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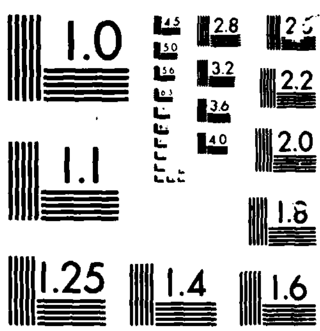

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MRC Technical Summary Report #2905

THE NEUMANN PROBLEM FOR NONLINEAR SECOND ORDER SINGULAR PERTURBATION PROBLEMS

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January 1986

(Received October 30, 1985)

Approved for public release
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Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Singularly perturbed second order elliptic partial differential equations with Neumann boundary conditions arise in many areas of application. These problems rarely have smooth limit solutions. In this paper, ~~we~~ the author characterize the limit solution for a wide class of such problems. ~~We~~ They also give an abstract rate of convergence theorem and apply the abstract theorem to certain finite difference approximations. Keywords;

AMS (MOS) Subject Classifications: 35F30, 65M15

Key Words: singular perturbation, viscosity solution; viscosity inequalities

Work Unit Number 1 (Applied Analysis)

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This work was completed while the author was at the Ecole Normale Superieure.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by the National Science Foundation under Grant No. MCS 82-00676.

THE NEUMANN PROBLEM FOR NONLINEAR SECOND
ORDER SINGULAR PERTURBATION PROBLEMS

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§1. INTRODUCTION.

In this paper, we study the singular perturbation problem for partial differential equations which have the form:

$$(NP_\varepsilon) \qquad -\varepsilon\Delta u_\varepsilon + H(x,u_\varepsilon,\nabla u_\varepsilon) = 0, \quad x \in \Omega ,$$

$$\frac{\partial u_\varepsilon}{\partial n}(x) = \gamma(x), \quad x \in \partial\Omega ,$$

where Ω is a bounded domain in $\mathbb{R}^d$, n is Ω's outward unit normal, u_ε is a scalar unknown and H is a continuous function on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d$. One application that motivates the study of singular perturbation problems of the form (NP_ε) is found in the theory of optimal stochastic control. There, H depends on the deterministic part of a stochastic ODE, a control space and a specified cost function. u_ε can be identified as the optimal cost function. The positive parameter ε in (NP_ε) can be regarded as the intensity of noise in the dynamics equation. Control problems whose trajectories reflect at a boundary, give rise to Neumann problems of the type studied here; see [1] or [17] for a detailed treatment of this topic. One could ask for instance, is the optimal cost function of a stochastic control problem related to the optimal cost of its associated deterministic problem? Are the two close in any way when the noise is small?

As $\varepsilon \downarrow 0$, it is well known that solutions of (NP_ε) do not generally converge to a classical solution of:

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and by the National Science Foundation under Grant No. MCS 82-00676.

$$(NP_0) \qquad H(x,u,\nabla u) = 0, \quad x \in \Omega ,$$
$$\frac{\partial u}{\partial n}(x) = \gamma(x), \quad x \in \partial\Omega .$$

Indeed, (NP_0) does not generally admit a C^1 solution, as can easily be seen by considering the simple example:

$$\frac{du}{dx} + u = 0, \quad x \in [0,1] ,$$

$$\frac{du}{dx}(0) = 0, \quad \frac{du}{dx}(1) = 1 .$$

This example is clearly overdetermined, and here the data at 0 is not compatible with the data at 1. For this reason, a more general class of solutions to (NP_0) must be sought.

A new notion of continuous weak solutions to equations of Hamilton-Jacobi type has recently been introduced. In [4] and [5], M. G. Crandall and P. L. Lions have developed techniques that have been extremely successful in establishing a number of new results concerning continuous, but not necessarily differentiable, weak solutions to first order, fully nonlinear, partial differential equations. In their work, Crandall and Lions have utilized the "vanishing viscosity method", so named because of the link to the classical technique of vanishing viscosity from fluid mechanics, and they show that the method of vanishing viscosity gives rise to a specific notion of a "viscosity" weak solution.

In [15], and here as well, the notion of a viscosity solution for the generally over determined Neumann problem (NP_0) is given and is shown to include all L^∞ ε-limits of solutions to (NP_ε). All L^∞ ε-limits are shown to satisfy the so-called viscosity inequalities of Section 2; additional details in this direction can be found in [15]. Remarkably, (with additional hypotheses of course), these viscosity inequalities uniquely determine all such limits. In Section 3, we introduce what we call "approximate viscosity inequalities" and we show there that any reasonable approximate viscosity solution is approximately equal to the viscosity solution of (NP_0). More precisely, we give an abstract minimal rate-of-convergence theorem, Theorem 2, for approximate viscosity solutions to (NP_0). We also show that this rate is essentially sharp. A particular application of Theorem 2 gives an easy to determine measure of how far the solution of (NP_ε) can be away from the viscosity solution of (NP_0). In Section 4, the abstract rate-of-convergence theorem of Section 3 is applied to numerical approximations which are obtained from a class of finite difference schemes. Moreover, we show in Section 4 that these schemes have "computable" solutions and we motivate how they can be obtained.

The reader is encouraged to see [2], [18] and [19] where similar results as those above are obtained for divergence form singular perturbation problems with mixed or Dirichlet boundary conditions. Also see [6], [7] and [21] for a further treatment of approximations for time dependent Hamilton-Jacobi equations without spatial boundaries.

§2 VISCOSITY LIMIT SOLUTIONS.

As mentioned in the previous section, as $\varepsilon \downarrow 0$, the corresponding solutions to (NP_ε) do not in general converge to a classical C^1 solution of (NP_0). In this section we offer a characterization of viscous limits to (NP_0) and we show that this characterization often allows for only one solution in the class of continuous functions. Throughout, we shall assume that Ω is a bounded domain in $\mathbb{R}^d$ which has a C^2 boundary $\partial\Omega$. The outward normal of Ω at a point $x \in \partial\Omega$ will be denoted by $n(x)$ and we write the outward normal derivative of φ at $x \in \partial\Omega$ as $\frac{\partial\varphi}{\partial n}(x)$.

