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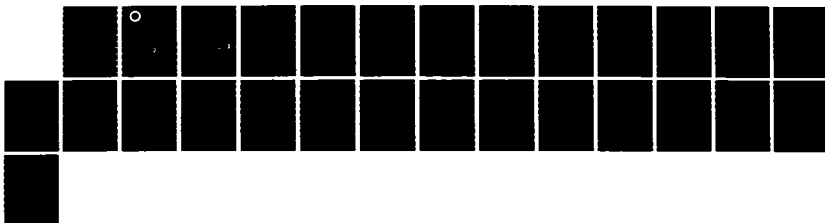
AN ALGORITHM FOR RESCALING A MATRIX POSITIVE DEFINITE
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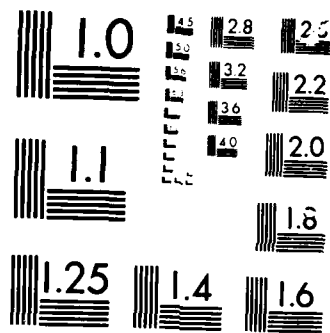
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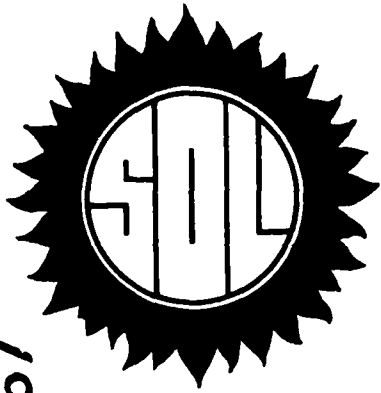




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by

Hui Hu

TECHNICAL REPORT SOL 86-10

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AN ALGORITHM FOR RESCALING A MATRIX POSITIVE DEFINITE

Hui Hu

Abstract

For a given square real matrix M , we present a general algorithm which decides the existence of a positive diagonal matrix D such that DM is positive definite and which constructs the D if it exists. It is shown that solving this matrix rescaling problem is equivalent to finding a solution of an infinite system of linear inequalities. The algorithm solves the infinite system of linear inequalities by generating and solving a sequence of linear programs.

1. Introduction

Given a square real matrix M , does there exist a positive diagonal matrix D such that DM is positive definite? If such a D exists, how can it be constructed? Such questions arise in mathematical economics and in the study of certain engineering systems [1]. A necessary and sufficient condition for the existence of such D for 3×3 matrices was given by Cross [4]. The existence of such D for Leontief matrices was proved by Tartar [12] and Dantzig [5]. A general necessary and sufficient condition for the existence of such D was given by Barker, Berman and Plemmons [1]. However, the condition is difficult to verify in practice. Methods for constructing such D for some special classes of matrices were discussed in [1] and [2].



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In this paper we present a general algorithm which decides the existence of such D and which constructs such a D if it exists. In Section 2 we explain notation and preliminaries. In Section 3 we specify the algorithm, prove its correctness and convergence, and discuss conditions that guarantee termination in a finite number of steps. Computational results are presented in Section 4. Finally, we discuss possible ways to accelerate the convergence in Section 5.

2. Notation and Preliminaries

An $n \times n$ real matrix M , not necessarily symmetric, is positive definite if $x^T M x > 0$ for all $0 \neq x \in R^n$.

If there exists a positive diagonal matrix D such that DM is positive definite, we say that M can be rescaled positive definite. Such matrices are called "diagonally stable" in [1] and "Volterra-Lyapunov stable" in [4].

Superscripts on vectors are used to denote different vectors, while subscripts are used to denote different components of a vector.

Let $S^{n-1} = \{x \in R^n : x^T x = 1\}$ denote the unit sphere in R^n and $S_+^{n-1} = \{x \in S^{n-1} : x \geq 0\}$ denote the set of nonnegative vectors in S^{n-1} .

$D(x)$ is a diagonal matrix with diagonal elements x_i for $i = 1, \dots, n$.

For a real symmetric matrix B , let $\lambda[B]$ stand for the smallest eigenvalue of B and $V[B]$ a corresponding eigenvector of unit length.

Given a mathematical programming problem (P) , $v(P)$ denotes the optimal objective function value of (P) .

