AD-A167 492 ON A THEOREM OF WEINSTEIN(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER P H RABINOWITZ FEB 86 MRC-TSR-2912 DAAG29-80-C-0041 1/1

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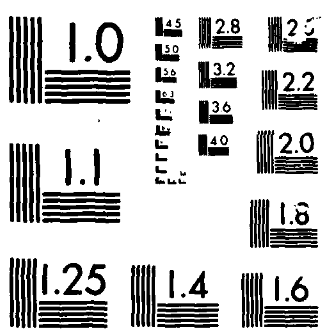

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AD-A167 492

MRC Technical Summary Report #2912

ON A THEOREM OF WEINSTEIN

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February 1986

(Received February 6, 1986)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

This paper contains an existence theorem for a special class of periodic solutions of Hamiltonian systems and generalizes an earlier result of the same nature due to Weinstein.

AMS (MOS) Subject Classifications: 34C25, 58E05, 58F22, 70H05, 70H25, 70K99

Key Words: periodic solution, Hamiltonian system, Minimax methods.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Weinstein studied a general class of mechanical systems and established the existence of special kinds of periodic solutions which he called brake orbits. Roughly speaking these solutions start at the boundary of a potential well with 0 velocity and return to the edge of the well, again with 0 velocity, a half period later. This paper contains a generalization of his results using minimax methods from the calculus of variations as the existence tool.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON A THEOREM OF WEINSTEIN

Paul H. Rabinowitz

§1. Introduction

In a study of periodic solutions of Hamiltonian systems, Weinstein [1] considered Hamiltonians of the form $H(p,q) = K(p,q) + V(q)$. Here the potential energy V satisfies

(V_1) $\mathcal{D} \equiv \{q \in \mathbb{R}^n | V(q) < 1\}$ is diffeomorphic to $\overline{B_1}$,

the closed unit ball in $\mathbb{R}^n$, and $V_q(q) \neq 0$ on $\partial\mathcal{D}$

while the kinetic energy satisfies

(K_1) $K(0,q) = 0$, K is even and strictly convex in p for fixed q, and

$K(\alpha p, q) \to \infty$ as $|\alpha| \to \infty$ uniformly for $p \in S^{n-1}$ and $q \in \mathcal{D}$.

Solutions of the corresponding Hamiltonian system:

$$\text{(HS)} \qquad \dot{p} = -H_q(p,q) \quad , \quad \dot{q} = H_p(p,q)$$

for which there exists a $T > 0$ and $Q_1, Q_2 \in \partial\mathcal{D}$ such that $p(0) = 0 = p(T)$ and $q(0) = Q_1$, $q(T) = Q_2$ were called brake orbits by Weinstein. Due to the evenness of K in p, by extending q as an even function and p as an odd function about 0 and T, the resulting functions (P,Q) satisfy (HS) and are $2T$ periodic. Thus brake orbits are special kinds of periodic solutions of (HS).

In [1] Weinstein proved:

Theorem 1.1: If $K, V \in C^2$ and satisfy (V_1) and (K_1), then (HS) possesses a brake orbit on $H^{-1}(1)$.

Theorem 1.1 generalizes an earlier result of Seifert [2] for $K(p,q) = \sum_{i,j=1}^{n} a_{ij}(q)p_ip_j$ with the matrix $(a_{ij}(q))$ uniformly positive definite in $\mathcal{D}$. Motivated by [1], it was proved in [3] that:

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__Theorem 1.2:__ If K, $V \in C^2$, V satisfies (V_1), and K satisfies

(K_2) $K(0,q) = 0$, $p \cdot K_p(p,q) > 0$ if $p \neq 0$, and $K(\alpha p,q) \to \infty$ as $|\alpha| \to \infty$ uniformly for $p \in S^{n-1}$ and $q \in \mathcal{D}$,

then (HS) possesses a periodic solution on $H^{-1}(1)$.