We should like to mention that previous to the writing of this paper P. L. Lions had introduced the same viscosity characterization of solutions to (NP0) as we give below; see [15]. For this reason, we borrow much of the notation and hypotheses of [15] and in this section we omit all proofs but those which motivate the results of the next sections.

We now state the vicosity characterization (see Proposition 1) of continuous weak solutions to (NP_0).

Definition 1. Suppose that $H(x,u,p) \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d)$ and $u(x) \in C(\bar{\Omega})$. We say that:

(a) $u(x)$ is a viscosity subsolution of (NP_0) if for all test functions $\varphi \in C^1(\mathbb{R}^d)$ with $\frac{\partial \varphi}{\partial n}(x) > \gamma(x)$, we have

$$H(x_0, u(x_0), \nabla\varphi(x_0)) \leq 0 ,$$

where $x_0 \in \bar{\Omega}$ satisfies

$$u(x_0) - \varphi(x_0) = \max_{x \in \bar{\Omega}} (u(x) - \varphi(x)) .$$

(b) $u(x)$ is a viscosity supersolution of (NP_0) if for all test functions $\varphi \in C^1(\mathbb{R}^d)$ with $\frac{\partial \varphi}{\partial n}(x) < \gamma(x)$, we have

$$H(x_0, u(x_0), \nabla\varphi(x_0)) \geq 0 ,$$

where $x_0 \in \bar{\Omega}$ satisfies

$$u(x_0) - \varphi(x_0) = \min_{x \in \bar{\Omega}} (u(x) - \varphi(x)) .$$

(c) $u(x)$ is a viscosity solution of (NP_0) if it satisfies both (a) and (b) above.

The fact that our test functions are required to satisfy $\frac{\partial \varphi}{\partial n}(x) > \gamma(x)$, (resp. $\frac{\partial \varphi}{\partial n}(x) < \gamma(x)$), in our definition of viscosity subsolution, (resp. supersolution), may at first seem superfluous. This is however, precisely the mechanism that "sees" the Neumann boundary conditions when vanishing viscosity is taken into account; (see Proposition 1 below).

Remark 2.1. Any C^1 solution of (NP_0) is also a viscosity solution. This fact is nontrivial only for the case when $\max(u - \varphi)$ or $\min(u - \varphi)$ is attained for some $x0 \in \partial\Omega$. To see that u must indeed be a viscosity subsolution, take an arbitrary $\varphi \in C^1(\mathbb{R}^d)$ with $\frac{\partial\varphi}{\partial n}(x) > \gamma(x)$. First choose a sequence $\{\varphi_m\}$, such that for every m, $\varphi_m(x_0) = \varphi(x_0)$, $\varphi_m(x) > \varphi(x)$ for $x \neq x_0$, $\frac{\partial\varphi_m}{\partial n}(x) > \frac{\partial\varphi}{\partial n}(x)$ and with $\varphi_m \to \varphi$ in C^1 as $m \to \infty$. We then have that $(u - \varphi)(x_0) = \max(u - \varphi_m)$ and x_0 is the point where the <u>strict</u> maximum of $u - \varphi_m$ is attained. Next, for any fixed m, choose a sequence $\{\varphi_m^n\}$ such that $\frac{\partial\varphi_m^n}{\partial n}(x) > \frac{\partial\varphi_m}{\partial n}(x)$ and with $\varphi_m^n \to \varphi_m$ in C^1 as $n \to \infty$. Denoting by x_n the points where $\max(u - \varphi_m^n)$ is attained, we must have that $x_n \to x_0$ as $n \to \infty$. This is true because $(u - \varphi_m)(x_0)$ is a strict maximum of $u - \varphi_m$. For $x \in \partial\Omega$, we also have that $\frac{\partial}{\partial n}(u - \varphi_m^n)(x) < 0$, which implies $x_n \in \Omega^0$. Therefore, since now x_n is an interior maximum of $u - \varphi_m^n$, $\nabla u(x_n) = \nabla\varphi_m^n(x_n)$, and so by taking limits we have

$$H(x_0,u(x_0),\nabla\varphi(x_0)) = \lim_{m,n} H(x_n,u(x_n),\nabla\varphi_m^n(x_n)) = 0 .$$

<u>Remark 2.2</u>. Obviously, the converse of Remark 2.1 is false. That is, a smooth viscosity solution need not satisfy the boundary conditions of (NP_0).

<u>Proposition 1</u>. Let $u_\varepsilon \in C^2(\bar\Omega)$ be a solution of (NP_ε) and suppose that $\varphi \in C^2(\bar\Omega)$. Then:

(a) For $\frac{\partial\varphi}{\partial n}(x) > \gamma(x)$ and $u_\varepsilon(x_0) - \varphi(x_0) = \max_{x\in\bar\Omega} (u_\varepsilon(x) - \varphi(x))$ we have that $H(x_0,u_\varepsilon(x_0),\nabla\varphi(x_0)) \le \varepsilon\Delta\varphi(x_0)$.

(b) For $\frac{\partial\varphi}{\partial n}(x) < \gamma(x)$ and $u_\varepsilon(x_0) - \varphi(x_0) = \min_{x\in\bar\Omega} (u_\varepsilon(x) - \varphi(x))$ we have that $H(x_0,u_\varepsilon(x_0),\nabla\varphi(x_0)) \ge \varepsilon\Delta\varphi(x_0)$.