Let e^i be the i -th unit vector of R^n and $e = e^1 + \dots + e^n$.

$\text{conv}(S)$ denotes the convex hull of S .

$\|x\|$ denotes the Euclidean norm of x .

Fact 1. M is positive definite if and only if $M + M^T$ is positive definite.

Fact 2. M is positive definite if and only if $x^T M x > 0$ for all $x \in S^{n-1}$, where $S^{n-1} = \{x \in \mathbb{R}^n : x^T x = 1\}$.

Fact 3. All principal minors of M remain sign invariant under a positive rescaling DM .

Fact 4. If M is positive definite, then all its principal minors are positive (see, e.g., Cottle [3]).

Fact 5. If M is positive definite, then $B^T M B$ is positive definite for any real nonsingular matrix B .

Fact 6. For any real symmetric matrix B ,
 $\lambda[B] = \min \{u^T B u : u \in S^{n-1}\}$ (see, e.g., Wilkinson [13], p. 98-99).

Fact 7. For any real symmetric matrix B , $\lambda[B]$ is a continuous function of the elements of B (see, e.g., Isaacson and Keller [9], p. 136).

Remark 1.

(1) Let DM (MD) be a positive rescaling of the rows (columns) of the matrix M . M can be column-rescaled positive definite if and only if M can be row-rescaled positive definite. Indeed, if DM is positive definite, where D is a positive diagonal matrix, then $(D^{-1})^T D M D^{-1} = M D^{-1}$ is also positive definite (Fact 5) and vice versa.

(2) We are only interested in rescaling nonsymmetric matrices because if a real symmetric matrix is not positive definite, then it can

not be rescaled positive definite. For given a real symmetric matrix M , if all its principal minors are positive, then M is positive definite. Otherwise, at least one of the principal minors of M is not positive. Therefore, DM has at least one nonpositive principal minor for any positive diagonal matrix D (Fact 3) and DM is not positive definite (Fact 4).

(3) If M can be rescaled positive definite, then it is easy to see that M is nonsingular and the diagonal elements of M are positive (by Fact 4, $d_i m_{ii} > 0$ for all i , which implies $m_{ii} > 0$ for all i). Therefore, without loss of generality we assume that the matrix M to be rescaled is nonsingular and has only positive diagonal elements.

3. The Algorithm and its Convergence

First we show that solving the matrix rescaling problem is equivalent to finding a solution of an infinite system of linear inequalities.

Theorem 1. Suppose that all diagonal elements of matrix $M = (m_{ij})$ are positive. Then M can be rescaled positive definite if and only if the following infinite system of linear inequalities

$$(ISLI): (D(u)Mu)^T x \geq 1 \text{ for all } u \in S^{n-1}$$

has a solution. Moreover, for any \bar{x} solving (ISLI), $D(\bar{x})$ rescales M positive definite.

Proof. If there exists a positive diagonal matrix $D(d)$ such that $D(d)M$ is positive definite, let $f(u) = u^T D(d)Mu$. $f(u)$ is a continuous function of u and S^{n-1} is a compact set. Hence, $f(u)$ achieves its infimum on S^{n-1} , i.e., there exists $\bar{u} \in S^{n-1}$ such that $f(u) \geq f(\bar{u}) > 0$ for all $u \in S^{n-1}$. Let $\bar{x}_i = d_i / f(\bar{u})$ for $i = 1, \dots, n$, we have $(D(u)Mu)^T \bar{x} = u^T D(d)Mu / f(\bar{u}) = f(u) / f(\bar{u}) \geq 1$ for all $u \in S^{n-1}$. Thus, (ISLI) has a solution. On the other hand, if \bar{x} is a solution of (ISLI), then we have $u^T D(\bar{x})Mu = (D(u)Mu)^T \bar{x} \geq 1 > 0$ for all $u \in S^{n-1}$. By Fact 2, $D(\bar{x})M$ is positive definite. To complete the proof, we only need to show that $\bar{x}_i > 0$ for all $i = 1, \dots, n$. Indeed, $(e^i)^T D(\bar{x})Me^i = m_{ii} \bar{x}_i > 0$ for all $i = 1, \dots, n$ since $e^i \in S^{n-1}$; also, all m_{ii} are positive by assumption. Consequently, all \bar{x}_i are positive. \square

Theorem 1 tells us that M can be rescaled positive definite if and only if (ISLI) has a solution; and moreover, for any \bar{x} solving (ISLI), $D(\bar{x})$ rescales M positive definite. The algorithm we are going to present is actually to decide whether (ISLI) has a solution or not and to find such a solution if it exists.

