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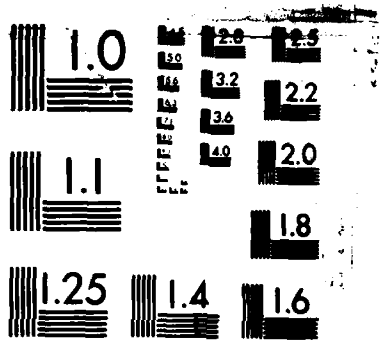

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MINIMAL SUPPORT FOR BIVARIATE SPLINES

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MINIMAL SUPPORT FOR BIVARIATE SPLINES

Carl de Boor[1] and Klaus Höllig[1,2]

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ABSTRACT

Let S denote the space of piecewise polynomials of degree $\leq k$ and smoothness ρ on the regular partition of $\mathbb{R}^2$ which is generated either by the three directions (1,0), (1,1), (0,1) or by the four directions (1,0), (1,1), (0,1), (-1,1). For the choice

$$\rho = \rho(k) := \max\{\rho : \dim S\big|_{[-N,N]^2} \neq o(N^2)\} \quad ,$$

(which is the maximal smoothness for which the space S is nondegenerate), we determine the functions which have minimal support in S. Moreover, we show that these functions form a basis for

$$S(\Omega) := \{f \in S : \operatorname{supp} f \subseteq \Omega\} \quad .$$

AMS (MOS) Subject Classifications: 41A15, 41A63

Key Words: bivariate, splines, minimal support

Work Unit Number 3 - Numerical Analysis and Scientific Computing

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SIGNIFICANCE AND EXPLANATION

In earlier works the authors

~~In [MRC #2415] we~~ applied ~~our~~ their results on box-splines [MRC #2320] to analyze the approximation properties of bivariate smooth piecewise polynomials on the three direction mesh. In this report ~~we~~ they obtain similar results for the other natural triangulation of R^2 which is generated by four directions. In particular ~~we~~ they extend ~~our~~ their results on minimality of support which are useful for constructing bases with good computational properties.

Keywords: numerical analysis

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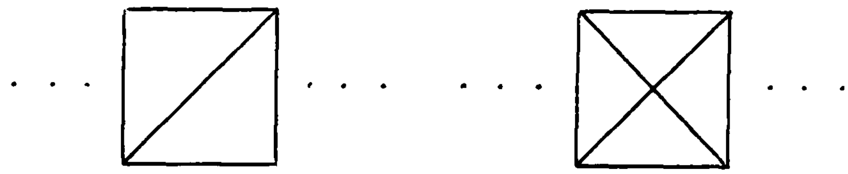
MINIMAL SUPPORT FOR BIVARIATE SPLINES

Carl de Boor[1] and Klaus Höllig[1,2]

1. Introduction and statement of results.

Let $S := \pi^{\rho}_{k,\Delta}$ denote the space of bivariate spline functions of smoothness ρ and (total) degree $\le k$ on a partition Δ of $\mathbb{R}^2$. In this note we determine the spline functions of **minimal support** for the two regular partitions Δ_1, Δ_2 which are generated by the unit vectors e_1, e_2 and their sum and difference $e_1 + e_2$, $e_2 - e_1$.



Δ_1 Δ_2

<Figure 1>

These minimal support elements provide a canonical basis for the subspace of functions in S with compact support. From a practical point of view, small support of basis functions is desirable for finite element approximations and quasi-interpolant schemes.

If the degree k of the spline space $S = \pi^{\rho}_{k,\Delta}$ is large compared to the smoothness ρ, elements of minimal support can be easily constructed using Hermite interpolation. However, in applications one often wants to achieve a certain smoothness with as few parameters as possible. When k is small compared to ρ, the smoothness requirements lead to nonlocal constraints which complicate the analysis. We consider in this note the extreme case of **minimal degree** $k(\rho)$, i.e. the smallest degree k for which the family of

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spaces $S_h := \{f(\cdot/h): f \in S\}$, $h > 0$, is dense in $C_0^\infty(\mathbb{R}^2)$. Obviously, the degree $k(\rho)$ is the most "economical" choice for a given smoothness ρ (if one wants to minimize the local dimension of S). For the two partitions in Figure 1 we have (c.f. [4] for Δ_1 and section 2 for Δ_2)

(1) $$k_\nu(\rho) = \lceil (2+\nu)(\rho+1)/(\nu+1) \rceil \quad , \quad \nu = 1,2,$$

where $\lceil x \rceil := \sup\{n \in \mathbb{Z}: n \leq x\}$. Roughly speaking, the (minimal) degree increases by $2+\nu$ if the smoothness increases by $1+\nu$. The first values of k_ν are listed in the table below.

