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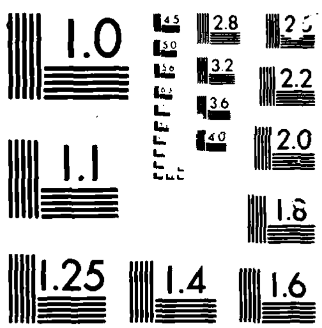

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ITERATIVE REFINEMENT OF THE METHOD OF MOMENTS

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ITERATIVE REFINEMENT OF THE METHOD OF MOMENTS

George Miel*

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ABSTRACT

It is observed that Vorobyev's method of moments, because it is a specialized form of Galerkin's method, is easily accelerated by the use of Sloan's trick of taking first iterates. The improvement, which is nearly costless since it requires no new quantities, is illustrated in a numerical example.

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SIGNIFICANCE AND EXPLANATION

Operator equations have countless realizations in miscellaneous engineering and scientific applications. The language of functional analysis provides a powerful and unifying tool for the study of phenomena associated with the computational solution of such equations. The functional analytic approach avoids tedious details and allows the analyst to focus on major issues. This methodology, though it may not produce actual software, can provide invaluable conceptual understanding.

We deal with the approximate solution of the equation

$$(J - K)u = v ,$$

where the operators $J - K$ and J are assumed invertible, by the use of Vorobyev's method. This technique uses the $n + 1$ "moments"

$$z_0 = J^{-1}v, \quad z_i = J^{-1}Kz_{i-1}, \quad 1 \le i \le n ,$$

in order to get an approximate solution of the form

$$y_n = \xi_0 z_0 + \xi_1 z_1 + \cdots + \xi_{n-1} z_{n-1} .$$

Under proper conditions, the sequence $\{y_n\}$ converges to the true solution $u^* = (J - K)^{-1}v$.

Sloan has shown that, given an approximate solution u_n to the equation, the first iterate

$$\bar{u}_n = J^{-1}Ku_n + z_0$$

is usually more accurate than u_n itself. Specifically, under adequate conditions, the sequence of first iterates $\{\bar{u}_n\}$ converges to u^* faster than the sequence $\{u_n\}$. The aim of our note is to report the simple but useful observation that Sloan's iterative refinement is applicable to Vorobyev's method and that the computation of the first iterate,

$$\bar{y}_n = z_0 + \xi_0 z_1 + \cdots + \xi_{n-1} z_n ,$$

involves only known quantities. A numerical example involving an integral equation illustrates this nearly costless improvement.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ITERATIVE REFINEMENT OF THE METHOD OF MOMENTS

George Miel*

Let E and F denote Hilbert spaces. Given bounded linear operators $J,K : E \to F$ and an element $v \in F$, consider the split equation of the first kind

$$(J - K)u = v . \tag{1}$$

Assume that J has a bounded inverse on F and that K is a compact operator. Equation (1) is equivalent to the equation of the second kind,

$$u = Mu + w , \tag{2}$$

where $M = J^{-1}K$ is a compact operator and $w = J^{-1}v$. The aim of this note is to report the observation that Vorobyev's method of moments [8] can be advantageously improved by Sloan's iterative refinement [6]. Since the quantities needed for the latter procedure are conveniently computed in the former method, the cost of the improvement is insignificant. After a brief description of the pertinent procedures, we illustrate the effect of the iterative refinement on a Fredholm integral equation with a Green's kernel.

Sloan's Iterative Refinement

We first need to describe Galerkin's method for equation (1). Let $F_n = JE_n$, where E_n is a finite-dimensional subspace of E, and let P_n be the orthogonal projection of F onto F_n. The Galerkin method consists of solving the approximate equation

$$(J - P_nK)u_n = P_nv, \quad u_n \in E_n .$$

This equation is equivalent to the equation

$$u_n = Q_nMu_n + Q_nw , \tag{3}$$

where $Q_n = J^{-1}P_nJ$ is the orthogonal projection of E onto E_n. A standard result

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states that if $J - K$ has a bounded inverse and $Q_n x \to x$ for every $x \in E$ as $n \to \infty$ then, for large enough n, the operator $I - Q_n M$ has a bounded inverse on E_n, the approximate solutions $u_n = (I - Q_n M)^{-1} Q_n w$ converge to the true solution $u^* = (J - K)^{-1} v$, and the error bounds

$$\|u^* - u_n\| \le \|(I - Q_n M)^{-1}\| \cdot \|u^* - Q_n u^*\| \tag{4}$$

are valid. We refer to Krasnosel'skii et al. [4] for a proof.

