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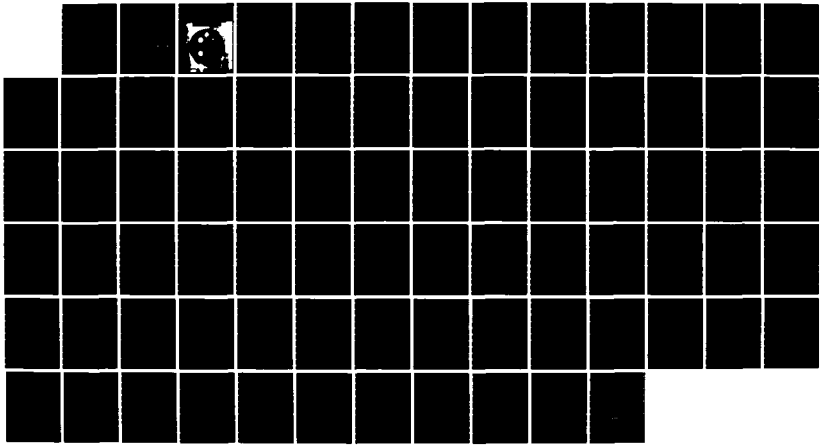

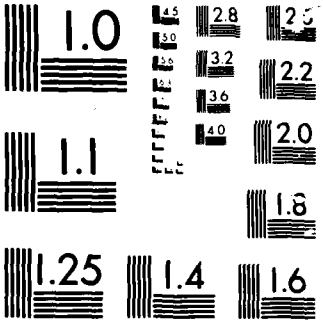

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Wendell H. Fleming*
Division of Applied Mathematics
Brown University
Providence, RI 02912 U.S.A.

Suresh P. Sethi**
Faculty of Management Studies
University of Toronto
Toronto, Ontario, Canada M5S 1V4

Halil Mete Soner***
Division of Applied Mathematics
Brown University
Providence, RI 02912 U.S.A.

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TURNPIKE SETS IN OPTIMAL STOCHASTIC PRODUCTION PLANNING PROBLEMS

by

W. H. Fleming, S. P. Sethi, H. M. Soner

Abstract. This paper considers an infinite horizon stochastic production planning problem with demand assumed to be a continuous-time Markov process. The problems with control (production) and state (inventory) constraints are treated. It is shown that a unique optimal feedback solution exists. The solution is characterized in terms of a turnpike set, toward which the optimal inventory level approach monotonically over time. Moreover, for nondeterministic demand the optimal inventory level reaches the turnpike set almost surely in a finite time and, thereafter, it wanders inside the set in response to the randomly fluctuating demand.

Key Words: production planning, stochastic optimal control, control constraints, state constraints, Markov Processes, turnpike sets.

Introduction.

Thompson and Sethi [16] consider a production-inventory model, which determines production rates over time to minimize an integral representing a discounted quadratic loss function. The model is solved both with and without nonnegative production constraints. It is shown that there exists a turnpike level of inventory, to which the optimal inventory levels approach monotonically over time. Of course, if the initial inventory level is the turnpike level, then it is optimal to produce just enough to satisfy the demand so that the inventory level stays at the turnpike level. The model was generalized by Sethi and Thompson [12] and Bensoussan, Sethi, Vickson, and Derzko [1] by incorporating an additive white noise term in the dynamics of the inventory process. It was shown that there exists a unique optimal solution. Moreover, there exists a turnpike level of inventory, in the neighborhood of which, the optimal inventory level stays most of the time.

In this paper, we generalize the Thompson-Sethi model in several different directions. First, we consider that the demand over time is a stochastic process, assumed to be either a jump Markov process or a reflected diffusion process. Second, we deal with fairly general convex costs that include the special case of quadratic costs. Finally, for jump Markov demand processes, we can incorporate a state constraint stating that the inventory level cannot fall below a prescribed level. We also note that our analysis of the case when the demand is a diffusion process introduces an approximation, which generalizes the

model of Bensoussan et. al. [1].

In order to provide a summary of the results obtained in the paper, we denote by $y(t)$, $p(t)$, $z(t)$ respectively the inventory level, production level, and randomly fluctuating demand at time t. Production is the control variable, subject to the constraint $p(t) \geq 0$. In §'s 1 - 6 we put no constraint on the inventory level $y(t)$, but in §7 the lower bound $y(t) \geq y_{min}$ is imposed. The demand $z(t)$ is modelled as a Markov process, which is bounded above and below ($\xi_0 \leq z(t) \leq \xi_1$, where $\xi_0 > 0$). We consider two cases: (i) $z(t)$ is a jump Markov process, with bounded generator L of the form (4.11); (ii) $z(t)$ is a Markov diffusion, reflected at ξ_0 and ξ_1 (5.1).

The control objective is to minimize an expected discounted cost of the form (1.2), which involves convex holding or shortage costs $h(y)$ and production costs $c(p)$. The value (or minimum cost) $v(y,z)$ defined in (1.3) for initial data $y(0) = y$, $z(0) = z$ obeys the dynamic programming equation (3.1). Special features of the model allow us to show that $v(\cdot,z)$ is strictly convex and that the quantities $\partial v/\partial y$ and $Lv(y,\cdot)$, which appear in the dynamic programming equation, exist and are continuous. The optimal feedback production law $p^*(y,z)$ is expressed as a function of $\partial v/\partial y$ by formula (3.2). We do not know that $p^*(\cdot,z)$ is Lipschitz continuous. However, since $p^*(\cdot,z)$ is a nonincreasing function of y, the differential equation

$$\frac{dy^*}{dt} = p^*(y^*(t),z(t)) - z(t), \; y^*(0) = y,$$

has a unique solution for the optimal inventory level $y^*(t)$.

In §4 we treat case (i), when the demand $z(t)$ is a jump Markov process. We make use of the concept of viscosity solution, introduced by Crandall-Lions [4] for nonlinear first-order partial differential equations, and by Soner [14] for dynamic programming equations of controlled jump Markov processes. The value function $v(y,z)$ is first shown to be a viscosity solution of the dynamic programming equation, and afterward a classical solution. In §5 we treat case (ii), when $z(t)$ is a reflected diffusion. Here we make the additional assumption that the holding cost $h(y)$ is twice continuously differentiable. A method to obtain regularity is to replace v by v^ε satisfying a dynamic programming equation to which a small term $\varepsilon\partial^2 v^\varepsilon/\partial y^2$ has been added. A crucial step in the argument is an à priori bound for $\|v^\varepsilon\|_{C^{1,\alpha}}$ independent of ε (Theorem 5.1). Once this is obtained, rather standard techniques show that $v^\varepsilon \to v$ as $\varepsilon \to 0$ and that $v(y,z)$ is a solution to the dynamic programming equation with the required regularity properties.

In §6 we show that in nontrivial cases, the optimal inventory level $y^*(t)$ reaches almost surely in finite time, a certain interval G, which we call the turnpike set.

The approach to G is monotonic over time. Moreover, once inside the turnpike set, the optimal inventory level stays inside the set forever. Of course, inside G, the inventory level keeps varying in response to the randomly fluctuating demand. We should note that the turnpike set represents a generalization of the single point turnpike level obtained in [16].

Then in §7 the analysis of §4 is modified to deal with a state-space constraint $y(t) \geq y_{min}$. Such a constraint imposes an inequality constraint (7.16) on $\partial v/\partial y$ at y_{min}. Dynamic programming equations and viscosity solutions for control problems with state-space constraints were discussed systematically in [14].

Section 8 concludes the paper with a brief discussion of some important extensions of the production planning problems that arise in automated manufacturing systems.

1. Notation, assumptions and the model

Consider a factory producing a homogeneous good in order to satisfy a stochastic demand over time. To formulate the optimization problem of the factory, we define the following quantities:

$(\Omega, \mathcal{F}, \mathbb{P})$: the underlying probability space

$Z = [\xi_0, \xi_1] \subset (0,\infty)$: the set of possible demand rates

$z(t)$: the demand process; which is a right-continuous Z-valued Markov process with infinitesimal generator L

$y(t)$: inventory level at time t (state variable)

$P = \{p(t) : t \geq 0\}$: production process, $p(t) \geq 0$ denotes the production rate at time t (control variable)

$$\mathbb{A} = \left\{ P : \Omega \times [0,\infty) \to [0,\infty) : \begin{array}{l} \text{(i) adapted to } \mathcal{F}_t = \sigma(z(s) : s \in [0,t]) \\ \text{(ii) } \sup\{p(t) : t \geq 0\} < +\infty \end{array} \right\}$$

: the set of feasible production trajectories

$c \in C([0,\infty) \to [0,\infty))$: the production cost function

$h \in C((-\infty,\infty) \to [0,\infty))$: the inventory cost function; on $(-\infty,0)$ it represents the shortage cost

$\alpha > 0$: the constant discount rate.

For any $P \in \mathbb{A}$, we define the controlled inventory trajectory $y(t)$ and the discounted cost associated with it, respectively, by

$$y(t) = y + \int_0^t [p(s)-z(s)]ds \tag{1.1}$$

$$J(y,z,P) = E\int_0^\infty e^{-\alpha t}[h(y(t))+c(p(t))]dt. \tag{1.2}$$

Note that the inventory trajectory $y(\cdot)$ depends on the production process P, the initial inventory level y and the initial demand rate z. For simplicity we suppressed these dependences in the notation.

The optimal control problem of the factory is to minimize J over all feasible production processes. Thus, we define the value function by

$$v(y,z) = \inf_{P\in A} J(y,z,P). \tag{1.3}$$

We assume the following throughout the paper:

(A1) h is continuous, convex, nonnegative on $(-\infty,\infty)$ with $h(0) = 0$.

(A2) c is continuous, convex, nonnegative on $[0,\infty)$ with $c(0) = 0$.

(A3) There are $K_1 \geq 0$ and $\gamma \geq 1$ such that

$$0 \leq h(y) \leq K_1(|y|^\gamma + 1) \quad \text{for all} \quad y \in (-\infty,\infty)$$

$$|h(y')-h(y)| \leq K_1(h(y)+1)|y'-y| \quad \text{whenever} \quad |y'-y| \leq 2\xi_1.$$

(A4) c is twice differentiable on $(0,\infty)$ with $c''(p) > 0$ for all $p > 0$ and $c'(0) = 0$.

(A5) There is $K_3 > 0$ and $\nu > 1$ such that

$$c(p) \geq K_3(|p|^{\nu}-1) \quad \text{for all} \quad p \geq 0.$$

(A6) There is $K_2 > \alpha c'(\xi_1)$ such that

$$h(y) \geq K_2(|y| - 1) \quad \text{for all} \quad y \in (-\infty,0].$$

Remark 1.1.

(i) Functions $h(y) = |y|^{\gamma}$ and $c(p) = p^{\gamma'}$ with $\gamma,\gamma' > 1$ satisfy the above assumptions.

(ii) Let $F(r)$ be given by (see figure 1)

$$F(r) = \inf_{p\geq 0} [c(p)+pr]. \tag{1.4}$$

Then the infimum is achieved at $p = 0$ if $r > 0$. If $r < 0$, the minimum is achieved at $p = (c')^{-1}(-r)$, where $(c')^{-1}$ is the inverse of c'. Note that the inverse function $(c')^{-1}$ is well-defined on account of (A2) and (A4).

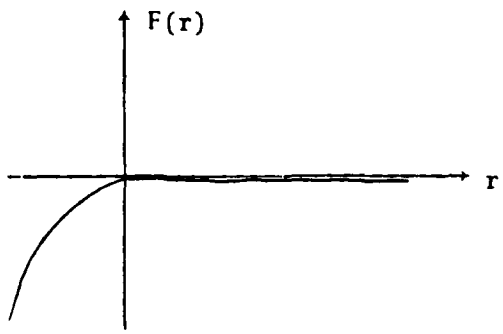

Figure 1 : The shape of function F

Also,

$$|F(r)-F(r')| \leq |r-r'|\max\{(c')^{-1}(-r \vee 0),(c')^{-1}(-r' \vee 0)\}$$

where $a \vee b = \max\{a,b\}$ and $F''(r) = -(1/c''[(c')^{-1}(-r)])$ for $r < 0$. As a function with domain $(-\infty,\infty)$, F is concave with a possible discontinuity of its second derivative at the origin.

(iii) Assumption (A4) implies that $(c')^{-1}$ is locally Lipschitz continuous on $(0,\infty)$. Hence F is locally Lipschitz continuous on $[0,\infty)$.

(iv) Most of the results that follow would hold without the assumption (A6) This assumption is innocuous as it merely serves to rule out the pathological cases, in which it is optimal, at least when the current demand is at its maximum, not to decrease the current level of shortage irrespective of how large that level is. In particular, it rules out the trivial case $h \equiv 0$. Moreover, with (A6) we can obtain more detailed characterization of the solution of the problem; see Remark 2.1 and Example 1 in Section 6.

2. Properties of the value function

In this preliminary section, we establish convexity in y of the optimal value function as well as a bound and a local Lipschitz estimate.

Lemma 2.1: For every $z \in Z$, $v(\cdot,z)$ is a convex function.

Proof: It suffices to show that $J(\cdot,z,\cdot)$ is jointly convex. For any $y,\bar{y} \in (-\infty,\infty)$, $P,\bar{P} \in \mathbb{A}$ and $z \in Z$ let $y(\cdot)$ and $\bar{y}(\cdot)$ be the inventory trajectories corresponding to y,z,P and $\bar{y},z,\bar{P}$, respectively. Then for any $\beta \in [0,1]$, we have

$$\begin{aligned}\beta J(y,z,P)+(1-\beta)J(\bar{y},z,\bar{P}) &= E\int_0^\infty e^{-\alpha t}\{[\beta h(y(t))+(1-\beta)h(\bar{y}(t))] \\ &\quad + [\beta c(p(t))+(1-\beta)c(\bar{p}(t))]\}dt \\ &\geq E\int_0^\infty e^{-\alpha t}[h(\tilde{y}(t))+c(\tilde{p}(t))]dt\end{aligned}$$

where $\tilde{p}(t) = \beta p(t)+(1-\beta)\bar{p}(t)$ and

$$\tilde{y}(t) = (\beta y+(1-\beta)\bar{y}) + \int_0^t [\tilde{p}(s)-z(s)]ds.$$

Hence, $\tilde{y}(\cdot)$ is the inventory trajectory that corresponds to $\tilde{y} = \beta y+(1-\beta)\bar{y}, z$ and $\tilde{P} = \beta P+(1-\beta)\bar{P}$. Now we rewrite the above inequality as

$$\beta J(y,z,P)+(1-\beta)J(\bar{y},z,\bar{P}) \geq J(\beta y+(1-\beta)\bar{y},z,\beta P+(1-\beta)\bar{P}).$$

□

Remark 2.1: In section 6, we shall show that under (A.6) and some additional assumptions $v(\cdot,z)$ is strictly convex.

Lemma 2.2. There are $C_1, C_2, C_3 \geq 0$ and $C_4 > c'(\xi_1)$ such that

$$(2.1) \qquad 0 \leq v(y,z) \leq \frac{1}{\alpha}h(y) + C_1 \qquad \forall\, (y,z) \in (-\infty,\infty) \times Z$$

$$(2.2) \qquad v(y,z) \geq C_4|y| - C_2 \qquad \forall\, z \in Z \text{ and } y \leq -C_3 .$$

Proof: Positivity of the value function is an immediate consequence of the positivity of h and c. To establish the upper bound, we use the production process $p_0(t) = z(t)$. Then the corresponding inventory is $y(t) = y$ for all t. Now majorize $v(y,z)$ as follows

$$v(y,z) \leq J(y,z,P_0) = E\int_0^\infty e^{-\alpha t}[h(y)+c(z(t))]dt$$

$$\leq \frac{1}{\alpha}\, h(y) + \int_0^\infty e^{-\alpha t}\, c(\xi_1)dt$$

where ξ_1 is the largest point in z. Hence, (2.1) holds with $C_1 = \frac{1}{\alpha}\, c(\xi_1)$.