Theorem 1.2 drops the evenness assumption of Theorem 1.1 and replaces the convexity condition by the milder restriction that $p \cdot K_p > 0$ if $p \neq 0$. The price paid for this added generality is that Theorem 1.2 asserts the existence of a periodic solution rather than a brake orbit. A natural question to pose is whether (HS) possesses a brake orbit if in Theorem 1.2, K is also assumed to be even in p. Our main goal here is to resolve this question and show:

__Theorem 1.3:__ If K, $V \in C^2$, V satisfies (V_1), K satisfies (K_2) and is even in p, then (HS) possesses a brake orbit on $H^{-1}(1)$.

The existence approach taken in [3] was to reduce the solution of (HS) to that of finding a critical point of a corresponding functional. This latter problem was solved by a finite dimensional approximation argument together with appropriate estimates which permitted passage to a limit. In finding a critical point of the finite dimensional problem, a key role was played by an S^1 symmetry which the functional possesses. In trying to prove Theorem 1.3, a natural approach is to work in a class of functions in which p and q have the desired form. However by doing so, the functional losses the S^1 symmetry and the corresponding existence mechanism used in proving Theorem 1.2 breaks down. Thus the main difficulty here is to find a new approach to the existence question which overcomes the loss of symmetry. This is provided by some minimax ideas used in a recent paper [4].

In §2, the variational problem which yields Theorem 1.3 will be formulated and the finite dimensional approximation carried out. Aside from the new existence mechanism, several of the steps and details here are quite close to those of [3]. Therefore we will be sketchy at times and refer to [3] as appropriate.

§2. The proof of Theorem 1.3

In Theorem 2.1 of [3], a canonical transformation is made which converts the potential well, $\mathcal{D}$, given by (V_1) to the closed unit ball $\overline{B}_1$. Observing that the transformed Hamiltonian still satisfies (K_2) with $\mathcal{D}$ replaced by $\overline{B}_1$ and is even in p, Theorem 1.3 reduces to proving:

Theorem 2.1: Let $H = K+V$ where $K, V \in C^1$ and satisfy

(V_1') $\{q \in \mathbb{R}^n \mid 0 < V < 1\} = B_1$, $V = 1$ and $V_q(q) \neq 0$ on ∂B_1,

(K_2') $K(0,q) = 0$, K is even in p, $p \cdot K_p(p,q) > 0$ if $p \neq 0$, and $K(\alpha p,q) \to \infty$ as $|\alpha| \to \infty$ uniformly for $p \in S^{n-1}$ and $q \in B_1$.

Then (HS) possesses a brake orbit in $H^{-1}(1)$.

Set $M = H^{-1}(1)$. Hypothesis (V_1') amd (K_2') imply M is a compact C^2 manifold in $\mathbb{R}^{2n}$ which bounds a neighborhood of 0 in $\mathbb{R}^{2n}$. Let $z = (p,q)$. When convenient we write $H(p,q) \equiv H(z)$. A standard lemma - see e.g. [1] or [3] for $H \in C^2$ or [5] for the C^1 case - states that if $\overline{H}$ is a new Hamiltonian such that $\overline{H}^{-1}(1) = M$ and $\overline{H}_z \neq 0$ on M, any solution of

$$\dot{\xi} = -\overline{H}_q(\xi,\eta) \qquad \dot{\eta} = \overline{H}_p(\xi,\eta) \tag{2.2}$$

on M is a reparametrization of a solution of (HS) on M. Thus if there is an $\overline{H}$ for which (2.2) has a brake orbit on M, we easily get a brake orbit for (HS) on M. A particular such $\overline{H}$ is constructed in [3] and will also be employed here. For the convenience of the reader, we recall its construction.

By (V_1'), there are constants $\delta, \alpha > 0$ such that

$$V_q(q) \cdot q > \alpha V(q) > \frac{\alpha}{2}$$

for $|q| \in [1-2\delta, 1+2\delta]$ and there is a constant $\mu = \mu(\delta) > 0$ such that if $V(q) > 1-\mu$ and $|q| < 1+2\delta$, then $|q| > 1-\delta$. Since $K(0,q) = 0$, there is a constant $\mu_1(\delta)$ such that $K(p,q) < \mu/2$ if $|p| < \mu_1$ and $|q| < 1+2\delta$. Since M is compact, there is a constant $M_1 > 0$ such that $M \subset B_{M_1}$. Using (K_2') it can further be assumed without loss of generality that $\min(K(p,q), |p|^2) > 1$ if $|p| > M_1$ and $|q| < 1$.