It is well known that deciding whether a finite system of linear inequalities has a solution is equivalent to solving a linear program (see, e.g., Dantzig [6], Chapter 5). In an analogous way, to solve the infinite system of linear inequalities (ISLI), we solve (DILP)--the dual of the infinite linear program (ILP) (see, e.g., Duffin, Jeroslow and Karlovitz [8]).

(ILP): minimize $x_1 + x_2 + \dots + x_n$
 subject to
 $(D(u)Mu)^T x \geq 1$ for all $u \in S^{n-1}$

(DILP): maximize $\sum_{i \in \Delta} y_i$
 subject to
 $\sum_{i \in \Delta} D(u^i)Mu^i y_i = e$
 $y_i \geq 0$ for all $i \in \Delta$
 Δ is finite and $\{u^i : i \in \Delta\} \subset S^{n-1}$

This column generating algorithm is the analogue of a cutting plane algorithm applied to (ILP). It generates and solves a sequence of linear programs (LP(k)) for $k = 1, 2, \dots$. If $v(LP(k))$ tends to infinity, then M can not be rescaled positive definite (Theorem 3). Otherwise, a positive vector \bar{x} which rescales M positive definite will be found after finitely many iterations (Theorem 2).

We assume that the input matrix M is nonsingular and has only positive diagonal elements (Remark 1).

Algorithm.

Step 1.

Let $k := 0$;

let (LP(k)) be the following linear program:

$$\text{maximize } \sum_{i=1}^n y_i$$

subject to

$$\sum_{i=1}^n D(e^i) M e^i y_i = e$$

$$y_i \geq 0 \quad \text{for } i = 1, \dots, n$$

let $\epsilon < 1$ be a small positive number (e.g., $\epsilon = 10^{-6}$).

Step 2.

Let x^k be an optimal dual solution of (LP(k));

find a λ^k satisfying $|\lambda^k - \lambda[D(x^k)M + M^T D(x^k)]| < (1/2)\epsilon$;

if $\lambda^k > (1/2)\epsilon$, go to Step 4.

Step 3.

Find a vector $u^{k+1} \in S^{n-1}$ such that

$$\|u^{k+1} - v[D(x^k)M + M^T D(x^k)]\| < \epsilon \quad \text{and} \quad (u^{k+1})^T D(x^k) M u^{k+1} < \epsilon;$$

form (LP(k+1)) by adjoining the column, $D(u^{k+1}) M u^{k+1}$, to the

constraint matrix of (LP(k)) with cost coefficient 1;

solve (LP(k+1));

if $v(\text{LP}(k+1)) = \infty$, then go to Step 5;

else $k := k + 1$, go to Step 2.

Step 4.

$D(x^k)M$ is positive definite, stop.

Step 5.

M can not be rescaled positive definite, stop.

Comments on the algorithm.

(1) Since we assume $m_{ii} > 0$ for all $i = 1, \dots, n$, (LP(0)) is feasible. Therefore, (LP(k)) is feasible for all k .

(2) Efficient algorithms for calculating eigenvalues and eigenvectors of a matrix are discussed in Wilkinson [13].

(3) A symmetric matrix B is positive definite if and only if $\lambda[B] > 0$; in general this is not true for nonsymmetric matrices.

Therefore, we calculate the smallest eigenvalue of the symmetric matrix $D(x^k)M + M^T D(x^k)$ in order to know whether $D(x^k)M$ is positive definite (Fact 1).

(4) If the algorithm does not stop at an iteration k , then it generates u^{k+1} satisfying $(D(u^{k+1})Mu^{k+1})^T x^k < \epsilon < 1$. While $(D(u^i)Mu^i)^T x^k \geq 1$ for all $i = 0, 1, \dots, k$ since x^k is an optimal dual solution of LP(k). Therefore, no column $D(u^k)Mu^k$ can be brought in more than once.

