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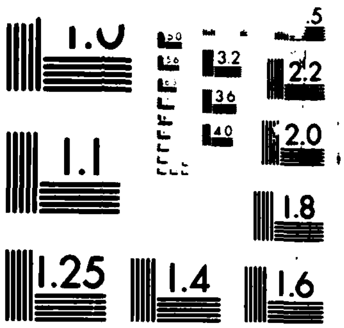

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General Solutions to Maxwell's Equations for a Transverse Field

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The general solution to the wave equation for a transverse field is obtained in terms of the geometry of the wavefront surfaces *S*. Every solution to Maxwell's equations is a solution to this wave equation, but the converse is not necessarily true. Indeed, by using results from differential geometry and topology, it is found that smooth, singularity-free, transverse solutions to Maxwell's equations cannot exist if *S* is a spheroid, a noncircular cylinder, or a surface of revolution. It is conjectured that smooth, singularity-free, transverse solutions to Maxwell's equations can only exist if *S* is a circular cylinder or a (flat) plane.

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CONTENTS

GENERAL SOLUTIONS TO MAXWELL'S EQUATIONS FOR A TRANSVERSE FIELD

INTRODUCTION AND STATEMENT OF RESULTS

Introduction

Suppose one is presented with a physical problem in which some surface or family of surfaces S plays a prominent role. If S is a sphere, then one generally finds it convenient to use spherical coordinates in the analysis; if S is a cylinder, one uses cylindrical coordinates; and so on. In the general case, we propose using a system of *surface-normal* coordinates, by which we mean a triple (u^1, u^2, z), where (u^1, u^2) is a system of surface coordinates on S and z is the perpendicular distance from a general point in space to S.

Our interest lies in obtaining solutions to certain scattering problems. As a first step we shall examine the propagation of a (strictly) transverse electromagnetic wave having a given wavefront surface S. When surface-normal coordinates are used, we find two things: First, that Maxwell's equations and the corresponding wave equations reduce to very simple forms involving space derivatives in only the single variable z. Second, Maxwell's equations also provide boundary conditions on S; namely, that the **E** and **B** fields are harmonic tangent fields on S.

According to certain well-known topological generalities, a closed spheroidal surface (topological sphere) cannot support a harmonic tangent field. Hence the boundary conditions have as an immediate consequence the fact that transverse fields with spheroidal wavefronts cannot exist.

We also find that solutions can exist only if the **E** and **B** fields satisfy a certain first order partial differential equation (PDE) whose coefficients involve the curvature of S. It then can be shown that this PDE cannot be satisfied if S is a noncircular cylinder or a surface of revolution. Hence transverse solutions to Maxwell's equations cannot exist in these cases, and we conjecture that transverse solutions can only exist if S is a circular cylinder or a (flat) plane.

In the foregoing discussion, it is assumed that the fields are smooth, singularity-free functions of time and position. If singularities are allowed, then the solution set is enlarged, but the solutions thus obtained do not appear to have any physical significance.

In this report, we describe our methods and results, but detailed proofs involving complicated calculations with tensors are presented elsewhere [1].

Statement of the Main Results

We shall consider an electromagnetic wave propagating in free space along lines normal to a family of wavefront surfaces, each of which is given by z = constant, where z is a function that satisfies the eikonal equation $|\nabla z|^2 = 1$. Then for a general point P, $z(P)$ is equal to the signed perpendicular distance from P to the wavefront S_0 given by $z = 0$ [2, p. 30]. Let $u = (u^1, u^2)$ be a system of surface coordinates on S_0, and let $\eta = \eta(u)$ be the unit normal vector to S_0, which will be taken to point along the direction of propagation. If **X** denotes the position vector of a general point on

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S_0, then $\mathbf{X} = \mathbf{X}\,(u)$ for some vector-valued function $X\,(u)$ with values in E^3 (= Euclidean 3-space). The position vector $\mathbf{R} = \mathbf{R}\,(u,z)$ of a general point P in E^3 can then be written in the form

$$\mathbf{R} = \mathbf{X}(u) + z\,\boldsymbol{\eta}\,(u)\,. \tag{1}$$

The correspondence that assigns to each point P the triple $(u,z) = (u^1,u^2,z)$ will be referred to as a surface-normal coordinate system and the wavefront obtained by setting z = constant in Fig. (1) will be denoted by S_z. Thus Eq. (1) has two uses: the first being a description of a surface-normal coordinate system for E^3, and the second—obtained by setting z = constant—being a parametric representation for the wavefront surface S_z.