Now for $a, b \in \mathbb{R}$ and $a < b$, let $\chi(s;a,b) \geq 1$ if $s \leq a$; $= 0$ if $s \geq b$; and $\frac{d\chi}{ds} < 0$ if $s \in (a,b)$. Four such cut-off functions will be used:

$$\chi_1(p) = 1 - \chi(|p|; \frac{\mu_1}{2}, \mu_1) \quad ,$$

$$\chi_2(q) = \chi(|q|; 1-2\delta, 1+2\delta) \quad ,$$

$$\chi_3(q) = \chi(|q|; 1, 1+\delta) \quad ,$$

$$\chi_4(p) = \chi(|p|; M_1+1, M_1+2) \quad .$$

Now the new Hamiltonian $\overline{H}$ is defined via

$$\overline{H}(p,q) = \overline{K}(p,q) + \overline{V}(p,q)$$

where

$$\overline{V}(p,q) = \chi_1(p)\chi_2(q)V(q) + (1-\chi_2(q))\chi_3(q)V(q) + \rho_1(1-\chi_3(q))|q|^2$$

and

$$\overline{K}(p,q) = \chi_4(p)\chi_3(q)K(p,q) + \rho_2(1-\chi_4(p))|p|^2 \quad .$$

The constants ρ_1 and ρ_2 are chosen large enough such that the following result holds:

Proposition 2.3: $\overline{H}$ possesses the following properties:

($\overline{H}_1$) $\overline{H}^{-1}(1) = M$,

($\overline{H}_2$) If H is even in p, so is $\overline{H}$,

($\overline{H}_3$) $p \cdot \overline{H}_p(p,q) \geq 0$ for all p,q,

($\overline{H}_4$) $q \cdot \overline{H}_q(0,q) \geq 0$ with strict inequality if $|q| > 1-\delta$,

($\overline{H}_5$) There are constants $a_1, a_2 > 0$ such that $\overline{H}(z) \geq a_1|z|^2 - a_2$ for all $z \in \mathbb{R}^{2n}$.

Proof: See Lemmas 2.14 and 2.17 of [3].

Above remarks justify proving Theorem 2.1 with H replaced by $\overline{H}$. By making the change of time scale $t \to \pi t T^{-1} \equiv \lambda^{-1} t$ where T is the unknown half period, T becomes π and (2.2) can be replaced by

(2.4) $$\dot{p} = -\lambda \overline{H}_q \quad , \quad \dot{q} = \lambda \overline{H}_p \quad .$$

Thus we seek $\lambda > 0$, p odd about 0 and π and q even about 0 and π such that

$(p(t), q(t))$ lies on $\bar{H}^{-1}(1)$ and satisfies (2.4). As in [3], it is convenient to make one further technical modification of (2.4). For $\varepsilon > 0$, set

$$\bar{H}_\varepsilon(z) = \bar{H}(z) + \varepsilon|p|^2 \quad .$$

Then

$$p \cdot \bar{H}_{\varepsilon p}(z) > 2\varepsilon|p|^2 \quad . \tag{2.5}$$

Our strategy is to find a solution of the desired type for

$$\dot{p} = -\lambda\bar{H}_{\varepsilon q} \quad , \quad \dot{q} = \lambda\bar{H}_{\varepsilon p} \tag{2.6}$$

on $\bar{H}_\varepsilon^{-1}(1)$. Then letting $\varepsilon \to 0$ will produce a solution of (2.4).

Let

$$X = \{z = (p,q) \in W^{1,2}(S^1,\mathbb{R}^{2n}) \mid p \text{ is odd and } q \text{ is even about } 0 \text{ and } \pi\} .$$

For $z \in X$, set

$$\Psi(z) \equiv \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_\varepsilon(z)dt$$

and

$$A(z) \equiv \int_0^{2\pi} p \cdot \dot{q} \, dt \quad .$$

We will find a critical point of A on

$$S \equiv \{z \in X \mid \Psi(z) = 1\}$$

and show that this provides a solution of (2.6) on $\bar{H}_\varepsilon^{-1}(1)$. Technical problems make it difficult to treat $A|_S$ directly. Therefore a finite dimensional approximation argument will be used.