If in addition, we have that $u_\varepsilon \to u$ in $L^\infty(\bar\Omega)$ for some sequence $\varepsilon \downarrow 0$, then:

(c) $u = \lim_\varepsilon u_\varepsilon$ is a viscosity solution, that is, u satisfies Definition 1c.

The proof of Proposition 1 can be found in [15], however the interested reader can easily reproduce its proof by taking limits as in Remark 2.1.

Before stating the main result of this section, we give a simple lemma.

Lemma 1. Let Ω be a bounded domain in R^d having a C^2 boundary $\partial\Omega$. Then:

(a) There exists a constant $C_\Omega < \infty$ such that for all $x \in \partial\Omega$,

$$C_\Omega \geq \sup_{y\in\Omega} \left(\frac{-(x - y)\cdot n(x)}{|x - y|^2}\right).$$

(b) There exists a function $w \in C^2(\bar{\Omega})$ such that

$$\frac{\partial w}{\partial n}(x) = \max(C_\Omega, 0), \quad x \in \partial\Omega ,$$

$$|\nabla w(x)| \leq \max(C_\Omega, 0), \quad x \in \bar{\Omega} .$$

Proof: $C_\Omega = w(x) \equiv 0$ would suffice in the case of convex Ω. For nonconvex Ω, (a) is shown in [12]. (b) can be shown by constructing a particular example. Under the hypotheses of the lemma, it is known that the distance function $d(x) = d(x;\partial\Omega)$ is C^2 in a neighborhood of $\partial\Omega$; [23], [10]. That is, $d(x) \in C^2(\Omega_\tau)$, where $\Omega_\tau = \{x \in \bar{\Omega} : d(x) < \tau\}$ and $\tau > 0$ is chosen sufficiently small. Set $0 < \tau_0 < \tau$ and verify that

$$w(x) = \begin{cases} \dfrac{C_\Omega}{3\tau_0^2} (\tau_0 - d(x))^3 & \text{if} \quad x \in \bar{\Omega}_{\tau_0} \\[2ex] 0 & \text{if} \quad x \in \bar{\Omega}\ \Omega_{\tau_0} , \end{cases}$$

is a particular example that satisfies (b).

Now, consider the following set of assumptions.

Assumption A: $H(x,u,p)$ is strictly increasing in u for all $x \in \bar{\Omega}$ and uniformly for $p \in R^d$. That is, for all $R > 0$ and $-R \leq v \leq u \leq R$, there exists a $\mu_R > 0$ such that

$$H(x,u,p) - H(x,v,p) \geq \mu_R(u - v) .$$

<u>Assumption B</u>: Let $\alpha,\beta \in R^d$ satisfy $|\alpha|,|\beta| \leq \max(C_\Omega,0)$, where C_Ω is as defined in the previous lemma. Then, for all such α,β and all $x \in \bar{\Omega}$, $x + \xi \in \bar{\Omega}$, all $|u| \leq R$ and any $\lambda > 1$, assume

$$|H(x + \xi,u,\lambda\xi + \frac{\lambda}{2}|\xi|^2\alpha + O(\xi)) - H(x,u,\lambda\xi + \frac{\lambda}{2}|\xi|^2\beta)| \leq \omega_R(\lambda|\xi|^2 + |\xi|) ,$$

where $\omega_R(s)$ is some function such that $\lim_{s\downarrow 0} \omega_R(s) = 0$.

<u>Remark 2.3</u>. Assumptions A and B are standard; see [5], [8] and [15]. In the following theorem, Assumption B may always be relaxed so that $\alpha = \beta \equiv 0$ and $O(\xi) \equiv 0$ except for x in a neighborhood of $\partial\Omega$. Assuming additional regularity on the class of solutions allows Assumption B to be neglected entirely.

<u>Theorem 1</u>. Suppose that $H(x,u,p) \in C(\bar{\Omega} \times R \times R^d)$ and that it satisfies Assumption A above. Let $u \in C(\bar{\Omega})$ be a viscosity subsolution of (NP_0) and let $v \in C(\bar{\Omega})$ be a viscosity supersolution of (NP_0). Finally, assume one from the following three sets of hypotheses:

(i) Ω is convex and Assumption B holds with $\alpha = \beta \equiv 0$.

(ii) Assumption B is satisfied.

(iii) Either u <u>or</u> v is Lipschitz continuous.

We then have that

$$\max_{x\in\bar{\Omega}} (u(x) - v(x)) \leq 0 .$$

Obviously, establishing this result would imply that a viscosity <u>solution</u> to (NP_0) is unique in the specified class of functions.

<u>Proof</u>: Given a $\delta > 0$, define the function $\phi^\delta(x,y)$ by

$$\phi^\delta(x,y) = \rho(x)\rho(y)|x - y|^2/\delta , \tag{2.1}$$

where $\rho(x) = \exp(w(x))$ and $w(x)$ satisfies the second conclusion of

Lemma 1. For $x \in \partial\Omega$ and any fixed $y_0 \in \bar{\Omega}$, observe that

$$\frac{\partial}{\partial n_x}\phi^\delta(x,y_0) = \phi^\delta(x,y_0)\{\frac{\partial w}{\partial n}(x) + \frac{2(x - y_0)\cdot n}{|x - y_0|^2}\} ,$$

and Lemma 1 implies that the bracketed term above is nonnegative. Therefore,

$$\frac{\partial}{\partial n_x}\phi^\delta(x,y_0) \geq 0 ,$$

and similarly, for $y \in \partial\Omega$ and any fixed $x_0 \in \bar{\Omega}$

$$\frac{\partial}{\partial n_y}\phi^\delta(x_0,y) \geq 0 .$$

Now, choose $\psi \in C^2(\bar{\Omega})$ such that $\frac{\partial\psi}{\partial n}(x) = \gamma(x)$. By the construction above, we have that for any fixed $y_0 \in \bar{\Omega}$

$$\phi_1(x) = \phi^\delta(x,y_0) + \psi(x) ,$$

is an admissible test function according to Definition 1a and similarly for fixed $x_0 \in \bar{\Omega}$

$$\phi_2(y) = -\phi^\delta(x_0,y) + \psi(y) ,$$

is admissible according to Definition 1b.

