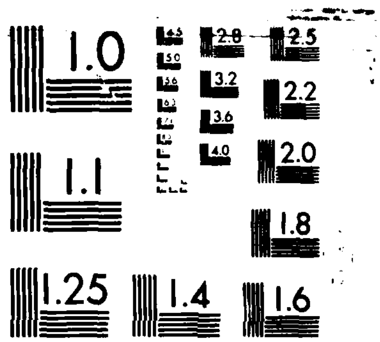
AD-A167 532 THE HAMILTON-JACOBI-BELLMAN EQUATION WITH A GRADIENT CONSTRAINT(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER N YAMADA MAR 86 MRC-TSR-2927 1/1

UNCLASSIFIED DAAG29-80-C-0041 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART

MRC Technical Summary Report #2927

THE HAMILTON-JACOBI-BELLMAN EQUATION WITH A GRADIENT CONSTRAINT

Naoki Yamada

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

March 1986

(Received March 10, 1986)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

THE HAMILTON-JACOBI-BELLMAN EQUATION WITH A GRADIENT CONSTRAINT

Naoki Yamada*

Technical Summary Report #2927

March 1986

ABSTRACT

The existence and uniqueness of solutions for the equation

$$\max\{L^1u - f^1,\dots,L^mu - f^m, |Du| - g\} = 0 \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = 0$$

are considered. Here L^p, $p = 1,\dots,m$ are second order uniformly elliptic operators, Du is the gradient of u and f^p, $p = 1,\dots,m$, g are non-negative functions. We approximate the equation by a system of penalized equations and prove the existence of solutions in the class $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$. The uniqueness of solutions is considered in the class $C^1(\Omega) \cap C(\bar{\Omega})$. Not only in the proof of the uniqueness but also in the existence proof, we use the notion of viscosity solutions.

Moreover we prove the uniqueness in the space $C(\bar{\Omega})$ in the case $m = 1$. We also prove the uniqueness of viscosity solutions of a minimax equation.

AMS (MOS) Subject Classifications: 35J65, 35J60

Key Words: Hamilton-Jacobi-Bellman equations, Obstacle problems, Viscosity solutions, Minimax equations, Variational inequalities.

Work Unit Number 1 - Applied Analysis

*Department of Mathematics, Kobe University, Rokko, Kobe, Japan.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

As the first order Hamilton-Jacobi equation is related to a control problem associated with ordinary differential equations, the Hamilton-Jacobi-Bellman (HJB) equation arises from a control problem with random noise. In the stationary problem, the HJB equation has the form

$$\sup_{\alpha \in A} \{L^{\alpha} u - f^{\alpha}\} = 0$$

where L^{α} are second order linear elliptic operators with parameter $\alpha \in A$.

In this paper, we are concerned with the HJB equation of the form

$$\max\{L^1 u - f^1, \ldots, L^m u - f^m, |Du| - g\} = 0$$

with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. We cannot expect the existence of smooth solutions because of the obstacle term $|Du| - g$. We prove the existence of solutions which satisfy the equation almost everywhere. Using the notion of weak solution (so called viscosity solution) we prove the uniqueness of the solution in the class of continuously differentiable functions.

The method of the uniqueness proof is also applicable to other obstacle problems. We prove uniqueness results in the class of continuous functions for two model problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE HAMILTON-JACOBI-BELLMAN EQUATION WITH A GRADIENT CONSTRAINT

Naoki Yamada*

1. Introduction

In this paper we are concerned with the existence and uniqueness of solutions of the Hamilton-Jacobi-Bellman (HJB) equation with a gradient constraint.

Let L^p, $p = 1,\ldots,m$ be second order linear elliptic operators defined in a bounded domain Ω in R^N. For given non-negative functions f^p, $p = 1,\ldots,m$ and g, we consider the Dirichlet problem

$$\text{(1.1)} \qquad \begin{aligned} &\max\{L^1u - f^1,\ldots,L^mu - f^m, |Du| - g\} = 0 \quad \text{in } \Omega ,\\ &u|_{\partial\Omega} = 0 . \end{aligned}$$

Here Du is the gradient of a function u.

Evans [2] was the first to treat the equation with a gradient constraint in the case $m = 1$ in (1.1). Relaxing the restrictions in [2], Ishii and Koike [9] have proved the existence of solutions in the space $W^{2,\infty}(\Omega)$ and the uniqueness in the class $W^{2,r}_{loc}(\Omega) \cap C(\bar{\Omega})$ with $r > N$.

On the other hand, the HJB equation has been treated by many authors. Using a system of variational inequalities Evans and Friedman [6], Lions [10], and Evans et Lions [7] have proved the existence of solutions in the space $W^{2,\infty}(\Omega)$ for uniformly elliptic HJB equations. Moreover Evans [4], [5] has proved the existence of classical solutions for uniformly elliptic HJB equations (see also Gilbarg and Trudinger [8] Chapter 17). By defining an appropriate notion of weak or viscosity solution, Lions [11] has obtained uniqueness in the space $C(\bar{\Omega})$, with the aid of stochastic representation of solutions. In [11] it is not assumed that the operators are uniformly elliptic, but rather that they contain zero-th order terms with strictly positive coefficients. Note that our equation (1.1) is a non-uniformly elliptic HJB equation without zero-th order term.

---

*Department of Mathematics, Kobe University, Rokko, Kobe, Japan.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

In section 2 we state our assumptions and main result. Since we use a penalty method to prove the existence of solutions, we introduce in section 3 our penalty systems and mention their solvability. In section 4 we establish a priori estimates for approximate solutions. Section 5 is devoted to finishing the proof of our main result. First we obtain existence of solutions in the class $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$. Then uniqueness of solutions is considered in the class $C^1(\Omega) \cap C(\bar{\Omega})$ by comparing an arbitrary viscosity solution with a limit of approximate solutions. Not only in the proof of the uniqueness but also in the existence proof we use the notion of viscosity solutions.