Let $e_1,\ldots,e_{2n}$ denote the usual basis in $\mathbb{R}^{2m}$ and set

$$X^0 \equiv \text{span } \{e_k \mid n+1 < k < 2n\}$$

$$X_m^+ \equiv \text{span } \{\varphi_{jk} = (\sin jt)e_k - (\cos jt)e_{k+n} \mid 1 < k < n,\ 1 < j < m\}$$

$$X_m^- \equiv \text{span } \{\psi_{jk} = (\sin jt)e_k + (\cos jt)e_{k+n} \mid 1 < k < n,\ 1 < j < m\} .$$

It is easily checked that for fixed m, these subspaces of X are mutually orthogonal in $L^2(S^1,\mathbb{R}^{2n})$. Let $X_m \equiv X_0 \oplus X_m^+ \oplus X_m^-$. For $z = z^0 + z^+ + z^- \in X_m$, a computation shows

$$(2.7)\qquad \begin{cases} A(z) = A(z^+) + A(z^-) \\ A(z^+) > \pi\|z^+\|^2_{W^{1/2,2}(S^1,\mathbb{R}^{2n})} \\ A(z^-) < -\pi\|z^-\|^2_{W^{1/2,2}(S^1,\mathbb{R}^{2n})} \end{cases}$$

where e.g. if $z \in X_m$ and $z = \Sigma\, a_{jk}\varphi_{jk}$, $\|z\|^2 = \Sigma j\, a_{jk}^2$. Let $S_m \equiv S \cap X_m$.

Lemma 2.8: There is a constant $\sigma_m > 0$ such that for each $|\sigma| < \sigma_m$, $\{z \in X_m \mid \psi(z) = 1+\sigma\}$ (and in particular S_m) is a compact C^1 manifold in X_m which bounds a neighborhood of 0.

Proof: This follows from (2.5), $(\bar{H}_4)$, $(\bar{H}_5)$, and a slight modification of the proof of Lemma 3.2 of [3].

If A and ψ are extended to $W^{1,2}(S^1,\mathbb{R}^{2n})$, they are invariant under the set of translations $g_\theta(z) = z(t+\theta)$ for $\theta \in \mathbb{R}$, i.e. $A(g_\theta z) = A(z)$, $\Psi(g_\theta(z)) = \Psi(z)$ for all such θ and z. This fact allows the use of an index theory for S^1 actions which plays a key role in proving Theorem 1.2. Unfortunately X is not invariant under $\{g_\theta\}$ and to make up for this loss of symmetry, a version of a minimax argument used in [4] will be employed. Towards this end, some comparison constants and sets must be introduced. Let

$$\alpha_m \equiv \inf_{z\in S_m \cap X_m^+} A(z) \quad .$$

Set

$$W_m \equiv X_0 \oplus \operatorname{span}\{\varphi\} \oplus X_m^-$$

where $\varphi = \varphi_{11}$ and let

$$\beta_m \equiv \sup_{z\in W_m \cap S_m} A(z) \quad .$$

Lemma 2.9: $0 < \alpha_m < \beta_m < \infty$.

Proof: By Lemma 2.8, for r_m sufficiently small, the ball B_{r_m} in X_m lies inside

S_m. Hence (2.7) shows

$$\alpha_m \geq \pi \inf_{z \in B_{r_m} \cap X_m^+} \|z\|^2_{W^{1/2,2}} > 0 .$$

Since S_m is compact, $\beta_m < \infty$. Finally since $W_m \cap X_m^+ = \mathrm{span}\{\varphi\}$,

$$\alpha_m \leq \inf_{\mathrm{span}\{\varphi\} \cap S_m} A \leq \beta_m .$$

Now let

$$\Gamma_m \equiv \{h \in C(X_m, X_m) \mid h \text{ satisfies } (h_1) - (h_3)\}$$

where

(h_1) $h = \mathrm{id}$ if $|\Psi(z) - 1| \geq \frac{1}{2}$ or if $A(z) \notin [0, \beta_m + 1]$

(h_2) $h : S_m \to S_m$

(h_3) h is 1-1.

