed as the limit of G_c as $c \to \infty$ in that

$G_c(x + \log c) \to G_\infty(x)$ as $c \to \infty$ (i.e. $X_c - \log c \to X_\infty$). It is known that

$$EX_\infty = E(-\log Y) = -\int_0^\infty e^{-y} \log y \, dy = \gamma,$$

where $\gamma = .5772...$ is the Euler-Mascheroni constant $\gamma = \lim_{m\to\infty} (\sum_{n=1}^{m} 1/n - \log m)$, and

$$E(e^{\alpha X_\infty}) = E(Y^{-\alpha}) = \Gamma(1-\alpha), \text{ for } \alpha < 1,$$

where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$. In comparison, for $0 < c < \infty$, we have

$$EX_c = E(\log(c + Y) - \log Y)$$
$$= e^c \int_0^\infty e^{-u} \log u \, du + \gamma.$$

Hence

$$EX_c = \gamma + e^c(\gamma_c - \gamma),$$

where $\gamma_c = -\int_0^c e^{-u} \log u \, du$, which can be interpreted as the Euler-Mascheroni constant on $[0,c]$ (recall that $\gamma_\infty = \gamma$). The γ_c can be computed by numerical integration; it is a positive continuous function in c that increases on [0,1] and decreases on $[1,\infty]$. Furthermore, using the binomial expansion, we have

$$E(\exp(\alpha X_c)) = E(1 + c/Y)^\alpha = \sum_{k=1}^{\infty} \binom{\alpha}{k} c^{\alpha-k} \Gamma(1 + k - \alpha), \quad \text{for } \alpha < 1.$$

We end this section with further insights into the irregular behavior of M_n. Can M_n have a limit distribution when ρ_n does not converge to one? Can M_n have a discrete limit distribution? The following results show that the answer to each of these questions is yes.

PROPOSITION 3.5. Suppose the birth and death processes satisfy (3.1), (3.2) and $\rho_n \to \rho < 1$. Let a_n be a sequence of integers. Then the distribution of $M_n - a_n$ converges weakly to a nondegenerate limit H as $n \to \infty$ if and only if

$$n^{-1} r_{ns_n} \rho_n^{s_n - a_n} \to \alpha > 0 \quad \text{as } n \to \infty. \tag{3.8}$$

In this case,

$$H(x) = \exp(-\alpha^{-1}(1-\rho)\rho^{[x]}), \quad x \in R,$$

where [x] denotes the integer part of x (H is concentrated on the integers).

Proof. With no loss in generality, we may assume that $\rho_n < 1$ for each n. By (3.6) with $m_n(x) = a_n + [x]$, we have

$$n(1 - F_n(a_n + x)) = (o(1) + n^{-1} r_{ns_n} (\rho_n^{s_n - a_n - [x]} - \rho_n)/(1 - \rho_n))^{-1}.$$

Recall that $n^{-1} r_{ns_n} \to 0$ because of (3.2). Then clearly $n(1 - F_n(a_n + x)) \to \alpha^{-1}(1-\rho)\rho^{[x]}$ if and only if (3.8) holds. Hence, the assertions follow by *Criterion* 2.2.

Proposition 3.5 may not be too useful for applications since (3.8), apparently, is rarely satisfied. Indeed, we know from Theorem 2.3 (i) that there do not exist a_n that satisfy (3.8) when λ_{nk}, μ_{nk}, s_n and ρ_n are independent of n. A subtle variation in these quantities as $n \to \infty$ is therefore needed for the existence of a_n that satisfy (3.8). Here is such an instance.