The **E** and **B** fields are always required to satisfy Maxwell's equations, viz.,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \qquad \nabla \cdot \mathbf{E} = 0. \qquad \text{(M)}$$
$$\nabla \times \mathbf{B} = (1/c^2)\dot{\mathbf{E}}, \qquad \nabla \cdot \mathbf{B} = 0.$$

In addition, it is assumed that the fields are transverse, and it turns out that a necessary (but not sufficient) condition for (M) is that, considered as tangent fields to S_z, the fields **E** and **B** are harmonic in the sense of Hodge theory. This implies that

$$\tilde{\nabla}^2\,\mathbf{E} = 0,\ \ \tilde{\nabla}^2\mathbf{B} = 0\,, \tag{2}$$

where $\tilde{\nabla}^2$ is the Hodge operator, a certain second order differential operator on the tangent vector fields to S_z that is the analogue to (but different from) the ordinary Laplace operator ∇^2 in 3-space. These results are formalized in the following two propositions.

Proposition A — Suppose there exists an open interval of z values such that, for each value of z in this interval, both $\mathbf{E}\,(u,z,t)$ and $\mathbf{B}\,(u,z,t)$ are tangent to S_z over some interval of t values. Then (M) is satisfied only if **E** and **B** are harmonic on each of the surfaces S_z.

Proposition B — For monochromatic fields, the tangency condition can be relaxed to require that it hold over some interval of z values, but only for a single value of t.

Suppose now that the wavefront surfaces are "closed" in the technical sense of modern differential geometry, i.e., they are compact (hence finitely extended) surfaces without boundary curves. Recall that the topological type of a closed surface is determined by its genus g (= number of holes), i.e., two surfaces with the same genus are "topologically equivalent" in the sense that one can be smoothly deformed into the other. A "spheroid" is any surface topologically equivalent to a sphere. For spheroids we have $g = 0$, for tori we have $g = 1$, for double tori we have $g = 2$, etc. Moreover, for closed surfaces it is known from deRham cohomology theory that the space of harmonic tangent vector fields has dimension $2g$. In particular, this dimension is zero if the wavefronts are spheroids; hence transverse fields with spheroidal wavefronts cannot satisfy Maxwell's equations (except for the trivial case when **E** and **B** are identically equal to 0).

We have already mentioned that the same result holds if the wavefronts are noncircular cylinders or surfaces of revolution, and we conjecture that transverse solutions to (M) exist only if the wavefronts are (flat) planes or circular cylinders. However, the fields in our discussion are always assumed to be smooth, singularity-free functions of time and position, and the wavefronts are always assumed to be closed submanifolds of 3-space. If either of these conditions is removed, then the solution set to (M) is enlarged and our results must be modified.

General Discussion

Comparison with Luneburg's Work

As discussed by Luneburg in Ref. 2, a "wavefront" is the furthest-on position of an expanding pulse of electromagnetic energy and is taken to correspond to a "sudden discontinuity" in the field quantities. At any instant of time, such a wavefront occupies only a single surface position, and Luneburg shows that at this instant, the **E** and **B** fields are tangent to this surface. The wavefronts discussed by Luneburg can be spheroidal; but there is no contradiction between Luneburg's results and ours since in our discussion, the field quantities are assumed to be smooth functions of time and position, and the tangency condition is required to hold over an interval of z values. The "wavefronts" discussed in this report are more akin to those discussed in Geometrical Optics [3] and perhaps might be described as the "ghosts" of a departed Luneburg front. The space in a neighborhood of an instantaneous wavefront surface position S will remain excited for some time after the Luneburg front passes through S; and if S is spheroidal, at least one of the fields **E** or **B** will lose its tangency property immediately after the time of passage. In fact, in the examples of spheroidal waves given by Luneburg in Ref. 2, (sec. 13), one of the fields remains tangent to S while the other does not.