The next step is to note the obvious inequality

$$(2.2)\qquad \max_{x\in\bar{\Omega}} (u(x) - v(x)) \leq \max_{\substack{x\in\bar{\Omega}\\ y\in\bar{\Omega}}} \left(u(x) - v(y) - (\phi^\delta(x,y) + \psi(x) - \psi(y))\right) .$$

We denote by x_δ, y_δ the points in $\bar{\Omega}$ where the right hand side of (2.2) is attained and we rewrite (2.2) as

$$(2.3)\qquad \max_{x\in\bar{\Omega}} (u(x) - v(x)) \leq u(x_\delta) - v(y_\delta) - (\phi^\delta(x_\delta,y_\delta) + \psi(x_\delta) - \psi(y_\delta)) .$$

Using (2.3), we easily arrive at

$$(2.4)\qquad \begin{aligned} \phi^\delta(x_\delta,y_\delta) &\leq |u(x_\delta) - u(y_\delta)| + |\psi(x_\delta) - \psi(y_\delta)| , \\ \phi^\delta(x_\delta,y_\delta) &\leq |v(x_\delta) - v(y_\delta)| + |\psi(x_\delta) - \psi(y_\delta)| , \end{aligned}$$

and recalling the definition of $\phi^{\delta}(x,y)$, (2.4) gives us that

$$|x_{\delta} - y_{\delta}| \le \text{const.}\sqrt{\delta} . \tag{2.5}$$

Furthermore, since u, (or v), and ψ are continuous, (2.4) combined with (2.5) shows that

$$\lim_{\delta \downarrow 0} \phi^{\delta}(x_{\delta},y_{\delta}) = 0 . \tag{2.6}$$

The object now is to show that the right hand side of (2.2) can be made arbitrarily small by choosing δ sufficiently small. From above, we see that the test functions defined as

$$\phi_1(x) = v(y_{\delta}) - \psi(y_{\delta}) + \phi^{\delta}(x,y_{\delta}) + \psi(x) ,$$

$$\phi_2(y) = u(x_{\delta}) - \psi(x_{\delta}) - \phi^{\delta}(x_{\delta},y) + \psi(y) ,$$

are admissible according to Definition 1a and Definition 1b respectively. Inserting these into Definition 1, and using the fact that u is a viscosity subsolution and v is a viscosity supersolution, allows us to conclude that

$$H(x_{\delta},u(x_{\delta}),\nabla_x\phi_1(x_{\delta})) \le 0 ,$$

and

$$H(y_{\delta},v(y_{\delta}),\nabla_y\phi_2(y_{\delta})) \ge 0 ,$$

because x_{δ} satisfies

$$u(x_{\delta}) - \phi_1(x_{\delta}) = \max_{x\in\bar{\Omega}} (u(x) - \phi_1(x)) ,$$

and y_{δ} satisfies

$$v(y_{\delta}) - \phi_2(y_{\delta}) = \min_{y\in\bar{\Omega}} (v(y) - \phi_2(y)) .$$

Combining the inequalities above and rearranging, we obtain

$$\begin{aligned} &H(x_{\delta},u(x_{\delta}),\nabla_x\phi_1(x_{\delta})) - H(x_{\delta},v(y_{\delta}),\nabla_x\phi_1(x_{\delta})) \\ &\quad \le H(y_{\delta},v(y_{\delta}),\nabla_y\phi_2(y_{\delta})) - H(x_{\delta},v(y_{\delta}),\nabla_x\phi_1(x_{\delta})) . \end{aligned} \tag{2.7}$$

By a direct calculation, the right hand side of (2.7) can be written as

$$\begin{aligned} &H(y_\delta, v(y_\delta), \lambda(x_\delta - y_\delta) + \frac{\lambda}{2}|x_\delta - y_\delta|^2\beta + \nabla\psi(y_\delta)) \\ (2.8)\quad & \\ &- H(x_\delta, v(y_\delta), \lambda(x_\delta - y_\delta) + \frac{\lambda}{2}|x_\delta - y_\delta|^2\alpha + \nabla\psi(x_\delta)) \ , \end{aligned}$$

where

$$\lambda = 2\rho(x_\delta)\rho(y_\delta)/\delta$$

$$\alpha = \nabla w(x_\delta)$$

$$\beta = \nabla w(y_\delta) \ .$$

(Recall from Lemma 1 that if Ω is convex we may assume that $\alpha = \beta \equiv 0$ and $\rho(x) = \rho(y) \equiv 1$). Assumption B allows (2.8) to be bounded above by

$$(2.9) \qquad \omega_R(\lambda|x_\delta - y_\delta|^2 + |x_\delta - y_\delta|) \ ,$$

where $R = \max(|u|_\infty, |v|_\infty)$.

To complete the proof of conclusions (i) and (ii), we again use inequality (2.3) to write

$$\max_{x\in\bar{\Omega}} (u(x) - v(x)) \leq u(x_\delta) - v(y_\delta) + |\psi(x_\delta) - \psi(y_\delta)| \ ,$$

which is bounded above by

$$(2.10) \qquad \max((u(x_\delta) - v(y_\delta)), 0) + |\psi(x_\delta) - \psi(y_\delta)| \ .$$

Assumption A applied to the left hand side of (2.7) combined with (2.8) and (2.9), allows us to bound (2.10) by

$$(2.11) \qquad \frac{1}{\mu_R}\,\omega_R(\lambda|x_\delta - y_\delta|^2 + |x_\delta - y_\delta|) + |\psi(x_\delta) - \psi(y_\delta)| \ .$$

Recalling that $\lambda|x_\delta - y_\delta|^2 = 2\phi^\delta(x_\delta, y_\delta)$, (2.6) along with (2.5) show that (2.11) tends to zero as δ tends to zero; therby proving

$$\max_{x\in\bar{\Omega}} (u(x) - v(x)) \leq 0 \ .$$

To establish (iii), observe that if u, (or v), is Lipschitz continuous, inequality (2.4) leads to an improvement of (2.5). That is, we may conclude that

$$|x_\delta - y_\delta| \leq \text{const}\ \delta\ . \tag{2.12}$$

This improved estimate implies that the p term of $H(\cdot,\cdot,p)$ in (2.8) remains bounded. Therefore, conclusion (iii) follows by noting the uniform continuity of $H(x,u,p)$ on a compact subset of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d$.