EXAMPLE 3.6. Consider the special case in which $\lambda_{nk}/\mu_{nk} = n^{-1/[\log n]}$, $k > s$, and $\prod_{\ell=1}^{s} \lambda_{n\ell}/\mu_{n\ell} \to \gamma > 0$ as $n \to \infty$. Then (3.1) and (3.2) hold and $\rho_n = \exp(-\log n/[\log n]) \to e^{-1}$. Let $a_n = [\log n]$. Then

$$n^{-1} r_{ns} \rho_n^{s-a_n} = r_{ns} \rho_n^{s} \to \gamma^{-1} e^{-s} \quad \text{as } n \to \infty.$$

Hence Proposition 3.5 yields

$$\lim_{n\to\infty} P(M_n - a_n \leq x) = \exp(-\beta e^{-[x]}), \quad x \in R,$$

where $\beta = \gamma e^{s}(1-e^{-1})$. This limit distribution is a discrete version of $G_\infty(x) = \exp(-e^{-x})$.

4. Extreme Values of Transient Processes

Consider a birth and death process, as in Section 2, with rates λ_k, μ_k. Assume that the process is transient, that is, the sum $B = \sum_{k=0}^{\infty} r_k$ is finite. Let M_n denote the maximum of the process up to the time T_n of its nth return to state 0. Because the process is transient, the M_n and T_n may be infinite: from (2.1) we know that $P(M_1 \leq x) = 1 - (\sum_{k=o}^{x} r_k)^{-1}$, and so $P(M_1 < \infty) = 1 - 1/B < 1$. Of interest, therefore, is the asymptotic behavior of M_n conditioned on $M_n < \infty$. Accordingly, we now consider the convergence of the conditional distribution

$$P((M_n - a_n)/b_n \leq x \mid M_n < \infty) = H(a_n + b_n x)^n, \tag{4.1}$$

where

$$H(x) = P(M_1 \leq x \mid M_1 < \infty) = (1 - (\sum_{k=o}^{x} r_k)^{-1})/(1-1/B). \tag{4.2}$$

Similar to the terminology above, we say that M_n conditioned on $M_n < \infty$ has a limit distribution or doesn't have one according to whether or not the distribution (4.1) converges weakly to a nondegenerate distribution. The following result is analogous to Theorem 2.3. Here we use $\rho = \liminf_{k\to\infty} \lambda_k/\mu_k$, $\bar{\rho} = \limsup_{k\to\infty} \lambda_k/\mu_k$, and $\beta_{mk} = \prod_{\ell=1}^{k} \mu_{m+\ell}/\lambda_{m+\ell}$.

THEOREM 4.1. (i) The M_n conditioned on $M_n < \infty$ does not have a limit distribution if and only if

$$(4.3) \qquad \liminf_{m\to\infty} \sum_{k=m}^{\infty} \beta_{mk} < \infty.$$

(ii) Inequality (4.3) holds when $\Sigma_{k=0}^{\infty} \limsup_{m\to\infty} \beta_{mk} < \infty$ or when $\rho > 1$.

(iii) Inequality (4.3) does not hold when $\Sigma_{k=0}^{\infty} \liminf_{m\to\infty} \beta_{mk} = \infty$ or when $\bar{\rho} \leq 1$. The condition $\bar{\rho} \leq 1$ is equivalent to $\lim_{k\to\infty} \lambda_k/\mu_k = 1$.

(iv) If $\beta_k = \lim_{m\to\infty} \beta_{mk}$ exists for each k, then inequality (4.3) holds if and only if $\Sigma_{k=0}^{\infty} \beta_k < \infty$.

Proof. From (4.2) and a little algebra, we get

$$(4.4) \qquad (H(x) - H(x-1))/(1 - H(x-1)) = \Big(\sum_{k=0}^{\infty} r_k / \sum_{k=0}^{m} r_k\Big)\Big(\sum_{k=m}^{\infty} \beta_{mk}\Big)^{-1}.$$

Then arguing as in the proof of Theorem 2.3 (i), it follows that M_n conditioned on $M_n < \infty$ does not have a limit distribution if and only if expression (4.4) does not converge to zero, which is equivalent to (4.3) (the first term on the right of (4.4) converges to one).

