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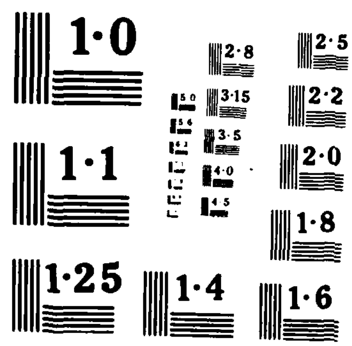
1.0
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(2)

Note on a characterization of exponential distributions

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ABSTRACT

Let U be uniformly distributed on (0,1) and let Y and $Y' \stackrel{d}{=} Y$ be random vectors with nonnegative components, U, Y and Y' independent. It is shown that the relation $Y \stackrel{d}{=} U(Y+Y')$ is satisfied if and only if the components of Y are multiples of a single exponentially distributed random variable.

1. One-dimensional case

In the solution to problem 159 in [3] the following question is answered. Let $U, Y^{(1)}$ and $Y^{(2)}$ be independent random variables, U uniformly distributed on (0,1), $Y^{(1)}$ and $Y^{(2)}$ distributed as Y. For what distributions of Y is it true that

$$(1) \quad Y \stackrel{d}{=} U(Y^{(1)} + Y^{(2)}) \qquad ?$$

There is a two-parameter family of solutions (cf. [3]), but under the additional assumption that Y is nonnegative, (1) characterizes the *exponential distributions*. We state this result as a proposition, and give a proof along the lines of the proof in [3].

Proposition 1. Let $Y \geqslant o$ with probability 1 and let Y satisfy condition (1). Then Y has an exponential distribution (possibly concentrated at zero).

Proof If ϕ denotes the Laplace-Stieltjes transform (LST) of the distribution of Y, i.e. $\phi(s) = E\, exp(-sY)$, then (1) is equivalent to

$$(2) \quad \phi(s) = \int_0^1 \phi^2(us)du = \frac{1}{s} \int_0^s \phi^2(t)dt .$$

Since ϕ and ϕ^2 are LST's, they are differentiable for $s > 0$. Differentiation of (2) yields

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(3) $s\phi'(s) + \phi(s) = \phi^2(s)$.

Substitution of $\phi = (f+1)^{-1}$ leads to

$$f'(s)/f(s) = 1/s$$

with solution $f(s) = as$ and so

(4) $$\phi(s) = \frac{1}{1+as} .$$

where $a \geqslant 0$. This proves the proposition □

In exactly the same way the following generalization can be proved.

Proposition 2. If $U, Y^{(1)}, \ldots, Y^{(N)}$ are independent, U uniformly distributed on (0,1) and the $Y^{(j)}$ distributed as Y, then

(5) $$Y \stackrel{d}{=} U(Y^{(1)}+\ldots+Y^{(N+1)})$$

if and only if the LST ϕ_N of Y is of the form

(6) $$\phi_N(s) = \frac{1}{1+a\, s^{1/N}},$$

where $a \geqslant o$.

Remark. Since $\exp(-s^{1/N})$ is an infinitely divisible (even stable; see [1], p. 448) LST it follows from Theorem 2 in [4] that ϕ_N in (6) is indeed the LST of a (infinitely divisible) probability distribution having no moments, of course. One can even make N a continuous variable: $S(1) \stackrel{d}{=} U\, S(t+1)$, where $S(\cdot)$ is a process with nonnegative stationary and independent increments. Then $S(1)$ must have an LST of the form $(1+a\, s^{1/t})^{-1}$ with $t > 0$.

An other generalization is considered in the next section.

2 Multi-dimensional case

Now let Y be an n-dimensional random vector with nonnegative components:

$$Y = (Y_1, \ldots, Y_n) .$$

and as before let $U, Y^{(1)}$ and $Y^{(2)}$ be independent with U uniform on (0,1) and $Y^{(1)}$ and $Y^{(2)}$ distributed as Y. Now let

$$Y \stackrel{d}{=} U\, (Y^{(1)}+Y^{(2)}).$$

where addition is component-wise, and let the n-dimensional LST ϕ be defined by

(7) $\phi(s_1, \ldots, s_n) = E\, exp[-\sum_{j=1}^{n} s_j Y_j]$.

