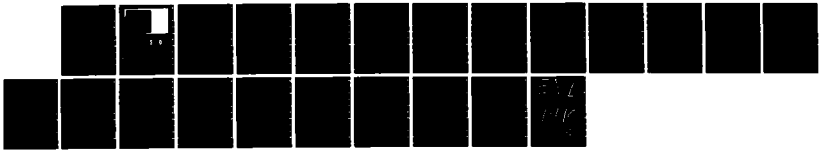
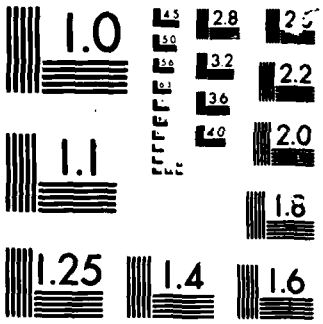
AD-A167 512 FINITE DIFFERENCE METHODS FOR POLAR COORDINATE SYSTEMS (U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER J C STRIKWERDA ET AL. APR 86 MRC-TSR-2934 1/1

UNCLASSIFIED DAAG29-80-C-0041 F/G 12/1 NL






MICROCOPY CHART

MRC Technical Summary Report #2934

# FINITE DIFFERENCE METHODS FOR POLAR COORDINATE SYSTEMS

John C. Strikwerda
and
Yvonne Nagel

**Mathematics Research Center**
**University of Wisconsin—Madison**
**610 Walnut Street**
**Madison, Wisconsin 53705**



April 1986

(Received February 3, 1986)



**Approved for public release**
**Distribution unlimited**

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science
Foundation
Washington, DC 20550

Office of Naval Research
Department of the Navy
Arlington, VA 22217

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

# FINITE DIFFERENCE METHODS FOR POLAR COORDINATE SYSTEMS

John C. Strikwerda and Yvonne Nagel[1]

Technical Summary Report #2934
April 1986

## ABSTRACT

We discuss finite difference methods for partial differential equations on polar and spherical coordinate systems. The distinctive feature of these coordinate systems is the coordinate system singularity at the origin. We show how to accurately and conveniently determine the solution at the origin for both scalar and vector fields. We also discuss the Fourier method to approximate derivatives with respect to the angular variable in polar coordinates. Computational examples are presented illustrating the accuracy and efficiency of the method for hyperbolic and elliptic equations, and also for the computation of vector fields at the origin.

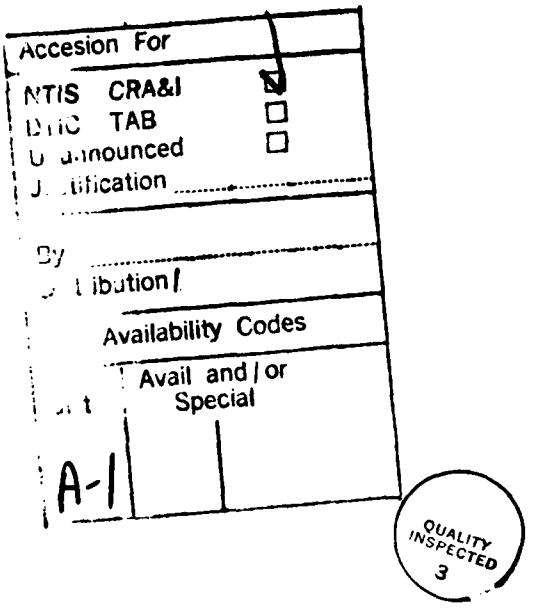
AMS (MOS) Subject Classifications: 65N05, 65M05
Key Words: Finite Difference Methods, Polar Coordinates
Work Unit Number 3 - Numerical Analysis and Scientific Computing

---

[1]Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. The first author was also supported in part by NSF grant MCS-8306880 and ONR grant N00014-84-K-0454.

## SIGNIFICANCE AND EXPLANATION

Many finite difference computations are done for regions with polar or spherical coordinate systems. The determination of the solution variables at the origin of such coordinate systems has been a source of much confusion. This is especially true for calculations of vector fields defined on polar or spherical grids. In this paper we show how to accurately and easily calculate the variables at the origin of such coordinate systems.




---

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

# FINITE DIFFERENCE METHODS FOR POLAR COORDINATE SYSTEMS

John C. Strikwerda and Yvonne Nagel[1]

## 1. Introduction

In this paper we consider the use of polar and spherical coordinates with finite difference methods, determining how to achieve accurate results with convenience. Although the use of finite difference methods with polar coordinates is not at all new there are several features of their use that are not well known among numerical analysts, computational scientists, and engineers. In particular, the accurate treatment of vector fields with polar coordinates is not widely known.

It is the aim of this paper to bring together the pertinent information and present it in an organized way. As such, this paper presents few new ideas, but it is hoped that it will be a useful addition to the literature on numerical methods.

Much of what is presented here also applies to axially symmetric problems in polar or spherical coordinates; the common feature of these problems is the singular nature of the coordinate system at the origin.

## 2. The Center Formulas

Consider the plane with a polar coordinate system. Each point is determined by its polar coordinates $(r, \phi)$ which, for points other than the origin, is unique up to integer multiples of $2\pi$ in $\phi$. However the origin has the coordinates $(0,\phi)$ for all angles $\phi$, and it is this lack of uniqueness in the coordinates of the origin that introduces difficulties for numerical methods. These difficulties are displayed in the Jacobian of the coordinate map which takes ordered pairs in $[0,\infty)\times \mathbb{R}$ to points in the plane. The Jacobian vanishes on $\{ 0 \} \times \mathbb{R}$. However it is important to realize that this singularity is present in the coordinate map and polar representations of functions and need not be present in the functions themselves. In this paper we shall only consider functions which are smooth in the domain being considered.