ρ	-1	0	1	2	3
$k_1(\rho)$	0	1	3	4	6
$k_2(\rho)$	0	1	2	4	5

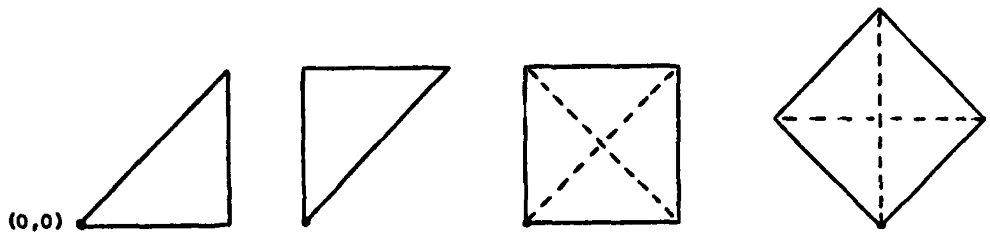
To state our results, we need addition notation. For a set $\Omega \subset \mathbb{R}^2$ we denote by $S(\Omega)$ the subspace of functions in S which have support in Ω. (Note that this differs from $S|_\Omega$, the restrictions of $f \in S$ to Ω.) By span F we denote the linear span of the set F. We say that <u>a function</u> M <u>has (unique) minimal support in</u> S <u>iff</u>

$$\text{span}\ \{f\}\ (=) \subseteq S(\text{supp}\ f)$$

(2) <u>and</u>

$$\Omega \subsetneq \text{supp}\ f \Longrightarrow \dim S(\Omega) = 0 \quad .$$

We write S_ν^ρ as abbreviation for $\pi^\rho_{k_\nu(\rho),\Delta_\nu}$. By $N_{\nu,\mu}$, $\mu = 1,2$, we denote the functions with unique minimal support in $S_\nu^{\nu-2}$, normalized by the condition $\|N\|_\infty = 1$. ($N_{1,\mu}$ is piecewise constant; $N_{2,\mu}$ is piecewise linear).



supp $M_{1,1}$ supp $M_{1,2}$ supp $M_{2,1}$ supp $M_{2,2}$

<Figure 2>

The simplest nontrivial examples of minimal support elements are the "<u>hat</u>"-<u>function</u> $M_1 \in S_1^0$ and the Zwart element [15] $M_2 \in S_2^1$; both functions are normalized to satisfy $\|M\|_\infty = 1$.

supp M_1 supp M_2

<Figure 3>

Further examples can be found in [14]. The element $M_{1,d}$, defined below, appeared in [11]; but the minimality of the support was not proved.

<u>Theorem 1</u>. Let $\nu = 1,2$, $d \in \mathbb{Z}_+$.

(i) The functions

$$M_{\nu,d} := \underbrace{M_\nu * \cdots * M_\nu}_{d\text{-times}}$$

have unique minimal support in

$$S_{\nu,d} := S_\nu^{d(\nu+1)-2} .$$

(ii) The functions

$$N_{\nu,\mu,d} := N_{\nu,\mu} * M_{\nu,d},\ \mu = 1,2,$$

have unique minimal support in

$$\tilde{S}_{\nu,d} := S_\nu^{d(\nu+1)+\nu-2} .$$

Here, $f * g = \int_{\mathbb{R}^2} f(\cdot - y)g(y)dy$ denotes the convolution of two functions f and g. Figure 4 below shows the supports of the minimal support elements.

Theorem 2. For any convex set $\Omega \subset \mathbb{R}^2$, the integer translates of the functions $M_{\nu,d}$ and $N_{\nu,\mu,d}$ with support entirely in Ω form a basis for the spaces $S_{\nu,d}(\Omega)$ and $\tilde{S}_{\nu,d}(\Omega)$ respectively.

We have not completed our investigations for the spaces S_2^ρ, $\rho = 2 \bmod 3$. One would expect that convolution of M_2 with the characteristic functions with minimal support in S_2^{-1} yields the sequence of minimal support elements. However, this is already false for S_2^2. F. Sablonniere [14] constructed a C^2 quartic element with the same support as M_2.