To establish (2.2), we obtain an estimate of v from below by the optimal value of a deterministic control problem. Fix $(y,z) \in (-\infty,0] \times Z$. Let τ be the exit time of $y(\cdot)$ from $(-\infty,0]$.

$$(2.3) \quad J(y,z,P) \geq E\int_0^\infty e^{-\alpha t}[K_2|y(t)|+K_3|p(t)|^\nu]dt - \frac{1}{\alpha}(K_2+K_3)$$

$$\geq E\int_0^\tau e^{-\alpha t}[K_2(|y+\int_0^t p(s)ds|-|\int_0^t z(s)ds|)+K_3|p(t)|^\nu]dt - \frac{1}{\alpha}(K_2+K_3)$$

$$\geq E\int_0^\tau e^{-\alpha t}[K_2|y+\int_0^t p(s)ds|+K_3|p(t)|^\nu]dt - \frac{1}{\alpha}(K_2+K_2\xi_1/\alpha+K_3).$$

The first inequality is obtained by using assumptions (A6) and (A5) with $\nu \geq 0$. The estimate $z(t) \leq \xi_1$ is used in the last inequality. Now let $C_3 = (\nu-1)[(\frac{1}{\alpha\nu})^\nu \frac{K_2}{K_3}]^{1/\nu-1}$ and define $\bar{v}(y)$ on $y \in (-\infty,-C_3]$ by

$$\bar{v}(y) = \inf_{P\in\bar{A}} \int_0^{\theta} e^{-\alpha t}[K_2|y+\int_0^t p(s)ds|+K_3|p(t)|^{\nu}]dt$$

where

$$\theta = \inf\{t \geq 0 : y + \int_0^t p(s)ds \geq -C_3\}$$

$$\bar{A} = \{P : [0,\infty) \to [0,\infty) : \text{(i) Borel measurable, (ii) } \mathrm{Sup}\{|p(t)|:t\geq 0\}<+\infty\}$$

Then $\bar{v}(y) = \frac{K_2}{\alpha}(-y-C_3)$ for $y \leq -C_3$ because it is a smooth solution of the following equation:

$$\begin{cases} \alpha\bar{v}(y) = \inf\limits_{p\in(-\infty,\infty)} [p\,\frac{d}{dy}\bar{v}(y)+K_3|p|^{\nu}] - K_2 y \ ; \ y < -C_3 \\ \bar{v}(-C_3) = 0 \ . \end{cases}$$

The inequality (2.3) yields that

$$J(y,z,P) \geq \bar{v}(y) - \frac{1}{\alpha}(K_2+K_3+K_2\xi_1/\alpha)$$

$$= \frac{1}{\alpha}(-K_2y-[K_2C_3+K_2+K_2\xi_1/\alpha+K_3]) \quad \text{for all } y \leq -C_3 \text{ and } z,P.$$

Choose $C_2 = K_2C_3+K_2+K_2\zeta_1/\alpha+K_3$ and $C_4 = K_2/\alpha$. The assumption (A6) implies that $C_4 > c'(\xi_1)$. Hence (2.2) holds.

□

<u>Lemma 2.3</u>: There is $C_5 > 0$ such that

(2.4) $|v(y,z)-v(y',z)| \leq C_5|y-y'|(h(y)+1), \ \forall z \in Z$ and $|y'-y| \leq 2\xi_1$

where ξ_1 is the largest point in Z.

<u>Proof</u>: Given $(y,z) \in (-\infty,\infty) \times Z$ and $\delta > 0$ choose $P_0 \in \mathbb{A}$ such that

$$J(y,z,P_0) \leq v(y,z) + \delta. \tag{2.5}$$

Let $y_0(\cdot)$ be the inventory process which corresponds to initial points y,z and the production process P_0. Now pick y' satisfying $|y'-y| \leq 2\xi_1$. Then the inventory process starting from y',z with production P_0 is given by $y_0'(t) = y_0(t)+y'-y$. Using these and assumption (A3) we obtain

$$\begin{aligned} J(y',z,P_0)-J(y,z,P_0) &= E\int_0^\infty e^{-\alpha t}[h(y_0'(t))-h(y_0(t))]dt \\ &\leq K_1 E\int_0^\infty e^{-\alpha t}|y_0'(t)-y_0(t)|[h(y_0(t))+1]dt \\ &= K_1|y'-y|\{E\int_0^\infty e^{-\alpha t}\ h(y_0(t))dt+(K_1/\alpha)\}\ . \end{aligned}$$

Since the production cost rate c is positive, the integral term above is less than $J(y,z,P_0)$. Also, use (2.5) and then (2.1) to obtain

$$\begin{aligned} J(y',z,P_0)-J(y,z,P_0) &\leq K_1|y'-y|\{(v(y,z)+\delta)+(K_1/\alpha)\} \\ &\leq K_1|y'-y|(h(y)/\alpha+C_1+\delta+K_1/\alpha). \end{aligned}$$

The following follows from (2.5), definition of v, and the above inequality

$$\begin{aligned} v(y',z)-v(y,z) &\leq J(y',z,P_0)-J(y,z,P_0) + \delta \\ &\leq C_5|y'-y|(h(y)+1)+\delta(K_1|y'-y|+1) \end{aligned}$$

where $C_5 = \max\{K_1/\alpha,\ K_1(C_1+K_1/\alpha)\}$. Since δ is arbitrary, the proof of the lemma is complete.

□

3. Bellman equation

Formally, it is known that the optimal value function is a solution of the following equation [6]:

$$(3.1)\quad \alpha v(y,z)=F(\frac{\partial}{\partial y}v(y,z))-z\frac{\partial}{\partial y}v(y,z)+[Lv(y,\cdot)](z)+h(y);\ \forall (y,z)\in(-\infty,\infty)\times Z$$

where F is as in (1.4). The purpose of this section is to obtain a sufficient condition for optimality. It is shown that a suitably behaved solution of (3.1) is the value function and that an optimal feedback production policy can be constructed from it. These results are obtained with no additional restrictions on the demand process. In sections 4 and 5, we shall see that under suitable restrictions on the demand process, the sufficient condition is also necessary.

In this section, we assume that there exists a solution of (3.1) in the space D_0 defined below. That such a solution exists will be shown in Sections 4 and 5.

Definition 3.1:

We say that a real valued function v with domain $(-\infty,\infty)\times Z$ is in D_0 if it satisfies the following

(i) v and $\frac{\partial}{\partial y}v$ are continuous and v is convex in y;

(ii) v satisfies the estimates (2.1),(2.2) and (2.4);

(iii) $v(y,\cdot)\in D(L)$ = domain of L.

Now we are ready to state the main result of this section which we call the "verification theorem". The method we use to prove this theorem is taken from [6].

Theorem 3.1.

Any solution $v \in D_0$ of equation (3.1) satisfies the following

$$v(y,z) = J(y,z,P^*) = \inf_{P \in \mathbf{A}} J(y,z,P)$$

where P^* is the feedback production policy defined by

$$(3.2)\quad p^*(y,z) = \begin{cases} (c')^{-1}(-\frac{\partial}{\partial y} v(y,z)) & \text{if } \frac{\partial}{\partial y} v(y,z) \leq 0 \\ 0 & \text{if } \frac{\partial}{\partial y} v(y,z) \geq 0 \end{cases}$$

Remark 3.1: Since v is convex in y and $(c')^{-1}$ is an increasing function, the feedback policy p^* defined above is nonincreasing in y. Therefore for any given demand trajectory $z(\cdot)$ and the initial inventory level y there is a unique solution of the following equation (see Theorem 6.2 in [7])

$$(3.3)\qquad y^*(t) = y + \int_0^t [p^*(y^*(s),z(s)) - z(s)]ds.$$

Now let $P^* = \{p^*(t) : t \geq 0\}$ where $p^*(t) = p^*(y^*(t),z(t))$. Then $p^*(\cdot)$ is adapted to $\{\mathcal{F}_t : t \geq 0\}$, the family of σ-algebras generated by $z(\cdot)$. Also, Lemma 3.2 below implies that it is bounded, therefore P^* is in $\mathbf{A}$.

Before we give the proof of Theorem 3.1, we prove some properties of the production policy defined by (3.2) and the corresponding inventory process $y^*(\cdot)$.

Lemma 3.2.

For any $y \in (-\infty,\infty)$ and the demand process $z(\cdot)$, there is a constant $K(y)$, independent of $z(\cdot)$, such that

$$\sup\{|y^*(t)| : t \geq 0\} \leq K(y) \tag{3.4}$$

where $y^*(\cdot)$ is the solution of (3.3).

Proof:

We start by defining the following set which is the set of the critical points of the differential equation related to (3.3).

$$G = \{y \in (-\infty,\infty) : p^*(y,z) = z \quad \text{for some} \quad z \in Z\}.$$

Since p^* is nonincreasing in y, the following statement is obvious

$$p^*(y,z) - z > 0 \quad \text{for all} \quad y < y_{min} \quad \text{and} \quad z \in Z$$
$$p^*(y,z) - z < 0 \quad \text{for all} \quad y > y_{max} \quad \text{and} \quad z \in Z$$

where $y_{min} = \inf\{y : y \in G\}$ and $y_{max} = \sup\{y : y \in G\}$. Now it is clear that to prove (3.4) it suffices to show that y_{min} and y_{max} are finite. But this is an easy consequence of the estimate (2.2) and convexity of $v(\cdot,z)$. □

Remark 3.2: In Section 6, we will examine the properties of $y^*(\cdot)$ and the set G defined above in more detail.

Proof of Theorem 3.1:

Fix $(y,z) \in (-\infty,\infty) \times Z$. First, we will show that $v(y,z)$ is less than $J(y,z,P)$ for every $P \in \mathbf{A}$. For this purpose, take any $P = \{p(t) : t \geq 0\} \in \mathbf{A}$. Then $p(\cdot)$ is bounded. So the corresponding inventory trajectory satisfies $|y(t)| \leq Kt+|y|$ for some K positive.

Also, $p(\cdot)$ is adapted to the family of σ-algebras generated by the demand process. Therefore, the following identity follows from an application of Dynkin's formula and the fundamental theorem of calculus.

$$(3.5)\quad \alpha v(y,z) = E\int_0^t e^{-\alpha s}[\alpha v(y(s),z(s))-(p(s)-z(s))\frac{\partial}{\partial y}v(y(s),z(s)) - (Lv(y(s),\cdot))(z(s))]ds + Ee^{-\alpha t}v(y(t),z(t))$$

$$\leq E\int_0^t e^{-\alpha s}[h(y(s))+c(p(s))]ds+Ee^{-\alpha t}v(y(t),z(t)).$$

The inequality is obtained by using equation (3.1). Now send t to infinity to conclude the following

$$\alpha v(y,z) \leq J(y,z,P) + \limsup_{t\to\infty} Ee^{-\alpha t}v(y(t),z(t)).$$

We complete the first step of the proof by the following chain of inequalities which are obtained first by using (2.1), then the assumption (A3) and the fact that $|y(t)| \leq Kt+|y|$.

$$\limsup_{t\to\infty} E\, e^{-\alpha t}v(y(t),z(t)) \leq \limsup_{t\to\infty} E\, e^{-\alpha t}[\frac{1}{\alpha}h(y(t))+C_1]$$

$$\leq \limsup_{t\to\infty} e^{-\alpha t}\,\frac{1}{\alpha}K_1(|Kt+|y||^{\gamma}+1)+C_1 = 0.$$

The equality $v(y,z) = J(y,z,P^*)$ is derived by the same argument as above. The only difference is the change of inequality to an equality in (3.5); and we are able to do that because of the following

$$F(\frac{\partial}{\partial y}v(y,z)) = c(p^*(y,z)) + p^*(y,z)\frac{\partial}{\partial y}v(y,z).$$

Recall that the above identity is proved in Remark 1.1 (ii). □

We have now obtained a sufficient condition for optimality. We must still show that there exists a solution v of (3.1) in D_0. In order to do that, we must assume specific stochastic models for the demand process. In the next two sections, we assume the nature of the demand process to be either a jump Markov process (§4) or a reflected diffusion process (§5). For each of these cases, we then show that the value function is in fact a solution of (3.1) in D_0. By Theorem 3.1, we can then construct an optimal feedback policy. In the later sections, we further characterize these optimal solutions.

4. Regularity with bounded infinitesimal generator L

In this section, we consider a jump Markov process with the infinitesimal generator L defined in (4.11) as the model of the random demand process. We then obtain the corresponding optimality condition by showing that the value function defined in (1.3) is a solution of the Bellman equation (3.1) in D_0.

To precisely formulate the random demand model we obtain, in Lemma 4.1 below, a condition on the demand process, which guarantees the continuity of the value function. This will serve as the motivation for assumption (A10) on the jump Markov process.

It should be obvious that the condition stipulated in Lemma 4.1 (and the consequent results) is not needed in the special case of Markov chains, when the set Z of possible demand rates contains only finitely many points.

Lemma 4.1:

Let $z(\cdot)$ and $z'(\cdot)$ be the demand processes starting at z and z', respectively. Suppose that there is $\rho > 0$ such that

$$E|z(t) - z'(t)| \leq \min\{|z - z'|e^{\rho t}, 2\xi_1\} \; ; \; \forall t \geq 0. \tag{4.1}$$

Then v is continuous.

Proof:

Fix $(y,z) \in (-\infty,\infty) \times Z$ and $\delta \in (0,1)$. Pick a production process $P \in \mathbf{A}$ such that

$$J(y,z,P) \leq v(y,z) + \delta. \tag{4.2}$$

Given any $z' \in Z$, define a stopping time τ and a production process $P' = \{p'(t) : t \geq 0\}$ by:

$$\tau = \inf\{t \geq 0 : y + \int_0^t [p(s)-z'(s)]ds \geq \ln(1/\delta)\} \tag{4.3}$$

$$p'(t) = \begin{cases} p(t) & \text{for } t \in [0,\tau] \\ 0 & \text{for } t > \tau \end{cases} .$$

Let $y'(\cdot)$ be the inventory process corresponding to the production process P', the initial inventory level y and the demand process $z'(\cdot)$ i.e.,

$$y'(t) = y + \int_0^t [p'(s)-z(s)]ds.$$

First, we will obtain an estimate that will be used later. Let $(y)^+ = \max\{y,0\}$. Then

$$\begin{aligned} E(y'(t)-y(t))^+ &= E[\int_0^t (p'(s)-p(s)-z'(s)+z(s))ds]^+ \\ &\leq E\int_0^t (p'(s)-p(s)-z'(s)+z(s))^+ ds \\ &\leq E\int_0^t |z'(s)-z(s)|ds \\ &\leq \int_0^t \min\{|z'-z|e^{\rho s}, 2\xi_1\}ds \\ &= \begin{cases} (1/\rho)|z'-z|(e^{\rho t}+1) \ ; \ t \leq \frac{1}{\rho}\ln(2\xi_1/|z'-z|) = T_0 \\ 2\xi_1(t-T_0+1/\rho)+(1/\rho)|z'-z| \ ; \ t \geq T_0 . \end{cases} \end{aligned}$$

We have used Jensen's ineq lity in the first step, then $p'(t) \leq p(t)$, and finally the hypothesis of the lemma. Now denote the last expression by $Y(t)$ and rewrite the above inequality as

$$(4.4) \qquad E(y'(t)-y(t))^+ \leq Y(t) \qquad \forall\, t \geq 0.$$

Since $p'(t) = p(t)$ for $t \leq \tau$, a similar argument would yield

$$(4.5) \qquad E|y'(t)-y(t)|\chi_{[0,\tau]} \quad (t) \leq Y(t) \ ; \ \forall\, t \geq 0$$

where χ_A is the indicator function of set A. Next, we will estimate the difference $J(y,z',P')-J(y,z,P)$. First, use $p' \leq p$ and then the assumption (A3) to obtain

$$\begin{aligned}(4.6) \quad J(y,z',P')-J(y,z,P) &\leq E\int_0^\infty e^{-\alpha t}[h(y'(t))-h(y(t))]dt \\ &\leq E\int_0^\tau e^{-\alpha t}K_1|y'(t)-y(t)|[h(y'(t))+1]dt \\ &\quad + E\int_\tau^\infty e^{-\alpha t}[h(y'(t))-h(y(t))]\chi_{[0,\infty)}(y'(t))dt \\ &\quad + E\int_\tau^\infty e^{-\alpha t}[h(y'(t))-h(y(t))]\chi_{(-\infty,0]}(y'(t))dt \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

I_1, I_2 and I_3 denote the first, second and third integrals in the last inequalities. Finally, we will majorize each term separately.