In section 6 we mention two remarks on the uniqueness of viscosity solutions of obstacle problems. First we show the uniqueness of viscosity solutions in the space $C(\bar{\Omega})$ when $m = 1$ in (1.1). Next we consider a minimax equation

$$\min\{\max\{-\Delta u + u - f,\ u - \psi_1\},\ u - \psi_2\} = 0 \quad \text{in } \Omega ,$$

$$u|_{\partial\Omega} = 0$$

and prove the uniqueness of viscosity solutions in the space $C(\bar{\Omega})$. In these proofs we do not use any probabilistic arguments.

This work was completed while the author was visiting the Mathematics Research Center, University of Wisconsin-Madison. The author would like to express his hearty gratitude to Professor M. G. Crandall for his kind advice.

2. Main result

Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$. Consider second order elliptic operators

$$L^p v = -a_{ij}^p v_{x_i x_j} + b_i^p v_{x_i} + c^p v, \quad p = 1,\ldots,m \quad , \tag{2.1}$$

where $m > 1$ is a given integer. We use the summation convention throughout this paper. We also follow normal usage to denote various function spaces such as $C^n(\Omega)$, $W^{n,r}(\Omega)$ or $W^{n,\infty}(\Omega)$ etc. $|Du|$ denotes the size of the gradient of u, i.e. $|Du|^2 = \sum_{i=1}^N u_{x_i}^2$.

We make the following assumptions on L^p:

$$a_{ij}^p \xi_i \xi_j > \theta |\xi|^2 \tag{2.2}$$

for some $\theta > 0$, all $\xi \in R^N$ and $p = 1,\ldots,m$,

$$a_{ij}^p,\ b_i^p,\ c^p \in C^2(\bar{\Omega}) \tag{2.3}$$

for $p = 1,\ldots,m$ and $1 < i,\ j < N$,

$$c^p > c_0 \tag{2.4}$$

for some constant $c_0 > 0$ in Ω, $p = 1,\ldots,m$,

$$a_{ij}^p = a_{ji}^p \tag{2.5}$$

for $p = 1,\ldots,m$, $1 < i,\ j < N$.

On given functions f^p, g on Ω, we impose the following assumptions:

$$f^p,\ g \in C^2(\bar{\Omega}) \tag{2.6}$$

for $p = 1,\ldots,m$,

$$f^p,\ g > 0 \tag{2.7}$$

in Ω for $p = 1,\ldots,m$.

Under these assumptions we may state our main theorem.

Theorem 2.1. (i) Under the assumptions (2.2) - (2.7), there exists a solution $u \in W_{\ell oc}^{2,\infty}(\Omega) \cap W^{1,\infty}(\Omega)$ of the equation

$$\begin{aligned} &\max\{L^1 u - f^1,\ldots,L^m u - f^m,\ |Du| - g\} = 0 \quad \text{a.e. in } \Omega \quad , \\ &u|_{\partial\Omega} = 0 \quad . \end{aligned} \tag{2.8}$$

(ii) If, in addition, $g > 0$ in Ω, then the solution of (2.8) is unique in the class $C^1(\Omega) \cap C(\bar{\Omega})$, where the solution is understood as a viscosity solution satisfying the boundary condition.

3. Approximate systems

In this section we construct approximate systems for (2.8). Let $\psi \in C^\infty(R)$ be a function such that

$$\psi(t) = 0 \text{ if } t < 0, \ \psi(t) = t-1 \text{ if } t > 2 \ ,$$
$$\psi'(t) > 0, \ \psi''(t) > 0 \text{ on } R \ . \tag{3.1}$$

For $\varepsilon > 0$ we put $\beta_\varepsilon(t) = \gamma_\varepsilon(t) = \psi(t/\varepsilon)$. Note that

$$\beta_\varepsilon(t) < t\,\beta_\varepsilon'(t) \text{ on } R. \tag{3.2}$$

We consider the following approximate systems:

$$L^p u_\varepsilon^p + \beta_\varepsilon(|Du_\varepsilon^p|^2 - g^2) + \gamma_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1}) = f^p \text{ in } \Omega \ ,$$
$$u_\varepsilon^p|_{\partial\Omega} = 0, \ p = 1,\dots,m, \text{ where } u_\varepsilon^{m+1} = u_\varepsilon^1 \ . \tag{3.3}$$

To prove the existence of solutions of (3.3), we use the method of successive approximation. In the sequel we omit the subscript ε for simplicity.

Define approximate solutions $u_{(n)}^p$, $p = 1,\dots,m$, $n = 1,2,\dots,$ for (3.3) by induction on n as follows:

First, let $u_{(1)}^p = 0$ for $p = 1,\dots,m$. If $u_{(n-1)}^p$, $p = 1,\dots,m$, have been determined, then we define $u_{(n)}^p$, $p = 1,\dots,m$, as the solution of

$$L^p u_{(n)}^p + \beta(|Du_{(n)}^p|^2 - g^2) + \gamma(u_{(n)}^p - u_{(n-1)}^{p+1}) = f^p \text{ in } \Omega \ ,$$
$$u_{(n)}^p|_{\partial\Omega} = 0, \ p = 1,\dots,m, \text{ where } u_{(n-1)}^{m+1} = u_{(n-1)}^1 \ . \tag{3.4}$$

It is known that there exist $u_{(n)}^p \in C^{3,\delta}(\bar\Omega)$, $p = 1,\dots,m$, for some $\delta \in (0,1)$ which solve (3.4).

If we see

$$\|u_{(n)}^p\|_{W^{1,\infty}(\Omega)} < \text{const} \tag{3.5}$$

for $p = 1,\dots,m$ and uniformly in n, then existence of the solution $u_\varepsilon^p \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ $(1 < r < \infty)$ of (3.3) follows from linear elliptic theory. Moreover, by applying regularity results for elliptic equations, we have $u_\varepsilon^p \in C^{2,\delta}(\bar\Omega)$, $p = 1,\dots,m$.

We defer the proof of (3.5) to the next section since its proof is quite similar to that for a priori estimates independent of ε.

4. A priori estimates

In this section we shall derive some a priori estimates for solutions u_ε^p, $p = 1,\dots,m$, of (3.3) which are independent of $\varepsilon > 0$. We always assume (2.2) - (2.7).