Next, we prove the correctness and convergence of the algorithm and discuss conditions that guarantee termination in a finite number of iterations.

Theorem 2. If there exists a positive diagonal matrix D such that DM is positive definite, then the algorithm can find such a D after finitely many iterations.

Proof. If M can be rescaled positive definite, then the corresponding program (ILP) is feasible by Theorem 1. Let \bar{x} be a feasible solution of (ILP). Then, for all feasible solutions of (DILP), we have:

$$\sum_{i \in \Delta} y_i \leq \sum_{i \in \Delta} y_i (D(u^i)Mu^i)^T \bar{x} = e^T \bar{x}.$$

Namely, the objective function of (DILP) is bounded from above by $e^T \bar{x}$. The algorithm generates a sequence of linear programs (LP(k)), $k = 0, 1, \dots$. Each (LP(k)) is feasible and is the restriction of (DILP) to certain columns. Hence, the objective function of (LP(k)) is also bounded from above by $e^T \bar{x}$. By the duality theory of linear programming, we can find an optimal dual solution x^k of (LP(k)) which satisfies

$$0 < (m_{ii})^{-1} \leq x_i^k < e^T x^k \leq e^T \bar{x} \quad \text{for all } i = 1, \dots, n.$$

Let $T = \{x \in R^n: 0 \leq x_i \leq e^T \bar{x} \text{ for all } i = 1, \dots, n\}$ and $F(x,u) = (D(u)Mu)^T x$. Then $F(x,u)$ is uniformly continuous on $T \times S^{n-1}$, i.e., for any $\delta > 0$, there exists $\eta > 0$ such that

$$(a) \quad \|(x,u) - (\bar{x}, \bar{u})\| < \eta \text{ implies } |F(x,u) - F(\bar{x}, \bar{u})| < \delta \text{ for all } (x,u) \text{ and } (\bar{x}, \bar{u}) \text{ in } T \times S^{n-1}.$$

In particular, for $\bar{\delta} = 1 - \epsilon > 0$, there exists $\bar{\eta} > 0$ such that (a) holds. If the algorithm goes on infinitely, then it generates $u^k \in S^{n-1}$ for $k = 0, 1, \dots$. Because S^{n-1} is compact, for the $\bar{\eta} > 0$, there exists u^i and u^j in the sequence satisfying $\|u^i - u^j\| < \bar{\eta}$.

Without loss of generality, we assume that $i < j$. Because the algorithm does not stop at iteration $j-1$, we have:

$$(b) \quad F(x^{j-1}, u^i) = (D(u^i)Mu^i)^T x^{j-1} \geq 1;$$

$$(c) \quad F(x^{j-1}, u^j) = (D(u^j)Mu^j)^T x^{j-1} < \epsilon.$$

However, (b) and (c) imply that $|F(x^{j-1}, u^i) - F(x^{j-1}, u^j)| > 1 - \epsilon = \bar{\delta}$ while $\|(x^{j-1}, u^i) - (x^{j-1}, u^j)\| < \bar{\eta}$, which contradicts the uniform continuity of $F(x, u)$ on $T \times S^{n-1}$. It follows that the algorithm must be finite in the case M can be rescaled positive definite. \square

Remark 2. We have proved the finiteness of the algorithm under the assumption that M can be rescaled positive definite. In fact, the boundedness of $v(LP(k)) = e^T x^k$ for $k = 1, 2, \dots$ is the only assumption we need for the proof. Since the feasible region of $(LP(k))$ can be considered as a subset of the feasible region of $(LP(k+1))$, we have $v(LP(k+1)) \geq v(LP(k))$ for all k . Therefore, in the case M can not be rescaled positive definite, the algorithm generates a sequence of feasible solutions of (DILP) whose objective function values tend increasingly to infinity. Hence, the following theorem is established.

Theorem 3. The following are equivalent:

- (1) M can not be rescaled positive definite;
- (2) $v(DILP) = \infty$;
- (3) $v(LP(k))$ tends to infinity as $k \rightarrow \infty$. \square

We have seen that the algorithm is finite in the case M can be rescaled positive definite. Next, we give a condition which ensures the finiteness of the algorithm in the case M can not be rescaled positive definite, i.e., there exists j such that $v(LP(j)) = \infty$.