The singular behavior of the polar coordinate system at the origin usually precludes the direct use of finite difference approximations to differential equations at that point. We consider therefore the use of interpolation formulas to accurately determine the solution at the origin. We begin by considering a function defined in the plane, without considering a coordinate system.

---

[1]Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. The first author was also supported in part by NSF Grant MCS-8306880 and ONR Grant N00014-84-K-0454.

Consider a smooth function $u$ defined in a neighborhood of a point $P$ in the plane. We wish to express $u(P)$ in terms of averages, $\bar{u}(P,\rho)$, on circles of radius $\rho$ centered at $P$. We begin by expanding $u$ in a Taylor series in cartesian coordinates with the origin at $P$,

$$u(x,y) = \sum_{k,l=0}^{N} \frac{x^k y^l}{k!\, l!} \frac{\partial^{k+l} u}{\partial x^k \partial y^l}(0,0) + R_N. \tag{2.1}$$

where $R_N$ is the remainder term. Then,

$$\bar{u}(P,\rho) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho\cos\phi, \rho\sin\phi) d\phi \tag{2.2}$$

and using (2.1)

$$\bar{u}(P,\rho) = \sum_{l=0}^{N} c_l \rho^{2l} \nabla^{2l} u(P) + \bar{R}_N \tag{2.3}$$

where $c_l = 1/4^l (l!)^2$. Formula (2.3), which is independent of a coordinate system, is the basis for determining a function value at the origin given values of the function at points nearby and given the differential equation satisfied by $u$.

Consider now a uniform finite difference grid with grid points $(r_i, \phi_j)$ for integers $i$ and $j$ with $i \geq 0$ and $0 \leq j \leq J-1$ where $r_i = i\Delta r$, $\phi_j = j\Delta\phi$ for $\Delta r > 0$ and $\Delta\phi = 2\pi/J$. For a function $u$ defined in a neighborhood of the origin $P$, we have

$$\bar{u}(l\Delta r) = \bar{u}(P, l\Delta r) = \frac{1}{J} \sum_{j=0}^{J-1} u_{l,j} + O(\Delta\phi^m) \tag{2.4}$$

where $u_{l,j} = u(r_l, \phi_j)$ and $m$ is a positive integer whose value will be considered in section 5. Using (2.3) we then have the relations

$$u(P) = \bar{u}(\Delta r) + O(\Delta r^2) + O(\Delta\phi^m) \tag{2.5}$$

and

$$u(P) = \frac{1}{3}(4\bar{u}(\Delta r) - \bar{u}(2\Delta r)) + O(\Delta r^4) + O(\Delta\phi^m) \tag{2.6}$$

which can be used with finite difference methods to determine values at the origin. Higher order formulas can be obtained by similar means.

## 3. The Laplacian with Polar Grids

When the differential equation being solved involves the laplacian operator then formula (2.3) can be used to a special advantage. We consider as examples the Poisson equation

$$\nabla^2 u = f \tag{3.1}$$

and the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \tag{3.2}$$

on a disk of unit radius.

For the Poisson equation (3.1) consider the semi–discrete finite difference approximation

$$\frac{1}{r_i \Delta r}\left(r_{i+\frac{1}{2}}\frac{u_{i+1}(\phi)-u_i(\phi)}{\Delta r} - r_{i-\frac{1}{2}}\frac{u_i(\phi)-u_{i-1}(\phi)}{\Delta r}\right) + \frac{1}{r_i^2}\frac{\partial^2 u_i(\phi)}{\partial \phi^2} = f_i(\phi) \text{ for } i>0, \tag{3.3}$$

where we discretize only the radial direction. Employing (2.3) we have at the origin

$$u_0 = \bar{u}(\Delta r) - \frac{\Delta r^2}{4}\nabla^2 u(0) + O(\Delta r^4)$$

or

$$u_0 = \bar{u}(\Delta r) - \frac{\Delta r^2}{4} f(0) + O(\Delta r^4). \tag{3.4}$$

This formula maintains the second–order accuracy of the scheme and is easy to use. However, even when the equation being solved involves the laplacian, (2.6) may be more accurate or convenient to use than formulas such as (3.4). Formula (3.4) has been used by Swarztrauber and Sweet [1973] for solving the Poisson equation in a disk, and by Swarztrauber [1974] for the Poisson equation on a sphere.

For the wave equation (3.2) we also consider a semi–discrete approximation in which only time and the radial direction are discretized with the angular variation continuous. Let $u_i^n(\phi)$ be the approximation to $u(n\Delta t, r_i, \phi)$. At the origin we have

$$\begin{aligned} u_0^n &= \bar{u}_0^n(\Delta r) - \frac{1}{4}\Delta r^2 \nabla^2 u_0^n + O(\Delta r^4) \\ &= \bar{u}_0^n(\Delta r) - \frac{1}{4}\Delta r^2 \frac{\partial^2 u}{\partial t^2}\Big|_0^n + O(\Delta r^4) \\ &= \bar{u}_0^n(\Delta r) - \frac{1}{4}\left(\frac{\Delta r}{\Delta t}\right)^2 (u_0^{n+1} - 2u_0^n + u_0^{n-1}) \\ &\quad + O(\Delta r^4) + O(\Delta r^2 \Delta t^2), \end{aligned}$$

using a central difference approximation in time. This gives the formula at the origin as

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} + 4\left(\frac{\Delta t}{\Delta r}\right)^2 (\bar{u}_0^n(\Delta r) - u_0^n) \tag{3.5}$$

which maintains the second–order accuracy of the scheme. Example 1 in section 7 shows that this formula gives accurate results. Similar methods can also be used with parabolic equations.