Now observe that $y-\xi_1 t \leq y'(t) \leq \ell n(1/\delta)$. Thus, (A.3) implies that $h(y'(t)) \leq K_1[\max\{[\ell n(1/\delta))]^\gamma, |y-\xi_1 t|^\gamma\} + 1]$. Use this and (4.5) to obtain

$$\begin{aligned}(4.7) \quad I_1 &\leq (K_1)^2\int_0^\infty e^{-\alpha t}[\max\{[\ell n(1/\delta)]^\gamma, |y-\xi_1 t|^\gamma\}+2]Y(t)dt \\ &\leq K_{\delta,1}(|z'-z|+|z'-z|^{\alpha/\rho})(|y|^\gamma+1+|\ell n|z'-z||^\gamma)\end{aligned}$$

where $K_{\delta,1}$ is a constant independent of z, z', y and P.

Since $h(y'(t)) - h(y(t)) \leq 0$ whenever $0 \leq y'(t) \leq y(t)$ and $h(y'(t)) - h(y(t)) \leq K_1[h(y'(t)) + 1](y'(t) - y(t))$ when $y'(t) \geq y(t)$, we obtain

$$[h(y'(t))-h(y(t))]\chi_{[0,\infty)}(y'(t))$$
$$\leq K_1[h(y'(t))+1](y'(t)-y(t))^+\chi_{[0,\infty)}(y'(t)).$$

First, use the above inequality and then (4.4) with the fact that $y'(t) \leq \ln(1/\delta)$ in I_2, to obtain

$$\begin{aligned} I_2 &\leq E\int_\tau^\infty e^{-\alpha t}K_1[h(y'(t))+1](y'(t)-y(t))^+\chi_{[0,\infty)}(y'(t))dt \\ &\leq \int_0^\infty e^{-\alpha t}K_1[h(\ln(1/\delta))+1]Y(t)dt \\ &\leq \int_0^\infty e^{-\alpha t}(K_1^2(\ln(1/\delta))^\gamma+K_1^2+K_1)Y(t)dt \\ &\leq K_{\delta,z}(|z'-z|+|z'-z|^{\alpha/\rho}). \end{aligned}$$

Again $K_{\delta,2}$ is a constant independent of z, z', y and P.

Observe that $y'(t) \geq 0$ for all $t \in [\tau, \tau + \ln(1/\delta)/\xi_1]$ because $p'(t) - z(t) \geq -\xi_1$ and $y'(\tau) = \ln(1/\delta)$. Now let $\theta = \inf\{t \geq \tau : y'(t) \leq 0\}$; the previous statement implies that $\theta \geq \tau + \ln(1/\delta)\xi_1$. Hence we have

$$\begin{aligned} I_3 &= E\int_\theta^\infty e^{-\alpha t}[h(y'(t))-h(y(t))]dt \\ &\leq E\, e^{-\alpha\theta}\int_0^\infty e^{-\alpha s}\, h(y'(s+\theta))ds. \end{aligned}$$

On $t \in [\theta,\infty)$, $p'(t) = 0$. Hence, $y'(s+\theta) = -\int_\theta^{s+\theta} z(t)dt \geq -\xi_1 s$. Use this in the above inequality to obtain

$$(4.9) \qquad I_3 \leq \int_0^\infty e^{-\alpha s}\, h(\xi_1 s)ds\; E\; e^{-\alpha\theta}$$

$$\leq K_1 \int_0^\infty e^{-\alpha s}[|\xi_1 s|^\gamma + 1]ds \cdot \exp[-\alpha \ln(1/\delta)/\xi_1]$$

$$= K_{,3}\, \delta^{(\alpha/\xi_1)}$$

where $K_{,3}$ is a constant independent of z, z', y and P. Combine (4.7)-(4.9) to obtain

$$(4.10) \quad J(y,z',P')-J(y,z,P)$$

$$\leq (K_{\delta,1}+K_{\delta,2})(|y|^\gamma+1)(|z'-z|+|z'-z|^{\alpha/\rho}) + K_{,3}\delta^{(\alpha/\xi_1)}.$$

The choice of P and (4.10) yield

$$v(y,z')-v(y,z) \leq w(|y|,|z'-z|,\delta)$$

where $w(|y|,|z'-z|,\delta) = (K_{\delta,1}+K_{\delta,2})(|y|^\gamma+1+(\ln|z-z'|)^\gamma)(|z'-z|+|z'-z|^{\alpha/\rho})$ $+\delta+K_{,3}\delta^{(\alpha/\xi_1)}$. Since all the constants are independent of P and z, the same argument as above would yield

$$|v(y,z')-v(y,z)| \leq w(|y|,|z'-z|,\delta).$$

Now the proof of the lemma is complete because for all $R > 0$

$$\lim_{\delta\searrow 0} \lim_{z'\to z} \sup_{|y|\leq R} w(|y|,|z'-z|,\delta) = 0.$$

Also, recall that continuity of v in y has been proved in Lemma 2.3.

□

In this section, we assume that the infinitesimal generator L has the following form:

$$(4.11) \qquad L\varphi(z) = \lambda(z) \int_Z [\varphi(z')-\varphi(z)]\pi(z,dz') \ ; \ \forall \varphi \in D(L)$$

where

$$D(L) = \{\varphi : Z \to (-\infty,\infty) : \text{bounded and Borel measurable}\}.$$

Also, the jump rate $\lambda(z)$ and the post-jump location distribution $\pi(z,\cdot)$ satisfy the following

(A8) λ is a nonnegative, bounded Borel measurable function with domain Z

(A9) the function $z \to \lambda(z)\pi(z,\cdot)$ maps Z into the set of positive measures on Z and it is weakly continuous i.e.,

$$\lim_{z' \to z} \lambda(z') \int \varphi(\bar{z})\pi(z',d\bar{z}) = \lambda(z) \int \varphi(\bar{z})\pi(z,d\bar{z}) \ ; \quad \forall \varphi \text{ continuous.}$$

(A10) For any z,z', there are pathwise realizations $z(\cdot)$, $z'(\cdot)$ starting at z and z', respectively; and $z(\cdot)$, $z'(\cdot)$ satisfy (4.1). See Remark 4.1 below.

Remark 4.1.

(i) A set of conditions (9.5) on λ and π that imply (A10) is given in Appendix 1. Also, in Appendix 1 a stochastic integral equation which yield solutions satisfying (A10) is discussed.

(ii) In view of Lemma 4.1, (A10) implies that the value function is continuous. But all that follow (with the exception of the continuity of $\frac{\partial}{\partial y} v$) can also be proved by having v to be simply a Borel

measurable function.

(iii) More precisely, (A10) means the following: For any z, z', there are Z-valued mappings $z(\cdot,\cdot)$, $z'(\cdot,\cdot)$ with domian $[0,\infty)\times\Omega$ satisfying

(i) $\mathbb{P}(\{w : z(0,w) = z\}) = \mathbb{P}(\{w : z'(0,w) = z'\}) = 1$

(ii) $\int_\Omega |z(t,w)-z'(t,w)|\, \mathbb{P}(dw) \leq \min\{|z'-z|e^{\rho t},\ 2\xi_1\}$

(iii) For any $\varphi \in D(L)$ the processes $M(\cdot)$ and $M'(\cdot)$ are $\mathcal{F}_t$-martingales, where

$$M(t,w) = \varphi(z) - \int_0^t (L\varphi)(z(s,w))ds$$

$$M'(t,w) = \varphi(z') - \int_0^t (L\varphi)(z'(s,w))ds .$$

We will first show that the value function is a "viscosity solution" and then using this information we will prove that v is in D_0. So let us define the notion of viscosity solutions for equation (3.1).

<u>Definition 4.1</u>.

(i) $v \in C((-\infty,\infty)\times Z)$ is said to be a <u>viscosity subsolution</u> of (3.1) if

$$\alpha v(y,z)-F(r)+zr-[Lv(y,\cdot)](z)-h(y) \leq 0 \quad \forall\ (y,z)\in(-\infty,\infty)\times Z \quad \text{and} \quad r\in D_y^+ v(y,z) \tag{4.12}$$

where

$$D_y^+ v(y,z) = \{r\in(-\infty,\infty) : \limsup_{\varepsilon\to 0}[v(y+\varepsilon,z)-v(y,z)-r\varepsilon]/|\varepsilon| \leq 0\}$$

(ii) $v \in C((-\infty,\infty) \times Z)$ is said to be a <u>viscosity supersolution</u> of (3.1) if

$$(4.13) \quad \alpha v(y,z)-F(r)+zr-[Lv(y,\ )](z)-h(y) \geq 0;\ \forall\ (y,z) \in (-\infty,\infty) \times Z \text{ and } r \in D_y^- v(y,z)$$

where

$$D_y^-(y,z) = \{r \in (-\infty,\infty) : \liminf_{\varepsilon \to 0} [v(y+\varepsilon,z)-v(y,z)-r\varepsilon]/|\varepsilon| \geq 0\}.$$

(iii) $v \in C((-\infty,\infty) \times Z)$ is a <u>viscosity solution</u> of (3.1) if it is both sub and supersolution of (3.1).

<u>Remark 4.2</u>.

(i) The above definition is a straightforward generalization of the original definition given by M. G. Crandall and P. L. Lions [4]. Also, see [5] for more information.

(ii) v is differentiable in y-direction at y,z if and only if $D_y^- v(y,z)$ and $D_y^+ v(y,z)$ are both singletons. Moreover, if v is convex in y, then $D_y^+ v(y,z)$ is empty unless $\frac{\partial}{\partial y} v$ exists at y,z. But when v is convex in y, the set $D_y^- v(y,z)$ is always nonempty and given by

$$D_y^- v(y,z) = [\frac{\partial^-}{\partial y} v(y,z),\ \frac{\partial^+}{\partial y} v(y,z)]$$

where $\frac{\partial^-}{\partial y}$, $\frac{\partial^+}{\partial y}$ are left and right derivatives, respectively.

We first approximate the value function v by $\{v_N : N = 1,2,\ldots\}$

$$v_N(y,z) = \inf_{P \in \mathbf{A}_N} J(y,z,P)$$

where

$$\mathbf{A}_N = \{P \in \mathbf{A} : |p(t)| \leq N \text{ for all } t \geq 0\}.$$

Arguing exactly as in Section 2 and in Lemma 4.1, we obtain that v_N is continuous, convex in y and v_N satisfies (2.1)-(2.4). Also, the corresponding Bellman equation is:

$$(3.1)_N \quad \alpha v_N(y,z) = F_N(\frac{\partial}{\partial y}v_N(y,z)) - z\frac{\partial}{\partial y}v_N(y,z) + [Lv_N(y,\cdot)](z) + h(y); \quad \forall\, (y,z) \in (-\infty,\infty) \times Z$$

where

$$F_N(r) = \inf_{0 \leq p \leq N} [c(p)+pr].$$

<u>Lemma 4.2</u>: The value function v_N is a viscosity solution of $(3.1)_N$.

<u>Proof</u>:

Fix $(y_0,z_0) \in (-\infty,\infty) \times Z$. It is well-known that v_N satisfies the dynamic programming relation [6], i.e., for any stopping time $\theta \geq 0$ we have

$$(4.14) \quad v_N(y_0,z_0) = \inf_{P \in \mathbf{A}_N} E\{\int_0^\theta e^{-\alpha t}[h(y)+c(p(t))]dt + e^{-\alpha\theta}v_N(y(\theta),z(\theta))\}.$$

Take any $r \in D_y^- v_N(y_0,z_0)$ and define a test function φ by

$$(4.15) \quad \varphi(y,z) = \begin{cases} v_N(y_0,z_0) + r(y-y_0) & ; \ \forall y \text{ and } z = z_0 \\ v_N(y,z) & ; \ \forall y \text{ and } z \neq z_0 \end{cases}$$

Since v_N is convex in y and $r \in D_y^- v_N(y_0, z_0)$, it is easy to show that $\varphi(y, z_0) \leq v_N(y, z_0)$ for all y. Hence, the definition of φ implies that $\varphi \leq v_N$. Now let T be any positive constant and τ_1 be the first jump-time of the demand process $z(\cdot)$. Choose $\theta = T \wedge \tau_1 = \min\{T, \tau_1\}$ in (4.14) and use the inequality $\varphi \leq v_N$ to obtain:

$$(4.16) \quad \varphi(y_0, z_0) \geq \inf_{P \in \mathbf{A}_N} E\left\{\int_0^{T\wedge\tau_1} e^{-\alpha t}[h(y(t))+c(p(t))]dt + e^{-\alpha(T\wedge\tau_1)}\varphi(y(T\wedge\tau_1), z(T\wedge\tau_1))\right\}.$$

Since $p(\cdot)$ is adapted to the family of σ-algebras generated by $z(\cdot)$, Dynkin's formula and the fundamental theorem of calculus yield that for any $P \in \mathbf{A}$

$$(4.17) \quad E\, e^{-\alpha(T\wedge\tau_1)}\psi(y(T\wedge\tau_1), z(T\wedge\tau_1)) = \psi(y_0, z_0) + E\left\{\int_0^{T\wedge\tau_1} e^{-\alpha t}[-\alpha\psi(y(t), z_0) + (p(t)-z_0)\frac{\partial}{\partial y}\psi(y(t), z_0) + (L\psi(y(t), \cdot))(z_0)]dt\right\}.$$

The above equality holds for every ψ such that (i) ψ is Borel measurable, (ii) ψ satisfies (2.1), (iii) $\frac{\partial}{\partial y}\psi(\cdot, z_0)$ is continuous and has polynomial growth. Since the test function φ satisfies (i)-(iii), we choose $\psi = \varphi$ in (4.17) and substitute this identity into (4.16) to obtain:

$$(4.18) \quad 0 \geq \inf_{P \in \mathbf{A}_N} E\left\{\int_0^{T\wedge\tau_1} e^{-\alpha t}[h(y)) + c(p(t)) - \alpha\varphi(y(t), z_0) + r(p(t)-z_0) + [L\varphi(y(t), \cdot)](z_0)]dt\right\}.$$

To derive (4.18) we also used the fact $\frac{\partial}{\partial y}\varphi(y,z_0) = r$. Now take $T = (1/m)$ in (4.18) and choose a $(1/m)^2$-optimal production process $P_m \in \mathcal{A}_N$, i.e.,

$$(1/m)^2 \geq E\left\{\int_0^{(1/m)\wedge\tau_1} e^{-\alpha t}[h(y_m(t))+c(p_m(t))-\alpha\varphi(y_m(t),z_0) + r(p_m(t)-z_0)+(L\varphi(y_m(t),\cdot))(z_0)]dt\right\}$$

where $y_m(\cdot)$ is the inventory trajectory corresponding to P_m. Now observe that $|y_m(t)-y_0| \leq (N+\xi_1)t$, because $P_m \in \mathcal{A}_N$ implies that $|p_m(t)| \leq N$. Use this and the Lipschitz continuity of φ in y to obtain:

$$(1/m)^2 \geq E\left\{\int_0^{(1/m)\wedge\tau_1} e^{-\alpha t}[h(y_0)+c(p_m(t))-\alpha\varphi(y_0,z_0)+r(p_m(t)-z_0) + (L\varphi(y_0,\cdot))(z_0)]dt\right\} - - K_1\int_0^{(1/m)} e^{-\alpha t}|N+\xi_1|t\,dt$$

where K_1 is a positive constant independent of m. Since $p_m(\cdot)$ is bounded the term $[h(y_0)+c(p_m(t))-....]$ is bounded by some constant K_2. Hence, we have

$$(4.19)\quad (1/m)^2[1+K_1(N+\xi_1)/2] + K_2\int_0^{(1/m)}(1-e^{-\alpha t})dt \geq E\left\{\int_0^{(1/m)\wedge\tau_1}[h(y_0)+c(p_m(t))-\alpha\varphi(y_0,z_0)+r(p_m(t)-z_0)+[L\varphi(y_0,\cdot)](z_0)]dt\right\}.$$