Lemma 4.1. We have

$$0 < u_\varepsilon^p < C \quad \text{in } \Omega \quad , \tag{4.1}$$

$$0 < \frac{\partial u_\varepsilon^p}{\partial n} < C \quad \text{on } \partial\Omega \quad . \tag{4.2}$$

Here and hereafter capital C denotes various constants depending on known constants and $\partial/\partial n$ denotes the inward normal derivative on $\partial\Omega$.

Proof. Let $w^p \in C^2(\bar{\Omega})$ be the solution of

$$\begin{aligned} L^p w^p &= f^p \quad \text{in } \Omega \quad , \\ w^p|_{\partial\Omega} &= 0 \quad . \end{aligned} \tag{4.3}$$

Since $L^p u_\varepsilon^p < f^p$, $u_\varepsilon^p|_{\partial\Omega} = 0$, applying the comparison theorem, we have $u_\varepsilon^p < w^p$ in $\bar{\Omega}$.

Let $x_0 \in \bar{\Omega}$, p_0 be such that

$$u_\varepsilon^{p_0}(x_0) = \min_{\substack{x\in\bar{\Omega} \\ p=1,\dots,m}} u_\varepsilon^p(x) \quad . \tag{4.4}$$

We shall see $u_\varepsilon^{p_0}(x_0) > 0$. We suppress the sub and superscripts ε, p_0 and denote $v = u_\varepsilon^{p_0+1}$. First, consider the case $x_0 \in \Omega$. In this case we have $Du(x_0) = 0$ and $u(x_0) < v(x_0)$. Hence, we have applying the maximum principle,

$$\begin{aligned} 0 &> -a_{ij}u_{x_i x_j}(x_0) \\ &= f(x_0) - \beta(|Du(x_0)|^2 - g(x_0)^2) - \gamma(u(x_0) - v(x_0)) \\ &\quad - b_i(x_0)u_{x_i}(x_0) - c(x_0)u(x_0) \\ &= f(x_0) - c(x_0)u(x_0) \quad . \end{aligned}$$

From (2.4) and (2.7), we get $u(x_0) > 0$.

In the case $x_0 \in \partial\Omega$, it is obvious from the boundary condition that $0 = u(x_0) \leq u(x)$. Therefore we have shown (4.1), and (4.2) is a consequence of (4.1).

Remark. To get $L^\infty(\Omega)$ and $W^{1,\infty}(\partial\Omega)$ estimates of the successive approximate solutions $u^p_{(n)}$, we need only make a minor modification. The upper bound $u^p_{(n)} \leq w^p(x)$ still holds. We shall see $0 \leq u^p_{(n)}(x)$ by induction. It is obvious in the case $n = 1$. Assume that $0 \leq u^p_{(n-1)}(x)$ for all p and $x \in \bar{\Omega}$. Let $x_0 \in \bar{\Omega}$, p_0 be such that (4.4) holds where u^p_ε is replaced by $u^p_{(n)}$. If $u^{p_0}_{(n)}(x_0) < 0$, then $x_0 \in \Omega$ and $\gamma_\varepsilon(u^{p_0}_{(n)}(x_0) - u^{p_0+1}_{(n-1)}(x_0)) = 0$. Therefore we have by the same calculation at x_0 as in the proof of Lemma 4.1,

$$0 > f^{p_0}(x_0) - c^{p_0}(x_0)u^{p_0}_{(n)}(x_0) \quad .$$

Since this contradicts the assumptions (2.4) and (2.7), we get $0 \leq u^p_{(n)}(x)$.

In the following we write $u_i, u_{ij}, a_{ij,k}, \ldots$ for $u_{x_i}, u_{x_i x_j}, a_{ij x_k}, \ldots$.

Lemma 4.2. We have

$$\|u^p_\varepsilon\|_{W^{1,\infty}(\Omega)} \leq C \quad . \tag{4.5}$$

Proof. Consider the function

$$w^p_\varepsilon(x) = |Du^p_\varepsilon|^2 - \lambda u^p_\varepsilon \tag{4.6}$$

where $\lambda > 0$ is a constant to be selected later on. Let $x_0 \in \bar{\Omega}$, p_0 be such that

$$w^{p_0}_\varepsilon(x_0) = \max_{\substack{x\in\bar{\Omega} \\ p=1,\ldots,m}} w^p_\varepsilon(x) \quad . \tag{4.7}$$

We again suppress the sub and superscripts ε, p_0 and denote $v = u^{p_0+1}_\varepsilon$, $\beta(\cdot) = \beta(|Du|^2 - g^2)$, $\gamma(\cdot) = \gamma(u-v)$ and etc.

First consider the case $x_0 \in \Omega$. In this case we have by the maximum principle,

$$\begin{aligned} 0 &\leq -a_{ij}w_{ij} \\ &= -2a_{ij}u_{ki}u_{kj} - 2a_{ij}u_{kij}u_k + \lambda\, a_{ij}u_{ij} \\ &= -2a_{ij}u_{ki}u_{kj} - 2u_k\beta'(\cdot)(2u_{k\ell}u_\ell - (g^2)_k) \\ &\quad - 2u_k\gamma'(\cdot)(u_k - v_k) + 2u_k(\tilde{D}^2u + f_k) \\ &\quad + \lambda\beta(\cdot) + \lambda\gamma(\cdot) + \lambda(\tilde{D}^1u - f) \quad \text{at} \quad x_0 \quad . \end{aligned}$$

Here $$\tilde{D}^2 u = a_{ij,k} u_{ij} - b_i u_{ki} - b_{i,k} u_i - c u_k - c_k u \quad ,$$

$$\tilde{D}^1 u = b_i u_i + cu$$

where we have used (3.3) and differentiated it once.