For the three-direction mesh $(\nu = 1)$ Theorems 1 and 2 have been proved in [4]. This case is included here for completeness. The analysis for the four-direction mesh Δ_2 is more complicated because of the two different types of vertices, $\mathbb{Z}^2$ and $\tau + \mathbb{Z}^2$, with $\tau := (-1/2, 1/2)$. However, some of the techniques developed in [4] are still applicable. If the necessary modifications are straightforward we shall only outline the arguments and refer to [4]. In particular the proof of Theorem 2 for $\nu = 2$ is completely analogous to the case of the three-direction mesh [4, Prop. 4.2] and will not be repeated here.

In section 2 we obtain a few general results about the spaces π_{k,Δ_2}^ρ. Sections 3 and 4 are devoted to the proof of Theorem 1 (for the four-direction mesh).

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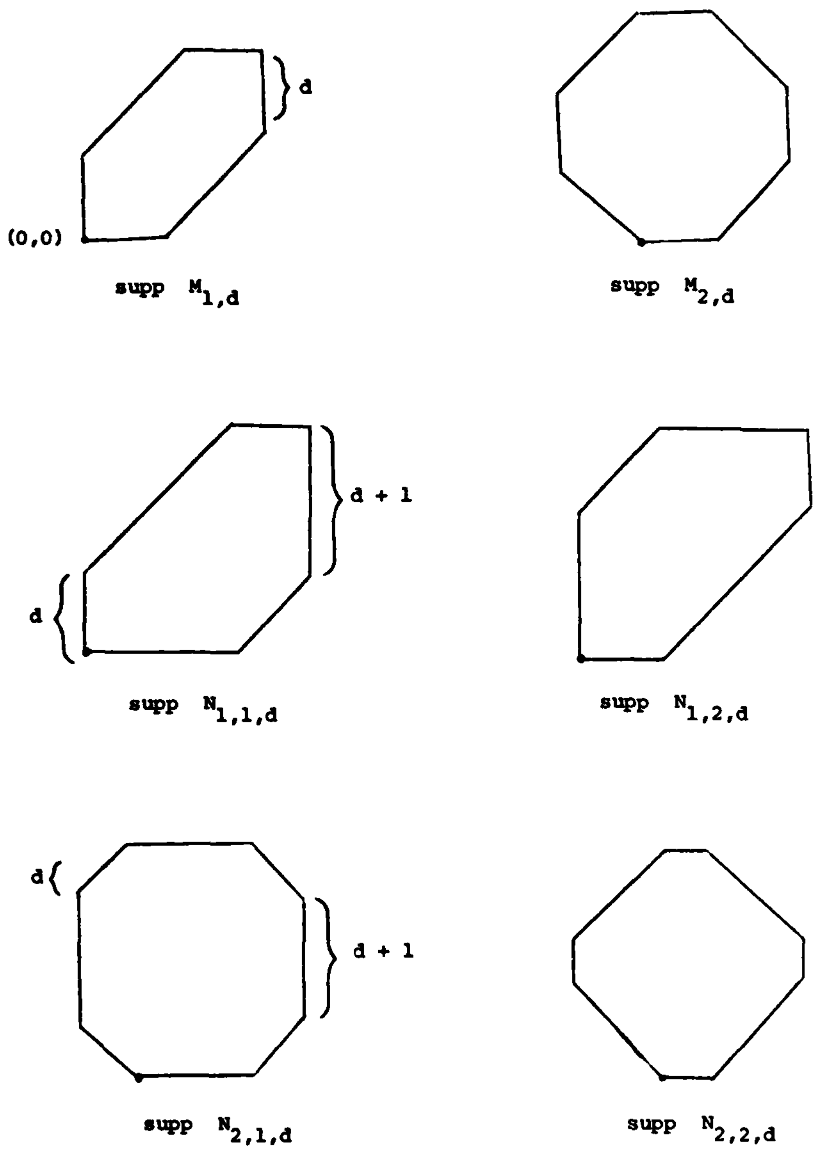

d
(0,0)
supp $M_{1,d}$
supp $M_{2,d}$
d + 1
d
supp $N_{1,1,d}$
supp $N_{1,2,d}$
d
d + 1
supp $N_{2,1,d}$
supp $N_{2,2,d}$

<Figure 4>

2. Auxiliary results.

In this section we obtain a representation of functions in $S := \pi^{\rho}_{k,\Delta_2}$ in terms of translates of truncated powers. The four-direction mesh Δ_2 has two vertex types, the one exemplified by 0 and the one exemplified by

$$\tau := (-1/2, 1/2) \quad .$$

The two differ in that the latter is "singular", i.e. formed as the intersection of two meshlines, hence is less likely to be on the boundary of the support of elements of S.