Clearly $\Gamma_m \neq \phi$ since $\mathrm{id} \in \Gamma_m$.

A critical value of $A|_{S_m}$ can now be defined. Let

$$c_m = \inf_{h \in \Gamma_m} \max_{z \in W_m \cap S_m} A(h(z)) . \tag{2.10}$$

Since $\mathrm{id} \in \Gamma_m$, $c_m \leq \beta_m < \infty$. To prove that c_m is indeed a critical value of $A|_{S_m}$, two preliminary results are needed. The first is a crucial intersection theorem.

<u>Proposition 2.11</u>: If $h \in \Gamma_m$, $h(W_m \cap S_m) \cap X_m^+ \neq \phi$.

<u>Proof</u>: Let $h \in \Gamma_m$. Note that $A \leq 0$ on $X^0 \oplus X_m^-$ and therefore by (h_1), $h = \mathrm{id}$ on this set. By $(\overline{H}_5)$,

$$\Psi(z) \geq a_3 \|z\|^2_{L^2} - a_4 \tag{2.12}$$

so $\Psi(z) \to \infty$ as $z \to \infty$. Hence by (2.12), there is an $R_m > 1$ such that for z outside the L^2 ball B_{R_m} in X_m, $\psi(z) > 1$. Therefore by (h_1) $h = \mathrm{id}$ on this set. Let

$$Q_m \equiv B_{R_m} \cap (X^0 \oplus \{r\varphi \mid r > 0\} \oplus X_m^-) .$$

We will find $z_m \in Q_m \cap S_m$ such that

$$h(z_m) \in X_m^+ \tag{2.13}$$

thereby establishing the Proposition. Let P^0, P_m^+, P_m^- denote respectively the (L^2) orthogonal projectors of X_m onto X^0, X_m^+, X_m^-. Satisfying (2.13) for $z_m \in S_m$ is equivalent to

$$(2.14)\qquad \begin{cases} \text{(i)} & \Psi(z_m) = 1 \\ \text{(ii)} & (P_0 + P_m^-)h(z_m) = 0. \end{cases}$$

If $z \in Q_m$, $z = y+r\varphi \equiv (y,r)$ where $y \in X^0 \oplus X_m^-$. Let

$$\Phi(y,r) \equiv ((P_0 + P_m^-)h(y+r\varphi),\ \Psi(y+r\varphi)) \quad .$$

Identifying W_m with $\mathbb{R}^{(n+1)m} \times \mathbb{R}$ and Q_m with a subset thereof, $\Phi : \overline{Q_m} \to \mathbb{R}^{(n+1)m} \times \mathbb{R}$. Consider the Brouwer degree of Φ with respect to the bounded open set Q_m and the point $(0,1)$. This degree will be denoted by $d(\Phi,Q_m,(0,1))$. It is defined provided that $\Phi \neq (0,1)$ on ∂Q_m. But if $(y,r) \in \partial Q_m$, either $r = 0$ in which case $A(y) < 0$, $h(y) = y$, and $\Phi(y,0) = (y,\Psi(y)) \neq (0,1)$, or $\|y+r\varphi\|_{L^2} = R_m$ so that $h(y+r\varphi) = y+r\varphi$ by the choice of R_m and $\Phi(y,r) = (y,\Psi(y+r\varphi))$ with $\Psi(y+r\varphi) > 1$. Thus $d(\Phi,Q_m,0)$ is defined.