## 4. Vector Fields with Polar Grids

In addition to the coordinate singularity at the origin, the polar coordinate representation of vector fields introduces an additional difficulty. Let $\vec{F}$ be a vector field defined on a domain on which there is a polar coordinate system. The polar coordinate representation assigns to each vector $\vec{F}(P)$ the component in the radial direction and the component in the direction of increasing angle. This representation is unique at all points other than the origin.

At the origin the vector $\vec{F}(0)$ has a different representation for each choice of the radial direction. This is best illustrated using the mapping between the polar and cartesian representations. Let $(U, V)$ be the usual cartesian representation of the vector field $\vec{F}$ which is uniquely determined, then the polar representation $(u, v)$ is given by

$$\begin{aligned} u &= U\cos\phi + V\sin\phi \\ v &= -U\sin\phi + V\cos\phi. \end{aligned} \tag{4.1}$$

Since at the origin the pair $(U, V)$ is single valued, (4.1) shows the multivalued nature of the polar representation.

Using a polar grid the vector field $\vec{F}$ will be represented, and approximated, by vectors $(u_{ij}, v_{ij})$ at each grid point $(r_i, \phi_j)$. At the origin, there is a representation $(u_{0j}, v_{0j})$ for each coordinate direction $(0, \phi_j)$. For consistency these representations must be related by the formulas (4.1). That is, there are values $(U_0, V_0)$ such that

$$\begin{aligned} u_{0j} &= U_0\cos\phi_j + V_0\sin\phi_j \\ v_{0j} &= -U_0\sin\phi_j + V_0\cos\phi_j. \end{aligned} \tag{4.2}$$

The values of $U_0$ and $V_0$ can be obtained by formulas such as (2.6). For example, on a uniform grid define

$$\begin{aligned} \bar{U}(i\Delta r) &= \frac{1}{J}\sum_{j=0}^{J-1} u_{ij}\cos\phi_j - v_{ij}\sin\phi_j \\ \bar{V}(i\Delta r) &= \frac{1}{J}\sum_{j=0}^{J-1} u_{ij}\sin\phi_j + v_{ij}\cos\phi_j. \end{aligned} \tag{4.3}$$

Then $U_0$ and $V_0$ can be approximated by

$$\begin{aligned} U_0 &= \frac{1}{3}(4\bar{U}(\Delta r) - \bar{U}(2\Delta r)) \\ V_0 &= \frac{1}{3}(4\bar{V}(\Delta r) - \bar{V}(2\Delta r)). \end{aligned} \tag{4.4}$$

These values can then be used in (4.2) to give the values of $(u_{0j}, v_{0j})$. Example 3 in section 7 demonstrates the accuracy of this method as applied to the Stokes equations. This method has been used in Strikwerda [1984a] and Nagel and Strikwerda [1985] with excellent results.

For finite difference grids which are not uniform in the angular variable formulas (4.3) should be replaced by

$$\bar{U}(i\Delta r) = \frac{1}{\sigma}\left(\sum_{j=0}^{J-1} u_{ij}(\sin\phi_{j+1} - \sin\phi_{j-1}) + v_{ij}(\cos\phi_{j+1} - \cos\phi_{j-1})\right) \tag{4.5}$$

$$\bar{V}(i\Delta r) = \frac{1}{\sigma}\left(\sum_{j=0}^{J-1} u_{ij}(\cos\phi_{j+1} - \cos\phi_{j-1}) + v_{ij}(\sin\phi_{j+1} - \sin\phi_{j-1})\right) \tag{4.6}$$

where

$$\sigma = 2\sum_{j=0}^{J-1} \sin(\phi_{j+1} - \phi_j).$$

The formulas (4.5) and (4.6) are exact for the case when the vector field has constant cartesian components in a neighborhood of the origin.

## 5. The Fourier Method

We now consider the Fourier method for the approximation of derivatives with respect to $\phi$. Consider a periodic discrete function $f_j$ defined on grid points $\phi_j = j\Delta\phi$ with $\Delta\phi = 2\pi/J$. The object of both the finite difference and Fourier methods is to obtain approximations to $\partial f/\partial\phi$ at the grid points. The Fourier method begins with the finite Fourier series representation of $f_j$, i.e. for the case when $J$ is an even integer

$$f_j = b_0 + \sum_{k=1}^{J/2-1} (a_k \sin k\phi_j + b_k \cos k\phi_j) + b_{\frac{J}{2}} \cos(\tfrac{J}{2}\phi_j). \tag{5.1}$$

Note that $\cos(\frac{J}{2}\phi_j) = (-1)^j$. Replacing $\phi_j$ in (5.1) by a continuous variable $\phi$ we can approximate $\partial f/\partial\phi$ at $\phi_j$ as

$$\left.\frac{\partial f}{\partial\phi}\right|_j \simeq \sum_{k=1}^{J/2-1} a_k k \cos k\phi_j - b_k k \sin k\phi_j \tag{5.2}$$

and similarly

$$\left.\frac{\partial^2 f}{\partial\phi^2}\right|_j \simeq -\sum_{k=1}^{J/2-1} (a_k k^2 \sin k\phi_j - b_k k^2 \cos k\phi_j) - (-1)^j(\tfrac{J}{2})^2 b_{\frac{J}{2}}.$$

The coefficients $a_k$ and $b_k$ are easily obtained by

$$b_k = \frac{2}{J}\sum_{j=0}^{J-1} f_j \cos k\phi_j \tag{5.3}$$
$$a_k = \frac{2}{J}\sum_{j=0}^{J-1} f_j \sin k\phi_j$$

for $0 < k < J/2$, and

$$b_0 = \frac{1}{J}\sum_{j=0}^{J-1} f_j \tag{5.4}$$
$$b_J = \frac{1}{J}\sum_{j=0}^{J-1} (-1)^j f_j.$$

The Fourier method has the advantage that it gives far higher accuracy for a given number of grid points than do finite difference methods (Gottlieb and Orzag [1977]). Alternatively to attain a given accuracy the Fourier method requires significantly fewer grid points than do finite difference methods. For example 2 of section 7, finite difference methods would require at least three times as many grid points in the angular direction to obtain comparable accuracy.