Condition 1. For every nonnegative and nonzero diagonal matrix D , there exists $u \in S^{n-1}$ such that $u^T D M u < 0$. Equivalently, there does not exist nonnegative and nonzero diagonal matrix D such that $D M$ is positive semidefinite.

Theorem 4. If M satisfies Condition 1, then the algorithm stops after finitely many iterations.

Proof. Define $G(x) = \minimize_{x \in S^{n-1}} u^T D(x) M u = \lambda[(D(x)M + M^T D(x))/2]$ (Fact 6).

It follows from Fact 7 that $G(x)$ is a continuous function of x . Since M satisfies Condition 1, we have $G(x) < 0$ for all $x \in S_+^{n-1}$, where $S_+^{n-1} = \{x \in S^{n-1} : x \geq 0\}$. Therefore,

$$\beta = \maximize_{x \in S_+^{n-1}} G(x) = \maximize_{x \in S_+^{n-1}} \minimize_{u \in S^{n-1}} u^T D(x) M u < 0.$$

Let $F(x, u) = (D(u) M u)^T x = u^T D(x) M u$. For $\delta = -(1/2)\beta > 0$, there exists $\eta > 0$ such that

(d) $\|(x, u) - (\bar{x}, \bar{u})\| < \eta$ implies $|F(x, u) - F(\bar{x}, \bar{u})| < \delta$ for all (x, u) and (\bar{x}, \bar{u}) in $S_+^{n-1} \times S^{n-1}$.

As indicated in the proof of Theorem 2, if the algorithm goes on infinitely, it generates u^i and u^j ($i < j$) satisfying $\|u^i - u^j\| < \eta$ and $F(x^{j-1}, u^i) \geq 1$. Thus,

$$(e) \quad F(x^{j-1}/\|x^{j-1}\|, u^i) \geq 1/\|x^{j-1}\| > 0.$$

$$\text{Let } w = V[D(x^{j-1})M + M^T D(x^{j-1})] = V\{(D(x^{j-1})/\|x^{j-1}\|)M + M^T D(x^{j-1})/\|x^{j-1}\|\}/2\}$$

and assume that $\epsilon < \eta$, where the ϵ is specified in Step 1 of the

algorithm. Then, $\|u^j - w\| < \epsilon < \eta$ (Step 3) and hence we have

$$|F(x^{j-1}/\|x^{j-1}\|, u^j) - F(x^{j-1}/\|x^{j-1}\|, w)| < \delta = -(1/2)\beta \quad \text{by the uniform continuity of } F(x, u) \text{ on } S_+^{n-1} \times S^{n-1}. \text{ Thus,}$$

$$\begin{aligned} (f) \quad & F(x^{j-1}/\|x^{j-1}\|, u^j) \\ & < -(1/2)\beta + w^T D(x^{j-1}/\|x^{j-1}\|)Mw \\ & = -(1/2)\beta + (1/2)w^T (D(x^{j-1})/\|x^{j-1}\|)M + M^T D(x^{j-1})/\|x^{j-1}\|)w \\ & = -(1/2)\beta + \lambda[(D(x^{j-1})/\|x^{j-1}\|)M + M^T D(x^{j-1})/\|x^{j-1}\|)/2] \\ & = -(1/2)\beta + G(x^{j-1}/\|x^{j-1}\|) \\ & \leq -(1/2)\beta + \beta \\ & = (1/2)\beta. \end{aligned}$$

However, (e) and (f) imply that

$$\begin{aligned} & |F(x^{j-1}/\|x^{j-1}\|, u^i) - F(x^{j-1}/\|x^{j-1}\|, u^j)| > -(1/2)\beta = \delta \quad \text{while} \\ & \|(x^{j-1}/\|x^{j-1}\|, u^i) - (x^{j-1}/\|x^{j-1}\|, u^j)\| = \|u^i - u^j\| < \eta, \end{aligned}$$

which contradicts the uniform continuity of $F(x, u)$ on $S_+^{n-1} \times S^{n-1}$.