Then in exactly the same way as in (2) we have

$$\phi(s_1, \ldots, s_n) = \int_0^1 \phi^2(us_1, \ldots, us_n)du,$$

or putting $s_j = \alpha_j s$,

$$s\phi(\alpha_1 s, \ldots, \alpha_n s) = \int_0^s \phi^2(\alpha_1 t, \ldots, \alpha_n t)dt.$$

Writing $\phi(\alpha_1 s, \ldots, \alpha_n s) = \phi_\alpha(s)$ for all $\alpha \in \mathbb{R}_+^n$ we obtain

$$s\phi_\alpha(s) = \int_0^s \phi_\alpha^2(t)dt,$$

the same equation as (2). It follows that for all $\alpha \in \mathbb{R}_+^n$ We have

(8) $\phi_\alpha(s) = \phi(\alpha_1 s, \ldots, \alpha_n s) = (1+a(\alpha)s)^{-1}$,

where $a(\alpha) = a(\alpha_1, \ldots, \alpha_n)$ by the definition of ϕ_α satisfies

(9) $a(s\alpha_1, \ldots, s\alpha_n) = s\, a(\alpha_1, \ldots, \alpha_n)$,

i.e. $a(\alpha)$ is *homogeneous of degree one*. We are now ready to prove

Proposition 3. A random vector $Y = (Y_1, \ldots, Y_n)$ with $Y^j \geq 0$ $(j = 1, \ldots, n)$ satisfies

(10) $Y \stackrel{d}{=} U(Y^{(1)} + Y^{(2)})$

with $U, Y^{(1)}$ and $Y^{(2)}$ independent, U uniform on (0,1) and $Y^{(1)} \stackrel{d}{=} Y^{(2)} \stackrel{d}{=} Y$ if and only if the LST ϕ of Y is of the form

(11) $\phi(s_1, \ldots, s_n) = \dfrac{1}{1 + a_1 s_1 + \cdots + a_n s_n}$,

where $a_1 \geq 0, \ldots, a_n \geq 0$.

Proof. From (7) and (8) with $s_j = \alpha_j s$ and (9) it follows that for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$

(12) $\alpha_1 Y_1 + \cdots + \alpha_n Y_n \stackrel{d}{=} a(\alpha_1, \ldots, \alpha_n)X$,

where X is exponentially distributed with expectation one and $Y_1, \ldots, Y_n$ are exponential with expectations $a(1,0, \ldots, 0), \ldots, a(0, \ldots, 0,1)$. Taking expectations in (11) we obtain

(13) $\alpha_1 a(1,0,\ldots,0)+\cdots+\alpha_n a\ (0,\ldots,0,1)=a(\alpha_1,\ldots,\alpha_n)$.

If we put $a(1,0,\ldots,0)=a_1,\ldots,a(0,\ldots,0,1)=a_n$, then (11) follows from (8) and (13).

Remark 1. From proposition 3 it follows that the only random vectors $Y=(Y_1,\ldots,Y_n)$ satisfying (10) are of the from

$$Y=(a_1X,\ldots,a_nX),$$

where X is an exponentially distributed random variable with expectation one. This means that Y has a (singular) exponential distribution concentrated on the ray with direction $(a_1,\ldots,a_n)$ through the origin. So, none of the classical multivariate distributions, such as described in [2] satisfy (10).

Remark 2. One could also generalize (5) to n-dimensional vectors; this leads to results similar to proposition 3.

Remark 3. It the condition $Y_1 \geq 0,\ldots,Y_n \geq 0$ is dropped than more general solutions then (11) are possible. For $n=2$, for instance, (10) is satisfied for Y with a characteristic function of the form

$$\Psi(t_1,t_2) = (1+a_0\sqrt{t_1^2+t_1^2}-a_1 it_1-a_2 it_2)^{-1},$$

with $a_0 \geq 0$, a_1 and a_2 real. A similar situation occurs for $n=1$ (cf. [3]).

References

[1] Feller, W. (1971) *An introduction probability theory and its applications*, vol. 2. Wiley, New York, etc.

[2] Johnson, N.L. and Kotz, S. (1972) *Distributions in Statistics: Continuous Multivariate Distributions*, Wiley, New York.

[3] Problem Section (1985). *Statistica Neerlandica* 39, 57-58.

[4] Steutel, F.W. (1968) A class of infinitely divisible mixtures. *Ann. Math. Statist.* 39, 1153-1158.

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