Remark 2.4. The combined results of Proposition 1 and Theorem 1 imply that *if the family* $\{u_\varepsilon\}_{\varepsilon>0}$ of solutions to (NP_ε) is relatively compact in L^∞, then $\lim_{\varepsilon\downarrow 0} u_\varepsilon$ exists in L^∞. For further results concerning the compactness of $\{u_\varepsilon\}_{\varepsilon>0}$, see [13] or [14].

§3. VISCOSITY APPROXIMATIONS AND A RATE OF CONVERGENCE.

In this section we consider the rate at which certain approximations converge to the viscosity solution of (NP_0). We show in a precise sense below, that if an approximation "almost" satisfies the viscosity inequalities of Definiton 1 then the approximation is "almost" equal to its associated viscosity limit solution. The abstract rate of convergence theorem given in this section is then applied in Section 4 to particular approximations generated by a class of numerical schemes.

Before making a precise statement of "almost satisfies the viscosity inequalities", recall the definitions of the test functions used in the proof of Theorem 1:

$$\phi^\delta(x,y) = \rho(x)\rho(y)|x - y|^2/\delta\ , \tag{3.1}$$

where $\delta > 0$, $\rho(x) = \exp(w(x))$ and $w(x)$ satisfies the second conclusion of Lemma 1. Also recall the function ψ, which satisfies

(3.2) $$\psi(x) \in C^2(\bar{\Omega})$$

$$\frac{\partial\psi}{\partial n}(x) = \gamma(x) \quad \text{on} \quad \partial\Omega ,$$

and the specific test functions

(3.3a) $$\phi_1(x) = \phi^\delta(x,y_0) + \psi(x) ,$$

(3.3b) $$\phi_2(y) = -\phi^\delta(x_0,y) + \psi(y) ,$$

where x_0, y_0 are arbitrary fixed points in $\bar{\Omega}$.

We now give:

Definition 2. Suppose that $H(x,u,p) \in C(\bar{\Omega} \times R \times R^d)$ and $u_\varepsilon(x) \in C(\bar{\Omega})$. We say that:

(a) u_ε is an approximate viscosity subsolution to (NP_0) if for all test functions ϕ_1, which have the particular form (3.3a), we have that

$$H(x_0,u_\varepsilon(x_0),\nabla_x\phi_1(x_0)) \leq \varepsilon\Delta_x\phi_1(x_0) + \underline{C}\varepsilon ,$$

where $x_0 \in \bar{\Omega}$ satisfies

$$u_\varepsilon(x_0) - \phi_1(x_0) = \max_{x\in\bar{\Omega}} (u_\varepsilon(x) - \phi_1(x)) ,$$

and $\underline{C}$ is some fixed constant.

(b) u_ε is an approximate viscosity supersolution to (NP_0) if for all test functions ϕ_2, which have the particular form (3.3b), we have that

$$H(y_0,u_\varepsilon(y_0), \nabla_y\phi_2(y_0)) \geq \varepsilon\Delta_y\phi_2(y_0) - \underline{C}\varepsilon ,$$

where $y_0 \in \bar{\Omega}$ satisfies

$$u_\varepsilon(y_0) - \phi_2(y_0) = \min_{x\in\bar{\Omega}} (u_\varepsilon(x) - \phi_2(x)) .$$

(c) u_ε is an approximate viscosity solution if it satisfies (a) and (b) above.

In the proof of Theorem 1 we showed that $\frac{\partial\phi_1}{\partial n_x}(x) > \gamma(x)$ and $\frac{\partial\phi_2}{\partial n_y}(y) < \gamma(y)$, therefore the statement of Proposition 1 implies that if u_ε is a C^2 solution of (NP_ε), then it is also an approximate viscosity solution of (NP_0) as defined above.

With Definition 2, we now state:

Theorem 2. In addition to Assumption A of Theorem 1, assume for ease of presentation that $\psi(x) \equiv 0$. Furthermore, assume that (NP_0) admits a Lipschitz continuous viscosity solution u, with say Lipschitz constant L, and assume that $H(x,u,p)$ is locally Lipschitz continuous. Then, for any approximate viscosity solution to (NP_0), say u_ε, we have that

$$|u_\varepsilon - u|_\infty < \frac{1}{\mu_{R_0}}\left[(8dL_H L(1 + 2C_\Omega L)\varepsilon)^{1/2} + O(\varepsilon)\right]$$

where

$$R_0 = \sup_{1>\varepsilon>0} |u_\varepsilon|_\infty ,$$

$$L_H = \sup_{\substack{x_1,x_2\in\bar\Omega \\ |u|<R_0 \\ |P_1|,|P_2|<3L}} \left[\frac{|H(x_1,u,P_1) - H(x_2,u,P_2)|}{|x_1 - x_2| + |P_1 - P_2|}\right] .$$

The definition of C_Ω is given in Lemma 1.

Remark 3.1. We may replace the assumption that (NP_0) admits a Lipschitz continuous viscosity solution by the assumption that u_ε is Lipschitz continuous, uniformly in $\varepsilon > 0$.