Multiply both sides of (4.19) by $[E(T_1 \wedge (1/m))]^{-1}$ and rewrite it as

$$(4.20)\qquad \alpha\varphi(y_0,z_0)-(L\varphi(y_0,\cdot))(z_0)+z_0r-C_m-r\mathcal{P}_m-h(y_0) \geq -K_m$$

where

$$K_m = [E(\tau_1 \wedge (1/m))]^{-1}[(1/m)^2(1+K_1(N+\xi_1)/2)+K_2\int_0^{(1/m)} e^{-\alpha t}dt]$$

$$C_m = [E(\tau_1 \wedge (1/m))]^{-1}E\left\{\int_0^{(1/m)\wedge\tau_1} c(p_m(t))dt\right\}$$

$$\mathfrak{P}_m = [E(\tau_1 \wedge (1/m))]^{-1}E\left\{\int_0^{(1/m)\wedge\tau_1} p_m(t)dt\right\} .$$

Let $CP_N = \{(c(p),p)$ for some $p \in [0,N]\}$. Then $(C_m,\mathfrak{P}_m) \in \overline{co}[CP_N]$, closed convex hull of CP_N. Since $\overline{co}[CP_N]$ is a compact set, there is $(C,\mathfrak{P}) \in \overline{co}[CP_N]$ such that $(C_m,\mathfrak{P}_m)$ converges to $(C,\mathfrak{P})$ on a subsequence of m. Pass to the limit in (4.20) to obtain:

$$\alpha\varphi(y_0,z_0)-C-r\mathfrak{P}+z_0r-(L\varphi(y_0,\cdot))(z_0)-h(y_0) \geq 0 .$$

Thus;

$$\alpha\varphi(y_0,z_0)-\inf\{C+r\mathfrak{P}:(C,\mathfrak{P})\in\overline{co}[CP_N]\}+z_0r-(L\varphi(y_0,\cdot))(z_0)-h(y_0) \geq 0.$$

But $\varphi(y_0,z_0) = v(y_0,z_0)$, $(L\varphi(y_0,\cdot))(z_0) = (Lv(y_0,\cdot))(z_0)$ and $\inf\{C+r\mathfrak{P}:(C,\ )\in\overline{co}[CP_N]\} = F_N(r)$. Therefore, v_N is a viscosity supersolution of $(3.1)_N$.

Now take any $r \in D_y^+v_N(y_0,z_0)$ and define φ as in (4.15). Since v_N is convex if $D_y^+v_N(y_0,z_0)$ is nonempty, then v_N is differentiable in y-direction at y_0,z_0. (See Remark 4.2(ii)).

Therefore $r = \frac{\partial}{\partial y} v_N(y_0, z_0)$ and there is a continuous function k with $k(0) = 0$ such that

$$(4.21) \qquad v_N(y, z_0) \leq \varphi(y, z_0) + |y - y_0| k(|y - y_0|).$$

Use (4.14) with $\theta = T \wedge \tau_1$ and (4.21) to obtain

$$(4.22) \quad \varphi(y_0, z_0) \leq \inf_{P \in \mathcal{A}_N} E\left\{ \int_0^{T \wedge \tau_1} e^{-\alpha t} [h(y(t)) + c(p(t))] dt + e^{-\alpha(T \wedge \tau_1)} \varphi(y(T \wedge \tau_1), z(T \wedge \tau_1)) + |y(T \wedge \tau_1) - y_0| k(|y(T \wedge \tau_1) - y_0|) \right\}$$

For any $p_0 \in [0, N]$ the constant production process $p(t) = p_0$ is in $\mathcal{A}_N$. Thus (4.22) yields that

$$\varphi(y_0, z_0) \leq E\left\{ \int_0^{T \wedge \tau_1} e^{-\alpha t} [h(y_0(t)) + c(p_0)] dt + e^{-\alpha(T \wedge \tau_1)} \varphi(y_0(T \wedge \tau_1), z(T \wedge \tau_1)) + |y(T \wedge \tau_1) - y_0| k(|y(T \wedge \tau_1) - y_0|) \right\}$$

where $y_0(\cdot)$ is the inventory trajectory corresponding to P_0 and $|y(t) - y_0| \leq (N + \xi_1)t$. Use this and (4.17) to obtain:

$$(4.23) \quad 0 \leq E \int_0^{T \wedge \tau_1} e^{-\alpha t} [h(y_0(t)) + c(p_0) - \alpha\varphi(y_0(t), z_0) + r(p_0 - z_0) + [L\varphi(y_0(t), \cdot)](z_0)] dt + (N + \xi_1) T \, k((N + \xi_1) T).$$

Divide both sides of (4.23) by T and then send T to zero to obtain

$$\alpha\varphi(y_0,z_0)-(p_0r+c(p_0))+zr-(L\varphi(y_0,\cdot))(z_0)-h(y_0)\leq 0 \ ; \ \forall p_0\in[0,N].$$

But $\varphi(y_0,z_0) = v_N(y_0,z_0)$ and $(L\varphi(y_0,\cdot))(z_0) = (Lv_N(y_0,\cdot))(z_0)$. Therefore v_N is also a viscosity subsolution of $(3.1)_N$.

□

<u>Proposition 4.3</u>: The value function v defined in (1.3) is a viscosity solution of (3.1).

<u>Proof</u>:

Clearly, $v_N \geq v_M \geq v$ for all $N \leq M$. Fix $(y_0,z_0)\in(-\infty,\infty)\times Z$ and $\delta > 0$. Pick $P_\delta \in \mathbb{A}$ such that

$$J(y_0,z_0,P_\delta) \leq v(y_0,z_0)+\delta \ . \tag{4.24}$$

Any $P\in\mathbb{A}$ is bounded. Therefore, there is M_δ such that $P_\delta \in \mathbb{A}_{M_\delta}$ and $J(y_0,z_0,P_\delta) \geq v_{M_\delta}(y_0,z_0)$. This and (4.24) yield that $v_{M_\delta}(y_0,z_0) \leq v(y_0,z_0)+\delta$. Since δ is arbitrary we conclude that $v_M(y_0,z_0)$ converges to $v(y_0,z_0)$ monotonically as M tends to infinity. Moreover, v_M and v are continuous. Hence, Dini's theorem yields

$$v_M \searrow v \text{ uniformly on every bounded subset of } (-\infty,\infty)\times Z. \tag{4.25}$$

Now take $r\in D_y^- v(y_0,z_0)$ and for small $\varepsilon > 0$, define ψ_ε by

$$\psi_\varepsilon(y,z) = \varphi(y,z) - \varepsilon(y - y_0)^2 \tag{4.26}$$

where φ is as in (4.14). Then the map $y \to v(y,z_0) - \psi_\varepsilon(y,z_0)$ has a strict minimum at y_0. Since v_N converges to v uniformly we have that

(i) the map $y \to v_N(y,z_0)-\psi_\varepsilon(y,z_0)$ has a local minimum of y_N

(ii) y_N converges to y_0 as N tends to infinity.

So $\frac{\partial}{\partial y}\psi_\varepsilon(y_N,z_0)\in D_y^- v_N(y_N,z_0)$ and the viscosity property of v_N yield

$$\alpha v_N(y_N,z_0)-F_N(\frac{\partial}{\partial y}\psi_\varepsilon(y_N,z_0))-z_0\frac{\partial}{\partial y}\psi_\varepsilon(y_N,z_0)-h(y_N)-(Lv_N(y_N,\cdot))(z_0)\geq 0.$$

First send N to infinity and then ε to zero to prove that v is a viscosity supersolution of (3.1).

Suppose that $r\in D_y^+ v(y_0,z_0)$. Then there is $\overline{\varphi}$ satisfying the following

(i) $v-\overline{\varphi}$ has a local maximum at (y_0,z_0)

(ii) $\overline{\varphi}(\cdot,z_0)\in C^1((-\infty,\infty))$

(iii) $\frac{\partial}{\partial y}\overline{\varphi}(y_0,z_0) = r$.

Now define $\overline{\psi}_\varepsilon$ as in (4.26) by using $\overline{\varphi}$ instead of φ. Then proceed exactly in the same way as before to prove that v is a viscosity subsolution of (3.1).

□

Theorem 4.4: The value function defined in (1.3) is in D_0 (Definition 3.1).

Proof:

Fix $(y_0,z_0) \in (-\infty,\infty) \times Z$. Since v is convex, right and left derivatives in y-direction at (y_0,z_0) exist. Denote $\frac{\partial^-}{\partial y} v(y_0,z_0)$ by d^- and $\frac{\partial^+}{\partial y} v(y_0,z_0)$ by d^+. Also, v is locally Lipschitz continuous in y. Therefore, there are sequences $\{\bar{y}_n : n = 1,2,\ldots\}$ and $\{y_n : n = 1,2,\ldots\}$ converging to y_0 from above and below, respectively, such that

$$(4.27)\quad \begin{cases} \text{(i)} & \frac{\partial}{\partial y} v \text{ exists at } (y_n,z_0) \text{ and } (\bar{y}_n,z_0) \\ \text{(ii)} & \frac{\partial}{\partial y} v(\bar{y}_n,z_0) \searrow d^+ \text{ as } n \text{ tends to infinity} \\ \text{(iii)} & \frac{\partial}{\partial y} v(y_n,z_0) \nearrow d^- \text{ as } n \text{ tends to infinity.} \end{cases}$$

Observe that $D_y^- v(\bar{y}_n,z_0) = D_y^+ v(\bar{y}_n,z_0) = \{\frac{\partial}{\partial y} v(\bar{y}_n,z_0)\}$ because of (4.27)(i). Thus, the viscosity property of v yields

$$(4.28)\quad \alpha v(\bar{y}_n,z_0) - F(\frac{\partial}{\partial y} v(\bar{y}_n,z_0)) + z\frac{\partial}{\partial y} v(\bar{y}_n,z_0) - h(y_n) - (Lv(y_n,\cdot))(z_0) = 0.$$

Recall that $(Lv(y,\cdot))(z_0) = \lambda(z_0)\int [v(y,z') - v(y,z_0)]\pi(z_0,dz')$. Hence, it is continuous in y because v is continuous. This and (4.27)(ii) imply that the limit of (4.28) as n tends to infinity is the following equation

$$(4.29)\qquad \alpha v(y_0,z_0) - F(d^+) + z_0 d^+ - h(y_0) - (Lv(y_0,\cdot))(z_0) = 0.$$

Similarly, one can obtain

$$(4.30)\qquad \alpha v(y_0,z_0) - F(d^-) + z_0 d^- - h(y_0) - (Lv(y_0,\cdot))(z_0) = 0.$$

Subtract (4.30) from (4.29) to obtain

$$F(d^+)-F(d^-) = z_0(d^+-d^-). \tag{4.31}$$

Suppose that $d^+ > d^-$. Since $z_0 > 0$, (4.31) implies that $F(d^+) > F(d^-)$. But $F(d^+) \leq 0$ and $F(r) = 0$ for all $r \geq 0$. Therefore, d^- must be negative and the strict convexity of F on $(-\infty,0)$ yields

$$F(\tfrac{1}{2}[d^++d^-]) > \tfrac{1}{2}F(d^+) + \tfrac{1}{2}F(d^-). \tag{4.32}$$

Combine (4.29),(4.30) and (4.32) to conclude that

$$\alpha v(y_0,z_0)-F(\tfrac{1}{2}[d^++d^-])+z_0\tfrac{1}{2}(d^++d^-)-h(y_0)-(Lv(y_0,\cdot))(z_0) < 0.$$

But $\frac{1}{2}(d^++d^-) \in D_y^- v(y_0,z_0)$, (see remark 4.2(ii)). So the above inequality contradicts with the fact that v is a viscosity supersolution of (3.1) and consequently we conclude that $d^+ = d^-$. Thus $\frac{\partial}{\partial y}v(\cdot,z)$ is a continuous function for every z.

Since v is convex in y, for every $z \in Z$ we have the following

$$v(y,z) \geq v(y_0,z)+(y-y_0)\frac{\partial}{\partial y}v(y_0,z);\ \forall y\ . \tag{4.33}$$

The estimate $|\frac{\partial}{\partial y}v(y_0,z)| \leq C_5(h(y_0)+1)$ follows from Lemma 2.3. Now take a sequence $\{z_n : n = 1,2,\ldots\} \subset Z$ such that z_n converges to z_0. Then there is a subsequence of n, denoted by n again, and a constant r such that

$$\lim_{n\to\infty} \frac{\partial}{\partial y}v(y_0,z_n) = r.$$

Pass to the limit in (4.33) and use the continuity of v to obtain

$$v(y,z_0) \geq v(y_0,z_0)+r(y-y_0) \ ; \ \forall y \ . \tag{4.34}$$

But v is convex and differentiable in y, therefore $r = \frac{\partial}{\partial y}v(y_0,z_0)$ and consequently $\frac{\partial}{\partial y}v$ is a continuous function.

□

<u>Corollary 4.5</u>. There is $C_6 > 0$ such that

$$|v(y,z)-v(y,z')| \leq C_6|z-z'|^{\beta}(h(y)+1)$$

where $\beta = \min(1,\alpha/\rho)$ and ρ is as in (4.1).

<u>Proof</u>:

Theorem 4.4, Theorem 3.3 and Lemma 3.2 imply that $y^*(\cdot)$ is bounded. The same argument as in Lemma 4.1 along with this information yields the result.

□

5. Regularity with unbounded L: The diffusion case

In this section, we assume that the demand process $z(\cdot)$ is a Markov diffusion process reflected at the boundary of Z. Then the infinitesimal generator of $z(\cdot)$ has the following form:

$$(5.1) \qquad L\varphi(z) = \frac{1}{2}\sigma^2(z)\frac{d^2}{dz^2}\varphi(z)+b(z)\frac{d}{dz}\varphi(z) \ ; \quad \forall\, \varphi \in D(L)$$

where

$$D(L) = \{\varphi \in C^2((\xi_0,\xi_1)) \cap C^1([\xi_0,\xi_1]) \ : \ \frac{d}{dz}\varphi(\xi_0)=\frac{d}{dz}\varphi(\xi_1)=0 \ .$$

See Section 2.4 in [8] for information about reflected diffusions. Also, we assume that σ and b satisfy the following

(A11) $\sigma, b \in C^3([\xi_0,\xi_1])$ and there is $\sigma_0 > 0$ such that

$$\sigma(z) \geq \sigma_0 \quad \text{for all} \quad z \in Z.$$

In addition to all the assumptions made in Section 1, we require that the holding cost h satisfy

(A12) There are $K_4 \geq 0$ and $\gamma' \in (0,1)$ such that

$$h(x)+h(y)-2h(\tfrac{x+y}{2}) \leq K_4(h(\tfrac{x+y}{2})+1)\,|x-y|^{1+\gamma'} \ ; \quad \forall\, x,y \in (-\infty,\infty).$$

Now we are ready to state the result of this section:

Theorem 5.1.

The value function v defined in (1.3) is in D_0 (Definition 3.1).