For a set of vectors $\Xi = \{\xi_1,\ldots,\xi_\ell\}$ the truncated power T_Ξ can be inductively defined by

$$T_\Xi := T_\xi * T_{\Xi\backslash\xi} \quad , \text{ with}$$

(3)

$$T_\xi\, \phi := \int_{R_+} \phi(\cdot - \lambda\xi)d\lambda \quad .$$

We denote by T_p, $p \in Z^4_+$, the truncated power corresponding to the directions $\xi = (\xi_1,\xi_2,\xi_3,\xi_4) := (e_1,e_1+e_2,e_2,e_2-e_1)$ occurring with multiplicities p_1, p_2, p_3, p_4 respectively. For example we have

$$T_{1,0,1,0} * \phi = \iint_{R^2_+} \phi(\cdot - (\lambda_1,\lambda_2))d\lambda \quad ,$$

i.e. for $\Sigma p_\mu = 2$, $p_\mu < 2$, T_p is the characteristic function of the cone spanned by the appropriate two directions. The second relation in (3) becomes

$$T_{p+p'} = T_p * T_{p'} \quad . \tag{4}$$

It is easy to see that <u>the truncated power</u> T_p <u>is a homogeneous piecewise polynomial of degree</u> $\Sigma p_\mu - 2$, <u>with smoothness</u> $\sum_{\mu\neq j} p_\mu - 2$ <u>across the ray generated by the</u> j-th <u>direction and with support in the cone generated by the vectors</u> $p_r\xi_r$, $r = 1,\ldots,4$.

Denote by C the cone generated by ξ_1, ξ_4 and let $W := \{(u,v): \|(u,v)\|_\infty := \max(|u|,|v|) \leq 1/2\}$. Then $S(C)_{|W}$ can be decomposed into its homogeneous components (cf. [4, Lemma 2]), i.e.

$$S(C)|_W = \bigoplus_{\ell<k} Q^\rho_\ell \tag{5}$$

where $Q^\rho_\ell := \{f \in S(C)|_W: f(\lambda\cdot) = \lambda^\ell f\}$. The restriction of functions in $S(C)$ to the segment $\Gamma := [\xi_1,\xi_4]/2$ is an isomorphism from Q^ρ_ℓ onto the univariate spline space Q'

of degree ℓ with the knot sequence $(0, 1/3, 1/2, 1)$, each knot occurring with multiplicity $\ell - \rho$ (i.e., Q' has smoothness ρ). From the smoothness and support of the truncated powers it is clear that their restrictions to Γ are B-splines and in Q'. We identify each B-spline with a vector $q \in \mathbb{Z}_+^4$ where q_ν is the multiplicity of the ν-th knot. Let Λ_ℓ^ρ denote the collection of all such vectors q for the standard B-spline basis for Q'; e.g. $\Lambda_3^1 = \{(2, 2, 1, 0), (1, 2, 2, 0), (0, 2, 2, 1), (0, 1, 2, 2)\}$. It follows that

$$Q_\ell^\rho = \bigoplus_{q\in\Lambda_\ell^\rho} \operatorname{span} T_q\big|_W . \tag{6}$$

Denote by $\tilde{C}$ the cone spanned by ξ_2, ξ_4, but with vertex $\tau = (-1/2, 1/2)$, and let $\tilde{W} := \tau + W$. In a similar manner one concludes that

$$S(\tilde{C})\big|_{\tilde{W}} = \bigoplus_{\ell<k} \tilde{Q}_\ell^\rho \tag{7}$$

$$\tilde{Q}_\ell^\rho = \bigoplus_{\tilde{q}\in\tilde{\Lambda}_\ell^\rho} \operatorname{span} T_{\tilde{q}}(\cdot - \tau)\big|_{\tilde{W}} \tag{8}$$

where $\tilde{\Lambda}_\ell^\rho := \{(0, \nu, 0, \mu): \nu,\mu < \ell - \rho,\ \nu + \mu = \ell + 2\}$.