The efficiency gained by the Fourier method over the finite difference method for the angular variation is due to the natural periodicity in the variable $\phi$ . Spectral methods can be used with the radial variation but not necessarily with the same gain in efficiency, Gottlieb and Orzag [1977].

Line successive-over-relaxation (LSOR) can easily be used to solve elliptic boundary value problems in polar coordinates in which the Fourier method is used to approximate the derivatives with respect to $\phi$. The basic formula for LSOR as applied to the semi-discrete approximation (3.3) is

$$\left(-\frac{2}{\Delta r^2} + \frac{1}{r_i^2}\frac{\partial^2}{\partial \phi^2}\right)(u_i^{\nu+1}(\phi) - u_i^{\nu}(\phi)) \tag{5.5}$$
$$= -\omega\left(\frac{1}{r_i \Delta r}\left(r_{i+\frac{1}{2}}\frac{u_{i+1}^{\nu}(\phi) - u_i^{\nu}(\phi)}{\Delta r} - r_{i-\frac{1}{2}}\frac{u_i^{\nu}(\phi) - u_{i-1}^{\nu+1}(\phi)}{\Delta r}\right) + \frac{1}{r_i^2}\frac{\partial^2 u_i^{\nu}}{\partial \phi^2} - f_i(\phi)\right).$$

In (5.5) the order of progression through the grid for the LSOR is in the direction of decreasing radius. When the Fourier method is used to approximate the derivatives with respect to $\phi$, the Fourier coefficients of the update, $u_i^{\nu+1} - u_i^{\nu}$ can be easily obtained from the coefficients of the right-hand side of (5.5). That is, the right-hand side of (5.5) is evaluated for each value of $j$, then the Fourier coefficients are calculated. Dividing the coefficients of the $k^{th}$ node by $-(2/\Delta r^2 + k^2/r_i^2)$ gives the coefficients of the update, from which the update is determined at each value of $j$. This method is used in examples 2 and 3 of section 7.

## 6. Quadrature Formulas

The approximation of integrals by sums arises in several contexts in the use of finite difference methods on polar grids. As we have seen the approximations at the origin (2.5) and (2.6) use integrals in $\phi$ at various values of $r$. Also, the accurate determination of integral quantities over the domain requires quadrature formulas in r and $\phi$.

We begin by considering integration in the angular variable only. We consider a $2\pi$-periodic function $f(\phi)$. We first consider the error resulting from approximating the integral

$$\int_0^{2\pi} f(\phi)d\phi \tag{6.1}$$

by the sum

$$\sum_{j=0}^{J-1} f(\phi_j)\Delta\phi \tag{6.2}$$

where $\Delta\phi = 2\pi/J$ and $\phi_j = j\Delta\phi$. By the theory of Fourier series we have

$$f(\phi) = \sum_{n=-\infty}^{\infty} a_n e^{in\phi} \tag{6.3}$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)e^{-in\phi}d\phi. \tag{6.4}$$

Thus

$$\sum_{j=0}^{J-1} f(\phi_j)\Delta\phi = \sum_{j=0}^{J-1}\sum_{n=-\infty}^{\infty} a_n\, e^{in\phi_j}\Delta\phi = \sum_{k=-\infty}^{\infty} a_{kJ},$$

since $\sum_{j=0}^{J-1} e^{in\phi_j}$ vanishes unless $n$ is a multiple of $J$. The integral (6.1) is precisely $a_0$ thus the error in the approximation (6.2) is

$$2\pi \sum_{k=-\infty, k\neq 0}^{\infty} a_{kJ}.$$

By the definition of the $a_n$ in (6.4) we have

$$a_n = \frac{1}{2\pi}(in)^{-m}\int_0^{2\pi} f^{(m)}(\phi)e^{-in\phi}d\phi$$

if $f$ is $m$ times differentiable. Thus

$$|a_n| \leq C_m|n|^{-m},$$

for some constant $C_m$ depending on $f$. Thus the error in the approximation (6.2) is bounded by

$$2C_m \sum_{k=1}^{\infty} |kJ|^{-m} = O(J^{-m}) = O(\Delta\phi^m). \tag{6.5}$$

We now consider quadrature for the unit circle using uniform spacing in $r$ and $\phi$. We consider

$$\int_0^1 \int_0^{2\pi} f(r,\phi)\, r\, dr\, d\phi = 2\pi \int_0^1 \bar{f}(r)\, r\, d\phi \tag{6.6}$$

where $\bar{f}(r)$ may be approximated to $O(\Delta\phi^m)$ as in (6.2). Using the trapezoid formula we have

$$\begin{aligned} 2\pi \int_0^1 f(r) r dr &= 2\pi \sum_{i=0}^{I-1} \frac{1}{2}(\bar{f}_i r_i + \bar{f}_{i+1} r_{i+1})\Delta r + O(\Delta r^2) \\ &= 2\pi \sum_{i=1}^{I-1} \bar{f}_i r_i \Delta r + \pi \bar{f}_I r_I \Delta r + O(\Delta r^2). \end{aligned}$$

Hence

$$\int_0^1 \int_0^{2\pi} f(r,\phi) r dr d\phi = \sum_{i=1}^{I-1} \sum_{j=0}^{J-1} f_{ij} r_i \Delta r \Delta\phi + \frac{1}{2} \sum_{j=0}^{J-1} f_{Ij} r_I \Delta r \Delta\phi + O(\Delta r^2, \Delta\phi^m). \tag{6.7}$$

## 7. Computational Results

In this section we present results of computations using the formulas discussed in the previous sections applied to three test problems. The first test problem is to solve the second-order wave equation. The two formulas (2.6) and (3.5) for determining the solution at the origin are compared. The second test problem is to solve an elliptic equation using the LSOR method given in section 4 to solve the discrete equations. The third test problem uses the Stokes equations to illustrate the use of the formulas for vector fields at the origin.