Therefore, if M satisfies Condition 1, then the algorithm terminates after finitely many iterations (assume that $\epsilon < \eta$). \square

In the rest of this section, we discuss other necessary and sufficient conditions for M to be rescaled positive definite and equivalent statements of Condition 1. First, we state a theorem that we are going to use in the following discussion.

Alternative Theorem. Let T be a set (T may be infinite), $h_i(t)$ for $i = 1, \dots, n$ be real-valued functions on T , and

$P = \{(h_1(t), \dots, h_n(t)) : t \in T\}$. If P is closed, then

- (1) the system $(h_1(t), \dots, h_n(t))^T x > 0$ for all $t \in T$ has a solution if and only if the origin is not contained in $\text{conv}(P)$, the convex hull of P ;
- (2) the system $(h_1(t), \dots, h_n(t))^T x \geq 0$ for all $t \in T$ has a non-trivial (i.e., nonzero) solution if and only if the origin is not contained in the interior of $\text{conv}(P)$ (Dines and McCoy [7]). \square

Lemma 1. M can not be rescaled positive definite if and only if

$$\{x : (D(u)Mu)^T x > 0 \text{ for all } u \in S^{n-1}\} = \emptyset.$$

Proof. Let $T = \{x : (D(u)Mu)^T x > 0 \text{ for all } u \in S^{n-1}\}$. If $T = \emptyset$, then (ISLI) has no solution. By Theorem 1, M can not be rescaled positive definite. On the other hand, if $T \neq \emptyset$, let $\bar{x} \in T$. Then, $u^T D(\bar{x})Mu = (D(u)Mu)^T \bar{x} > 0$ for all $u \in S^{n-1}$. Hence, $D(\bar{x})M$ is positive definite and \bar{x} is a positive vector (assume that all diagonal elements of M are positive). \square

Lemma 2. M can not be rescaled positive definite if and only if the origin is contained in $\text{conv}(D(u)Mu: u \in S^{n-1})$.

Proof. It is easy to show that $\{D(u)Mu: u \in S^{n-1}\}$ is closed. The lemma then follows easily from the Alternative Theorem and Lemma 1. \square

Condition 2. $0 \in \text{int}\{\text{conv}(D(u)Mu: u \in S^{n-1})\}$

Lemma 3. Condition 1 and Condition 2 are equivalent.

Proof. Although Conditions 1 and 2 look different, they are in fact equivalent. Indeed, if M satisfies Condition 2, then the system $u^T D(x)Mu = (D(u)Mu)^T x \geq 0$ for all $u \in S^{n-1}$ has no non-trivial solutions (by the Alternative Theorem). This implies for any $0 \neq d \geq 0$, there exists $u \in S^{n-1}$ such that $u^T D(d)Mu < 0$, i.e., M satisfies Condition 1. On the other hand, if M does not satisfy Condition 2, then (by the Alternative Theorem) there exists $\bar{x} \neq 0$ such that $u^T D(\bar{x})Mu = (D(u)Mu)^T \bar{x} \geq 0$ for all $u \in S^{n-1}$. In particular, $(e^i)^T D(\bar{x})Me^i = \bar{x}_i m_{ii} \geq 0$ for all $i = 1, \dots, n$. This implies $\bar{x}_i \geq 0$ for all $i = 1, \dots, n$ (assume that all diagonal elements of M are positive). Hence, M does not satisfy Condition 1. \square

Remark 3. Notice that $G(x) = \underset{x \in S^{n-1}}{\text{minimize}} u^T D(x) M u$ is a continuous

function of x , it is not hard to show the following.

(1) M can be rescaled positive definite if and only if

$$\underset{x \in S_+^{n-1}}{\text{maximize}} \underset{u \in S^{n-1}}{\text{minimize}} u^T D(x) M u > 0.$$

(2) Condition 1 or 2 is equivalent to

$$\text{Condition 3. } \underset{x \in S_+^{n-1}}{\text{maximize}} \underset{u \in S^{n-1}}{\text{minimize}} u^T D(x) M u < 0.$$

We summarize all necessary and sufficient conditions for M to be rescaled positive definite in Theorem 5.

Theorem 5. Suppose that all diagonal elements of M are positive.