Therefore we get

$$0 \le -2\theta |D^2 u|^2 - \beta'(\cdot)\ [4\, u_{k\ell} u_k u_\ell - 2u_k (g^2)_k]$$

$$- \gamma'(\cdot)\ [2u_k (u_k - v_k)] + C|Du| + C|Du|\ |D^2 u|$$

$$+ \lambda[\tilde{D}^1 u - f + \beta(\cdot) + \gamma(\cdot)] \quad .$$

Continuing the calculation, we obtain

$$0 \le -\theta |D^2 u|^2 + C|Du|^2 + \lambda C(|Du| + 1)$$

$$- \beta'(\cdot)\ [4\, u_{k\ell} u_k u_\ell - 2u_k (g^2)_k - \lambda(u_\ell u_\ell - g^2)] \tag{4.8}$$

$$- \gamma'(\cdot)\ [2u_k (u_k - v_k) - \lambda(u - v)] \quad ,$$

where we have used (3.2). Since

$$2u_k (u_k - v_k) - \lambda(u - v) \ge w_\varepsilon^{p_0}(x_0) - w_\varepsilon^{p_0+1}(x_0) \ge 0 \quad ,$$

we have

$$-\gamma'(\cdot)\ [2u_k (u_k - v_k) - \lambda(u - v)] \le 0 \quad . \tag{4.9}$$

On the other hand, since $w_\ell(x_0) = 0$ we get

$$4\, u_{k\ell} u_k u_\ell = 2\lambda u_\ell u_\ell \quad .$$

Therefore we have

$$-\beta'(\cdot)\ [4u_{k\ell} u_k u_\ell - 2u_k (g^2)_k - \lambda(u_\ell u_\ell - g^2)]$$

$$= -\beta'(\cdot)\ [\lambda |Du|^2 - 2u_k (g^2)_k + \lambda g^2] \quad . \tag{4.10}$$

Combining, we get

$$0 \le C|Du|^2 + \lambda C(|Du| + 1)$$

$$-\beta'(\cdot)\ [\lambda |Du|^2 - 2u_k (g^2)_k + \lambda g^2] \quad .$$

We may assume that $\beta'(\cdot) > 1$ at x_0, because otherwise we can immediately derive a bound for $|Du|^2$ and therefore for w. Thus we have

$$0 < (C-\lambda)|Du|^2 + \lambda C(|Du|+1) + 2u_k(g^2)_k - \lambda g^2 \quad . \tag{4.11}$$

Now we choose λ large enough to obtain a bound for $|Du|^2$ and therefore for w at x_0.

Next consider the case $x_0 \in \partial\Omega$. In this case, we get a bound for w at once from $W^{1,\infty}(\partial\Omega)$ estimate.

Remark. To obtain a $W^{1,\infty}(\Omega)$ bound for the successive approximate solutions we proceed as follows. Consider the function $w^p_{(n)}$ defined by (4.6) where u^p_ε is replaced by $u^p_{(n)}$ and let p_0, x_0 be as in (4.7). We may assume $x_0 \in \Omega$. We shall show the boundedness of $\|Du^p_{(n)}\|_{L^\infty(\Omega)}$ by induction. By the same calculation in Lemma 4.2 we have (4.8). By the induction assumption we may assume $w^{p_0}_{(n)}(x_0) > w^{p_0+1}_{(n-1)}(x_0)$. Then we have (4.9). (4.10) is also satisfied in this case. Hence we get (4.11), with constants C independent of n.

Lemma 4.3. We have

$$\|u^p_\varepsilon\|_{W^{2,\infty}_{loc}(\Omega)} < C \quad . \tag{4.12}$$

Proof. Let ζ be a function in $C^\infty_0(\Omega)$ such that $0 < \zeta < 1$. We shall derive a bound of

$$K = \max_{\substack{x\in\overline{\Omega} \\ p=1,\ldots,m}} \zeta(x)|D^2u^p_\varepsilon(x)| \quad .$$

Without loss of generality, we may assume that $K > 1$. Let $x_0 \in \Omega$, p_0 be such that $K = \zeta(x_0)|D^2u^{p_0}_\varepsilon(x_0)|$ and consider the function

$$w^p_\varepsilon(x) = \zeta^2(x)|D^2u^p_\varepsilon(x)|^2 + \lambda K\zeta(x)a^{p_0}_{k\ell}(x_0)u^p_{\varepsilon,k\ell}(x) + \mu|Du^p_\varepsilon(x)|^2$$

where $\lambda, \mu > 1$ are constants to be selected later on. Let $x_1 \in \overline{\Omega}$, p_1 be such that

$$w^{p_1}_\varepsilon(x_1) = \max_{\substack{x\in\overline{\Omega} \\ p=1,\ldots,m}} w^p_\varepsilon(x) \quad .$$

We may assume that $x_1 \in \Omega$.

In the following we suppress the sub and superscripts ε, p_1 and denote $v = u^{p_1+1}_\varepsilon$, $a_{k\ell} = a^{p_0}_{k\ell}(x_0)$, $\beta(\cdot) = \beta(|Du|^2 - g^2)$, $\gamma(\cdot) = \gamma(u - v)$ and etc. Using the maximum principle and the differentiated equation of (3.3), we have at x_1

$$0 \leq -\zeta^2 a_{ij} w_{ij}$$

$$
\begin{aligned}
(4.13)\qquad &= -2\zeta^4 a_{ij} u_{k\ell i} u_{k\ell j} - 2\mu\zeta^2 a_{ij} u_{ki} u_{kj} \\
&\quad -\zeta^2(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})\{\beta''(\cdot)(|Du|^2 - g^2)_k(|Du|^2 - g^2)_\ell + \gamma''(\cdot)(u-v)_k(u-v)_\ell\} \\
&\quad -\zeta^2\beta'(\cdot)\{(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})(|Du|^2 - g^2)_{k\ell} + 2\mu u_k(|Du|^2 - g^2)_k\} \\
&\quad -\zeta^2\gamma'(\cdot)\{(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})(u-v)_{k\ell} + 2\mu u_k(u-v)_k\} \\
&\quad +\zeta^2(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})(\tilde{D}^3 u + f_{k\ell}) + 2\mu\zeta^2 u_k(\tilde{D}^2 u + f_k) \\
&\quad -4\zeta^2(\zeta^2)_i a_{ij} u_{k\ell} u_{k\ell j} - 2\lambda K\zeta^2\zeta_i \alpha_{k\ell} a_{ij} u_{k\ell j} \\
&\quad -\zeta^2(\zeta^2)_{ij} a_{ij} u_{k\ell} u_{k\ell} - \lambda K\zeta^2\zeta_{ij}\alpha_{k\ell} a_{ij} u_{k\ell} \quad .
\end{aligned}
$$

Here $\tilde{D}^i$ is an i-th order differential operator.