Remark 3.2. The interested reader can easily modify the following proof to include inhomogenious boundary data to obtain the same $\sqrt{\varepsilon}$ rate of convergence. Relaxing the hypothesis on $H(x,u,p)$ and the regularity of u can also be done to obtain a more general, (and slower), rate of convergence. This however, will not be done here.

<u>Proof of Theorem 2</u>: Mimicking the proof of Theorem 1, we arrive that the analogue of inequality (2.7):

$$(3.4)\quad H(x_\delta,u_\varepsilon(x_\delta),\nabla_x\phi_1(x_\delta)) - H(x_\delta,u(y_\delta),\nabla_x\phi_1(x_\delta))$$

$$\leq H(y_\delta,u(y_\delta),\nabla_y\phi_2(y_\delta)) - H(x_\delta,u(y_\delta),\nabla_x\phi_1(x_\delta)) + \varepsilon\Delta_x\phi_1(x_\delta) + \underline{C}\varepsilon ,$$

where $\nabla_x\phi_1$ and $\nabla_y\phi_2$ are given by (3.3), (with $\psi \equiv 0$), and x_δ and y_δ are as in (2.3). Recalling (2.4) and using the fact that u, (or u_ε), is Lipschitz continuous, we have that

$$(3.5)\qquad \phi^\delta(x_\delta,y_\delta) \leq |u(x_\delta) - u(y_\delta)| \leq L|x_\delta - y_\delta| ,$$

or

$$(\phi^\delta(x_\delta,y_\delta) \leq |u_\varepsilon(x_\delta) - u_\varepsilon(y_\delta)| \leq L|x_\delta - y_\delta|) ,$$

which, with the definition of ϕ^δ, gives us that

$$(3.6)\qquad |x_\delta - y_\delta| \leq \frac{L}{\rho(x_\delta)\rho(y_\delta)}\,\delta .$$

Furthermore, a direct calculation shows that

$$|\nabla_x\phi_1(x) - \nabla_y\phi_2(y)| \leq 2C_\Omega\phi^\delta(x,y) ,$$

where C_Ω is as in Lemma 1. This inequality, along with (3.5) shows that

$$(3.7)\qquad |\nabla_x\phi_1(x_\delta) - \nabla_y\phi_2(y_\delta)| \leq 2C_\Omega L|x_\delta - y_\delta| .$$

Returning now to inequality (3.4), we use (3.7), Assumption A and the fact that H is (locally) Lipschitz continuous to obtain

$$\mu_{R_0}\max((u_\varepsilon(x_\delta) - u(y_\delta)),0)$$

$$\leq L_H(1 + 2C_\Omega L)\cdot|x_\delta - y_\delta| + \varepsilon\Delta_x\phi_1(x_\delta) + \underline{C}\varepsilon .$$

Calculating $\Delta_x \phi_1$ and inserting (3.6) into the right hand side above, we find that

(3.8) $$\mu_{R_0} \max((u_\varepsilon(x_\delta) - u(y_\delta)),0)$$

$$\le \left[2d\varepsilon\left(\frac{\rho(x_\delta)\rho(y_\delta)}{\delta}\right) + \hat{L}L\left(\frac{\delta}{\rho(x_\delta)\rho(y_\delta)}\right)\right] + \text{const.}(\varepsilon + \varepsilon\delta) ,$$

where $\hat{L} = L_H(1 + 2C_\Omega L)$. By setting

$$\frac{\delta}{\rho(x_\delta)\rho(y_\delta)} = \left(\frac{2d}{\hat{L}L}\,\varepsilon\right)^{1/2} ,$$

which can be done for $\hat{L}L \neq 0$ by the continuity of the left hand side with respect to δ, we minimize the bracketed term in (3.8). This yields

$$\mu_{R_0} \max((u_\varepsilon(x_\delta) - u(y_\delta)),0) \le (8d\hat{L}L\varepsilon)^{1/2} + \text{const.}(\varepsilon + \varepsilon^{3/2}) ,$$

and using the fact that

$$\max_{x\in\bar{\Omega}} (u_\varepsilon(x) - u(x)) \le u_\varepsilon(x_\delta) - u(y_\delta) ,$$

as done in the proof of Theorem 1, we have established the desired result for $\max(u_\varepsilon - u)$.

An identical estimate can be obtained for $\max(u - u_\varepsilon)$ by a similar argument and so the proof of Theorem 2 is complete.

Remark 3.3. When the domain Ω is convex and $\psi \equiv 0$, the term $O(\varepsilon)$ in the estimate of Theorem 2 is precisely $\underline{C}\varepsilon$. In addition, if the approximate viscosity solution is the solution of (NP_ε), the constant $\underline{C}$ is zero.

Next, we show that the order of the rate of convergence obtained above can not in general be improved. To see this, consider the example

$$(3.9)\qquad \begin{gathered} -\varepsilon \frac{d^2u_\varepsilon}{dx^2} + u_\varepsilon = 0 \\ \frac{du_\varepsilon}{dx}(0) = 0, \qquad \frac{du_\varepsilon}{dx}(1) = 1 . \end{gathered}$$

The exact solution of (3.9) is given by

$$u_\varepsilon(x) = \sqrt{\varepsilon}\, \cosh(x/\sqrt{\varepsilon})/\sinh(1/\sqrt{\varepsilon})$$

and its an easy exercise to show that $u_\varepsilon \to 0$ uniformly as $\varepsilon \downarrow 0$. In fact, one easily finds that

$$|u_\varepsilon - 0|_\infty = \sqrt{\varepsilon}\,\{1 + O(\exp(-2/\sqrt{\varepsilon}))\} ,$$

which is exactly the order obtained by Theorem 2. We should mention however, that the rate constant of Theorem 2 is not the best possible.