Part (iii) follows since, by Fatou's lemma and the product form of β_{mk}, we have

$$\liminf_{m\to\infty} \sum_{k=m}^{\infty} \beta_{mk} \geq \sum_{k=0}^{\infty} \liminf_{m\to\infty} \beta_{mk} \geq \sum_{k=0}^{\infty} b^{-k},$$

and the second assertion in (iii) is proved the same way its analogue in

Theorem 2.3 (iii) was. Part (ii) follows by a similar argument, and part (iv) is a consequence of (ii) and (iii).

Theorem 4.1 says that for a typical transient process with $\liminf_{k\to\infty} \lambda_k/\mu_k > 1$, there does not exist a limit distribution for M_n conditioned on $M_n < \infty$, but there might be a limit for a process with $\lim_{k\to\infty} \lambda_k/\mu_k = 1$. When it is possible for the limit to exist, then the asymptotic behavior of M_n conditioned on $M_n < \infty$ is analogous to Properties 2.4. Here we have a further simplification.

REMARK 4.2. The distribution (4.1) has the same limiting behavior as the distribution $\tilde{H}(a_n + b_n x)^n$, where $\tilde{H}(x) = B^{-1} \sum_{k=0}^{x} r_k$. This follows since

$$1 - H(x) = (1 - \tilde{H}(x))/((1 - B^{-1}) \sum_{k=0}^{x} r_k),$$

and so ratios of the form $(1 - H(x_n)/(1 - H(y_n))$, like (2.6) and (2.7), have the same limiting behavior as $(1 - \tilde{H}(x_n))/(1 - \tilde{H}(y_n)$ when $x_n, y_n \to \infty$.

The preceding observations lead to the study of M_n when the ratio λ_k/μ_k depends on n and is nearly unity for large n. Accordingly, consider the maximum M_n, as in Section 3, for a sequence of birth and death processes with rates λ_{nk}, μ_{nk} that depend on n as well as k. Then

$$P(M_n \leq x \mid M_n < \infty) = H_n(x)^n$$

where

$$H_n(x) = (1 - (\sum_{k=0}^{x} r_{nk})^{-1})/(1 - 1/B_n),$$

and $r_{n0} = 1$,

$$r_{nk} = (\mu_{n1} \cdots \mu_{nk})/(\lambda_{n1} \cdots \lambda_{nk}), \quad B_n = \sum_{k=0}^{\infty} r_{nk}.$$

Assume that (3.1) and (3.2) hold and that $\rho_n > 1$ for each n. Then each birth and death process is transient since B_n is finite:

$$B_n = r_{ns_n} \sum_{m=0}^{\infty} \rho_n^{-m} = r_{ns_n}/(1 - 1/\rho_n). \tag{4.5}$$

The following result, analogous to Theorems 3.1 and 3.2 combined, says that when $\rho_n \to 1$, the possible limit distributions for M_n conditioned on $M_n < \infty$ are G_c, $0 \leq c \leq \infty$. Here we let

$$c_n = n(1 - 1/\rho_n)r_{ns_n}^{-1} \qquad c = \lim_{n\to\infty} c_n$$

and

$$a_n = s_n - 1 \qquad b_n = nr_{ns_n}^{-1} \qquad \text{when } c = 0$$

$$a_n = s_n - 1 \qquad b_n = 1/\log\rho_n \qquad \text{when } 0 < c < \infty$$

$$a_n = s_n - 1 - \log(1/c_n - 1/n)/\log\rho \qquad b_n = 1/\log\rho_n \qquad \text{when } c = \infty.$$

This notation is that of Section 3 with ρ_n replaced by $1/\rho_n$ and the last a_n changed slightly.

THEOREM 4.3. Suppose (3.1) and (3.2) hold, $\rho_n > 1$ for each n, and $\rho_n \to 1$. Then M_n conditioned on $M_n < \infty$ has a limit distribution if and only if $c_n \to c$ where $0 \leq c \leq \infty$. In this case

$$\lim_{n\to\infty} P(M_n - a_n)/b_n \leq x \mid M_n < \infty) = G_c(x), \ x \in R, \tag{4.6}$$

where a_n, b_n are defined above.