If we choose λ such that $\lambda > 2N^2/\theta$, then we have

$$(4.14)\qquad (2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})\xi_k\xi_\ell \geq (\lambda K\zeta\theta - 2\zeta^2|u_{k\ell}|)|\xi|^2 \geq 0$$

for any $\xi \in R^N$.

On the other hand, since $w_i = 0$ at x_1 we have

$$
\begin{aligned}
&(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})(|Du|^2 - g^2)_{k\ell} + 2\mu u_k(|Du|^2 - g^2)_k \\
(4.15)\qquad &= -2(\zeta^2)_i u_{k\ell} u_{k\ell} u_i - 2\lambda K\zeta_i\alpha_{k\ell} u_{k\ell} u_i + 4\zeta^2 u_{k\ell} u_{ki} u_{\ell i} \\
&\quad +2\lambda K\zeta\alpha_{k\ell} u_{ki} u_{\ell i} - 2\zeta^2(g^2)_{k\ell} u_{k\ell} - \lambda K\zeta(g^2)_{k\ell}\alpha_{k\ell} - 2\mu(g^2)_k u_k \\
&\geq K\{2(\lambda\theta - 2)\zeta|D^2u|^2 - \lambda C|D^2u| - (\lambda + \mu)C\} \quad .
\end{aligned}
$$

We also have

$$
\begin{aligned}
&(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})(u-v)_{k\ell} + 2\mu u_k(u-v)_k \\
(4.16)\qquad &\geq w_\varepsilon^{p_1}(x_1) - w_\varepsilon^{p_1+1}(x_1) \geq 0 \quad .
\end{aligned}
$$

The last six terms in (4.13) are estimated as follows:

$$\zeta^2(2\zeta^2 u_{k\ell} + \lambda K\zeta\alpha_{k\ell})(\tilde{D}^3 u + f_{k\ell}) + 2\mu\zeta^2 u_k(\tilde{D}^2 u + f_k)$$

$$- 4\zeta^2(\zeta^2)_i a_{ij} u_{k\ell} u_{k\ell j} - 2\lambda K\zeta^2 \zeta_i \alpha_{k\ell} a_{ij} u_{k\ell j}$$

(4.17)

$$- \zeta^2(\zeta^2)_{ij} a_{ij} u_{k\ell} u_{k\ell} - \lambda K\zeta^2 \zeta_{ij} \alpha_{k\ell} a_{ij} u_{k\ell}$$

$$\leq \theta\zeta^4 |D^3 u|^2 + \lambda^2 CK^2 + \lambda CK + \mu^2 C \quad .$$

Substituting (4.14) - (4.17) into (4.13) and using (2.2) we have

$$0 \leq -2\mu\theta\zeta^2 |D^2 u|^2 + \lambda^2 CK^2 + \lambda CK + \mu^2 C$$

(4.18)

$$- \zeta^2 K\beta'(\cdot)\{2(\lambda\theta - 2)\zeta |D^2 u|^2 - \lambda C|D^2 u| - (\lambda+\mu)C\}$$

at x_1 where $\lambda > 2N^2/\theta$.

First consider the case

$$2\mu\theta\zeta^2 |D^2 u|^2 \leq \lambda^2 CK^2 + \lambda CK + \mu^2 C \quad \text{at} \quad x_1 \quad .$$

Then we have

$$w_\varepsilon^{p_1}(x_1) \leq \lambda^2 C\zeta^2(x_1)|D^2 u_\varepsilon^{p_1}(x_1)|^2 + \frac{1}{4}K^2 + \mu C$$

$$\leq \left(\frac{\lambda^4 C}{2\mu\theta} + \frac{1}{4}\right)K^2 + \frac{\lambda^3 C}{2\mu\theta} K + \frac{\lambda^2 \mu}{2\theta} C + \mu C \quad .$$

If we choose μ so large that $4\lambda^4 C \leq 2\mu\theta$, then we have

$$w_\varepsilon^{p_1}(x_1) \leq \frac{1}{2}K^2 + CK + C \quad .$$

On the other hand since

$$\alpha_{k\ell} u_{k\ell}^{p_0}(x_0) \geq b_k^{p_0}(x_0) u_k^{p_0}(x_0) + c^{p_0}(x_0) u^{p_0}(x_0) - f^{p_0}(x_0)$$

$$\geq -C \quad ,$$

we have

(4.19)
$$w_\varepsilon^{p_0}(x_0) \geq K^2 - \lambda CK - \mu C \quad .$$

Therefore we get

(4.20) $$\frac{1}{2} K^2 \leq CK + C$$

which implies the boundedness of K.

Next consider the case

$$2(\lambda\theta - 2)\zeta|D^2u|^2 \leq \lambda C|D^2u| + (\lambda + \mu)C \quad \text{at} \quad x_1 \quad .$$

If we choose λ satisfying $1 \leq 2(\lambda\theta - 2)$, then we have

$$\zeta^2|D^2u|^2 \leq \lambda C\zeta|D^2u| + (\lambda + \mu)C \quad \text{at} \quad x_1$$

which implies the boundedness of $\zeta(x_1)|D^2u_\varepsilon^{p_1}(x_1)|$. Hence we have

$$w_\varepsilon^{p_1}(x_1) \leq CK + C \quad .$$

By (4.19), we also have (4.20) in this case. Therefore the proof is completed.

5. Proof of the main result

In this section we shall prove the existence of solutions in $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$ and the uniqueness of the viscosity solution in the class $C^1(\Omega) \cap C(\bar{\Omega})$, which completes the proof of Theorem 2.1.

Lemma 5.1. There exists a solution u of (2.8) belonging to $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$.