We conclude this section by analyzing the specific example:

$$(3.10)\qquad \begin{gathered} -\varepsilon \frac{d^2u_\varepsilon}{dx^2} + \left(\frac{1}{2}\frac{du_\varepsilon}{dx}\right)^2 + u_\varepsilon = 0 , \\ \frac{du_\varepsilon}{dx}(0) = \gamma_0, \qquad \frac{du_\varepsilon}{dx}(1) = -\gamma_0 . \end{gathered}$$

Setting $\varepsilon = 0$ and solving the reduced differential equation, we find that $u = \lim_\varepsilon u_\varepsilon$ should be built from functions having the form $-(x - c)^2$ and 0. The objective now is to piece things together in a way such that the constructed function satisfies the viscosity inequalities of Definition 1.

We have three basic cases, (which depend on γ_0). Set

$$u_L(x) = -\left(x - \frac{\gamma_0}{2}\right)^2 ,$$

$$u_R(x) = -\left(x + \frac{\gamma_0}{2} - 1\right)^2 ,$$

and note that u_L satisfies the left boundary condition of (3.10) and u_R satisfies the right boundary conditions.

Case 1. For $1 > \gamma_0 > 0$, consider the candidate limit solution:

$$u_1(x) = \begin{cases} u_L(x) & 0 < x < \frac{\gamma_0}{2} \\ 0 & \frac{\gamma_0}{2} < x < 1 - \frac{\gamma_0}{2} \\ u_R(x) & 1 - \frac{\gamma_0}{2} < x < 1 . \end{cases}$$

The analysis of this case is trivial since $u_1(x)$ is a classical C^1 solution to the reduced problem. By Remark 2.1, it must therefore be a viscosity limit solution.

Case 2. For $\gamma_0 > 1$, consider the candidate limit solution:

$$u_2(x) = \begin{cases} u_L(x) & 0 < x < 1/2 \\ u_R(x) & 1/2 < x < 1 . \end{cases}$$

Obviously, we need only check the viscosity inequalities at $x_0 = 1/2$, the corner of u_2. In this case however, $\min(u_2 - \phi)$ can not occur at $x_0 = 1/2$ for any C^1 function ϕ. If $\max(u_2 - \phi)$ occurs at $x_0 = 1/2$, its easy to check that we must have $(u_L)_x(1/2) > \phi_x(1/2) > (u_R)_x(1/2)$. Computing these derivatives, we have that all possible values of $\phi_x(1/2)$ lie in the interval $[1 - \gamma_0, \gamma_0 - 1]$, in which case

$$\left(\tfrac{1}{2}\,\phi_x\left(\tfrac{1}{2}\right)\right)^2 + u_2\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\,\phi_x\left(\tfrac{1}{2}\right)\right)^2 - \left(\tfrac{1}{2}\,(1 - \gamma_0)\right)^2 < 0 .$$

Therefore, u_2 is a viscosity solution.

Case 3. For $\gamma_0 < 0$, consider the candidate:

$$u_3(x) = 0 .$$

Here, u_3 does not take on its boundary condition at $x = 0$ or at $x = 1$. However, $\max(u_3 - \phi)$ can not occur at $x = 0$ for any admissible test function, $\left(\frac{\partial \phi}{\partial n}(0) > -\gamma_0\right)$. If on the other hand, $\min(u_\varepsilon - \phi)$ is attained at $x_0 = 0$, we must have that $\phi_x(0)$ lies in $[\gamma_0, 0]$ and in this case

$$\left(\frac{1}{2}\phi_x(0)\right)^2 + u_3(0) \geq 0 .$$

A similar argument shows that u_3 satisfies the viscosity inequalities if $\max(u_3 - \phi)$ is attained at $x_0 = 1$.

In these specific examples, we have demonstrated that these candidate limit solutions are viscosity limit solutions of (3.10) since they satisfy Definition 1c. They are furthermore Lipschitz continuous and so by Proposition 1 and Remark 3.2 of Theorem 2, they satisfy $|u_\varepsilon - u|_\infty \leq \text{const.}\sqrt{\varepsilon}$ where u_ε is the exact solution of problem (3.10). However, for these examples, (as well as other nonlinear examples) there is evidence that indicates a convergence rate faster than the $\sqrt{\varepsilon}$, [3]. We believe that there is a yet undiscovered mechanism that links certain nonlinearities in H to diffusion which often gives rise to a faster rate of convergence than Theorem 2 predicts.

§4. NUMERICAL APPROXIMATIONS.

In this section, we introduce and analyze a class of numerical schemes that generate approximations of the viscosity limit solution to the one-dimensional version of (NP_ε), which we write here as:

$$(4.1)\qquad \begin{aligned} &-\varepsilon \frac{d^2u_\varepsilon}{dx^2} + H\left(x, u_\varepsilon, \frac{du_\varepsilon}{dx}\right) = 0 \\ &-\frac{du_\varepsilon}{dx}(0) = \gamma_0 , \qquad \frac{du_\varepsilon}{dx}(1) = \gamma_1 . \end{aligned}$$

Throughout this section, we make the following assumptions concerning $H(x,u,p)$, which for ease of presentation only, is assumed C^1 smooth.

Assumption A': For all $x \in [0,1]$, $|u| = R$ and $|p| \leq K$, there exists a $\mu_K > 0$ and an $0 \leq \eta_1 < 1$, such that

$$\frac{\partial}{\partial u} H(x,u,p) \geq \mu_K/(\max(R,1))^{\eta_1}$$

Assumption B': For all $x \in [0,1]$ and $|p| = K$, there exists an $0 \leq \eta_2 < 1$ and a constant $C(|u|)$ such that

$$\left|\frac{\partial}{\partial x} H(x,u,p)\right| \leq \mu_K(\max(K,1))^{\eta_2} C(|u|) .$$

Assumption A' is merely a refined version of Assumption A of Section 2. Assumption B' guarantees that the viscosity limit solution of (4.1) is Lipschitz continuous and therefore supercedes Assumption B of Section 2.