Comparison with Geometrical Optics

Geometrical optics provides a description of how a field attenuates as it recedes from its source. Assuming that the power density is proportional to $|\mathbf{E}|^2$ (or $|\mathbf{B}|^2$), and that energy is conserved along tubes of rays that span two surface patches on S_0 and S_z, one obtains the representation of Ref. 3.

$$\mathbf{E}\,(u,z,t) = \frac{\mathbf{E}_0(u)\exp(i\,k\,z)}{\left[1 + 2\tilde{H}(u)\,z + \tilde{K}(u)\,z^2\right]^{1/2}} \tag{3}$$

where $\mathbf{E}_0 = \mathbf{E}_0(u)$ are the values of **E** on S_0, k is the wavenumber, $\tilde{H} = \tilde{H}(u)$ is the mean curvature on S_0, and $\tilde{K} = \tilde{K}(u)$ is the total (or Gaussian) curvature on S_0. A similar expression holds for **B**, and we now ask whether these fields can be made to satisfy (M) by an appropriate choice of the boundary data $\mathbf{E}_0$ and $\mathbf{B}_0$. It turns out that the answer is "no." However, it will be shown that when transverse fields with a given wavefront exist, Eq. (3) is essentially correct in the far field, i.e., for values of z that are large with respect to the radii of curvature and the wavelength.

Physical Considerations

One does not expect to find strictly transverse fields generated by a physically realizable system of sources located in a finite region of space. Such fields generally contain longitudinal components whose amplitudes vary as $1/r^2$, whereas the transverse components vary as $1/r$ (where r is range from source to observer). For example, considering this matter at its most "elementary" level, we note that Feynman's formula for the field generated by a moving charge contains two longitudinal components, (one being the Coulomb field [4, vol. I sec. 28-2, vol. II sec. 21-1]). However, theoretical fields generated by infinitely extended sources, such as Sommerfeld's solution to scattering from an infinite half plane, can still provide insight into real physical problems. These classical solutions can also be used to estimate high-order scattering effects from finitely extended bodies. For a recent account of this subject, we refer the reader to Ref. 5.

METHODS AND ANALYSIS

Vector Operators in General Coordinates

The formalism of tensor analysis will be used to obtain formulas for the calculation of vector operators in a general coordinate system. In the discussion below, we follow the notation and conventions given in Refs. 6,7.

Let $u = (u^1,u^2,u^3)$ be a general coordinate system whose metric tensors and Christoffel symbols are denoted by g_{ij}, g^{ij}, and Γ^i_{jk}. Let $\mathbf{R} = \mathbf{R}(u)$ denote the position vector of a general point in 3-space, set $\mathbf{R}_i = \partial\,\mathbf{R}/\partial\,u^i$, and let $\{\mathbf{R}^1, \mathbf{R}^2, \mathbf{R}^3\}$ be the basis that is dual to $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$, so that

$$\mathbf{R}^i \cdot \mathbf{R}_j = \delta^i_j \ (= \text{Kroenecker delta})$$

Then employing the tensor summation convention for a general vector field $\mathbf{E} = \mathbf{E}(u)$ we have

$$\nabla \cdot \mathbf{E} = \mathbf{R}^i \cdot \frac{\partial \mathbf{E}}{\partial u^i}, \tag{4}$$

$$\nabla \times \mathbf{E} = \mathbf{R}^i \times \frac{\partial \mathbf{E}}{\partial u^i}, \tag{5}$$

and

$$\nabla^2 \mathbf{E} = g^{ij} \left[\frac{\partial^2 \mathbf{E}}{\partial u^i \, \partial u^j} - \Gamma^k_{ij} \frac{\partial \mathbf{E}}{\partial u^k} \right]. \tag{6}$$

In Ref. 1, we give the specialization of these formulas for transverse fields in surface-normal coordinates. These specializations are obtained by differentiating Eq. (1) with respect to the surface coordinates and by using the classical differential geometry of surfaces to express the derivatives of η in terms of the second fundamental form of the surface S_0.