Proof. From a priori estimates in the preceding section, we can choose a sequence ε_j (which we simply denote ε) such that

$$u^p_\varepsilon \to u^p, \; Du^p_\varepsilon \to Du^p \text{ uniformly in } \bar{\Omega} \; ,$$

(5.1)

$$D^2u^p_\varepsilon \to D^2u^p \text{ weakly in } L^r_{\ell oc}(\Omega) \text{ with } r < \infty \; .$$

Since $\gamma_\varepsilon(u^p_\varepsilon - u^{p+1}_\varepsilon)$ are locally bounded, it follows that the u^p defined in (5.1) satisfy $u^1 = \cdots = u^m \equiv u \in W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$. We shall prove that u solves (2.8).

First we note that $L^pu^p_\varepsilon - f^p \le 0$ a.e. in Ω. Hence we have $L^pu - f^p \le 0$ a.e. in Ω, $p = 1,\ldots,m$. Since $\beta_\varepsilon(|Du^p_\varepsilon|^2 - g^2)$ are also locally bounded, we get

$$\max\{L^1u - f^1,\ldots,L^mu - f^m, |Du| - g\} \le 0 \text{ a.e. in } \Omega \; . \tag{5.2}$$

To prove the inequality in the opposite direction, it is sufficient to show that u is a viscosity supersolution of (2.8). Let $\varphi \in C^2(\Omega)$ and assume that $u - \varphi$ takes its local strict minimum at $x_0 \in \Omega$. We shall show

$$\max_{p=1,\ldots,m} \{-a^p_{ij}\varphi_{ij} + b^p_i\varphi_i + c^pu - f^p, |D\varphi| - g\} \ge 0 \text{ at } x_0 \; . \tag{5.3}$$

Since $|D\varphi(x_0)| \ge g(x_0)$ implies (5.3), we may assume $|D\varphi(x_0)| < g(x_0)$. Since u^p_ε converges to u uniformly, there exists a sequence $\{x^p_\varepsilon\} \subset \Omega$ such that

(i) $\lim_{\varepsilon\to 0} x^p_\varepsilon = x_0$ for any $p = 1,\ldots,m$,

(ii) $u^p_\varepsilon - \varphi$ attains its local minimum at x^p_ε ,

(iii) $|D\varphi(x^p_\varepsilon)| < g(x^p_\varepsilon)$.

For each ε, let $p(\varepsilon)$ be such that

$$(u^{p(\varepsilon)}_\varepsilon - \varphi)(x^{p(\varepsilon)}_\varepsilon) = \min_{p=1,\ldots m} (u^p_\varepsilon - \varphi)(x^p_\varepsilon) \; . \tag{5.4}$$

Since p varies in a finite set there exists $\bar{p}$ which appears infinitely many times in (5.4). Consider such $\bar{p}$ and ε such that $p(\varepsilon) = \bar{p}$. Then we have $\beta_\varepsilon(|Du_\varepsilon^{\bar{p}}|^2 - g^2) = 0$ and $\gamma_\varepsilon(u_\varepsilon^{\bar{p}} - u_\varepsilon^{\bar{p}+1}) = 0$ at $x_\varepsilon^{\bar{p}}$. Since $u_\varepsilon^{\bar{p}}$ is also a viscosity supersolution of (3.3), we get

$$-a_{ij}^{\bar{p}}\varphi_{ij} + b_i^{\bar{p}}\varphi_i + c^{\bar{p}}u_\varepsilon^{\bar{p}} \geq f^{\bar{p}} \quad \text{at} \quad x_\varepsilon^{\bar{p}} .$$

Passing to the limit as $\varepsilon \to 0$, along which we take $\bar{p} = p(\varepsilon)$, we have (5.3).

Lemma 5.2. Assume $g > 0$ in Ω. Then the viscosity solution of (2.8) is unique in the class $C^1(\Omega) \cap C(\bar{\Omega})$.

Proof. By Lemma 5.1 we have a solution u belonging to $W_{loc}^{2,\infty}(\Omega) \cap W^{1,\infty}(\Omega)$ and approximate solutions u_ε^p which converge to u along a subsequence. In the following we fix such a u and convergent approximate solutions $u_{\varepsilon_j}^p$ (simply we denote u_ε^p).

Let v be any viscosity solution of (2.8) which belongs to $C^1(\Omega) \cap C(\bar{\Omega})$.

First we claim that $v \leq u$ in Ω. If not, there exist $x_0 \in \Omega$ and p_0 such that

$$(v - u_\varepsilon^{p_0})(x_0) = \max_{\substack{x\in\bar{\Omega} \\ p=1,\dots,m}} (v - u_\varepsilon^p)(x) > 0 . \tag{5.5}$$

Since v is a viscosity subsolution, we have

$$\begin{aligned} & - a_{ij}^{p_0}u_{\varepsilon,ij}^{p_0} + b_i^{p_0}u_{\varepsilon,i}^{p_0} + c^{p_0}v \leq f^{p_0} \quad \text{at} \quad x_0 , \\ & |Du_\varepsilon^{p_0}(x_0)| \leq g(x_0) . \end{aligned} \tag{5.6}$$

The second inequality in (5.6) implies $\beta_\varepsilon(|Du_\varepsilon^{p_0}|^2 - g^2) = 0$ at x_0 and (5.5) implies $\gamma_\varepsilon(u_\varepsilon^{p_0} - u_\varepsilon^{p_0+1}) = 0$ at x_0. Hence we have

$$- a_{ij}^{p_0}u_{\varepsilon,ij}^{p_0} + b_i^{p_0}u_{\varepsilon,i}^{p_0} + c^{p_0}u_\varepsilon^{p_0} = f^{p_0} \quad \text{at} \quad x_0 .$$

Subtracting this from the first inequality in (5.6), we get

$$c^{p_0}(x_0)(v - u_\varepsilon^{p_0})(x_0) \leq 0 ,$$

which is a contradiction.