The first test problem was to solve the second-order wave equation

$$u_{tt} = \nabla^2 u \tag{7.1}$$

in the unit disk for $0 \le t \le 1$. The exact solution we used was

$$u(t,x,y) = \cos(t - .6x - .8y). \tag{7.2}$$

The equation for the time advancement is

$$u_{ij}^{n+1} = 2u_{ij}^{n} - u_{ij}^{n-1} + (\Delta t)^2 \nabla_h^2 u_{ij}^{n}, \tag{7.3}$$

where the discrete laplacian, $\nabla_h^2$, is given by the left-hand side of (3.3) and the derivatives with respect to $\phi$ are approximated by the Fourier method. The formula for the first time-step is based on a Taylor series in time and is

$$u_{ij}^{1} = u_{ij}^{0} + \Delta t (u_t)_{ij}^{0} + \frac{1}{2}(\Delta t)^2 \nabla_h u_{ij}^{0}, \tag{7.4}$$

where $u_{ij}^{0}$ and $(u_t)_{ij}^{0}$ were obtained from the exact solution.

Both the interpolation formula (2.6) and the formula (3.5) were used to determine the solution at the origin. The interpolation formula was applied using $\bar{u}(\Delta r)$ and $\bar{u}(2\Delta r)$ at the given time level to compute $u$ at the origin for that same time level. The results of four test cases are displayed in Table 1, where $I$ and $J$ are the number of radial and angular grid points, respectively, and $K$ is the number of time steps. Both the $L^2$ norm of the error and the error at the origin are shown for each case. The two formulas are seen to be comparable in accuracy, but the interpolation formula is slightly more accurate. This was also observed in all the other cases in which these two formulas were compared. Since formulas (2.6) and (3.5) yielded comparable results, we used the simpler formula (2.6) for all subsequent runs.

**Table 1. Two Center Methods**

| GRID | | | FORMULA 1 | | FORMULA 2 | |
|---|---|---|---|---|---|---|
| $I$ | $J$ | $K$ | $\|u_{err}\|$ | $c_{err}$ | $\|u_{err}\|$ | $c_{err}$ |
| 21 | 16 | 160 | .5586(-4) | .1010(-3) | .5685(-4) | .1069(-3) |
| 41 | 20 | 220 | .1662(-4) | .3177(-4) | .1623(-4) | .3105(-4) |

A list of cases using formula (2.6) is given in Table 2. The data show that errors are relatively insensitive to $J$, the number of angular grid points, for the chosen values of $J$. The number of time steps must be chosen so that the scheme is stable. No attempt was made to determine the stability condition for this example. Because the accuracy of the scheme depends on the three parameters $I, J$, and $K$ it is difficult to discern the order of accuracy of the scheme.

Tests were made using a non-uniform radial grid for this test case. We found that in all cases it degraded the accuracy. Further study is needed to determine if non-uniform grids can be used to give good accuracy and less restrictive stability limitations on the time step.

**Table 2. Wave Equation Results**

| $I$ $J$ $K$ | $\|u_{err}\|$ | $c_{err}$ |
|---|---|---|
| 11 12 40 | .2050(-3) | .3511(-3) |
| 21 12 80 | .5314(-4) | .9447(-4) |
| 21 16 100 | .5294(-4) | .9525(-4) |
| 21 20 120 | .5397(-4) | .9727(-4) |
| 41 12 160 | .1595(-4) | .2998(-4) |
| 41 16 180 | .1623(-4) | .3117(-4) |
| 41 20 320 | .1662(-4) | .3177(-4) |

The second test problem was to solve the elliptic equation

$$\nabla^2 u - c(x,y)u = 0. \tag{7.5}$$

on the unit disk with $u$ specified on the boundary. The exact solution was given by

$$u(x,y) = \exp((x-0.1)(y-0.5)) \tag{7.6}$$

with

$$c(x,y) = (x-0.1)^2 + (y-0.5)^2. \tag{7.7}$$

The equation (7.1) was approximated using the left-hand side of (3.3) for the laplacian with the Fourier method being used to approximate the derivatives with respect to $\phi$. The solutions were obtained with the LSOR method discussed at the end of section 5. If finite difference methods are used in the angular variable, then a direct solver such as that of Swarztrauber and Sweet [1973] can be used.

The results of several test runs are displayed in Table 3. The number of grid points is given along with the iteration parameter $\omega$ and the tolerance on the updates. The iterative procedure was stopped when

$$\|u^{n+1} - u^n\|/\omega \leq tol. \tag{7.8}$$

The number of iterations required for convergence is seen to be dependent on the number of radial grid points and not on the number of angular grid points. The accuracy at the origin is seen to be relatively independent of the value of $J$, as is expected from the analysis of section 5. This was also noted for test problem 1. The norm of the error is, however, dependent on $J$ for small values of $J$. If $J$ is sufficiently large then the second-order accuracy of the scheme is seen.

A center formula similar to (3.4) gave results comparable to those obtained by (2.6). The results shown were obtained by using center formula (2.6).