Proof. We will prove that $c_n \to c$ implies (4.6). Then the rest of the assertion will follow by the argument we used in the proof of Theorem 3.2. Similar to (4.5), we can write, for $m \geq s_n$,

$$\sum_{k=0}^{m} r_{nk} = r_{ns_n} \sum_{\ell=0}^{m-s_n} \rho_n^{-\ell} = r_{ns_n}(1 - \rho_n^{-m+s_n-1})/(1 - 1/\rho_n). \tag{4.7}$$

Let $m_n(x)$ denote the integer part of $a_n + b_n x$. Then using (4.5) and (4.7), we have

$$(4.8)\qquad n(1 - H_n(a_n + b_n x)) = n(B_n/\sum_{k=0}^{m_n(x)} r_{nk} - 1)/(B_n - 1)$$
$$= c_n((1 - \rho_n^{-m_n(x)+s_n-1})^{-1} - 1)/(1 - c_n/n)$$
$$= 1/((1/c_n - 1/n)(\rho_n^{m_n(x)-s_n+1} - 1)).$$

Suppose $c_n \to c$, where $0 \leq c < \infty$. Then by Cases 1 and 2 in the proof of Theorem 3.1, with ρ_n replaced by ρ_n^{-1}, it follows that

$$(4.9)\qquad \lim_{n\to\infty} n(1 - H_n(a_n + b_n x)) = \lim_{n\to\infty} c_n/(\rho^{m_n(x)-s_n} - 1)$$
$$= x^{-1} \qquad \text{when } c = 0$$
$$= c/(e^x - 1) \qquad \text{when } 0 < c < \infty.$$

Next, suppose $c_n \to \infty$. Here

$$m_n(x) = s_n - 1 - \log(1/c_n - 1/n)/\log\rho_n + x/\log\rho_n + 0(1).$$

Using this in (4.8), we have

$$(4.10)\qquad \lim_{k\to\infty} n(1 - H_n(a_n + b_n x)) = \lim_{n\to\infty} 1/(1/c_n - 1/n)\rho_n^{m_n(x)-s_n+1}$$
$$= \lim_{n\to\infty} \exp(-(m_n(x) - s_n + 1)\log\rho_n - \log(1/c_n - 1/n))$$
$$= e^{-x}.$$

Thus, (4.9), (4.10) and Criterion 2.2 yield (4.6).

REMARK 4.4. The analogue of Approximation 3.3 for the birth and death process with $\lambda_k/\mu_k = \rho$, $k \geq s$, and $\rho > 1$ is as follows

$$(4.11)\qquad P((M_n - s + 1)\log\rho_n \leq x \mid M_n < \infty) \cong \exp(c_n/(e^x - 1)),\ x \geq 0,$$

where $c_n = n(1 - 1/\rho)\prod_{k=1}^{s} \lambda_k/\mu_k$.

5. Extreme Values of Queues

We now apply the preceding results to the M/M/s and related queues. The M/M/s queueing process described in Section 1 is a birth and death process with birth rate $\lambda_k = \lambda$ (the Poisson arrival rate of customers) and death rate $\mu_k = \mu \min\{k,s\}$ (the rate at which k customers depart from the s servers).

For our first result, we suppose M_n is the maximum of this M/M/s queue up to the nth time the system becomes empty. The limiting behavior of M_n depends on the queue's traffic intensity $\rho = \lambda/s\mu$. The queueing process is positive recurrent when $\rho < 1$, null recurrent when $\rho = 1$, and transient when $\rho > 1$. The following is an immediate consequence of Theorems 2.3, 2.5 and 4.2.

COROLLARY 5.1. If $\rho < 1$, then M_n does not have a limit distribution. If $\rho = 1$, then

$$\lim_{n\to\infty} P(m_n/nb \leq x) = e^{-x^{-1}}, \quad x \geq 0, \tag{5.1}$$

where $b = (\lambda/\mu)^s s!$. If $\rho > 1$, then M_n conditioned on $M_n < \infty$ does not have a limit distribution.