Then the following are equivalent:

(1) M can be rescaled positive definite;

(2) (ISLI): $(D(u)Mu)^T x \geq 1$ for all $u \in S^{n-1}$ has a solution;

(3) $(D(u)Mu)^T x > 0$ for all $u \in S^{n-1}$ has a solution;

(4) the origin is not contained in $\text{conv}(D(u)Mu: u \in S^{n-1})$;

(5) $\underset{x \in S_+^{n-1}}{\text{maximize}} \underset{u \in S^{n-1}}{\text{minimize}} u^T D(x) M u > 0. \square$

Summary. The algorithm solves the matrix rescaling problem correctly and completely. It is finite if M can be rescaled positive definite or M satisfies Condition 1 (or 2, 3). It is possibly infinite only if M can not be rescaled positive definite but still can be rescaled positive semidefinite (i.e., there exists $0 \neq d \geq 0$ such that $D(d)M$ is positive semidefinite).

4. Example and Computational Results

Example:

$$M = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -17 \\ 4 & 0 & 1 \end{pmatrix}$$

We use revised simplex method to solve (LP(k)) and follow the notation in Dantzig [6]. For convenience, we do not normalize u^k . We use B to denote the optimal basis of LP(k) and \bar{c}_{k+1} the relative cost factor of the entering column for LP(k+1).

Tableau of (LP(0)) (k = 0)

Basis inverse	y_1	y_2	y_3	r.h.s.
1 0 0	1	0	0	1
0 1 0	0	1	0	1
0 0 1	0	0	1	1
0 0 0	1	1	1	0
	.	.	.	

Basis inverse	y_1	y_2	y_3	r.h.s.
1 0 0	1	0	0	1
0 1 0	0	1	0	1
0 0 1	0	0	1	1
-1 -1 -1	0	0	0	-3
	.	.	.	

$$x^0 = (1 \ 1 \ 1)^T$$

$$u^1 = (0 \ 1 \ 1)^T$$

$$B^{-1}D(u^1)Mu^1 = (0 \ -16 \ 1)^T$$

$$\bar{c}_1 = 1 - x^0 D(u^1)Mu^1 = 16$$

Tableau of (LP(1)) (k = 1)

Basis inverse	y_1	y_2	y_3	y_4	r.h.s.
1 0 0	1	0	0	0	1
0 1 0	0	1	0	-16	1
0 0 1	0	0	1	1	1
-1 -1 -1	0	0	0	16	-3

. . . *

Basis inverse	y_1	y_2	y_3	y_4	r.h.s.
1 0 0	1	0	0	0	1
0 1 16	0	1	16	0	17
0 0 1	0	0	1	1	1
-1 -1 -17	0	0	-16	0	-19

. . .

$$x^1 = (1 \ 1 \ -17)^T$$

$$u^2 = (-1 \ 1 \ 1)^T$$

$$B^{-1}D(u^2)Mu^2 = (2 \ -65 \ -3)^T$$

$$\bar{c}_2 = 1 - x^1 D(u^2)Mu^2 = 67$$

Tableau of (LP(2)) (k = 2)

Basis inverse	y_1	y_2	y_3	y_4	y_5	r.h.s.
1 0 0	1	0	0	0	2	1
0 1 16	0	1	16	0	-65	17
0 0 1	0	0	1	1	-3	1
-1 -1 -17	0	0	-16	0	67	-19

. . . *

Basis inverse	y_1	y_2	y_3	y_4	y_5	r.h.s.
0.5 0 0	0.5	0	0	0	1	0.5
32.5 1 16	32.5	1	16	0	0	49.5
1.5 0 1	1.5	0	1	1	0	2.5
-34.5 -1 -17	33.5	0	-16	0	0	-52.5

. . .

$$x^2 = (34.5 \ 1 \ 17)^T$$

$$u^3 = (-1 \ -4 \ 0)^T$$

$$B^{-1}D(u^3)Mu^3 = (-1.5 \ -77.5 \ -4.5)^T$$

$$\bar{c}_3 = 1 - x^2 D(u^3)Mu^3 = 84.5$$

Since $B^{-1}D(u^3)Mu^3 \leq 0$ and $\bar{c}_3 > 0$, we know that $v(LP(3)) = \infty$. Therefore, M can not be rescaled positive definite.