**TABLE 3. Poisson equation results**

| $I$ $J$ | iter | $\omega$ | $tol$ | $\|u_{err}\|$ | $c_{err}$ |
|---|---|---|---|---|---|
| 11 12 | 42 | 1.5 | 1(-5) | .13(-2) | -.11(-2) |
| 11 16 | 42 | 1.5 | 1(-5) | .13(-2) | -.11(-2) |
| 21 8 | 53 | 1.8 | 1(-6) | .39(-2) | -.63(-3) |
| 21 12 | 52 | 1.8 | 1(-6) | .38(-3) | -.28(-3) |
| 21 16 | 52 | 1.8 | 1(-6) | .35(-3) | -.29(-3) |
| 41 12 | 131 | 1.9 | 1(-7) | .19(-3) | -.74(-4) |
| 41 16 | 131 | 1.9 | 1(-7) | .90(-4) | -.75(-4) |
| 41 20 | 131 | 1.9 | 1(-7) | .90(-4) | -.75(-4) |
| 61 12 | 281 | 1.9 | 1(-7) | .17(-3) | -.26(-4) |
| 61 16 | 283 | 1.9 | 1(-7) | .32(-4) | -.27(-4) |
| 61 20 | 281 | 1.9 | 1(-7) | .30(-4) | -.26(-4) |
| 61 24 | 286 | 1.9 | 1(-7) | .32(-4) | -.27(-4) |

The third test problem was to solve the Stokes equations

$$\begin{aligned} \nabla^2 u - \frac{u}{r^2} - 2\frac{1}{r^2}\frac{\partial v}{\partial \phi} - \frac{\partial p}{\partial r} &= 0 \\ \nabla^2 v - \frac{v}{r^2} + 2\frac{1}{r^2}\frac{\partial u}{\partial \phi} - \frac{\partial p}{r\partial \phi} &= 0 \\ \frac{1}{r}\frac{\partial ru}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \phi} &= 0 \end{aligned} \tag{7.9}$$

on the unit disk with the velocity components $u$ and $v$ given on the boundary.

The exact solution was given by

$$\begin{aligned} u(r,\phi) &= r\sin\phi(r\cos\phi - a)(r - a\cos\phi)/R^4 + \frac{1}{2}(r - a\cos\phi)/R^2 \\ v(r,\phi) &= a\, r\sin^2\phi(r\cos\phi - a)/R^4 + \frac{1}{2}a\sin\phi/R^2 \\ p(r,\phi) &= 2r\sin\phi(r\cos\phi - a)/R^4 \end{aligned}$$

with

$$R^2 = r^2 + a^2 - 2ar\cos\phi \tag{7.10}$$

where $a$ had the value 1.5. Notice that for this solution the polar representation of the solution at the origin is multiply valued with

$$(u(0,\phi), v(0,\phi)) = (-\frac{\cos\phi}{2a}, \frac{\sin\phi}{2a}), \tag{7.11}$$

which corresponds to a vector of magnitude $(2a)^{-1}$ in the direction of the negative x-axis.

The system (7.9) was approximated using the discrete laplacian as given in (3.3) with the Fourier approximation of derivatives with respect to $\phi$. The system was solved with the iterative method as given in Strikwerda [1984b] with the LSOR method used to update the velocities. Explicitly the formulas are:

$$\begin{aligned} &( -\frac{2}{\Delta r^2} + \frac{1}{r_i^2}\frac{\partial^2}{\partial\phi^2} )(\Delta u_{i,j}^{\nu}) \\ &\quad = -\omega(\frac{1}{r_i\Delta r}\left(r_{i+1/2}\frac{(u_{i+1,j}^{\nu} - u_{i,j}^{\nu})}{\Delta r^2} - r_{i-1/2}\frac{(u_{i,j}^{\nu} - u_{i-1,j}^{\nu+1})}{\Delta r^2}\right) \\ &\quad + \frac{1}{r_i^2}\frac{\partial^2 u_{i,j}^{\nu}}{\partial\phi^2} - \frac{u_{i,j}^{\nu}}{r_i^2} - \frac{2}{r_i^2}\frac{\partial v_{i,j}^{\nu}}{\partial\phi} \\ &\quad - \left(\frac{p_{i+1,j}^{\nu} - p_{i-1,j}^{\nu}}{2\Delta r} - \frac{p_{i+2,j} - 3p_{i+1,j} + 3p_{i,j} - p_{i-1,j}}{6\Delta r}\right)) \end{aligned} \tag{7.12}$$

and

$$\begin{aligned}
&\left( -\frac{2}{\Delta r^2} + \frac{1}{r_i^2}\frac{\partial^2}{\partial \phi^2} \right)(\Delta v_{i,j}^{\nu}) \qquad (7.13)\\
&= -\omega\Big(\frac{1}{r_i \Delta r}\left(r_{i+1/2}\frac{(v_{i+1,j}^{\nu} - v_{i,j}^{\nu})}{\Delta r^2} - r_{i-1/2}\frac{(v_{i,j}^{\nu} - v_{i-1,j}^{\nu+1})}{\Delta r^2}\right)\\
&+ \frac{1}{r_i^2}\frac{\partial^2 v_{i,j}^{\nu}}{\partial \phi^2} - \frac{v_{i,j}^{\nu}}{r_i^2} + \frac{2}{r_i^2}\frac{\partial u_{i,j}^{\nu}}{\partial \phi} - \frac{1}{r}\frac{\hat{\partial} p_{i,j}}{\partial \phi}\Big)
\end{aligned}$$