Harmonicity of Transverse Fields

The elements of Hodge theory are discussed in many texts in modern differential geometry, but we shall only refer to Ref. 8 for definitions and notation. Roughly speaking, a tangent vector field is harmonic (in the sense of Hodge theory) if locally (in the neighborhood of any point) it is the gradient of a harmonic function. More precisely, a tangent field is harmonic if its surface curl and surface divergence vanish. These two conditions together imply Eq. (2), but the converse is only true on compact surfaces.

To prove the harmonicity of transverse fields, we write out (M) in surface-normal coordinates. It turns out that the surface divergence of a transverse field is equal to its ordinary space divergence, and hence the former vanishes by the divergence equations in (M). By decomposing the curl equations in (M) into normal and tangential components, we find that surface curl of a transverse field also vanishes. Hence it follows that transverse fields satisfying (M) are necessarily harmonic. The details of the derivation are given in Ref. 1, and we conclude this discussion by emphasizing that the derivation requires that *both* the **E** *and* the **B** fields be tangent to the surfaces S_z over some interval of z values.

The Wave Equations

For the remainder of this report, we only consider the case of monochromatic radiation, but our results can be applied to the more general case via the Fourier transform. A vector field is said to be "harmonic" if it is a harmonic tangent field to S_z for every z within a given interval of z values, (which by convention is assumed to contain the value $z = 0$). Hence, from the results of the previous paragraph, every transverse solution to (M) is harmonic.

In standard field theory, the solutions to (M) are shown to satisfy the wave equations, which in the monochromatic case take the form

$$\nabla^2 \mathbf{E} = -k^2 \mathbf{E}, \ \nabla^2 \mathbf{B} = -k^2 \mathbf{B}. \tag{W}$$

Conversely, every solution to (W) is a solution to (M), *provided* that one of the fields—for example—**E** satisfies the divergence equation $\text{div}(\mathbf{E}) = 0$.

Using surface-normal coordinates to express the Laplacians in terms of surface invariants, (W) becomes

$$-k^2 \mathbf{E} = \tilde{\nabla}^2 \mathbf{E} + [2\tilde{K}\,\mathbf{I} - 2\tilde{H}\,\mathbf{Q}]\,\mathbf{E} + 2\tilde{H}\frac{\partial \mathbf{E}}{\partial z} + \frac{\partial^2 \mathbf{E}}{\partial z^2}$$

with a similar equation for **B**, where $\tilde{\nabla}^2$ is the Hodge operator that occurs in Eq. (2), $\tilde{K}$ is the Gaussian (total) curvature of S_z, $\tilde{H}$ is the mean curvature, **I** is the identity matrix, and **Q** is the second fundamental form—the matrix operator, whose eigenvalues are the principal curvatures of S_z and whose eigenvectors are the principal directions. But the first term on the right-hand side vanishes for harmonic fields, and therefore transverse solutions to (M) satisfy the transverse wave equations

$$-k^2\mathbf{E} = [2\tilde{K}\,\mathbf{I} - 2\tilde{H}\,\mathbf{Q}]\,\mathbf{E} + 2\tilde{H}\frac{\partial \mathbf{E}}{\partial z} + \frac{\partial^2 \mathbf{E}}{\partial z^2}$$

$$-k^2\mathbf{B} = [2\tilde{K}\,\mathbf{I} - 2\tilde{H}\mathbf{Q}]\,\mathbf{B} + 2\tilde{H}\frac{\partial \mathbf{B}}{\partial z} + \frac{\partial^2 \mathbf{B}}{\partial z^2}. \qquad \text{(WT)}$$

Conversely, harmonic solutions to (WT) are solutions to (M), but as we shall later see, solutions to (WT) are not necessarily harmonic, even if harmonic initial data are prescribed on S_0.