For the next result, we suppose that M_n, as in Sections 3 and 4, is the maximum for an M/M/s_n queue with arrival rate $\lambda_{nk} = \lambda(n)$, service rate $\mu_{nk} = \mu(n)\min\{k,s_n\}$, and number of servers s_n. The traffic intensity of the queue is $\rho_n = \lambda(n)/(s_n\mu(n))$. Clearly $\lambda_{nk}/\mu_{nk} = \rho_n$, for $k \geq s_n$. We will use

$$r_{ns_n} = \prod_{\ell=1}^{s_n} \mu_{n\ell}/\lambda_{n\ell} = s_n!(\mu(n)/\lambda(n))^{s_n}.$$

COROLLARY 5.2. Suppose $n^{-1} r_{ns_n} \to 0$ and $\rho_n \to 1$.

(i) If $\rho_n < 1$ for each n, then M_n has a limit distribution if and only if $n(1 - \rho_n)r^{-1}_{ns_n} \to c$, where $0 < c < \infty$. In this case, the limit distribution is G_c and appropriate norming constants are as in Theorem 3.1.

(ii) If $\rho_n > 1$ for each n, then M_n conditioned on $M_n < \infty$ has a limit distribution if and only if $n(1 - 1/\rho_n)r^{-1}_{ns_n} \to c$, where $0 < c < \infty$. In this case, the limit distribution is G_c and appropriate norming constants are as in Theorm 4.3.

<u>Proof</u>. The two assertions are special cases of Theorems 3.1, 3.2 and Theorem 4.3, respectively. Note that condition (3.1) is satisfied, and so is (3.2) since

$$n^{-1} \sum_{k=0}^{s_n} r_{nk} < 2n^{-1} r_{ns_n} \to 0.$$

REMARKS 5.3. (a) The number of customers in an M/M/∞ queueing system over time is a birth and death process with rates $\lambda_k = \lambda$ (the Poisson arrival rate of customers) and $\mu_k = k\mu$ (where μ is the service rate of each of the infinite servers). The traffic intensity is $\rho = \lambda/\mu$. The first and third assertions of Corollary 5.1 also hold for this queue, but there are apparently no analogues of (5.1) or Corollary 5.2.

b) Consider a service system in which the number of customers in the system over time is a birth and death process with rates λ_k, μ_k that represent customer arrival and departure rates when k customers are present. We assume that $\lambda_\mu/\mu_k = \rho$ for $k > s$, where s is a specific state, but we place no other restriction on the rates. We call this an M/M/GR-s queueing process, where GR stands for general rates. General rates are used for modeling such phenomena as balking and reneging of customers; non-standard service disciplines; dynamically changing rates under a control policy that minimizes the system cost; and simultaneous customer processing, where μ_k is the total workrate when k customers are present and customer i receives p_i of the workrate ($p_1+\ldots+p_m = 1$). Corollaries 5.1 and 5.2 readily extend to M/M/GR-s queues.

(c) The approximations (3.7) and (4.11) apply to the M/M/s and M/M/GR-s queues.

References

Anderson, C.W. (1970). Extreme value theory for a class of discrete distribution with applications to some stochastic processes. *J. Appl. Prob.* 7, 99-113.

Chung, K.L. (1967). *Markov Chains With Stationary Transition Probabilities.* Second ed. Springer-Verlog.

Cohen, J.W. (1982). *The Single Server Queue.* North-Holland.

Galambos, J. (1978). *The Asymptotic Theory of Extreme Order Statistics.* Wiley.

Heyde, C. (1971). On the growth of the maximum queue length in a stable queue. *Operations Res.* 19, 447-452.

Iglehart, D.L. (1972). Extreme values in the GI/G/1 queue. *Ann. Math. Statist.* 43, 627-635.

Leadbetter, M.R., Lindgren, G. and R. Rootzén (1983). *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag.

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