Remark 5. It follows easily from Facts 3 and 4 that if M can be rescaled positive definite, then M is a P-matrix (i.e., all principal minors of M are positive). The above 3×3 matrix is a P-matrix but it can not be rescaled positive definite.

Computational Results.

We have coded the algorithm in FORTRAN. We use the subroutine MINOS (from the Systems Optimization Laboratory, Department of Operations Research, Stanford University) to solve $(LP(k))$ and the subroutine F02ABF (from NAG Library, Stanford University) to calculate eigenvalues and eigenvectors. The data were randomly generated and the program was executed on a DEC 20 computer with the following results, see Table 1.

Table 1

Problem dimension	number of iterations	CPU time (second)
3 × 3	5	3.15
5 × 5	14	7.18
6 × 6	9	4.69
8 × 8	9	6.26
16 × 16	8	6.23

5. Accelerating the Convergence

This column generating algorithm is the analogue of a cutting plane algorithm applied to (ILP). It solves the infinite linear program (DILP) by generating and solving a sequence of linear programs (LP(k)), $k = 0, 1, \dots$. If the algorithm does not stop at iteration k , then a new column $D(u^{k+1})Mu^{k+1}$ is generated and brought in. This is a cut, $(D(u^{k+1})Mu^{k+1})^T x \geq 1$, on (ILP). If we want to accelerate the convergence of the algorithm, we have to find ways to generate more efficient cuts.

Let's look at the problem geometrically. Suppose the algorithm does not stop at iteration k . Let $f_k(u) = u^T D(x^k)Mu$. Then, $f_k(u^i) > 0$, $i = 1, \dots, k$ and $f_k(u^{k+1}) < 0$. Since $f_k(u)$ is a continuous function, for each u^i , $i = 1, \dots, k$, there exists a relatively open neighborhood $N_k(u^i) \subseteq S^{n-1}$ such that $f_k(u) > 0$ for all $u \in N_k(u^i)$. $f_k(u^{k+1}) < 0$ means that $\bigcup_{i=1}^k N_k(u^i)$ does not

cover S^{n-1} . Suppose M can be rescaled positive definite and the algorithm stops at iteration k . That means $\bigcup_{i=1}^k N_k(u^i)$ covers S^{n-1} . Therefore, we want to make $N_k(u^i)$ bigger so that we need fewer $N_k(u^i)$ to cover S^{n-1} . Let's consider the cut $(D(u^{k+1})Mu^{k+1})^T x \geq \alpha$ for some $\alpha > 1$, and hope that it will give a bigger $N_{k+1}(u^i)$. However, since we are solving linear programs, changing all the cost coefficients to α will result in a solution αx^{k+1} and therefore has no influence on choosing u^{k+2} . If we go over the proofs of Theorems 1 and 2, we find that if we change $(D(u^k)Mu^k)^T x \geq 1$ to $(D(u^k)Mu^k)^T x \geq \alpha_k$, where $0 < \delta \leq \alpha_k \leq L$ for all k , then Theorems 1 and 2 still hold. Since

$$\begin{aligned} f_k(u^{k+1}) &= (1/2)u^{k+1} [D(x^k)M + M^T D(x^k)] u^{k+1} \\ &= (1/2)\lambda [D(x^k)M + M^T D(x^k)], \end{aligned}$$

if $f_k(u^{k+1}) < 0$, a natural way to choose α_{k+1} is:

$$\alpha_{k+1} = -\theta f_k(u^{k+1}) = -\theta \lambda_1 [D(x^k)M + M^T D(x^k)],$$

where θ is a positive constant (if $-\theta f_k(u^{k+1}) < \delta$, just let $\alpha_{k+1} = \delta$).

A number of randomly generated problems were computed using the above idea ($\theta = 2$) and compared with $\alpha_k \equiv 1$, see Table 2.

Table 2

problem dimension	no. of iterations ($\alpha_k \equiv 1$ for all k)	no. of iterations ($\alpha_k = -2f_k(u^{k+1})$)
5 × 5	14	2
6 × 6	9	8
8 × 8	9	2
16 × 16	8	3

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