Analysis of the Transverse Wave Equation

We shall now hold the u variables fixed and consider how the solutions to (WT) vary with z. We write (WT) in terms of components by resolving the E vector along the principal directions $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$:

$$\mathbf{E} = E_1\,\boldsymbol{\xi}_1 + E_2\,\boldsymbol{\xi}_2$$

The principal directions are orthogonal since they are the eigenvectors of **Q**. By direct substitution into (WT), we get the following result, where apostrophes are used to denote differentiation with respect to z and γ_1, γ_2 denote the principal curvatures,

$$-k^2E_1 = (\gamma_1\gamma_2 - \gamma_1^2)\,E_1 + (\gamma_1 + \gamma_2)\,E'_1 + E''_1$$

$$-k^2E_2 = (\gamma_1\gamma_2 - \gamma_2^2)E_2 + (\gamma_1 + \gamma_2)E'_2 + E''_2. \qquad (7)$$

The variation of the principal curvatures with z is given by

$$\gamma_\alpha(z) = \gamma_\alpha(0)/[1 + z\,\gamma_\alpha(0)],\ (\alpha = 1,2),$$

where the dependence of the curvatures on the u variables has been suppressed in the notation. There are then three cases to consider, depending on whether both, one, or none of the principal curvatures vanishes.

Case 1. Both curvatures vanish — In this case Eq. (7) reduces to the scalar wave equation in the variable z, and **E** propagates as a pure sinusoid in kz with no attenuation with increasing z.

Case 2. $\gamma_1 = 0$ *and* $\gamma_2 \neq 0$ — In this case, Eq. (7) reduces to the system

$$E''_1 + \frac{1}{\rho_2}E'_1 + k^2E_1 = 0,$$

$$\rho_2^2\,E''_2 + \rho_2E'_2 + (k^2\rho_2^2 - 1)\,E_2 = 0,$$

where $\rho_\alpha(z) = 1/\gamma_\alpha(z)$ are the radii of curvature. Then expressing E_1 and E_2 as functions of ρ_2, the general solution is given by

$$E_1 = b_1\,J_0(k\,\rho_2) + c_1\,Y_0(k\,\rho_2),$$

$$E_2 = b_2J_1(k\rho_2) + c_2\,Y_1(k\rho_2),$$

where the b's and c's are independent of z and the J's and Y's are Bessel functions of the indicated order and type. From the asymptotic properties of the Bessel functions, it follows that the components of E vary as $\exp(\pm\, i\, k\, z)/\sqrt{z}$ for large values of z.

Case 3. Neither curvature vanishes — Although we suspect that transverse solutions to (M) do not exist when neither of the principal curvatures is identically zero, a discussion of this case is included for theoretical completeness. Setting

$$F_1 = \rho_1 E_1, \; F_2 = \rho_2 E_2$$

and substituting into Eq. (7), we eventually get

$$F''_1 - \frac{2a}{\rho_1\rho_2} F'_1 + k^2 F_1 = 0,$$

$$F''_2 + \frac{2a}{\rho_1\rho_2} F'_2 + k^2 F_2 = 0,$$

where $2a = \rho_2(0) - \rho_1(0)$ is a constant. The solutions are easily shown to be sinusoidal in the far field, from which it follows that in the far field the components of **E** have the form exp $(\pm i\, k\, z)/z$.

Solutions with Planar or Circular-Cylindrical Wavefronts

The discussion in the last paragraph was concerned with the z-variation of solutions for fixed values of the u coordinates. We shall now allow u to vary and consider the global behavior of fields whose wavefronts are planes or circular cylinders. Special solutions E to (W) in these cases are usually obtained by the method of separation of variables, and solutions to (M) are then obtained by imposing the additional condition, div(**E**) = 0. Using the structure theory for harmonic fields, one can establish rigorously that these are the only transverse solutions to (M) in these cases. Details are given in Ref. 1.

For the planar case, let S be the plane $x^3 = 0$. Then the most general solutions to (M) must be of the form

$$\mathbf{E}\,(x^1,x^2,x^3 t) = e^{ikx^3}\, grad\,(U) + e^{-ikx^3}\, grad\,(V),$$

where $U = U\,(x^1,x^2)$ and $V = V\,p(x^1,x^2)$ are harmonic functions on S_0 and *grad* denotes the surface gradient. In Ref. 1 we also illustrate with an example how the solution set to (M) must be enlarged and our conclusions modified if the fields are allowed to have singularities or if the wavefronts are not required to be closed submanifolds of 3-space. In the example, S_0 is an annular region in the $x^1\,x^2$ -plane that excludes the origin; setting

$$r = [(x^1)^2 + (x^2)^2]^{1/2},$$

the most general solution is now given by

$$\mathbf{E}\,(x^1,x^2,x^3) = e^{ikx^3}\, grad\,(U) + e^{-ikx^3}\, grad\,(V) + (a/r^2)\,(x^1\,\mathbf{e}_2 - x^2\mathbf{e}_1 p),$$

where a is an arbitrary constant and $\mathbf{e}_1,\mathbf{e}_2$ are unit basis vectors in the plane.