Proof: Take any two points $x,y\in(-\infty,\infty)$. For $\delta>0$ and $z\in Z$ choose a production process P such that

$$(5.1)\qquad v(\tfrac{x+y}{2},z)\geq J(\tfrac{x+y}{2},z,P)-\delta.$$

Then the definition of v yields

$$(5.2)\quad v(x,z)+v(y,z)-2v(\tfrac{x+y}{2},z)\leq J(x,z,P)+J(y,z,P)-2J(\tfrac{x+y}{2},z,P)+2\delta$$
$$= E\{\int_0^\infty e^{-t}[h(y_x(t))+h(y_y(t))-2h(y(t))]dt\}+2\delta$$

where $y_x(\cdot)$, $y_y(\cdot)$ and $y(\cdot)$ are inventory levels corresponding to P starting at x,y and $(x+y/2)$, respectively. Then, in view of the assumption (A12) we obtain

$$(5.3)\quad v(x,z)+v(y,z)-2v(\tfrac{x+y}{2},z)\leq E\{\int_0^\infty e^{-t}K_4[h(y(t))+1]dt\}|x-y|^{1+\gamma'}+2\delta$$
$$\leq K_4(v(\tfrac{x+y}{2},z)+1+\delta)|x-y|^{1+\gamma'}+2\delta.$$

Send δ to zero in (5.3) and also use the convexity of v in y to obtain

$$(5.4)\quad 0\leq v(x,z)+v(y,z)-2v(\tfrac{x+y}{2},z)\leq c(|x+y|^{\gamma}+1)|x-y|^{1+\gamma'}.$$

The above estimate implies that $\frac{\partial}{\partial y}v(\cdot,z)$ is Hölder continuous with Hölder exponent γ' (see Proposition 7 on page 142 in [15]).

Note that we have obtained the Hölder continuity of the nonlinear term in the equation (3.1). Now, it is standard to show that v is a solution of (3.1) and consequently that $v\in D_0$. □

Remark 5.1.

An assumption analogous to (A12) is used in [2] to study the regularity properties of the value function of certain control problems.

6. Optimal trajectories and turnpike sets

Recall that we have constructed an optimal feedback control in section 3 by using the value function. The corresponding optimal inventory process $y^*(\cdot)$ starting at the initial inventory level y is given by

$$(6.1) \qquad y^*(t) = y+\int_0^t [p^*(y(s),z(s))-z(s)]ds$$

where p^* is as in (3.2). Observe that if the demand process is deterministic (i.e., $z(s) = z_0$ for $s \geq 0$), then $y^*(t)$ converges to the set on which $p^*(y,z_0) - z_0 = 0$ or equivalently, the distance of $y^*(t)$ to the set $G(z_0)$ tends to zero, where $G(z_0)$ is defined by

$$(6.2) \qquad G(z_0) = \{y \in (-\infty,\infty) : \frac{\partial}{\partial y}v(y,z_0) = -c'(z_0)\}.$$

Such a $G(z_0)$ is called a <u>turnpike</u> associated with demand z_0 [16]. Since v is convex in y and is bounded as in (2.2), $G(z_0)$ is a bounded interval.

Now define G by

$$(6.3) \qquad G = \overline{co}[\bigcup_{z\in Z} G(z)] = [y_1,y_2].$$

Again, in view of (2.2) G is a bounded interval and the monotonicity of $p^*(y,z)$ in y yields that

$$(6.4) \qquad \begin{cases} \text{(i)} \quad p^*(y,z)-z < 0 \quad ; \ \forall (y,z) \in (y_2,\infty) \times Z \\ \text{(ii)} \quad p^*(y,z)-z > 0 \quad ; \ \forall (y,z) \in (-\infty,y_2) \times Z. \end{cases}$$

Hence, for stochastic demand processes G, is an attractor set for the optimal inventory trajectories, i.e.,

$$\text{(6.5)} \qquad \operatorname{dist}(y^*(t),G) \searrow 0 \quad \text{as} \quad t \nearrow +\infty .$$

The set G is thus an appropriate generalization of the turnpike concept for the stochastic case. We shall, therefore, term set G to be the <u>turnpike set</u>.

In what follows, we will study the important properties of these sets.

<u>Theorem 6.1.</u>

(i) There is a nonpositive element in G

(ii) G is a bounded nondegenerate interval if

a) For every $y \in (-\infty,\infty)$, $\{-\alpha c'(z) : z \in Z\} \not\subset D^-h(y)$, when L is given by (4.11), or

b) h satisfies (A12), when L is given by (5.1)

where $D^-h(y)(= D_y^-h(y))$ is as in Definition 4.1.

<u>Proof</u>: Define $\underline{y}(z)$ by

$$\text{(6.7)} \qquad \underline{y}(z) = \inf\{y \in (-\infty,\infty) : \frac{\partial}{\partial y}v(y,z) \geq -c'(z)\}.$$

Then we have

$$\text{(6.8)} \qquad y_1 = \inf\{\underline{y}(z) : z \in Z\} = \inf\{y : y \in G\}.$$

Since $\frac{\partial}{\partial y}v$ and c are continuous, $\underline{y}(\cdot)$ is a lower semi-continuous function from Z into $(-\infty,\infty)$. Hence, there is $z_0 \in Z$ such that $y_1 = \underline{y}(z_0)$. Now suppose that $\underline{y}(z_0) > 0$. Let y^* be the solution of

(6.1) with $y^*(0) = 0$ and $z(0) = z_0$. Construct $P' = \{p'(t) : t \geq 0\}$ as follows

$$p'(t) = z(t)\chi_{[0,\tau]}(t) + p^*(y^*(t), z(t))\chi_{(\tau,\infty)}(t) \tag{6.9}$$

where τ is the stopping time defined by

$$\tau = \inf\{t \geq 0 : y^*(t) \geq (1/2)\underline{y}(z_0)\}.$$

Now observe that $y^*(t) \geq y^*(\tau) = (1/2)\underline{y}(z_0)$ for $t \geq \tau$, on account of (6.5). Also, $y^*(t) - y^*(\tau) = \int_\tau^t [p^*(y^*(s), z(s)) - z(s)]ds$. So we have

$$\int_\tau^t [p'(s) - z(s)]ds \geq 0 \quad \text{for} \quad t \geq \tau. \tag{6.10}$$

Let $y'(\cdot)$ be the inventory trajectory corresponding to P' starting at $y = 0, z_0$. Then

$$y'(t) = \begin{cases} 0 & ; \quad t \in [0,\tau] \\ \int_\tau^t [p'(s) - z(s)]ds & ; \quad t \geq \tau \end{cases}$$

Therefore, (6.10) yields that $y'(t) \geq 0$. Also, $p'(s) \leq p^*(y^*(s), z(s))$ with strict inequality for $s \leq \tau$, implies that $y'(t) \leq y^*(t)$ for all $t \geq 0$. Use these to obtain

$$J(0, z_0, P') < J(0, z_0, P^*) = v(0, z_0). \tag{6.11}$$

This contradiction with the optimality of p^* implies that $\underline{y}(z_0) \leq 0$. This proves (i).

(ii) Suppose that G is degenerate, i.e., $G = \{y_0\}$ for some y_0. The equation (3.1) yields that for any $\varepsilon, \delta \geq 0$ and $z \in Z$:

$$(6.12)\quad \alpha[v(y_0+\varepsilon,z)-v(y_0-\delta,z)]=[(F(\frac{\partial}{\partial y}v(y_0+\varepsilon,z))-z\frac{\partial}{\partial y}v(y_0+\varepsilon,z))-$$
$$-(F(\frac{\partial}{\partial y}v(y_0-\delta,z))-z\frac{\partial}{\partial y}v(y_0-\delta,z))]+$$
$$+[L(v(y_0+\varepsilon,\cdot)-v(y_0-\delta,\cdot))](z)+[h(y_0+\varepsilon)-h(y_0-\delta)].$$

Since $\frac{\partial}{\partial y}v(y_0,z) = -c'(z)$ for $z \in Z$, $-c'(z) \in [\frac{\partial}{\partial y}v(y_0-\delta,z),\frac{\partial}{\partial y}v(y_0+\varepsilon,z)]$. Also, the map $r \to F(r) - zr$ achieves its maximum at $-c'(z)$ and it is strictly concave at $r = -c'(z)$. Therefore, there are $\varepsilon^n(z),\delta^n(z) \in (0,1/n)$ such that

$$F(\frac{\partial}{\partial y}v(y_0+\varepsilon^n(z),z))-z\frac{\partial}{\partial y}v(y_0+\varepsilon^n(z),z)=F(\frac{\partial}{\partial y}v(y_0-\delta^n(z),z))$$
$$-z\frac{\partial}{\partial y}v(y_0-\delta^n(z),z).$$

Use this in equation (6.12) to obtain

$$(6.13)\quad \alpha[v(y_0+\varepsilon^n(z),z)-v(y_0-\delta^n(z),z)] = L(v(y_0+\varepsilon^n(z),\cdot)$$
$$-v(y_0-\delta^n(z),\cdot))(z)+[h(y_0+\varepsilon^n(z))-h(y-\delta^n(z))].$$

<u>Case (1)</u> <u>L is as in (4.11)</u>:

Divide both sides of (6.13) by $(\varepsilon^n(z)+\delta^n(z))$ and then send n to infinity to obtain:

$$(6.14)\quad \alpha\frac{\partial}{\partial y}v(y_0,z) = L(\frac{\partial}{\partial y}v(y_0,\cdot))(z) + \lim_{n\to\infty}[h(y_0+\varepsilon^n(z))$$
$$-h(y-\delta^n(z))](\varepsilon^n(z)+\delta^n(z))^{-1}.$$

We have used the boundedness of the operator L to obtain (6.14). Now the limit in (6.14) is in $D^- h(y_0)$, thus we have the following:

$$\begin{cases} \alpha\frac{\partial}{\partial y}v(y_0,z) \le L(\frac{\partial}{\partial y}v(y_0,\cdot))(z)+\sup\{k : k\in D^- h(y_0)\} \\ \alpha\frac{\partial}{\partial y}v(y_0,z) \ge L(\frac{\partial}{\partial y}v(y_0,\cdot))(z)+\inf\{k : k\in D^- h(y_0)\}. \end{cases}$$

The above inequalities yield:

$$\inf\{k : k\in D^- h(y_0)\} \le \alpha\frac{\partial}{\partial y}v(y_0,z) \le \sup\{k : k\in D^- h(y_0)\}.$$

On the other hand, $\alpha\frac{\partial}{\partial y}v(y_0,z) = -\alpha c'(z)$. So the above inequality contradicts with the hypothesis (a). Hence, G is not degenerate.

<u>Case (2)</u> <u>L is as in (5.1)</u>:

Formally, $\frac{\partial}{\partial y}[F(\frac{\partial}{\partial y}v(y,z))-z\frac{\partial}{\partial y}v(y,z)]=(F'(\frac{\partial}{\partial y}v(y,z))-z)\frac{\partial^2}{\partial y^2}v(y,z)$. At $y = y_0$ we know that $F'(\frac{\partial}{\partial y}v(y_0,z))-z = 0$. Moreover, $\frac{\partial^2}{\partial y^2}v(y,z)$ is bounded on account of (5.11). So we have:

$$(6.15)\quad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}[F(\frac{\partial}{\partial y}v(y_0+\varepsilon,z))-z\frac{\partial}{\partial y}v(y_0+\varepsilon,z) - F(\frac{\partial}{\partial y}v(y_0-\varepsilon,z))+z\frac{\partial}{\partial y}v(y_0-\varepsilon,z)] = 0.$$

Choose $\varepsilon = \delta$ in (6.12), divide both sides by 2ε, and then use (6.15) to pass to the limit as ε tends to zero. As a result, we obtain the following:

$$(6.16)\quad \alpha\frac{\partial}{\partial y}v(y_0,z) = L\frac{\partial}{\partial y}v(y_0,z) + \frac{d}{dy}h(y_0) \quad \text{in } \mathcal{D}'([\xi_0,\xi_1])$$

where $\mathcal{D}'$ is the set of distributions on $[\xi_0,\xi_1]$. Note that (6.16) is a second order linear differential equation in z with Neumann boundary conditions at $z = \xi_0$ and $z = \xi_1$. Thus $\frac{\partial}{\partial y}v(y_0,z) = (1/\alpha)\frac{d}{dy}h(y_0)$ is the only solution of (6.16) which is an obvious contradiction with $\frac{\partial}{\partial y}v(y_0,z) = -c'(z)$ for all $z \in Z$.

Corollary 6.2:

Suppose that hypotheses a) or b) of Theorem 6.1 hold. Let $A(z)$ be given by

$$(6.17)\quad A(z) = \{z' \in Z : \mathbb{P}(z(t) \in [z'-\varepsilon, z'+\varepsilon] \mid z(0) = z) > 0; \text{ for all } t,\varepsilon > 0\}.$$

If $A(z) \neq \{z\}$ for all $z \in Z$, then there is a random time $\tau_0 < \infty$ such that $\mathrm{dist}(y^*(\tau_0),G) = 0$.

Proof: The second part of the previous theoren can be modified to conclude that for all $z_0 \in Z$ there is no $y \in (-\infty,\infty)$ such that $\frac{\partial}{\partial y}v(y,z) = -c'(z)$ for all $z \in A(z_0)$. Now let $y^*(\cdot)$ be the optimal inventory starting at (y,z). Suppose that $y < \inf\{y' : y' \in G\} = y_1$. Then there exists $z \in A(z_0)$ and $\varepsilon > 0$ such that

$$p^*(y,z')-z' < 0 \quad \text{for all} \quad |z'-z| \leq \varepsilon.$$

The above inequality together with (6.17) implies the claimed result. A similar argument yields the result when $y > \sup\{y' : y' \in G\}$.

□

Proposition 6.3: For the case of a constant deterministic demand, i.e., $Z = \{\xi\}$

$$G = I(\xi) := \{y \in (-\infty,\infty) : -\alpha c'(\xi) \in D^-h(y)\}.$$

Proof: Rewrite the Bellman equation (3.1) as

$$\alpha v(y,z) = c(p^*(y,z))+(p^*(y,z)-\xi)\frac{\partial}{\partial y} v(y,z)+h(y)$$

where p^* is as in (3.2). Formally taking the y-derivative in above we obtain

$$\alpha\frac{\partial}{\partial y}v(y,z) = (p^*(y,z)-\xi)\frac{\partial^2}{\partial y^2}v(y,z)+h'(y)$$

which at $y \in I(\xi)$ becomes

$$\alpha\frac{\partial}{\partial y}v(y,z) = (p^*(y,z)-\xi)\frac{\partial^2}{\partial y^2}v(y,z)-\alpha c'(\xi) \ ; \ \forall y \in I(\xi).$$

Now, if $p^*(y,z) > \xi$, then the above equation, in view of the fact that v is convex in y, gives $\frac{\partial}{\partial y}v(y,z) \geq -c'(\xi)$. This is equivalent to $p^*(y,z) \leq \xi$ which is a contradiction with $p^*(y,z) > \xi$. A similar contradiction can be obtained if $p^*(y,z) < \xi$. Thus, $p^*(y,z) = \xi$ and $y \in G$.

The above argument can be made rigorous by using the technique developed in the proof of Theorem 6.1(ii). Also, the same technique

yields that $G \subset I(\xi)$. Therefore, the proof of the proposition is complete.

□

Remark 6.1. An economic interpretation of Proposition 6.3 is useful to provide. Assume for convenience that $h(\cdot)$ is differentiable. Let $y = y_0$ be a turnpike point. Then $p^*(t) = \xi$. Let $p(t) = \xi+\varepsilon$, $t \in [\theta,\delta]$, δ, $\varepsilon > 0$ and $p(t) = p^*(t) = \xi$, $t \in (\delta,\infty)$. Then

$$\text{the marginal production cost} = c'(\xi)\varepsilon\delta + o(\varepsilon\delta)$$

$$\text{the marginal inventory cost} = \int_\delta^\infty e^{-\alpha t} h'(y_0)\varepsilon\delta dt + o(\varepsilon\delta) = \frac{h'(y_0)\varepsilon\delta}{\alpha} + o(\varepsilon\delta).$$

Setting the total marginal production cost to zero and dividing through by $\varepsilon\delta$ gives the relation $-\alpha c'(\xi) = h'(y_0)$ for y_0.