We claim Φ is homotopic to the identity map on ∂Q_m, the homotopy avoiding $(0,1)$. Then

$$(2.15)\qquad d(\Phi,Q_m,(0,1)) = d(id,Q_m,(0,1)) = 1$$

since $R_m > 1$ and therefore $(0,1) \in Q_m$. To verify the claim, observe that the argument showing that $d(\Phi,Q_m,(0,1))$ is defined implies $\Phi(y,r) = (y,\Psi(y+r\varphi))$ for $(y,r) \in \partial Q_m$. Thus we only need construct an appropriate homotopy of $\Psi(y+r\varphi)$ to r on ∂Q_m. For $\theta \in [0,1]$, let

$$\Phi_\theta(y,r) = (y,\ \theta\ \Psi(y+r\varphi) + (1-\theta)r) \quad .$$

If $\Phi_\theta(y,r) = (0,1)$ on ∂Q_m, then $y = 0$. Therefore $r = 0$ or $r = R_m$. But if $r = 0$, $\Phi_\theta(0,0) = (0,0) \neq (0,1)$ while if $r = R_m$, $\Phi_\theta(0,R_m) = (0,\theta\Psi(R_m\varphi) + (1-\theta)R_m) \neq (0,1)$ since $\Psi(R_m\varphi)$, $R_m > 1$. Thus (2.15) holds and by the properties of Brouwer degree, there exists a $z_m \in Q_m$ satisfying (2.14). Finally since $h(z_m) \in S_m$, $(h_1) - (h_2)$ imply $z_m \in S_m$ and the proposition is proved.

Corollary 2.16: $c_m > \alpha_m$.

Proof: If $h \in \Gamma_m$, by Proposition 2.11,

(2.17) $$\max_{z \in W_m \cap S_m} A(h(z)) > \inf_{w \in X_m^+} A(w) \equiv \alpha_m$$

Since (2.17) holds for all $h \in \Gamma_m$, $c_m > \alpha_m$.

Next a version of a standard Deformation Theorem is required.

Proposition 2.18: If c_m is not a critical value of $A|_{S_m}$, there is an $r > 0$ and $\eta \in C([0,1] \times X_m, X_m)$ such that

1°. $\eta(s,\cdot)$ is a homeomorphism of X_m onto X_m for each $s \in [0,1]$,

2°. $\eta(1,z) = z$ if $A(z) \in [0,\beta_m+1]$ or if $|\Psi(z)-1| > \frac{1}{2}$,

3°. $\eta(s,\cdot) : S_m \to S_m$ for each $s \in [0,1]$,

4°. Letting $A_\sigma \to \{x \in S_m \mid A(x) < \sigma\}$,

$$\eta(1, A_{c_m+r}) \subset A_{c_m-r}.$$

Proof: This result is essentially in the literture - see e.g. [6 - 8] - so we will be rather sketchy. Suppose first that $\tilde{H} \in C^2$. Then η is determined as the solution of an ordinary differential equation in X_m of the form:

(2.19) $$\frac{d\eta}{ds} = -\omega(\eta)\,[A'(\eta) - \lambda(\eta)\Psi'(\eta)]$$
$$\eta(0,z) = z$$

where

$$\lambda(\eta) = A'(\eta) \cdot \Psi'(\eta)\|\Psi'(\eta)\|^{-2} .$$

In (2.19), ω is a locally Lipschitz continuous function satisfying $0 < \omega(\cdot) < 1$, $\omega(\cdot) = 0$ outside of a small neighborhood of S_m, $\omega(\cdot) = 0$ if $A(z) \notin [0,\beta_m+1]$, and ω is such that the right hand side of (2.19) is < 1 in norm. In the construction of ω, use is made of the fact that $c_m \in [\alpha_m,\beta_m]$ with $\alpha_m > 0$ as has been verified above. Now since $\Psi' \neq 0$ near S_m (via the proof of Lemma 3.3 of [3]), the right hand side of (2.19) is well defined and locally Lipschitz continuous. Therefore 1° - 3° of Proposition 2.18

follow immediately from our above remarks. Finally 4° is a consequence of a standard argument [7 - 8].

If $\overline{H}$ is merely C^1, $V(\eta) = A'(\eta) - \lambda(\eta)\Psi'(\eta)$ must be replaced by a corresponding pseudo-gradient vector field for V on a neighborhood of S_m which is tangential to S_m. Then the result follows essentially as in the C^2 case. See [7 - 8].

These preliminaries now yield

Proposition 2.20: c_m is a critical value of $A|_{S_m}$.