Other Cases for Which Transverse Fields Do Not Exist

We have seen that the nonexistence of transverse fields with spheroidal wavefronts is a simple consequence of the fact that a spheroidal surface cannot support a nonzero harmonic tangent field. We recall that a solution to (WT) is also a solution to (M) if and only if it is harmonic. As previously mentioned, a field $\mathbf{E} = \mathbf{E}(u,z)$ is said to be harmonic if it is a harmonic tangent field on each S_z for some interval of z values. We shall now give examples of surfaces S_0 that do support harmonic tangent fields, but for which (WT) has no harmonic solution. In other words, one can assign harmonic initial data for (WT) with no guarantee that the solution will propagate as a harmonic field. The examples are noncircular cylinders and surfaces of revolution; as previously mentioned, we suspect that with the proviso given at the end of the section titled, Statement of the Main Results, transverse solutions to (M) are only possible when the wavefronts are planes or circular cylinders.

To establish these results, we resolve **E** into components with respect to the tangent basis vectors $\mathbf{R}^1$, $\mathbf{R}^2$, so that we now write

$$\mathbf{E} = E_1 \mathbf{R}^1 + E_2 \mathbf{R}^2 .$$

Then (WT) resolves into two equations—one for $\mathbf{E}_1$ and one for $\mathbf{E}_2$. Differentiating the first equation with respect to u^2 and the second with respect to u^1, subtracting the results, and assuming that **E** is harmonic and therefore closed, one obtains

$$d_2 E'_1 + d_1 E'_2 = 0 ,$$

where

$$d = \gamma_2 - \gamma_1, \; d_1 = \partial d / \partial u^1, \text{ and } d_2 = \partial d / \partial u^2 .$$

But for the special coordinate systems described above, the principal curvatures and hence d depend on only one of the surface coordinates. If we call this coordinate u^2, then $d_1 = 0$; from the above equation, we get

$$d_2 \, E'_1 = 0 .$$

For the cylinder, d is the curvature of the base curve and is nonconstant for noncircular cylinders. Hence $d_2 = 0$ in this case, and it can be shown that the same result holds for surfaces of revolution. We therefore conclude that $E'_1 = 0$, from which it follows that $E_1 = 0$ since (WT) has no nonzero solutions whose derivatives are identically zero in z. The same arguments apply to the B vector, and we conclude that $B'_1 = 0$ and $B_1 = 0$. But using (M), the vanishing of E'_1 implies the vanishing of B_2, and the vanishing of B'_1 implies the vanishing of E_2. Hence $E_1 = E_2 = B_1 = B_2 = 0$. In other words, for noncircular cylinders and surfaces of revolution, we have shown that $\mathbf{E} = 0$ and $\mathbf{B} = 0$ are the only solutions to (M).

SUMMARY

- We consider a transverse electromagnetic wave propagating along rays normal to a given wavefront surface S.

- By using surface-normal coordinates, Maxwell's equations and the corresponding wave equ[illegible]n reduce to forms involving derivatives in only a single space variable z, where z is the perpendicular distance from a general point in space to S.

- We also show that the **E** and **B** fields are harmonic (in the sense of Hodge theory).

- It is known that a (closed) spheroidal surface cannot support a harmonic tangent field; it follows that smooth, singularity-free transverse fields with spheroidal wavefronts cannot exist.

- An analysis of the wave equation shows that the same result holds if S is a noncircular cylinder or a surface of revolution.

From these results and from general physical principles, we conjecture that smooth singularity-free transverse solutions to Maxwell's equations can only exist if S is a plane or a circular cylinder.

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