The subspace $S(C)$ *of elements of* $S = \pi_{h,\Delta_2}^\rho$ having support entirely in C is infinite-dimensional, but we can specify a truncated power basis for it in the spirit familiar from univariate spline theory. Explicitly, we can specify a sequence of truncated powers with the property that every $f \in S(C)$ has a unique expansion in terms of this sequence, with the expansion converging uniformly (in fact finitely) on any bounded set. The formal statement below, in Lemma 1, is to be interpreted in this sense.

<u>Lemma 1.</u>

$$\begin{aligned} S(C) = \bigoplus \operatorname{span} (&\{T_q(\cdot - j): q \in \Lambda_\ell^\rho,\ \ell < k,\ j \in \mathbb{Z}^2 \cap C\} \\ &\cup \{T_{\tilde{q}}(\cdot - \tau - j): \tilde{q} \in \tilde{\Lambda}_\ell^\rho,\ \ell < k,\ j \in \mathbb{Z}^2 \cap C\}) . \end{aligned} \tag{9}$$

<u>Proof</u>. In outline, the proof is as follows. Associate with each vertex v in C the cone

$$C_v := v + \begin{cases} C, & v \in \mathbb{Z}^2 \\ \tilde{C}, & v \in \tau + \mathbb{Z}^2 \end{cases} .$$

This induces a partial order

$$w > v := w \in C_v \quad ,$$

and we give a <u>linear</u> ordering of the vertices in C which refines this one. The promised truncated power basis consists of the relevant truncated powers for each vertex, ordered according to this vertex ordering.

Obviously, the truncated powers, $T_p(\cdot - i)$, $(p,i) = (q,j)$ or $(\tilde{q},j+\tau)$, appearing on the right hand side of (9) are elements of $S(C)$. Their linear independence follows from (5)-(8) and the fact that

$$(i + W) \cap \operatorname{supp} T_p(\cdot - i) \cap \operatorname{supp} T_{p'}(\cdot - i') = \emptyset \quad ,$$

if $i_2 < i_2'$ or if $(i_2 = i_2'$ and $i_1 < i_1')$.

Let $f \in S(C)$. We claim that there exist functions $f_\nu \in \oplus \operatorname{span} \{T_q(\cdot - \nu\xi_1): q \in \Lambda_\ell^\rho, \ell < k\}$, $\nu \in \mathbb{Z}_+$, such that the support of $g := f - \Sigma f_\nu$ is contained in the union Ω of the cones $\nu\xi_1 + \tilde{C}$, $\nu \in \mathbb{Z}_+$. To show this, we assume that $f_0,\dots,f_{\nu-1}$ have been defined and that $g_\nu := f - \sum_{\mu=0}^{\nu-1} f_\mu$ has support in $\Omega \cup (\nu\xi_1 + C)$. It is clear that $g_\nu(\cdot + \nu\xi_1)|_W \in S(C)|_W$ and we define f_ν as the extension of the truncated power representation for $g_\nu|_{\nu\xi_1+W}$.

From the definition of Ω we see that $g(\cdot + \nu\xi_1)|_{\tilde{W}} \in S(C)|_{\tilde{W}}$. Therefore by (7) and (8) there exist functions

$$h_\nu \in \oplus \operatorname{span} \{T_{\tilde{q}}(\cdot - \tau - \nu\xi_1): \tilde{q} \in \tilde{\Lambda}_\ell^\rho, \ell < k\}, \nu \in \mathbb{Z}_+ \quad ,$$

such that $g - \Sigma h_\nu$ has support in $\xi_4 + C$.

By repeating the above procedure we can find inductively linear combinations of truncated powers which agree with f on the cones $\mu\xi_4 + C$, $\mu = 1,2,\dots$. This completes the proof of the Lemma.

It is clear from the above proof that translates of any functions which agree with the truncated powers near zero and have smaller support also provide a basis for $S(C)$. Moreover, an analogous version of Lemma 1 is valid for any cone which is the image of C

under an affine mapping which leaves the partition Δ_2 invariant.