Proof: If not, let r and η be as given by Proposition 2.18. Choose $h \in \Gamma_m$ such that

(2.21) $$\max_{z \in S_m \cap W_m} A(h(z)) < c_m + r$$

By (2.21) and 3° - 4° of Proposition 2.18,

(2.22) $$\max_{z \in S_m \cap W_m} A(\eta(1,h(z))) < c_m - r \quad .$$

But 1° - 3° of Proposition 2.18 imply $\eta(1,h) \in \Gamma_m$. Thus (2.22) contradicts (2.10).

Remark 2.23: If z_m is a critical point of $A|_{S_m}$ corresponding to c_m, then $\lambda_m = \lambda(z_m)$ is positive. Indeed since

(2.24) $$(A'(z_m) - \lambda\ \Psi'(z_m))\zeta = 0$$

for all $\zeta \in X_m$, choosing $\zeta = (p_m, 0)$ where $z_m = (p_m, q_m)$ shows

(2.25) $$c_m = \int_0^{2\pi} p_m \cdot \dot{q}_m\, dt = \lambda_m \int_0^{2\pi} p_m \cdot \overline{H}_{\epsilon p}(z_m) dt \quad .$$

Since $c_m > 0$, (2.5) and (2.25) imply $\lambda_m > 0$.

The idea now is to let $m \to \infty$ and show that (λ_m, z_m) converge along a subsequence to (λ, z) satisfying (2.6). Some estimates are required to carry this out.

Proposition 2.26: There are constants $0 < \underline{c} < \overline{c} < \infty$ such that

(2.27) $$\underline{c} < c_m < \overline{c}$$

for all $m \in \mathbb{N}$.

Assuming Proposition 2.26 for now, the proof of Lemma 3.22 of [3] - slightly modified since c_m, $\lambda_m > 0$ here while c_m, $\lambda_m < 0$ in [3] - shows there are constants

$0 < \underline{\gamma} < \overline{\gamma} < \infty$ and independent of m such that

$$\underline{\gamma} < \lambda_m < \overline{\gamma} \tag{2.28}$$

for all $m \in \mathbb{N}$. Moreover Lemma 3.32 of [3] then proves the sequence (z_m) is bounded in $W^{1,2}(S^1,\mathbb{R}^{2n})$. Hence, along a subsequence, λ_m converges to $\lambda > 0$ and z_m converges weakly in $W^{1,2}(S^1,\mathbb{R}^{2n})$ and strongly in $C(S^1,\mathbb{R}^{2n})$ to $z \in W^{1,2}(S^1,\mathbb{R}^{2n})$ satisfying

$$(A'(z) - \lambda \Psi'(z))\zeta = 0 \tag{2.29}$$

for all $\zeta \in X$. Setting $\zeta = (u,v)$, equation (2.29) is equivalent to

$$\begin{aligned} &\int_0^{2\pi}[(p \cdot \dot{v}) + (\dot{q}\cdot u) - \lambda[(\overline{H}_{\varepsilon p}(z) \cdot u) + (\overline{H}_{\varepsilon q}(z) \cdot v)]]dt \\ &= \int[((-\dot{p} - \lambda \overline{H}_{\varepsilon q}(z)) \cdot v) + ((\dot{q} - \lambda\overline{H}_{\varepsilon p}(z)) \cdot u)]dt = 0 \end{aligned} \tag{2.30}$$

for all $(u,v) \in X$. Since p and q are continuous, $(H_p(p,q), H_q(p,q)) \in Y$, the closure of X in $L^2(S^1,\mathbb{R}^{2n})$. It then follows from (2.30) that $z = (p,q)$ satisfies (2.6) and $z \in C^1(S^1,\mathbb{R}^{2n})$. Since (2.6) is a Hamiltonian system, $\overline{H}_\varepsilon(z) \equiv$ constant. Hence $z \in S$ implies z lies on $H^{-1}(1)$.