Next we shall show that $\rho u \leq v$ in Ω for $0 < \rho < 1$. If not, there exist $\rho \in (0,1)$ and $x_0 \in \Omega$ such that

$$(v - \rho u)(x_0) = \min_{x \in \overline{\Omega}} (v - \rho u) < 0 \quad . \tag{5.7}$$

Since $v \in C^1(\Omega)$ we have $|Dv(x_0)| = \rho|Du(x_0)| < g(x_0)$. Then there exists a ball U with center x_0 satisfying

$$|Dv| < g \quad \text{in} \quad U \quad . \tag{5.8}$$

This implies that v is a viscosity supersolution of

$$\max_{p=1,\ldots,m} \{L^p v - f^p\} = 0 \quad \text{in} \quad U \quad . \tag{5.9}$$

Consequently v is a viscosity solution of (5.9) in U. Considering (5.9) with boundary condition $\phi = u|_{\partial U}$, it is known (Evans [4], [5], Gilbarg and Trudinger [8] Chapter 17) that (5.9) has a smooth solution. On the other hand it is also known (Lions [11]) that the viscosity solution of (5.9) is unique. Therefore we can conclude that v is the smooth solution of (5.9) in U.

By a selection lemma, there exists a measurable function $p : U \to \{1,\ldots,m\}$ such that

$$L^{p(x)}v - f^{p(x)} = 0 \text{ a.e. in } U \quad .$$

Since u is a subsolution of (5.9) we have

$$L^{p(x)}(v - \rho u)(x) - (1-\rho)f^{p(x)} \geq 0 \text{ a.e. in } U \quad . \tag{5.10}$$

On the other hand by Bony's maximum principle we get

$$\lim_{x \to x_0} \operatorname{ess\,inf} \, (-a_{ij}^{p(x)}(v-\rho u)_{ij} + b_i^{p(x)}(v-\rho u)_i) \leq 0 \quad . \tag{5.11}$$

Combining (5.10) and (5.11) we have

$$c^{p(x_0)}(x_0)(v-\rho u)(x_0) - (1-\rho)f^{p(x_0)}(x_0) \geq 0$$

which contradicts (5.7).

Since ρ is arbitrary in $(0,1)$ we have $v \equiv u$ in Ω. This completes the proof.

By Lemmas 5.1 and 5.2 we have completed the proof of Theorem 2.1.

Since Lemma 5.2 asserts that any viscosity solution in $C^1(\Omega) \cap C(\bar{\Omega})$ can be approximated by solutions u_ε^p of the approximate system, we have the following comparison result.

Corollary 5.3. Let $u \in C(\bar{\Omega})$ be a viscosity solution of

$$\tag{2.8} \begin{aligned} &\max_{p=1,\ldots,m} \{L^p u - f^p, |Du| - g\} = 0 \quad \text{in } \Omega \ , \\ &u|_{\partial\Omega} = 0 \end{aligned}$$

and $\tilde{u} \in C^1(\Omega) \cap C(\bar{\Omega})$ be a viscosity solution of

$$\tag{2.$\tilde{8}$} \begin{aligned} &\max_{p=1,\ldots,m} \{L^p\tilde{u} - \tilde{f}^p, |D\tilde{u}| - \tilde{g}\} = 0 \quad \text{in } \Omega \ , \\ &\tilde{u}|_{\partial\Omega} = 0 \ . \end{aligned}$$

If $0 \le f^p \le \tilde{f}^p$, $p = 1,\ldots,m$ and $0 \le g \le \tilde{g}$ in Ω, then $u \le \tilde{u}$ in Ω.

6. Remarks on the uniqueness of viscosity solution for obstacle problems.

In this section we show that the uniqueness of viscosity solutions for some other classes of obstacle problems can be proved by the same method as in the previous sections. To avoid needless repetition, we consider the modeled operator $Lu = -\Delta u + u$. Note that the following proofs are based on the convergence of approximate solutions of penalized equations and do not use any probabilistic arguments.

6.1. The case $m = 1$.

Consider the equation

$$\begin{aligned} &\max\{-\Delta u + u - f,\ |Du| - g\} = 0 \text{ a.e. in } \Omega \ , \\ &u|_{\partial\Omega} = 0 \end{aligned} \tag{6.1}$$

where f and g are smooth and $f > 0$, $g > 0$ in Ω. This equation has been considered in Evans [2], [3] and Ishii and Koike [9]. They proved the uniqueness of solutions in the class $W^{2,r}_{\ell oc}(\Omega) \cap C(\bar{\Omega})$ with $r > N$. We proved in Lemma 5.2 that for more general equations the viscosity solution of (6.1) is unique in the class $C^1(\Omega) \cap C(\bar{\Omega})$. Moreover in this case we have the uniqueness in the space $C(\bar{\Omega})$.

Theorem 6.1. The viscosity solution of (6.1) is unique in the class $C(\bar{\Omega})$.

Proof. Let u_ε be a solution of the approximate equation

$$\begin{aligned} &-\Delta u_\varepsilon + u_\varepsilon + \beta_\varepsilon(|Du_\varepsilon|^2 - g^2) = f \text{ in } \Omega \ , \\ &u_\varepsilon|_{\partial\Omega} = 0 \ . \end{aligned} \tag{6.2}$$

By the same argument as in sections 3, 4 and Lemma 5.1 (or by Evans [2]), there exist $u \in W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$ and a subsequence u_{ε_j} (simply we denote $\varepsilon_j = \varepsilon$) which converges to u as in (5.1).

Let $v \in C(\bar{\Omega})$ be any viscosity solution. By the same argument as in the first claim in Lemma 5.2 we have $v \le u$ in Ω.

We shall show that $\rho u \le v$ in Ω for $0 < \rho < 1$. If not, we can find $\rho \in (0,1)$, x_{ε_j} (again we denote $\varepsilon_j = \varepsilon$) and $x_0 \in \Omega$ such that

$$\begin{aligned}
&\text{(i)} \quad x_\varepsilon \to x_0 \quad \text{as} \quad \varepsilon \to 0 \ , \\
(6.3) \quad &\text{(ii)} \quad (v - \rho u_\varepsilon)(x_\varepsilon) = \min_{x \in \overline{\Omega}} (v - \rho u_\varepsilon)(x) \ , \\
&\text{(iii)} \quad (v - \rho u)(x_0) = \min_{x \in \overline{\Omega}} (v - \rho u)(x) < 0 \ .
\end{aligned}$$

Since $\rho|Du(x_0)| < g(x_0)$ we have $\rho|Du_\varepsilon(x_\varepsilon)| < g(x_\varepsilon)$ for small ε. Since v is a viscosity supersolution, this implies

$$-\rho\Delta u_\varepsilon + v \geq f \quad \text{at} \quad x_\varepsilon \ . \tag{6.4}$$

From (6.2) we have

$$-\rho\Delta u_\varepsilon + \rho u_\varepsilon \leq \rho f \quad \text{in} \quad \Omega \ .$$

Subtracting this from (6.4) and letting $\varepsilon \to 0$ we have

$$(v-\rho u)(x_0) - (1-\rho)f(x_0) \geq 0 \ .$$

This contradicts (6.3 - iii).