Note that $\xi > 0$ implies $y_0 < 0$. So if the initial inventory $y = 0$, then it pays to produce less than the demand until $y(t) = y_0$. This results in savings in production cast. This is exactly offset by increased shortage cost above the optimal path. Note that the discounting plays an essential role in this balancing act. In fact, in the absence of discounting, i.e., $\alpha = 0$, the turnpike point $y_0 = 0$.

When the value function is strictly convex, there is only one $y(z)$ such that $p^*(y(z),z)-z = 0$. The next result gives a set of sufficient conditions for the strict convexity.

Proposition 6.4.

The value function $v(\cdot,z)$ is strictly convex in y if one of the following holds

(i) $A(z) \neq \{z\}$ where $A(z)$ is as in (6.17).

(ii) $I(z)$ is a singleton, where $I(z)$ is given by

$$I(z) = \{y \in (-\infty,\infty) : -\alpha c'(z) \in D^- h(y)\}.$$

Proof: Suppose that there are $y_1 < y_2$ and $z_0 \in Z$ such that

$$\frac{\partial}{\partial y} v(y_1,z_0) = \frac{\partial}{\partial y} v(y_2,z_0).$$

Let $y_1^*(\cdot)$, $y_2^*(\cdot)$ be the optimal inventory trajectories starting at (y_1,z_0) and (y_2,z_0), respectively. Then we claim that for every $t \geq 0$, we have

$$\mathbb{P}\left(\frac{\partial}{\partial y} v(y_1^*(t),z(t)) = \frac{\partial}{\partial y} v(y_2^*(t),z(t))\right) = 1. \tag{6.18}$$

Suppose to the contrary that there is $t_0 \geq 0$ such that (6.18) does not hold. Since $\frac{\partial}{\partial y}v$ is continuous, there exists a constant $\beta > 0$ such that

$$(6.19) \qquad \mathbb{P}(p^*(y_1^*(t),z(t)) \neq p_1^*(y_2^*(t),z(t)) \text{ for } t \in [t_0,t_0+\beta]) > 0.$$

Let $P_i^* = \{p_i^*(t) : t \geq 0\}$ be defined by $p_i^*(t) = p^*(y_i^*(t),z(t))$ for $i = 1,2$. Then (6.19) together with the convexity of h and the strict convexity of c yields

$$(1/2)[J(y_1,z_0,P_1^*)+J(y_2,z_0,P_2^*)] > J(\tfrac{1}{2}(y_1+y_2),z_0,\tfrac{1}{2}(P_1^*+P_2^*))$$

which implies that $\frac{1}{2}(v(y_1,z_0)+v(y_2,z_0)) > v(\frac{1}{2}(y_1+y_2),z_0)$. But $v(\cdot,z_0)$ is assumed to be linear on $[y_1,y_2]$. Therefore (6.18) holds.

Next, we claim the following

$$(6.20) \quad \mathbb{P}(p^*(y_i^*(t),z(t)) = z(t)) = 1 \ ; \ \forall t \geq 0 \text{ and } i = 1,2.$$

Before we give the proof of (6.20), let us complete the proof of the proposition. It is clear that $y_i^*(t) = y_i$ for $t \geq 0$ and $i = 1,2$. So the value function at (y,z_0) for $y \in [y_1,y_2]$ is given by:

$$v(y,z_0) = (1/\alpha)h(y) + E\int_0^\infty e^{-\alpha t}c(z(t))dt \ ; \ \forall y \in [y_1,y_2].$$

The integral term does not depend on y, thus we have

$$\frac{\partial}{\partial y}v(y,z_0) = (1/\alpha)\frac{d}{dy}h(y) = -c'(z_0) \quad ; \ \forall y \in (y_1,y_2).$$

Also, (6.18) implies that $\frac{\partial}{\partial y}v(y_1,z') = \frac{\partial}{\partial y}v(y_2,z')$ for all $z' \in A(z_0)$. So the same argument as the above yields

$$(6.21) \qquad \frac{\partial}{\partial y}v(y,z') = (1/\alpha)\frac{d}{dy}h(y) = -c'(z') \ , \ \forall\, y \in (y_1,y_2) \text{ and } z' \in A(z_0).$$

Using either one of the hypotheses of the proposition, we obtain a contradiction. Hence it suffices to prove (6.20).

Now suppose that $p^*(y_2,z_0) > z_0$. Then there is $\bar{\varepsilon} > 0$ such that

$$(6.22) \qquad p^*(y_2,z) > z \ , \ \forall\ z \text{ such that } |z-z_0| \le \bar{\varepsilon}.$$

Also, for every ε and $T > 0$, $\mathbb{P}(|z(t)-z_0| \le \varepsilon,\ t \in [0,T]) > 0$. On the set $|z(t)-z_0| \le \varepsilon$ for $\varepsilon < \bar{\varepsilon}$, the optimal inventory trajectories move towards the point $y_\varepsilon = \inf\{\underline{y}(z) : |z-z_0| \le \varepsilon\}$, where $\underline{y}(z)$ is as in (6.7). Therefore, given $\varepsilon,\delta > 0$, there is $T(\varepsilon,\delta) > 0$ and a random time $\tau \le T(\varepsilon,\delta)$ such that

$$|y_2^*(\tau)-y_\varepsilon| \le \delta \text{ on the event } A(\varepsilon,\delta)$$

where $A(\varepsilon,\delta) = \{|z(t)-z_0| \le \varepsilon,\ \forall\, t \in [0,T(\varepsilon,\delta)]\}$. Monotonicity of $\frac{\partial}{\partial y}v$ implies the following:

$$(6.23) \qquad \frac{\partial}{\partial y}v(y_2^*(\tau),z(\tau)) \ge \inf\{\frac{\partial}{\partial y}v(y_\varepsilon-\delta,z) : |z-z_0| \le \varepsilon\} \text{ on } A(\varepsilon,\delta).$$

Since the event $A(\varepsilon,\delta)$ has a positive probability, (6.18) implies that

$$(6.24) \qquad \frac{\partial}{\partial y}v(y_1^*(\tau)),z(\tau)) = \frac{\partial}{\partial y}v(y_2^*(\tau),z(\tau)) \text{ on } A(\varepsilon,\delta)$$

and $y_1^*(\tau) = y_2^*(\tau)-y_2+y_1 \le y_\varepsilon-y_2+y_1+\delta$. Use this, (6.23), and (6.24) to obtain

$$(6.25)\quad \sup\{\frac{\partial}{\partial y}v(y_\varepsilon-y_2-y_1+\delta,z):|z-z_0|\leq\varepsilon\}\geq\inf\{\frac{\partial}{\partial y}v(y_\varepsilon,z):|z-z_0|\leq\varepsilon\}.$$

Since $\underline{y}(z)$ is lower semi-continuous, y_ε converges to $\underline{y}(z_0)$ as ε tends to zero. Hence, (6.25) implies that

$$\frac{\partial}{\partial y}v(\underline{y}(z_0)-y_2+y_1,z_0)\geq\frac{\partial}{\partial y}v(\underline{y}(z_0),z_0).$$

Clearly, this contradicts with the definition of $\underline{y}(z_0)$ in (6.7). Hence, (6.22) is not true. Now arguing similarly, one can show that the assumption $p^*(y_1,z_0) < z_0$ yields a contradiction. Since $p^*(y_1,z_0) = p^*(y_2,z_0)$, we have the following:

$$p^*(y_1,z_0) = p^*(y_2,z_0) = z_0.$$

Recall that $\frac{\partial}{\partial y}v(y_1,z') = \frac{\partial}{\partial y}v(y_2,z')$ for $z'\in A(z_0)$. So the same argument as the above yields that $p^*(y_1,z') = p^*(y_2,z')$ whenever $z'\in A(z_0)$. This proves (6.20) because $\mathbb{P}(z(t)\in A(z_0)|z(0)=z_0) = 1$ for $t\geq 0$.

□

<u>Remark 6.2</u>: If $h(\cdot)$ is strictly convex, then $I(z)$ is a singleton. Also, for deterministic problems $G = I(\xi)$, where ξ denotes the constant demand.

<u>Remark 6.3</u>: Condition (i) is weaker than the assumption of ergodicity, which requires $A(z) = Z$, $\forall z\in Z$. However, condition (i) implies that z is not absorbing.

In Proposition 6.4 above, we have proved the strict convexity of $v(\cdot,z)$ under conditions that are weaker than the assumption that $h(\cdot)$ is strictly convex. In view of (i), we need (ii) only when $A(z) = z$. In that case, $v(\cdot,z)$ is the same as the value function for a deterministice problem with the constant demand z.

In the following example, we are able to explicitly obtain the value function, for a deterministic problem with $Z = \{\xi\}$ and thus illustrate Proposition 6.4(ii) and also Proposition 6.3. When the shortage cost, which is linear, is too low, Assumption (A6) does not hold and $I(\xi) = \phi$ or $I(\xi) = (-\infty,0]$. In these cases, $v(\cdot,\xi)$ is not strictly convex. When the assumed linear shortage cost is not too low, Assumption (A6) holds and $I(\xi) = \{0\}$, which is a singleton. The value function $v(\cdot,\xi)$ is easily seen to be strictly convex.

<u>Example 1</u>: $Z = \{\xi\}$, ξ a constant, $C(p) = p^2$

$$h(y) = \begin{cases} -Ky & y \leq 0 \\ 0 & y > 0 \end{cases}$$

<u>Case (a)</u>: <u>Low Shortage Cost</u>: <u>$K \leq 2\alpha\xi$</u>

For $K \leq 2\alpha\xi$, the value function

$$v(y,\xi) = \begin{cases} -\frac{Ky}{\alpha} + \frac{K}{\alpha^2}[\xi - \frac{K}{4\alpha}], & y \leq 0 \\ \frac{K}{2\alpha^2}[2e^{-\alpha T(y)}\xi - e^{-2\alpha T(y)}\frac{K}{2\alpha}], & y > 0, \end{cases}$$

where $T(y)$ is the first time at which the optimal inventory $y^*(T(y)) = 0$ given the initial level $y \geq 0$ and, it is the unique positive solution of

$$\frac{Ke^{-\alpha T(y)}}{2\alpha^2} + \xi T(y) = y + \frac{K}{2\alpha^2}, \quad y > 0.$$

Furthermore,

$$I(\xi) = \begin{cases} \phi & \text{for } K < 2\alpha\xi \\ (-\infty,0] & \text{for } K \geq 2\alpha\xi \end{cases}.$$

We note that for $K < 2\alpha\xi$, $v_y \geq -\frac{K}{\alpha} > -2\xi$ with the consequence that there exists no turnpike set i.e., $G = \phi = I(\xi)$. Although, we can think of $\{-\infty\}$ to be the turnpike point in the extended sense, as the optimal inventory level approach $-\infty$ as $t \to \infty$.

The case $K = 2\alpha\xi$ is the critical case. In this case,

$$v(y,\xi) = \begin{cases} -2\xi y + \xi^2/\alpha & , y \leq 0 \\ \frac{\xi^2}{\alpha}[2e^{-\alpha T(y)} - e^{-2\alpha T(y)}], & y > 0 \end{cases}$$

where $T(y)$ is the unique positive solution of

$$\frac{e^{-\alpha T(y)}}{\alpha} + T(y) = \frac{y}{\xi} + \frac{1}{\alpha}, \quad y > 0.$$

Since $v_y = -2\xi$ for $y \in (-\infty,0]$, the turnpike set $G = (-\infty,0] = I(\xi)$. From $y > 0$, the optimal inventory level reaches 0 at $t = T(y)$ and then it stays there. From $y \leq 0$, the optimal production $p^*(t) \equiv \xi$ and $y^*(t) \equiv y$, for all $t \in [0,\infty)$.

<u>Case (b)</u>: <u>High Shortage Cost</u>: <u>$K > 2\alpha\xi$</u>

For $K > 2\alpha\xi$, the value function

$$v(y,\xi) = \begin{cases} -\frac{Ky}{\alpha} - \frac{K^2}{4\alpha^3} + \frac{K\xi}{\alpha^2} + (\frac{\xi^2}{\alpha} + \frac{K^2}{4\alpha^3} - \frac{K\xi}{\alpha^2})[2e^{-\alpha\theta(y)} - e^{-2\alpha\theta(y)}], & y \leq 0 \\ \frac{\xi^2}{\alpha}[2e^{-\alpha T(y)} - e^{-2\alpha T(y)}], & y > 0 , \end{cases}$$

where $T(y)$ is as defined in the critical case above and $\theta(y)$ is the first time the optimal inventory level $y^*(\theta(y)) = 0$ from a given initial inventory level $y \leq 0$ and it is given by the unique positive solution of

$$\frac{e^{-\alpha\theta(y)}}{\alpha} + \theta(y) = \frac{-y}{\frac{K}{2\alpha} - \xi} + \frac{1}{\alpha}, \quad y \leq 0.$$

In this case, the turnpike set $G = \{0\} = I(\xi)$.

The value functions obtained in this example are sketched in Figure 2.

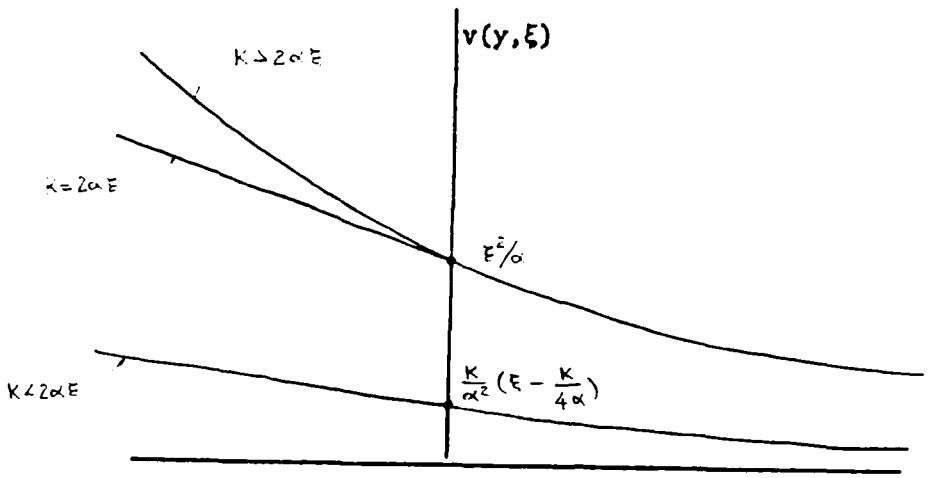

Figure 2. Value function $v(y,\xi)$ for different values of K.

The value functions satisfy $v_y \leq -K/2$ in all the cases. $K > 2\alpha\xi$, the value function is strictly convex and $v_y \to -K/\alpha$ as $y \to -\infty$. We note that in this case, Assumption (A6) and hypothesis (ii) of Proposition 6.4 hold. The value function for $K \leq 2\alpha\xi$ are linear on $(-\infty,0]$ with the slope of $-K/\alpha$. For these cases, Assumption (A6) does not hold. Furthermore, both cases a) and b) illustrate Proposition 6.3.

In the rest of this section, we will examine the monotonicity of $\underline{y}(\cdot)$ defined in (6.7), when the demand process is a Markov chain with a finite state space. We start with two point chains.

Lemma 6.5.

Suppose that $L\varphi(z)$ at z_1 and z_2 is given by

$$(6.26)\qquad \begin{cases} L\varphi(z_1) = \lambda_1[\varphi(z_2)-\varphi(z_1)] \\ L\varphi(z_2) = \lambda_2[\varphi(z_1)-\varphi(z_2)] \end{cases}$$

Let $z_1 < z_2$. Then $y(z_1) > y(z_2)$.