Sloan [6] has shown that the sequence of first iterates

$$\bar{u}_n = M u_n + w \tag{5}$$

converges more rapidly to the true solution. Moreover, once u_n is known, the iterate $\bar{u}_n$ can be computed with little cost. Indeed, letting $\varphi_1, \ldots, \varphi_n$ constitute a basis for E_n, since the Galerkin method requires the quantities $\bar{\varphi}_i = M\varphi_i$ in the setting-up of the $n \times n$ linear system needed to find $u_n = \sum_1^n \xi_i \varphi_i$, the iterate $\bar{u}_n = \sum_1^n \xi_i \bar{\varphi}_i + w$ is easily obtained.

To see the improvement of $\bar{u}_n$ over u_n, we proceed as follows. Define an operator by

$$M_n = M Q_n .$$

Then, (3) and (5) imply that $\bar{u}_n$ is a solution of the equation

$$\bar{u}_n = M_n \bar{u}_n + w .$$

Verify now that

$$(I - M_n)(u^* - \bar{u}_n) = (M - M_n)(u^* - Q_n u^*) .$$

From this expression, we get the error bound,

$$\|u^* - \bar{u}_n\| \le \|(I - M_n)^{-1}\| \cdot \|M - M_n\| \cdot \|u^* - Q_n u^*\| ,$$

which shows improvement over the error bound in (4). See Sloan [6] for details.

Recent applications of the iterative refinement to integral equations can be found in Chandler [1], Graham [3], and Spence and Thomas [7]. A functional analytic overview of the iterated Galerkin procedure is given in Chatelin and Lebbar [2] and in Schock [5].

Vorobyev's Method of Moments

We turn our attention back to equation (1) and its equivalent formulation (2). Consider the $n + 1$ "moments"

$$z_0 = w, \quad z_1 = Mz_0, \; \ldots \;, z_n = Mz_{n-1} \; . \tag{6}$$

The usual Neumann iterates $x_n = Mx_{n-1} + w$, starting with $x_0 = w$, are given by

$$x_n = z_0 + z_1 + \cdots + z_n \; .$$

In Vorobyev's method, one uses the $n + 1$ moments (6) in order to find a linear combination of the first n moments,

$$y_n = \xi_0 z_0 + \xi_1 z_1 + \cdots + \xi_{n-1} z_{n-1} \; . \tag{7}$$

The algorithm proceeds like this:

1. Set-up and solve the $n \times n$ linear system

$$\sum_{j=0}^{n-1} \langle z_i, z_j \rangle \alpha_j = -\langle z_i, z_n \rangle, \quad 0 \le i \le n-1 \; , \tag{8}$$

   for $\alpha_0, \ldots, \alpha_{n-1}$.

2. Evaluate recursively the coefficients

$$\xi_0 = 1 - c\alpha_0, \;\; \xi_i = \xi_{i-1} - c\alpha_i, \quad 1 \le i \le n-1 \; ,$$

   where $c = 1/(\alpha_0 + \cdots + \alpha_{n-1} + 1)$.

3. Evaluate y_n using (7).

It turns out that the algorithm is equivalent to Galerkin's method

$$y_n = Q_n M y_n + w, \qquad y_n \in E_n \; , \tag{9}$$

where Q_n is the orthogonal projection of E onto $E_n = \text{span}(z_0, \ldots, z_{n-1})$. The elements of E_n are of the form $y_n = q(M) z_0$, where

$$q(t) = \xi_0 + \xi_1 t + \cdots + \xi_{n-1} t^{n-1} \tag{10}$$

is an arbitrary polynomial of degree $\le n - 1$. The residual is given by

$$z_0 - y_n + My_n = p(M) z_0 \; ,$$

where

$$p(t) = 1 - (1 - t) q(t) = c(\alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1} + t^n) \; . \tag{11}$$

To see that the approximate operator equation (9) represents Vorobyev's method, take $t = 1$ in (11), thereby showing that $c = 1/(\alpha_0 + \cdots + \alpha_{n-1} + 1)$, and then substitute (10) and (11) and compare coefficients of like powers of t to get the recurrence relation for ξ_i.

If the elements $z_0,\ldots,z_n$ are linearly independent then the linear system (8) is uniquely solvable. Otherwise, the operators M and Q_nM coincide on the subspace E_n and there is then no gain to be made by using the method. If the operator $J - K$ has a bounded inverse then, for sufficiently large n, equation (9) has a unique solution y_n and the sequence $\{y_n\}$ converges to the solution $(J - K)^{-1}v$ faster than any geometric progression. We refer to Vorobyev [8] for details. Miscellaneous applications of the method can be found in Chapters V and VII of the cited text.