with

$$u_{i,j}^{\nu+1} = u_{i,j}^{\nu} + \Delta u_{i,j}^{\nu}$$

and

$$v_{i,j}^{\nu+1} = v_{i,j}^{\nu} + \Delta v_{i,j}^{\nu}$$

The pressure was updated by

$$\begin{aligned}
p_{i,j}^{\nu+1} &= p_{i,j}^{\nu} - \gamma\left(\frac{1}{r_i}\frac{(r_{i+1}u_{i+1,j}^{\nu+1} - r_{i-1}u_{i-1,j}^{\nu+1})}{2\Delta r}\right. \qquad (7.14)\\
&- \left.\frac{u_{i+1,j}^{\nu+1} - 3u_{i,j}^{\nu+1} + 3u_{i-1,j}^{\nu+1} - u_{i-2,j}^{\nu+1}}{6\Delta r} + \frac{1}{r_i}\frac{\hat{\partial} v_{i,j}^{\nu+1}}{\partial \phi}\right),
\end{aligned}$$

where $\gamma$ is an iteration parameter, as described in Strikwerda [1984b] The third-order differences with respect to $\gamma$ in (7.12) and (7.14) are necessary to preserve the regularity of the scheme and hence the smoothness of the solution, e.g. Bube and Strikwerda [1983] and Strikwerda [1984a]. The derivatives with respect to $\phi$ which are marked with a carat in (7.13) and (7.14) are computed as in the Fourier method with the addition of the term

$$\pm\, b_{J/2}(\tfrac{J}{4})(-1)^j, \qquad (7.15)$$

where the plus sign is used in (7.14) and the minus sign in (7.13).These terms are included to ensure the regularity of the scheme and hence the smoothness of the solution. Without terms such as these the solution would contain Fourier modes with wavelength $2\Delta\phi$ of sufficient amplitude to affect the accuracy of the solution.

The results of test problem 3 are displayed in Tables 4 and 5. In Table 4 the $l^2$ norms of the error are displayed for the velocity components and the pressure. That is,

$$\|u_{\mathrm{err}}\| = \left(\frac{1}{\pi}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}|u(r_i,\phi_j) - u_{i,j}|^2 r_i \Delta r \Delta\phi\right)^{1/2}$$

where the initial factor of $\pi^{-1}$ is included to normalize by the area of the disc. The error for $v$ is computed similarly. The expression $u(r_i, \phi_j)$ is the exact solution evaluated at $(r_i, \phi_j)$ and $u_{i,j}$ is the computed solution at that grid point.

Because the pressure is defined only to within an additive constant we use

$$\|p_{\text{err}}\| = \left( \frac{1}{\pi} \sum_{i=1}^{I} \sum_{j=0}^{J-1} | \, p(r_i, \phi_j) - p_{i,j} - \overline{\Delta p} \, |^2 \, \varepsilon_i r_i \Delta r \Delta \phi \right)^{1/2} \tag{7.16}$$

where

$$\varepsilon_i = \begin{cases} 1/2, & \text{if } i = 0 \quad \text{or} \quad I; \\ 1, & \text{otherwise,} \end{cases}$$

as required by the trapezoid rule (6.7), and $\overline{\Delta p}$ is the average value of $p(r_i, \phi_j) - p_{i,j}$ computed over the disc, i.e.

$$\overline{\Delta p} = \frac{1}{\pi} \sum_{i=1}^{I} \sum_{j=0}^{J-1} (p(r_i, \phi_j) - p_{i,j}) \varepsilon_i r_i \Delta r \Delta \phi. \tag{7.17}$$

**Table 4. Norm Errors for Problem 3**

| $I$ | $J$ | $\|u_{\text{err}}\|$ | $\|v_{\text{err}}\|$ | $\|p_{\text{err}}\|$ | $\delta_h$ |
|---|---|---|---|---|---|
| 11 | 16 | 1.1(-3) | 9.1(-4) | .34 | -1.5(-3) |
| 11 | 24 | 3.9(-5) | 3.3(-5) | 3.2(-2) | -5.9(-5) |
| 21 | 32 | 2.1(-6) | 2.1(-6) | 4.0(-3) | -2.3(-6) |
| 41 | 40 | 1.1(-7) | 1.2(-7) | 3.6(-4) | -1.5(-7) |

Also displayed in Table 4 is the value of $\delta_h$ which is the average of the discrete approximation to the divergence of the velocity field. The finite difference and Fourier scheme does not enforce the condition

$$\vec{\nabla} \cdot \vec{u} = 0$$

on the discrete solution, rather the iterative method converges to a solution with

$$\vec{\nabla}_h \cdot \vec{u}_{i,j} = \delta_h \tag{7.18}$$

where $\delta_h$ is the average of the left-hand side of (7.18). Thus the value of $\delta_h$ is an

*a posteriori* indicator of the accuracy of the discrete solution. As seen in Table 4 the numerical method gives very good solutions to the Stokes equation.

**Table 5. Errors at the center for Problem 3**

| $I$ | $J$ | $u$ | $v$ | $p$ |
|---|---|---|---|---|
| 11 | 16 | 2.8(-3) | 8.1(-4) | -1.8(-7) |
| 11 | 24 | 8.6(-5) | 1.8(-3) | 1.7(-7) |
| 21 | 32 | 7.3(-5) | 4.0(-4) | 9.9(-8) |
| 41 | 40 | 7.2(-6) | 1.1(-4) | 1.2(-7) |

Table 5 displays the errors at the center for test problem 3. The errors displayed for $u$ and $v$ are the errors for $\phi$ equal to 0 at the origin. The error in the pressure is

$$p(0) - p_{i,j} - \overline{\Delta p}$$

where $\overline{\Delta p}$ is the average of the difference between $p(r_i, \phi_j)$ and $p_{i,j}$ taken over the whole disc given by (7.17). Note that the computation of $\overline{\Delta p}$ does not use the values at the origin.