Proof: Suppose to the contrary. The argument used in the proof of Theorem 6.1(ii) implies that $y(z_1) \neq y(z_2)$. So the interval $I = [y(z_1),y(z_2)]$ is nondegenerate and on this interval the following inequalities hold:

$$(6.27)\qquad \frac{\partial}{\partial y}(y,z_1) \geq -c'(z_1) \geq -c'(z_2) \geq \frac{\partial}{\partial y}v(y,z_2)\ ;\quad \forall y \in I.$$

The specific form of the infinitesimal generator L yields

$$(6.28)\qquad \begin{cases} \frac{\partial}{\partial y}(Lv(y,\cdot))(z_1) \leq 0 \quad ; \ \forall y \in I \\ \frac{\partial}{\partial y}(Lv(y,\cdot))(z_2) \geq 0 \quad ; \ \forall y \in I \end{cases}$$

Now observe that v is also the value function of the following deterministic control problems:

$$J(y,z_i,P) = \int_0^{\infty} e^{-\alpha t}[h(y(t))+c(p(t))+Lv(y(t),z_i)]dt\ ;\quad i = 1,2$$

$$v(y,z_i) = \inf_{P\geq 0} J(y,z_i,P)$$

Moreover, the "verification theorem" (Theorem 3.1) holds. Therefore,

$$(6.29)\qquad v(y(z_1),z_2) = \int_0^{\infty} e^{-\alpha t}[h(y^*(t))+c(p^*(y^*(t),z_2))+Lv(y^*(t),z_2)]dt$$

where $y^*(t) = y(z_1)+\int_0^t [p^*(y^*(s),z_2)-z_2]ds$. Since $p^*(y,z_2)-z_2 \geq 0$ for $y \in I$, $y^*(t) \in I$ for $t \geq 0$. Rewrite (6.29) as follows:

$$\alpha v(y(z_1),z_2) = \int_0^\infty e^{-\alpha t}[h(y^*(t))+c(p^*(y^*(t),z_2))+Lv(y^*(t),z_1)]dt$$
$$+ \int_0^\infty e^{-\alpha t}[Lv(y^*(t),z_2)-Lv(y^*(t),z_1)]dt$$

Relation (6.28) implies that $Lv(y,z_2)-Lv(y,z_1) \geq Lv(y(z_1),z_2)-Lv(y(z_1),z_1)$ for $y \in I$. Since $y^*(t) \in I$, we obtain

$$\alpha v(y(z_1),z_2) \geq \int_0^\infty e^{-\alpha t}[h(y^*(t))+c(p^*(y^*(t),z_2))+Lv(y^*(t),z_1)]dt$$
$$+ (1/\alpha)[Lv(y(z_1),z_2)-Lv(y(z_1),z_1)]. \tag{6.30}$$

Now define $P' = \{p'(t);\ t \geq 0\}$ by $p'(t) = p^*(y^*(t),z_2)+z_2-z_1$. Then $P' \geq 0$, the strict convexity of c and (6.30) yield

$$v(y(z_1),z_2) > J(y(z_1),z_1,P')+(1/\alpha)[c(z_2)-c(z_1)+Lv(y(z_1),z_2)$$
$$-Lv(y(z_1),z_1)]$$
$$\geq v(y(z_1),z_1)+(1/\alpha)[c(z_2)-c(z_1)+Lv(y(z_1),z_2) \tag{6.31}$$
$$-Lv(y(z_1),z_1)].$$

We know that $v(y(z_1),z_1) = (1/\alpha)[c(z_1)+h(y(z_1))+L(v(y(z_1),z_1)]$. Substitute this into (6.31)

$$v(y(z_1),z_2) > (1/\alpha)[h(y(z_1))+c(z_2)+Lv(y(z_2),z_1)] = J(y(z_1),z_2,\overline{P})$$

where $\overline{p}(t) = z_2$ for $t \geq 0$. But the above inequality contradicts with the definition of v, hence $y(z_1) > y(z_2)$.

□

Next, we give examples to show that $y(\cdot)$ is not monotone, in general.

Example 2.

Take $c(p) = p^2$, $h(y) = y^2$, $\alpha = 1$ and $Z = \{1,2,3\}$. Define L by

$$L\varphi(1) = 0$$
$$L\varphi(2) = \lambda_2[\varphi(1)-\varphi(2)]$$
$$L\varphi(3) = \lambda_3[\varphi(1)-\varphi(3)]$$

where λ_i's are positive constants. Let v_{λ_2,λ_3} be the value function. We will show that for certain values of λ_2,λ_3 the monotonicity of $y(z)$ breaks down. First, observe that $v_{\lambda_2,\lambda_3}(y,1)$ is independent of λ_2 and λ_3, so it will be denoted by $v(y,1)$. Also, we have the following:

$$\lim_{\lambda_i\to+\infty} v_{\lambda_2,\lambda_3}(y,i) = v(y,1) \quad ; \quad i = 2,3 \text{ uniformly in bounded } y$$

$$\lim_{\substack{\lambda_2\to\lambda_2^0\\ \lambda_3\to\lambda_3^0}} v_{\lambda_2,\lambda_3}(y,i) = v_{\lambda_2^0,\lambda_3^0}(y,i) \quad ; \quad i = 1,2,3 \text{ uniformly in bounded } y$$

where λ_2^0,λ_3^0 are constants. Also, v_{λ_2,λ_3} is strictly convex because h and c are so. Therefore, there is only one point $y_{\lambda_2,\lambda_3}(i)$ satisfying $\frac{\partial}{\partial y}v_{\lambda_2,\lambda_3}(y,i) = -c'(i)$. So, the uniform convergence of v_{λ_2,λ_3} implies the following:

$$(6.32)\qquad \begin{cases} \lim\limits_{\lambda_i\to+\infty} y_{\lambda_2,\lambda_3}(i) = y(1) \quad ; \quad i = 2,3 \\ \lim\limits_{\lambda_2\to 0} y_{\lambda_2,\lambda_3}(2) = y_{0,\lambda_3}(2) \\ \lim\limits_{\lambda_3\to 0} y_{\lambda_2,\lambda_3}(3) = y_{\lambda_2,0}(3) \end{cases}$$

Since $v(y,1)$, $v_{0,\lambda_3}(y,2)$ and $v_{\lambda_2,0}(y,3)$ are value functions of deterministic control problems, Corollary 6.2 implies the following

$$(6.33)\qquad y(1) = -1\ ,\quad y_{0,\lambda_3}(2) = -2\ ,\quad y_{\lambda_2,0}(3) = -3,\quad ;\forall\lambda_2,\lambda_3 \geq 0.$$

Since $y_{0,\lambda_3}(3)$ is continuous in λ_3 with $y_{0,0}(3) = -3$ and $\lim\limits_{\lambda_3\to\infty} y_{0,\lambda_3}(3) = -1$, there is $\lambda_3^0 > 0$ such that $y_{0,\lambda_3^0}(3) = -3/2$. Observe that for the pair $(0,\lambda_3^0)$, the monotonicity of $y(z)$ breaks down.

<u>Example 3</u>.

Again take c,h,α,Z to be the same as in Example 2. Define L_ε by

$$L_\varepsilon\varphi(1) = \varepsilon[\varphi(3)+\varphi(2)-2\varphi(1)]$$

$$L_\varepsilon\varphi(2) = \varepsilon[\varphi(3)+\varphi(1)-2\varphi(2)]$$

$$L_\varepsilon\varphi(3) = \varepsilon[\varphi(1)+\varphi(2)-2\varphi(3)] + \lambda_3^0[\varphi(1)-\varphi(3)]$$

where λ_3^0 is as in the previous example. Arguing as in Example 2, we can conclude that for small but positive ε, the monotonicity does not hold.

Observe that L_ε generates an ergodic chain on $\{1,2,3\}$.

Now, suppose that L is of the following form

$$L\varphi(z_1) = \lambda_1[\varphi(z_2)-\varphi(z_1)]$$
$$L\varphi(z_n) = \lambda_n[\varphi(z_{n+1})-\varphi(z_n)]+\mu_n[\varphi(z_{n-1})-\varphi(z_n)] \ ; \quad n = 2,\ldots,N-1$$
$$L\varphi(z_N) = \lambda_N[\varphi(z_{N-1})-\varphi(z_N)]$$

where $Z = \{z_1,\ldots,z_N\}$ and $z_1 < z_2 < \ldots < z_N$. Let $z_n(\cdot)$ be the process starting at z_n. Then $z_n(t) \leq z_m(t)$ if $n < m$. Thus it is reasonable to expect monotonicity of $y(\cdot)$ in this case but we were unable to prove this assertion.

7. Inventory Constraint with Jump Markov Demand

In this section, in addition to the nonnegative production constraint we impose the constraint that the inventory level cannot fall below a certain prescribed level y_{min}. So the set of feasible production processes $\mathbb{A}(y,z)$ is given by

$$(7.1) \quad \mathbb{A}(y,z) = \{P \in \mathbb{A} : y + \int_0^t [p(s)-z(s)]ds \geq y_{min} \text{ for } t \geq 0\} \; ; \quad \forall (y,z) \in [y_{min},\infty) \times Z$$

Then the corresponding value function is defined by

$$(7.2) \quad v(y,z) = \inf_{P \in \mathbb{A}(y,z)} J(y,z,P) \; ; \; \forall (y,z) \in [y_{min},\infty) \times Z \; .$$

We will only consider the case when demand process is a jump Markov process.

Define h^ε by:

$$(7.3) \quad h^\varepsilon(y) = [(1/\varepsilon)-1]\max\{y_{min}-y,0\}+h(y) \; ; \; \forall y \in (-\infty,\infty), \; \varepsilon \in (0,1].$$

Let v^ε be the value of the unconstrained problem for a given $\varepsilon \in (0,1]$. Since $h^\varepsilon = h$ on $[y_{min},\infty)$, $v^\varepsilon \leq v$ on $[y_{min},\infty) \times Z$.

Theorem 7.1.

Suppose L is as in (4.11). Then for every $R > 0$ and small $\varepsilon > 0$, there is $K_R > 0$ such that

$$(7.4) \quad 0 \leq v(y,z)-v^\varepsilon(y,z) \leq K_R\sqrt{\varepsilon} \; ; \quad \forall (y,z) \in [y_{min},R] \times Z.$$

<u>Proof</u>: For $y \le y_{min}$ use the production process $p(t) = z(t)+\chi_{[0,y_{min}-y]}(t)$ to obtain:

$$(7.5) \qquad v^{\varepsilon}(y,z) \le \frac{1}{\varepsilon}(y-y_{min})^2 + \kappa(h(y)+1) \ ; \ \forall y \le y_{min}$$

where in this proof κ is a constant independent of ε. Equation (3.1) for v^{ε} has the following form:

$$(7.6) \qquad F(\frac{\partial}{\partial y}v^{\varepsilon}(y,z))-z\frac{\partial}{\partial y}v^{\varepsilon}(y,z) = \alpha v^{\varepsilon}(y,z)-Lv^{\varepsilon}(y,z)-h^{\varepsilon}(y).$$

At $y = y_{min}-\sqrt{\varepsilon}$, equation (7.6) and (7.5) yield that

$$\begin{aligned}(7.7) \quad F(\frac{\partial}{\partial y}v^{\varepsilon}(y_{min}-\sqrt{\varepsilon},z))-z\frac{\partial}{\partial y}v^{\varepsilon}(y_{min}-\sqrt{\varepsilon},z) &\le \alpha+\alpha\kappa(h(y_{min}-\sqrt{\varepsilon})+1) \\ &\quad + \kappa\|v^{\varepsilon}(y_{min}-\sqrt{\varepsilon},\cdot)\|_{\infty} - \frac{1}{\sqrt{\varepsilon}} \\ &\le \kappa - 1/\sqrt{\varepsilon}.\end{aligned}$$

In the first inequality, we have used the fact that $(Lv^{\varepsilon}(y,\cdot))(z) \le \kappa\|v^{\varepsilon}(y,\cdot)\|_{\infty}$. Recall that the map $r \to F(r) - zr$ is concave. Therefore, (7.7) implies that

$$(7.8) \qquad \frac{\partial}{\partial y}v^{\varepsilon}(y_{min}-\sqrt{\varepsilon},z) \in (-\infty,c_1^{\varepsilon}(z)] \cup [c_2^{\varepsilon}(z),\infty)$$

where

$$F(c_i^{\varepsilon}(z)) - z\, c_i^{\varepsilon}(z) = \kappa - 1/\sqrt{\varepsilon}.$$

Then for sufficiently small $\varepsilon > 0$, $c_1^{\varepsilon}(z) < -c'(z)$ and $c_2^{\varepsilon}(z) = ((1/\sqrt{\varepsilon})-\kappa)/z$.

Again use (7.6) at $y = y_{min}$ to obtain

$$F(\frac{\partial}{\partial y}v^{\varepsilon}(y_{min},z)) - z\frac{\partial}{\partial y}v^{\varepsilon}(y_{min},z) \geq -\kappa.$$

A similar argument to that for (7.8) yields that

$$\sup_{\substack{\varepsilon>0 \\ z\in Z}} |\frac{\partial}{\partial y}v^{\varepsilon}(y_{min},z)| \leq \kappa. \tag{7.9}$$

Since v^{ε} is convex, $\frac{\partial}{\partial y}v^{\varepsilon}(y_{min}-\sqrt{\varepsilon},z) \leq \frac{\partial}{\partial y}v^{\varepsilon}(y_{min},z) \leq \kappa$. Recall that $c_2^{\varepsilon}(z) = [(1/\sqrt{\varepsilon})-\kappa]/z$, thus (7.8) yields that

$$\frac{\partial}{\partial y}v^{\varepsilon}(y_{min}-\sqrt{\varepsilon},z) \leq c_1^{\varepsilon}(z) < -c'(z) \ ; \ \forall z \in Z \text{ and } \varepsilon \text{ small}. \tag{7.10}$$

Now, fix $(y,z) \in [y_{min},R] \times Z$. Let $p_{\varepsilon}^{*}(t)$ be the optimal production process constructed in section 3. Then (7.10) implies that the optimal inventory trajectory $y_{\varepsilon}^{*}(\cdot)$ satisfies the following

$$y_{\varepsilon}^{*}(t) \geq y_{min}-\sqrt{\varepsilon} \quad ; \forall t \geq 0.$$

Therefore, P_{ε} defined below is in $\mathbb{A}(y,z)$.

$$p_{\varepsilon}(t) = \begin{cases} z(t) + 1 & t \in [0,\sqrt{\varepsilon}] \\ p_{\varepsilon}^{*}(t-\sqrt{\varepsilon})+z(t)-z(t-\sqrt{\varepsilon}) \ ; & t \geq \sqrt{\varepsilon} \end{cases}$$

Also, $y_{\varepsilon}(t) = (y+t)\chi_{[0,\sqrt{\varepsilon}]}(t)+(y_{\varepsilon}^{*}(t-\sqrt{\varepsilon})+\sqrt{\varepsilon})\chi_{[\sqrt{\varepsilon},\infty)}(t)$. Thus, we have the following:

$$(7.11)\qquad v(y,z)-v^{\varepsilon}(y,z) \le J(y,z,P_{\varepsilon})-J(y,z,P^{*})$$
$$\le E\int_0^{\sqrt{\varepsilon}} e^{-\alpha t}[h(y+t)+c(z(t)+1)]dt$$
$$+\ e^{-\alpha\sqrt{\varepsilon}}\Big\{E\int_0^{\infty} e^{-\alpha t}[h(y_{\varepsilon}^{*}(t)+\sqrt{\varepsilon})-h(y_{\varepsilon}^{*}(t))]dt\ +$$
$$E\int_0^{\infty} e^{-\alpha t}[c(p_{\varepsilon}^{*}(t)+z(t+\sqrt{\varepsilon})-z(t)-c(p_{\varepsilon}^{*}(t))]dt\Big\}.$$

As in Lemma 3.2, one can prove that for $R > 0$, there is $K_R > 0$ such that

$$(7.12)\qquad |y_{\varepsilon}^{*}(t)|+|p_{\varepsilon}^{*}(t)| \le K_R\ ;\ \forall\, t \ge 0 \ \text{ and } \ y_{\varepsilon}^{*}(0) \in [y_{min},R].$$

Now use (A4) and (7.12) to obtain

$$(7.13)\qquad E[c(p_{\varepsilon}^{*}(t)+z(t+\sqrt{\varepsilon})-z(t))-c(p_{\varepsilon}^{*}(t))] \le K_5(K_R^{\overline{\gamma}}+1)E|z(t+\sqrt{\varepsilon})-z(t)|$$
$$\le K_R\,\mathbb{P}(z(t+\sqrt{\varepsilon})\ne z(t)) \le K_R\sqrt{\varepsilon}.$$

Similarly,

$$(7.14)\qquad E[h(y_{\varepsilon}^{*}(t)+\sqrt{\varepsilon})-h(y_{\varepsilon}^{*}(t))] \le K_R\sqrt{\varepsilon}.$$

Substitute (7.13) and (7.14) into (7.11) to complete the proof of (7.4).