<u>Proof of Proposition 2.26</u>: By a remark following (2.10) and Corollary 2.16, $\alpha_m \le c_m \le \beta_m$. If $\alpha_m \to 0$ as $m \to \infty$ along some subsequence, there is a corresponding sequence (w_m) such that $w_m \in X_m^+ \cap S_m$ and $A(w_m) \to 0$. Therefore by (2.7), $w_m \to 0$ in $W^{1,2}(S^1,\mathbb{R}^{2n})$ and a fortiori in $L^2(S^1,\mathbb{R}^{2n})$. Since $\overline{H}_\varepsilon(z)$ grows at a quadratic rate for large z, Ψ is continuous on $L^2(S^1,\mathbb{R}^{2n})$ and

$$\Psi(w_m) \to \frac{1}{2\pi}\int_0^{2\pi} \overline{H}_\varepsilon(0)dt = 0 \quad .$$

But $\Psi(w_m) = 1$ for all m so α_m must be bounded away from 0 and there is a positive $\underline{c}$ as desired.

Next to show that (β_m) is bounded away from infinity, recall that

$$\beta_m = \sup_{z \in W_m \cap S_m} A(z) \quad .$$

Let $W \equiv X^0 \oplus \mathrm{span}\{\varphi\} \oplus X^-$. Since $W_m \subset W$ and $S_m \subset S$,

$$\beta_m \le \sup_{z \in W \cap S} A(z) \equiv \overline{c} \quad .$$

Thus it suffices to prove that $\overline{c} < \infty$. But if $z \in W$, $z = z^0 + \rho(z)\varphi + z^-$ and

$$(2.27) \qquad A(z) = \rho^2(z)A(\varphi) + A(z^-) < \pi\,\rho^2(z) \quad .$$

By $(\overline{H}_5)$, if $z \in W \cap S$,

$$\Psi(z) > 1 > \frac{1}{2\pi}\int_0^{2\pi}(a_1|z|^2 - a_2)dt = \frac{a_1}{2\pi}\,\|z\|^2_{L^2} - a_2$$

$$> a_1\,\rho(z)^2 - a_2$$

or

$$(2.28) \qquad \rho(z)^2 < \frac{1+a_2}{a_1} \quad .$$

Combining (2.27) and (2.28) yields

$$\overline{c} < \pi\,\frac{1+a_2}{a_1}$$

and the proof is complete.

Remark 2.29: As examination of the above argument shows that $\underline{c}$, $\overline{c}$ can be chosen independently of ε. Therefore $\underline{c} < c_\varepsilon = A(z_\varepsilon) < \overline{c}$ where z_ε is a solution of (2.6) obtained via the finite dimmensional approximation argument.

Completion of the proof of Theorem 2.1: We must show (2.4) has a solution of the desired type. Such solutions $(\lambda_\varepsilon, z_\varepsilon)$ have already been obtained for (2.6). It suffices to prove that $\{\lambda_\varepsilon\}$ is bounded away from 0 and ∞ in $\mathbb{R}$ and $\{z_\varepsilon\}$ is bounded in $W^{1,2}(S^1,\mathbb{R}^{2n})$ for then we can easily pass to a limit to find a solution of (2.4) on $H^{-1}(1)$. But the ε independent bounds for c_ε of Remark 2.29 and Lemma 3.35 of [3] yield the necessary bounds for $\{\lambda_\varepsilon\}$, $\{z_\varepsilon\}$.

Remark 2.30: Gluck and Ziller [9] have proved there are brake orbits for (HS) under more general conditions on V than (V_1). (See also Hayashi [10] and Benci [11] for a special case.) We suspect that Theorem 1.3 holds in their generality with respect to V. The difficulty in proving such a result via the approach given here is to find bounds for the periods of approximate solutions.

Remark 2.31: The existence of a special kind of brake orbit which passes through the origin as well as multiplicity results for (HS) for a subclass of Hamiltonians of the type considered here was proved by van Groesen [12].

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2912	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON A THEOREM OF WEINSTEIN		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Paul H. Rabinowitz		8. CONTRACT OR GRANT NUMBER(s) MCS-8110556. DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE February 1986
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

periodic solution, Hamiltonian system, Minimax methods

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper contains an existence theorem for a special class of periodic solutions of Hamiltonian systems and generalizes an earlier result of the same nature due to Weinstein.

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