6.2. Variational inequalities

Consider a minimax equation

$$\begin{aligned}
&\min\{\max\{-\Delta u + u - f, u - \psi_1\}, u - \psi_2\} = 0 \text{ a.e. in } \Omega \ , \\
(6.5) \quad & \\
&u|_{\partial\Omega} = 0
\end{aligned}$$

where f, ψ_1 and ψ_2 are smooth functions satisfying $\psi_2 \leq \psi_1$ in Ω and $\psi_2|_{\partial\Omega} < 0 < \psi_1|_{\partial\Omega}$.

It is convenient to formulate the notion of viscosity solutions of (6.5) in the following manner. We say that $u \in C(\overline{\Omega})$ is a viscosity solution of (6.5) if both (0) and (1) hold:

(0) $\psi_2 \leq u \leq \psi_1$ in Ω, $u|_{\partial\Omega} = 0$.

(1) Let $\varphi \in C^2(\Omega)$

(6.6) (i) if $u - \varphi$ attains its local maximum at $x_0 \in \Omega$ and $\psi_2(x_0) < u(x_0)$, then $-\Delta\varphi + u \leq f$ at x_0 ,

(ii) if $u - \varphi$ attains its local minimum at $x_0 \in \Omega$ and $u(x_0) < \psi_1(x_0)$, then $-\Delta\varphi + u \geq f$ at x_0.

Since (6.5) is equivalent to the following variational inequalities with bilateral constraints:

$$\psi_2 \le u \le \psi_1 \quad \text{in} \quad \Omega,\ u|_{\partial\Omega} = 0 \ ,$$
$$-\Delta u + u = f \quad \text{on} \quad \{x \in \Omega \mid \psi_2 < u < \psi_1\} \ ,$$
$$-\Delta u + u \ge f \quad \text{on} \quad \{x \in \Omega \mid u = \psi_2\} \ ,$$
$$-\Delta u + u \le f \quad \text{on} \quad \{x \in \Omega \mid u = \psi_1\} \ ,$$

it is known (Bensoussan et Lions [1], Chapter 3, Section 5) that there exists a solution $u \in W^{2,r}(\Omega)$ with $r > N$ which is a limit of solutions u_ε of the penalized equation

$$\text{(6.7)} \qquad \begin{aligned} &-\Delta u_\varepsilon + u_\varepsilon + \beta_\varepsilon(u_\varepsilon - \psi_1) - \beta_\varepsilon(\psi_2 - u_\varepsilon) = f \quad \text{in} \quad \Omega \ , \\ &u_\varepsilon|_{\partial\Omega} = 0 \ . \end{aligned}$$

Theorem 6.2. Let $v \in C(\overline{\Omega})$ be a viscosity solution of (6.5). Then we have $u = v$ in $\overline{\Omega}$.

Proof. We prove only $v \le u$ in Ω because the inequality in the opposite direction can be proved similarly.

If $v \le u$ in Ω does not hold, there exist x_{ε_j} (again we denote $\varepsilon_j = \varepsilon$) and $x_0 \in \Omega$ such that

$$\text{(6.8)} \qquad \begin{aligned} &\text{(i)} \quad x_\varepsilon \to x_0 \quad \text{as} \quad \varepsilon \to 0 \ , \\ &\text{(ii)} \quad (v - u_\varepsilon)(x_\varepsilon) = \max_{x \in \overline{\Omega}} (v - u_\varepsilon)(x) \ , \\ &\text{(iii)} \quad (v - u)(x_0) = \max_{x \in \overline{\Omega}} (v - u)(x) > 0 \ . \end{aligned}$$

Since $\psi_2 \le u < v$ at x_0, we have $\psi_2 < v$ and $u_\varepsilon < v \le \psi_1$ near x_0. Then (6.6 - i) implies

$$\text{(6.9)} \qquad -\Delta u_\varepsilon + v \le f \quad \text{at} \quad x_\varepsilon \ .$$

We also have $\beta_\varepsilon(u_\varepsilon - \psi_1) = 0$. Hence from (6.7) we get

$$\text{(6.10)} \qquad -\Delta u_\varepsilon + u_\varepsilon \ge f \quad \text{at} \quad x_\varepsilon \ .$$

Combining (6.9), (6.10) and letting $\varepsilon \to 0$, we have

$$(v - u)(x_0) \le 0$$

which is a contradiction.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2927	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE HAMILTON-JACOBI-BELLMAN EQUATION WITH A GRADIENT CONSTRAINT		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Naoki Yamada		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE March 1986
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hamilton-Jacobi-Bellman equations, Obstacle problems, Viscosity solutions, Minimax equations, Variational inequalities

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The existence and uniqueness of solutions for the equation

$$\max\{L^1u - f^1, \ldots, L^m u - f^m, |Du| - g\} = 0 \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = 0$$

are considered. Here L^p, $p = 1,\ldots,m$ are second order uniformly elliptic

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

operators, Du is the gradient of u and f^p, $p = 1,\ldots,m$, g are non-negative functions. We approximate the equation by a system of penalized equations and prove the existence of solutions in the class $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$. The uniqueness of solutions is considered in the class $C^1(\Omega) \cap C(\overline{\Omega})$. Not only in the proof of the uniqueness but also in the existence proof, we use the notion of viscosity solutions.

Moreover we prove the uniqueness in the space $C(\overline{\Omega})$ in the case $m = 1$. We also prove the uniqueness of viscosity solutions of a minimax equation.

END

DTIC

6-86