From Lemma 1 we obtain what may be called the "local dimension" of S by counting the number of elements in the sets Λ. We have

$$\text{(10)} \qquad \text{local dimension at } v := \begin{cases} \# \Lambda_\ell^\rho = (4(\ell - \rho) - \ell - 1)_+, & v \in \mathbb{Z}^2 \\ \# \tilde{\Lambda}_\ell^\rho = (2(\ell - \rho) - \ell - 1)_+, & v \in \tau + \mathbb{Z}^2 \end{cases} .$$

It follows in particular that $\dim S(C) = 0$ iff $4(k - \rho) - k - 1 < 0$. This yields formula (1) for k_2 since a nonzero local dimension is necessary and sufficient for the denseness of S_h in $C_0^\infty(\mathbb{R}^2)$ [2].

We now specialize the above results for the spaces $S_\nu = \pi^\rho_{k_\nu(\rho),\Delta_2}$ of <u>minimal</u> degree. We have

$$\text{(11)} \qquad \begin{aligned} \Lambda^{3d-2}_{k_2(3d-2)} &= \{(d,d,d,d)\} , \\ \Lambda^{3d}_{k_2(3d)} &= \{(d+1,d+1,d+1,d), (d,d+1,d+1,d+1)\} \end{aligned}$$

and denote the corresponding truncated powers by t_d and $t_{\mu,d}$, $\mu = 1,2$, respectively. In both cases,

$$\# \tilde{\Lambda}_\ell^\rho = 0 \quad \text{for} \quad \ell < k_2(\rho)$$

$$\# \Lambda_\ell^\rho = 0 \quad \text{for} \quad \ell < k_2(\rho) .$$

In particular, the "secondary" vertices, i.e., $v \in \tau + \mathbb{Z}^2$, are not active. Therefore identity (9) reduces to

$$\text{(9')} \qquad \begin{aligned} S_{2,d}(C) &= \oplus \operatorname{span} \{t_d(\cdot - j): j \in \mathbb{Z}^2 \cap C\} \\ \tilde{S}_{2,d}(C) &= \oplus \operatorname{span} \{t_{\mu,d}(\cdot - j): \mu = 1,2, j \in \mathbb{Z}^2 \cap C\} . \end{aligned}$$

From (4) and the definitions of M, N and t one sees that for $x \in W$,

$$t_d(x) = M_{2,d}(x) \quad ,$$

(12)

$$t_{\mu,d}(x) = N_{2,\mu,d}(x) \quad .$$

Therefore we can replace the truncated powers in (9') by the corresponding elements M and N respectively.

For later reference we note that for $(u,v) \in C$, $v \downarrow 0$,

$$t_d(u,v) = \alpha u^{d-1} v^{3d-1} + O(v^{3d}) \quad ,$$

(13)

$$t_{1,d}(u,v) = \beta u^d v^{3d+1} + \delta u^{d-1} v^{3d+2} + O(v^{3d+3}) \quad ,$$

$$t_{2,d}(u,v) = \gamma u^{d-1} v^{3d+2} + O(v^{3d+3})$$

where α, β and γ are <u>positive</u> constants.

3. Proof of Theorem 1 (i)

Denote by conv A the convex hull of the set A. We first prove

Lemma 2. For $\ell \in \mathbb{Z}_+$ we set $\Omega_1 := \text{conv}\{0, \ell\xi_1, \ell\xi_1 + \xi_2, \xi_4\}$, $\Omega_2 :=$ $\text{conv}\{0, \ell\xi_2, \ell\xi_2 + \xi_3, -\xi_1\}$ and define $Z_1 := \{f|_{\Omega_1}: f \in S_{2,d}, \text{supp } f \subseteq \{(u,v): v > 1\} \cup \Omega_1\}$, $Z_2 := \{f|_{\Omega_2}: f \in S_{2,d}, \text{supp } f \subseteq \{(u,v): u - v < 1\} \cup \Omega\}$. Then we have

$$\dim Z_i = (\ell + 1 - d)_+, \ i = 1,2 \quad .$$

The cases $i = 1,2$ are not geometrically equivalent since the pattern of the mesh for Ω_1 and Ω_2 is slightly different.