**Table 6. Iteration Parameters for Problem 3**

| $I$ | $J$ | $\gamma$ | $\omega$ | *tol* | iterations |
|---|---|---|---|---|---|
| 11 | 16 | .4 | 1.6 | 10(-4) | 73 |
| 11 | 24 | .4 | 1.6 | 10(-4) | 65 |
| 21 | 32 | .2 | 1.7 | 10(-4) | 188 |
| 41 | 40 | .1 | 1.8 | 10(-5) | 596 |

Table 6 gives the iteration parameters and the resulting number of iterations for each of the cases reported in Tables 4 and 5. The iterative method was considered to have converged when the successive changes in $u$ and $v$ were less than than *tol* times $\omega$ and when the changes in $p$ deviated from its average value by less than *tol* times $\gamma$. That is, when the value of

$$\left( \sum_{i=0}^{I} \sum_{j=0}^{J-1} (\Delta p_{i,j}^{\nu} - \overline{\Delta p^{\nu}})^2 \epsilon_i r_i \Delta r \Delta \phi \right)^{1/2} \tag{7.19}$$

was less than *tol* times $\gamma$ the solution was considered converged.

The values of the expressions (7.16) and (7.19) was computed in one pass through the data by the following modification of West's algorithm [West (1979)]. To compute the value of

$$Q = \sum_{i,j=0}^{I,J-1} (X_{ij} - \overline{X})^2 \alpha_{ij}$$

where

$$\overline{X} = \sum_{i,j=0}^{I,J-1} X_{ij} \alpha_{ij} / \sum_{i,j=0}^{I,J-1} \alpha_{ij}$$

initialize with

$$\begin{aligned} A &= 0 \\ Q &= 0 \\ \overline{X} &= 0 \end{aligned}$$

Then for $i = 0$ to $I$, and $j = 0$ to $J - 1$,

$$\begin{aligned} Q &\longleftarrow Q + A\alpha_{ij}(X_{ij} - \overline{X})^2/(A + \alpha_{ij}) \\ \overline{X} &\longleftarrow (A\overline{X} + \alpha_{ij} X_{ij})/(A + \alpha_{ij}) \\ A &\longleftarrow A + \alpha_{ij}. \end{aligned} \tag{7.20}$$

Thus, to compute the expression (7.19) we have

$$X_{ij} = \Delta p_{i,j}^{\nu}$$

and

$$\alpha_{ij} = \begin{cases} r_i \Delta r \Delta \phi, & i \neq I \\ \frac{1}{2} r_i \Delta r \Delta \phi, & i = I, 0. \end{cases}$$

This algorithm makes the computation of expressions (7.16) and (7.19) only slightly more difficult than the computation of the usual norm.

## Conclusions

The results of the test problems show that the methods presented in this paper can be used to compute accurate solutions to equations on domains with polar grids. The basic

formulas can be used with most numerical procedures. The use of the Fourier method, while not essential to the center formulas, is very convenient and efficient to use with polar grids.

## References


1. K. Bube and J. Strikwerda (1983), Interior regularity estimates for elliptic systems of difference equations, SIAM J. Num. Anal., 20, pp. 639-656.

2. D. Gottlieb and S.A. Orszay, (1977) Numerical Analysis of Spectral Methods: Theory and Applications, SIAM, Philadelphia, PA.

3. Y. Nagel and J.C. Strikwerda (1986), A numerical study of the flow in a spinning and coning cylinder, to appear.

4. J.C. Strikwerda (1984a), Finite difference methods for the Stokes and Navier-Stokes equations. SIAM J. Sci. Stat. Comput. 5, pp. 56-68.

5. J.C. Strikwerda (1984b), An iterative method for solving finite difference approximations to the Stokes equations. SIAM J. Numer. Anal. 21, pp. 447-458.

6. P.N. Swarztrauber and R.A. Sweet (1973), The direct solution of the discrete Poisson equation on a disk, SIAM J. Numer. Anal. 10, pp. 900-907.

7. P.N. Swarztrauber (1974), The direct solution of the discrete Poisson equation on the surface of a sphere, J. Comp. Phys. 15, pp. 46-54.

8. D.H.D. West (1979), Updating mean and variance estimates: An improved method, Comm. ACM, 22, pp. 532-535.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | |
|---|---|
| 1. REPORT NUMBER: #2934 | 2. GOVT ACCESSION NO.: AD-A167512 |
| 3. RECIPIENT'S CATALOG NUMBER: | |
| 4. TITLE (and Subtitle): Finite Difference Methods for Polar Coordinate Systems | 5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period |
| | 6. PERFORMING ORG. REPORT NUMBER: |
| 7. AUTHOR(s): John C. Strikwerda and Yvonne Nagel | 8. CONTRACT OR GRANT NUMBER(s): MCS-8306880; N00014-84-K-0454 DAAG29-80-C-0041 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53705 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 3 - Numerical Analysis and Scientific Computing |
| 11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below | 12. REPORT DATE: April 1986 |
| | 13. NUMBER OF PAGES: 17 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report): UNCLASSIFIED |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

Office of Naval Research
Dept. of the Navy
Arlington, VA 22217

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite Difference Methods, Polar Coordinates

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We discuss finite difference methods for partial differential equations on polar and spherical coordinate systems. The distinctive feature of these coordinate systems is the coordinate system singularity at the origin. We show how to accurately and conveniently determine the solution at the origin for both scalar and vector fields. We also discuss the Fourier method to approximate derivatives with respect to the angular variable in polar coordinates. Computational examples are presented illustrating the accuracy and efficiency of the method for hyperbolic and elliptic equations, and also for the computation of vector fields at the origin.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

DTIC

6-86