□

<u>Theorem 7.2</u>.

The value function v for the constrained problem is in D_0 and it is the only solution of the following equation:

$$(7.15)\qquad \alpha v(y,z) = F(\tfrac{\partial}{\partial y}v(y,z))-z\tfrac{\partial}{\partial y}v(y,z)+(Lv(y,\cdot))(z)+h(y)\quad ;$$
$$\forall\,(y,z) \in [y_{min},\infty)\times Z.$$

$$(7.16) \qquad \frac{\partial}{\partial y}v(y_{min},z) \geq -c'(z) \quad ; \quad \forall z \in Z.$$

Proof: As a consequence of the previous theorem, we know that v is continuous, convex in y and satisfies (2.1) and (2.4).

Now using the method developed in Lemma 4.2, we obtain the following (also, see Theorem 1.1 in [14]):

$$(7.17) \quad \begin{cases} (1)\ \alpha v(y,z)-F(r)+zr-[Lv(y,\cdot)](z)-h(y) \leq 0 \ ; \\ \qquad\qquad \forall r \in D_y^+v(y,z) \quad \text{and} \quad (y,z) \in (y_{min},\infty) \times Z \\ (2)\ \alpha v(y,z)-F(r)+zr-[Lv(y,\cdot)](z)-h(y) \geq 0 \ ; \\ \qquad\qquad \forall r \in D_y^-v(y,z) \quad \text{and} \quad (y,z) \in [y_{min},\infty) \times Z. \end{cases}$$

Now proceed as in Theorem 4.3 to conclude that $\frac{\partial}{\partial y}v$ is continuous and (7.15) holds. To prove (7.16), observe that

$$(7.18) \qquad D_y^-v(y_{min},z) = (-\infty,\frac{\partial}{\partial y}v(y_{min},z)] .$$

Therefore, (7.17) and (7.18) imply that

$$(7.19) \quad \alpha v(y_{min},z)-(Lv(y_{min},\cdot))(z)-h(y_{min}) \geq F(r)-zr \quad ; \quad \forall r \leq \frac{\partial}{\partial y}v(y_{min},z).$$

Equation (7.15) and (7.19) yield

$$F(\frac{\partial}{\partial y}v(y_{min},z))-z\frac{\partial}{\partial y}v(y_{min},z) \geq F(r)-zr \quad ; \quad \forall r \leq \frac{\partial}{\partial y}v(y_{min},z).$$

Since the map $r \to F(r)-zr$ achieves its maximum at $r = -c'(z)$ only, the above condition is equivalent to (7.6).

Uniqueness follows from the verification theorem. Observe that the optimal feedback policy P^* constructed in (3.2) is in $\mathbb{A}(y,z)$ on account of (7.16).

□

If the value function v^ε of any of the unconstrained problems satisfy (7.6), then it is also the value function of the state constrained problem. But clearly there are values of y_{min} such that this does not happen. The following result deals with this case.

<u>Proposition 7.3</u>.

Let Y_ε be given by

$$(7.20) \quad Y_\varepsilon = \inf\{y \in (-\infty,\infty) : \frac{\partial}{\partial y} v^\varepsilon(y,z) = -c'(z) \text{ for some } z \in Z\}.$$

Then either one of the following hold for any $\varepsilon > 0$:

(i) $Y_\varepsilon > y_{min}$ and $\inf\{y : y \in G\} > y_{min}$

(ii) $Y_\varepsilon \leq y_{min}$ and $G = [y_{min}, y_{min}+a]$ for some $a > 0$

where $G = \overline{co}\{y \in [y_{min},\infty) : \frac{\partial}{\partial y} v(y,z) = -c'(z) \text{ for some } z \in Z\}$.

<u>Proof</u>: Take a sequence $\{(y_\varepsilon, z_\varepsilon) : \varepsilon \geq 0\} \subset (-\infty,\infty) \times Z$ such that $(y_\varepsilon, z_\varepsilon)$ converges to $(y_0, z_0) \in [y_{min},\infty) \times Z$ as ε tends to zero. The convexity of v^ε implies the following

$$v^\varepsilon(y,z_\varepsilon) \geq v^\varepsilon(y_\varepsilon,z_\varepsilon) + (y-y_\varepsilon)\frac{\partial}{\partial y} v^\varepsilon(y_\varepsilon,z_\varepsilon) \quad ; \forall y .$$

Now suppose that $\frac{\partial}{\partial y}v^{\varepsilon}(y_{\varepsilon},z_{\varepsilon})$ converges to r on a subsequence. Then Theorem 7.1 yields that

$$v(y,y_0) \geq v(y_0,z_0)+(y-y_0)r \quad ; \ \forall y \geq y_{min}.$$

Again, the convexity and differentiablity of v implies that

$$(7.21) \quad \begin{cases} \lim\limits_{\varepsilon\to 0} \frac{\partial}{\partial y}v^{\varepsilon}(y_{\varepsilon},z_{\varepsilon}) = \frac{\partial}{\partial y}v(y_0,z_0) & , \text{ if } y_0 > y_{min} \\ \limsup\limits_{\varepsilon\to 0} \frac{\partial}{\partial y}v^{\varepsilon}(y_{\varepsilon},z_{\varepsilon}) \leq \frac{\partial}{\partial y}v(y_{min},z_0), & \text{ if } y_0 = y_{min} . \end{cases}$$

Now let $(y_{\varepsilon},z_{\varepsilon})$ be such that $|Y_{\varepsilon}-y_{\varepsilon}| \leq \varepsilon$ and $\frac{\partial}{\partial y}v^{\varepsilon}(y_{\varepsilon},z_{\varepsilon}) = -c'(z_{\varepsilon})$. Observe that $\sup\limits_{\varepsilon}|Y_{\varepsilon}|$ is finite and $z_{\varepsilon} \in Z$. So there is (y_0,z_0) such that $(y_{\varepsilon},z_{\varepsilon}) \to (y_0,z_0)$ on a subsequence. Also, (7.10) implies that $y_{\varepsilon} \geq y_{min}-\sqrt{\varepsilon}$. Hence, $(y_0,z_0) \in [y_{min},\infty) \times Z$. If $y_0 = y_{min}$, then $\frac{\partial}{\partial y}v(y_{min},z_0) \geq -c'(z_0)$ on account of (7.21). But (7.16) yields $\frac{\partial}{\partial y}v(y_{min},z_0) = -c'(z_0)$. Therefore, we have $\frac{\partial}{\partial y}v(y_0,z_0) = -c'(z_0)$ and consequently the following

$$(7.22) \qquad \liminf_{\varepsilon\downarrow 0} Y_{\varepsilon} \geq \inf\{y : y \in G\}.$$

Similarly, one can prove that Y_{ε} is lower semi-continuous in ε.

Suppose that $Y_{\varepsilon} \leq y_{min}$. Then (7.22) and the lower semi-continuity of Y_{ε} imply that one of the following holds:

(1) $\inf\{y : y \in G\} = y_{min}$

(2) there is $\varepsilon_0 > 0$ such that $Y_{\varepsilon_0} = y_{min}$.

But for the second case $v^{\varepsilon_0} = v$ and $\inf\{y : y \in G\} = Y_{\varepsilon_0} = y_{min}$. Also, an argument as in the proof of Theorem 6.1(ii) yields that G is a non-degenerate interval. Hence the second part of the proposition is proved.

To prove the first part, we suppose that $Y_\varepsilon > y_{min}$. Then $v' = v$ and $\inf\{y : y \in G\} = Y_\varepsilon > y_{min}$.

□

8. Extensions and Concluding Remarks

We have now completed our study of infinite horizon stochastic production planning problems with demand assumed to be either a jump Markov process or a reflected Markov diffusion process. Problems with and without the inventory constraints are treated. We have shown the existence of optimal feedback production policies. These policies exhibit an appropriately generalized version of the so-called turnpike behavior.

An important extension of the problem involves production processes, which are bounded from above by a stochastic process representing the capacity of the production system. The capacity process over time may be modeled as a jump Markov process or a piece-wise deterministic process [5]. Moreover, there may be several different products competing for a variety of scarce capacities. This is an important problem faced by flexible manufacturing systems [11], upon which the methods developed in this paper have some bearing.

In our ongoing work, we use these methods to deal with the special case of the above problem, namely, when the demand is constant. The capacity process is assumed to be a vector jump Markov process. For a manufacturing system consisting of several machines, such a capacity process results from random machine breakdowns and subsequent repairs.

Appendix 1

Consider the following stochastic integral equation whose solutions are jump Markov processes.

$$z(t) = z+\int_0^t\int_{-\infty}^{\infty} \ell(z(s),x)\pi(ds\times dx) \tag{9.1}$$

where π is a random measure defined on the Borel subsets of $[0,\infty)\times[(-\infty,0)\cup(0,\infty)]$ satisfying

(9.2)
- (i) For all Borel subsets A of $(-\infty,0)\cup(0,\infty)$ let $\lambda(A) = \int_A dx/|x|^2$. Then if $\lambda(A)$ is finite $\eta_A(t) := \pi([0,t]\times A)$ is a Poisson process with parameter $\lambda(A)$.
- (ii) η_A is independent of η_B whenever $A\cap B = \emptyset$.

Also, let ℓ satisfy the following:

(9.3)
- (i) $\ell(z,x)+z\in Z$; $\forall z\in Z$ and $x\in(-\infty,0)\cup(0,\infty)$
- (ii) There is $N_1 \geq 0$ such that
$$\int_{-\infty}^{\infty}|\ell(z,x)-\ell(z',x)|dx/|x|^2 \leq N_1|z-z'|$$
- (iii) There is $r_0 > 1$ such that
$$\ell(z,x) = 0 \text{ whenever } |x| \leq r_0^{-1} \text{ or } |x| \geq r_0.$$

Then there is a unique solution of the equation (9.1)(Theorem 1, page 47 in [13]). Moreover, the solution $z(\cdot)$ is a strong Markov process with infinitesimal generator L given by

$$(9.4) \qquad L\varphi(z) = \int_{-\infty}^{\infty} [\varphi(z+\ell(z,x))-\varphi(z)]\frac{dx}{|x|^2} .$$

So the jump rate and post-jump distribution $\pi(z,dx)$ are given by

$$\lambda(z) = M(\{x \in (-\infty,0) \cup (0,\infty) : \ell(z,x)+z \in Z \setminus \{z\}),$$

where $M(A) = \int_A dx/|x|^2$. If $\lambda(z) = 0$, then $\pi(z,dx)$ is arbitrary. But if $\lambda(z) \neq 0$, then for any Borel subset A of $Z-\{z\}$

$$\pi(z,A) = \lambda(z)^{-1} M(\{x \in (-\infty,0) \cup (0,\infty) : (z,x)+z \in A\}).$$

Straightforward calculations show that the condition (9.3) implies the assumptions (A8)-(A.10). (See section 2.3 in [13] also.) Note that it also implies $\pi(z,Z \setminus \{z\}) = 1$ for all $z \in Z$.

Remark 9.1

(i) The condition (9.3)(i) implies that $z(t) \in Z$ for all $t \geq 0$.

(ii) The random measure π can be constructed from the jumps of a Cauchy process $\xi(t)$ as follows

$$\pi([0,t] \times A) = \sum_{s \leq t} \chi_{([\xi(s)-\xi(s^-)] \in A)} \; ; \; \forall A \subset (-\infty,0) \cup (0,\infty)$$

where ξ is an independent increment process with its characteristic function given by

$$E\{\exp(i\beta[\xi(t+s)-\xi(t)])\} = \exp\{s[\int_{|x| \leq 1} (e^{i\beta x}-i\beta x-1)\frac{dx}{|x|^2} + \int_{|x|>1} (e^{i\beta x}-1)\frac{dx}{|x|^2}]\}.$$

For more information see section 2.4 in [13].

Next we will show that if λ and π satisfy (9.5) in addition to (A8) and (A9), then (A10) holds.

$$(9.5)\quad \begin{cases} \text{There is } N_2 \geq 0 \text{ such that for all } A \subset Z \\ \lambda(z)\pi(z,A) - \lambda(z')\pi(z', A_{N_2|z'-z|}) \leq N_2|z'-z| \ . \end{cases}$$

where $A_c = \{z \in Z : \text{there is } z' \in A : |z-z'| \leq c\}$.

Since λ is bounded (assumption (A8)), there is $\varepsilon_0 > 0$ such that

$$(9.6)\qquad \int_{\varepsilon_0}^{\infty} dx/|x|^2 \geq 2 \sup\{\lambda(z) : z \in Z\} \ .$$

Now, define F and $\tilde{\ell}$ by

$$(9.7)\qquad F(z,\xi) = \lambda(z)\pi(z,[\xi_0,\xi]) \quad ; \ \forall z \in Z \text{ and } \xi \in [\xi_0,\xi_1] = Z$$

$$(9.8)\quad \tilde{\ell}(z,x) = \begin{cases} \inf\{\xi : F(z,\xi) \geq x - \varepsilon_0\} & ; \text{ if } x \in [\varepsilon_0, \varepsilon_0 + F(z,\xi_1)) \\ z & ; \text{ if } x \in (-\infty,\varepsilon_0) \cup [\varepsilon_0 + F(z,\xi_1),\infty) \ . \end{cases}$$

Straightforward calculations imply that for any $\varphi \in D(L)$

$$L\varphi(z) = \int_{(-\infty,\infty)} [\varphi(\tilde{\ell}(z,x)) - \varphi(z)]dx$$

where $L\varphi$ is given by (4.11). Change of variables in above integral yield

$$(9.9)\quad L\varphi(z) = \int_{(-\infty,\infty)} [\varphi(\tilde{\ell}(z,\varepsilon_0 + (1/\varepsilon_0) - (1/x)) - \varphi(z)]dx/|x|^2 \ .$$

Define ℓ by

$$(9.10)\quad \ell(z,x) = \begin{cases} \tilde{c}(z,\varepsilon_0+(1/\varepsilon_0)-1/x))-z; & \forall x\in[\varepsilon_0,[(1/\varepsilon_0)-F(z,\xi_1)]^{-1}) \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Note that choice ε_0 yields that $\varepsilon_0 < [(1/\varepsilon_0)-F(z,\xi_1)]^{-1} \leq 2\varepsilon_0$, so ℓ is well-defined. Combine (9.9) and (9.10) to obtain

$$L\varphi(z) = \int_{-\infty}^{\infty} [\varphi(z+\ell(z,x))-\varphi(z)]dx/|x|^2 .$$

It is clear that ℓ defined by (9.10) satisfies (9.3)(i) and (iii). A technical argument which we choose to omit implies that (9.5) implies (9.3)(ii). Therefore (A.10) holds true.

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