Proof. Consider, e.g., the case $i = 1$. Let

$$\theta := \ell\xi_1 + \text{conv}\{0, \xi_2/2, \xi_1\} \quad .$$

Since $\text{supp } t_d(\cdot - j) = j + C$, it follows from (9') that

$$Z_1 = \{f = \sum_{\nu=0}^{\ell} a_\nu t_d(\cdot - \nu\xi_1): a_\nu \in \mathbb{R}, f|_\theta \equiv 0\} \quad .$$

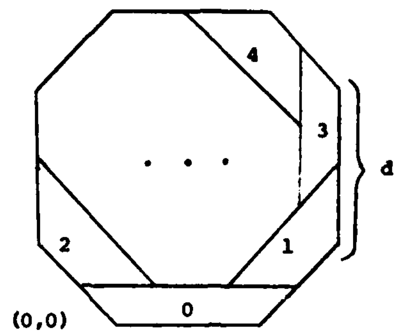
Since f vanishes on θ we obtain from (13) that

$$\sum_{\nu=0}^{\ell} a_\nu \alpha(u - \nu)^{d-1} = 0, \ \ell < u < \ell + 1 \quad .$$

These are $\min\{d,\ell+1\}$ linearly independent constraints on the coefficients a_ν which implies $\dim Z_1 \le (\ell + 1 - d)_+$. The reverse inequality follows since $M_{2,d}(\cdot - \nu\xi_1)|_{\Omega_1}$, $\nu = 0,\ldots,\ell-d$, are linearly independent and in Z_1.

To prove that $M_{2,d}$ has unique minimal support in $S_{2,d}$, assume that $\text{supp } f \subseteq \text{supp } M_{2,d}$ for some $f \in S_{2,d}$. Lemma 2, with $\ell = d$, implies that $f = c\,M_{2,d}$ on the set $A_0 := \text{conv}\{0, d\xi_1, d\xi_1+\xi_2, \xi_4\}$. We define inductively a sequence of sets $A_1, A_2, \ldots$ as follows. For $i = 1,2,\ldots$ we choose a shortest segment Γ_i with respect to $\|\ \|_\infty$ of the piecewise linear boundary of $B_i := \text{supp } M_{2,d} \setminus \bigcup_{\nu=0}^{i-1} A_\nu$. Then we define $A_i := \{x \in B_i: \text{dist}_\infty(x,\Gamma_i) < 1/2\}$. This procedure is illustrated in Figure 5 below for $d = 2$.

The sets A_i, $i > 0$, are contained in sets of the type described in Lemma 2 with $\ell < d$. Therefore, we inductively conclude that $f - c\,M_{2,d}$ vanishes on $A_1, A_2, \ldots$.



<Figure 5>

4. Proof of Theorem 1 (ii)

We need two lemmas.

Lemma 3. Let Ω_1 and Z_1 be defined as in Lemma 2 but with $S_{2,d}$ replaced by $\tilde{S}_{2,d}$. Then we have

$$\dim Z_1 = (\ell - d)_+ + (\ell + 1 - d)_+ \quad .$$

Proof. Similarly as in the proof of Lemma 2 we conclude that

$$Z_1 = \{f = \sum_{\substack{\nu=0,\ldots,\ell \\ \mu=1,2}} a_{\nu,\mu} t_{\mu,d}(\cdot - \nu\xi_1): a_{\nu,\mu} \in \mathbb{R},\ f|_\theta = 0\}$$

where θ is defined as before. Comparing the coefficients of v^{3d+1} and v^{3d+2} in the expression for f on the triangle θ we obtain, using (13), for $\ell < u < \ell + 1$

$$\sum_{\nu=0}^{\ell} a_{\nu,1}\beta(u - \nu)^d = 0 \quad ,$$

$$\sum_{\nu=0}^{\ell} (a_{\nu,1}\delta + a_{\nu,2}\gamma)(u - \nu)^{d-1} = 0 \quad .$$

These are $\min\{\ell+1,d+1\} + \min\{\ell+1,d\}$ linearly independent constraints on the coefficients $a_{\nu,\mu}$ which implies $\dim Z_1 < (\ell - d)_+ + (\ell + 1 - d)_+$. The reverse inequality follows since $N_{2,1,d}(\cdot - \nu\xi_1)|_{\Omega_1}$, $\nu = 0,\ldots,(\ell - 1 - d)_+$, and $N_{2,2,d}(\cdot - \nu\xi_1)|_{\Omega_1}$, $\nu = 0,\ldots,(\ell - d)_+$, are linearly independent and in Z_1.

Lemma 4. Let $\Omega := \mathrm{conv}\{0,\ d\xi_1+\xi_2,\ d\xi_4+\xi_3,\ d\xi_4\}$ and $\Omega' := \{(u,v): v > 1,\ u + v > 2\}$. If $f \in \tilde{S}_{2,d}(\Omega \cup \Omega')$, then $f|_{\Omega\setminus\Omega'} = 0$.