Refinement of the Method of Moment

Since Vorobyev's method is a Galerkin method on the subspace $E_n = \mathrm{span}(z_0, Mz_0, \ldots, M^{n-1}z_0)$, Sloan's iterative procedure can be applied. In view of (6) and (7), the desired iterate is given by

$$\bar{y}_n = \xi_0 z_1 + \xi_1 z_2 + \cdots + \xi_{n-1} z_n + w \,. \qquad (12)$$

The refinement is particularly advantageous since the computation of (12) requires only already known quantities.

Example. Let $E = F = L^2[0,1]$ and $J = I$. Consider the Fredholm integral equation of the second kind

$$u(s) = Ku(s) + s/2 \,,$$

$$K : E \to E, \quad Ku(s) = \int_0^1 k(s,t)u(t)dt \,,$$

where the kernel is given by

$$k(s,t) = \begin{cases} \frac{\beta}{2}(2 - t)s, & s < t \\ \frac{\beta}{2}(2 - s)t, & s > t \end{cases}, \quad \beta = \frac{\pi^2}{4} \,.$$

The exact solution is $u^*(s) = \sin\frac{\pi}{2}s$. We apply Vorobyev's method with three moments ($n = 2$):

$$z_0(s) = \frac{s}{2},$$

$$z_1(s) = Kz_0(s) = \frac{\beta}{12}(2s - s^3),$$

$$z_2(s) = Kz_1(s) = \frac{\beta^2}{720}(3s^5 - 20s^3 + 31s).$$

We find that:

$$\langle z_0,z_0\rangle = \frac{1}{12}, \qquad \langle z_0,z_1\rangle = \frac{\beta}{12}\cdot\frac{7}{30},$$

$$\langle z_0,z_2\rangle = \langle z_1,z_1\rangle = \frac{\beta^2}{12}\cdot\frac{71}{1260}, \qquad \langle z_1,z_2\rangle = \frac{\beta^3}{12}\cdot\frac{517}{37800},$$

$$\begin{cases} \alpha_0 + \dfrac{7\beta}{30}\alpha_1 = \dfrac{71}{1260}\beta^2 \\[2ex] 7\beta\alpha_0 + \dfrac{71\beta^2}{42}\alpha_1 = \dfrac{517}{1260}\beta^3 \end{cases}, \qquad \alpha_0 = \frac{8\beta^2}{945}, \quad \alpha_1 = \frac{-5\beta}{18},$$

$$\xi_0 = \frac{7560 - 525\pi^2}{7560 - 525\pi^2 + 4\pi^4}, \qquad \xi_1 = \frac{7560}{7560 - 525\pi^2 + 4\pi^4}.$$

Consequently, the desired approximation is

$$y_2(s) = \xi_0 z_0(s) + \xi_1 z_1(s) = as - bs^3,$$

where

$$a = \frac{\xi_0}{2} + \frac{\xi_1\pi^2}{24} = 1.552749,$$

$$b = \frac{\xi_1\pi^2}{48} = 0.561564.$$

Sloan's refinement,

$$\bar{y}_2(s) = Ky_2(s) + v(s),$$

requires the quantities $Kz_0 = z_1$ and $Kz_1 = z_2$ which are already known. We find that

$$\bar{y}_2(s) = \xi_0 z_1(s) + \xi_1 z_2(s) + v(s)$$
$$= cs - ds^3 + es^5,$$

where

$$c = \frac{\xi_0 \pi^2}{24} + \frac{31\xi_1 \pi^4}{11520} + \frac{1}{2} = 1.569244$$

$$d = \frac{\xi_0 \pi^2}{48} + \frac{\xi_1 \pi^4}{576} = 0.6385423$$

$$e = \frac{\xi_1 \pi^4}{3840} = 0.06928021 .$$

TABLE

s	$u^*(s)$	$y_2(s)$	$\bar{y}_2(s)$	$u^* - y_2$	$u^* - \bar{y}_2$
0.00	0.00000	0.00000	0.00000	0.00000	0.00000
0.20	0.30902	0.30606	0.30876	0.00296	0.00026
0.40	0.58779	0.58516	0.58754	0.00263	0.00025
0.60	0.80902	0.81035	0.80901	-0.00133	0.00001
0.80	0.95106	0.95468	0.95116	-0.00362	-0.00010
1.00	1.00000	0.99118	0.99998	0.00882	0.00002

The above table illustrates the nearly free improvement of the iterate $\bar{y}_2$ over y_2.

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It is observed that Vorobyev's method of moments, because it is a specialized form of Galerkin's method, is easily accelerated by the use of Sloan's trick of taking first iterates. The improvement, which is nearly costless since it requires no new quantities, is illustrated in a numerical example.

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