Proof. On the set $\Omega\setminus\Omega'$, the function f can be written as linear combination of truncated powers,

$$f = a_{0,1}t_{1,d} + a_{0,2}t_{2,d} + \sum_{\substack{0<\nu<d \\ \mu=1,2}} (a_{\nu,\mu}t_{\mu,d}(\cdot - \nu\xi_1) + a'_{\nu,\mu}t_{\mu,d}(\cdot - \nu\xi_4) \quad .$$

The truncated powers $t_{\mu,d}(\cdot - \nu\xi_4)$ as well as the function f vanish on the triangle θ. By using (13) the coefficient of v^{3d+1}, for $(u,v) \in \theta$, is

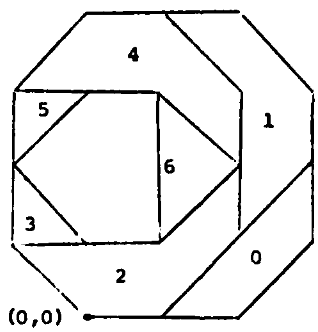
$$\sum_{0<\nu<d} a_{\nu,1}\beta(u-\nu)^d = 0 \quad .$$

This implies $a_{0,1} = \cdots = a_{d,1} = 0$. Applying the analogous argument for the triangle $\theta' := d\xi_4 + \mathrm{conv}\{0, \tau, \xi_3\}$, we conclude that $a_{0,2} = a'_{1,2} = \cdots = a'_{d,2} = 0$. Using in particular that $a_{0,2} = 0$ and again relation (13) it follows that for $(u,v) \in \theta$,

$$f(u,v) = \sum_{0<\nu<d} a_{\nu,2}\gamma(u-\nu)^{d-1}v^{3d+2} + O(v^{3d+3}) \equiv 0 \quad .$$

This implies that $a_{1,2} = \cdots = a_{d,2} = 0$ and finally, by using the analogous argument for θ', that $a'_{1,1} = \cdots = a'_{d,1} = 0$.

To prove that $N := N_{2,\mu d}$ has minimal support in $\tilde{S}_{2,d}$, assume that $\mathrm{supp}\, f \subseteq \mathrm{supp}\, N$ for some $f \in \tilde{S}_{2,d}$. Let Γ be a segment of the piecewise linear boundary of $\mathrm{supp}\, N$ with $\mathrm{diam}_\infty \Gamma = d/2$. The set $A_0 := \{x \in \mathrm{supp}\, N: \mathrm{dist}_\infty(x,\Gamma) < 1/2\}$ is of the type considered in Lemma 3 with $\ell = d$ and we conclude that $f = c\,N$ on A_0. We define inductively a sequence of sets $A_1, A_2, \ldots$ as follows. If $B_i := \mathrm{supp}\, N \setminus \bigcup_{\nu=0}^{i-1} A_\nu$ has a corner y with angle $< \pi/2$ we set $A_i := \{x: \|x - y\|_\infty < 1/2\}$. Otherwise we choose two adjacent segments Γ, Γ' of the boundary of B_i with diameter $< d/2$ and set $A_i := \{x \in B_i: \mathrm{dist}_\infty(x, \Gamma \cup \Gamma') < 1/2\}$. This procedure is illustrated in Figure 6 below for $N_{2,1,1}$.



<Figure 6>

The sets A_i are contained in sets of the type considered in Lemma 3 with $\ell = d - 1$ (3,5,6 in Figure 6) or Lemma 4. In either case we inductively conclude that $f - cN$ vanishes on $A_1, A_2, \ldots$.

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Let S denote the space of piecewise polynomials of degree $\leq k$ and smoothness ρ on the regular partition of $\mathbb{R}^2$ which is generated either by the three directions (1,0), (1,1), (0,1) or by the four directions (1,0), (1,1), (0,1), (-1,1). For the choice

$$\rho = \rho(k) := \max\{\rho : \dim S|_{[-N,N]^2} \neq o(N^2)\} ,$$

ABSTRACT (continued)

(which is the maximal smoothness for which the space S is nondegenerate), we determine the functions which have minimal support in S. Moreover, we show that these functions form a basis for

$$S(\Omega) := \{f \in S : \operatorname{supp} f \subseteq \Omega\} .$$

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