The numerical approximation that are considered here are built from a piece-wise linear interpolation of grid values $\{u_j\}_{j=0}^{J}$. That is, we partition the interval $[0,1]$ as $\bigcup_{j=0}^{J-1} [x_j, x_{j+1}]$, where we shall assume that

$$2(x_j - x_{j-1}) \geq (x_{j+1} - x_j) \geq \frac{1}{2}(x_j - x_{j-1}) ,$$

and then define $u^\Delta(x)$ by

$$u^\Delta(x) = \sum_{j=0}^{J} u_j T_j(x) , \tag{4.2}$$

where

$$T_j(x) = \begin{cases} (x - x_{j-1})/(x_j - x_{j-1}) & \text{if } x \in [x_{j-1}, x_j] \\ (x_{j+1} - x)/(x_{j+1} - x_j) & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases} .$$

In (4.2) the superscript Δ is to represent a measure of grid refinement and we set it equal to $\max_{0 \le j \le J-1} (x_{j+1} - x_j)$. For each $0 \le j \le J$, the grid values u_j are required to satisfy the difference scheme

$$\bar{H}(x_j, u_j, D^+u_j, D^-u_j) = 0 \tag{4.3}$$

$$-D^-u_0 = \gamma_0, \quad D^+u_J = \gamma_1 ,$$

where $D^+u_j = (u_{j+1} - u_j)/(x_{j+1} - x_j)$, $D^-u_j = (u_j - u_{j-1})/(x_j - x_{j-1})$, and $\bar{H}(x,u,p_1,p_2)$ is some difference operator that does not explicitly depend on any grid parameter. $\bar{H}(x,u,p_1,p_2)$ is assumed to be locally Lipschitz continuous and it is also assumed to satisfy three basic properties:

<u>Property 1</u>. $\bar{H}(x,u,p_1,p_2)$ is consistent with $H(x,u,p)$. That is, $\bar{H}(x,u,p,p) = H(x,u,p)$.

<u>Property 2</u>. $\bar{H}(x,u,p_1,p_2)$ is nonincreasing in the p_1 argument and nondecreasing in the p_2 argument.

<u>Property 3</u>. For all $|p_1| \le K$ and $|p_2| \le K$, $\bar{H}(x,u,p_1,p_2)$ satisfies Assumption A' above.

Of course, Property 3 simply says that $\bar{H}(x,u,p_1,p_2)$ is strictly increasing in u at the rate prescribed by Assumption A'. We now give:

<u>Theorem 3</u>. With Assumptions A' and B' above, suppose that u^Δ comes from scheme (4.3), where $\bar{H}(x,u,p_1,p_2)$ satisfies properties 1, 2 and 3. Then, (4.3) generates a unique approximate solution u^Δ and moreover, u^Δ converges to $u = \lim_\varepsilon u_\varepsilon$ at least as fast as

$$|u^\Delta - u|_\infty \le \text{const.}\sqrt{\Delta} ,$$

where above, u is the viscosity <u>limit</u> solution of (4.1) and

$$\Delta = \max_{0 \le j \le J-1} (x_{j+1} - x_j).$$

Before proving Theorem 3, we give two examples of finite difference operators which satisfy properties 1,2 and 3. Furthermore, we show that the rate above is the best possible under the hypotheses of Theorem 3.

Example 1. The Lax-Friedrichs difference operator [11], [20], is based upon approximating $H\left(x,u,\frac{du}{dx}\right)$ by a convex combination of $H(x,u,D^{+}u)$ and $H(x,u,D^{-}u)$ along with the introduction of an artificial numerical viscosity term. To be more specific, $\bar{H}$ is given by

$$\bar{H}(x,u,p_1,p_2) = \theta H(x,u,p_1) + (1-\theta)H(x,u,p_2) - c(p_1 - p_2) ,$$

where θ is chosen in $[0,1]$ and

$$c \geq \max(\theta \sup H_p, (\theta - 1)\inf H_p, 0) .$$

Clearly, this difference operator satisfies properties 1, 2 and 3 above. Moreover, if $H_p \geq 0$, (resp. $H_p \leq 0$), we could have chosen $\theta = 0$, (resp. $\theta = 1$), and $c = 0$; thus giving a scheme based on backward, (resp. forward), differencing.

Example 2. The Godunov difference operator, [9], [20], is given by

$$\bar{H}(x,u,p_1,p_2) = \begin{cases} \min\limits_{v\in[p_2,p_1]} H(x,u,v) & \text{if } p_2 \leq p_1 \\ \max\limits_{v\in[p_1,p_2]} H(x,u,v) & \text{if } p_1 \leq p_2 . \end{cases}$$

This difference operator clearly satisfies properties 1 and 2, and a straightforward exercise will verify that it satisfies property 3 as well. Again, when $H(x,u,p)$ is monotone in p, the scheme reduces to either a backward or a forward difference scheme.

Next, we show that the rate of convergence of Theorem 3 is sharp. We again consider the trivial example (3.9) and we approximate its viscosity limit solution, $(u = 0)$, by the Lax-Friedrichs difference scheme - however, we intentionally add too much numerical viscosity, (we take $c = 1$ rather than the allowable $c = 0$). Setting $x_{j+1} - x_j = h$, where $h = 1/J$, u_j is required to satisfy:

$$\text{(4.4)} \qquad -(D^+u_j - D^-u_j) + u_j = 0$$
$$D^-u_0 = 0, \qquad D^+u_J = 1 .$$

One easily computes the exact solution of (4.4)

$$u_j = \frac{h}{\alpha_2^{J+1} - \alpha_1^{J+1}} \left[\frac{1}{1 - \alpha_1} \alpha_1^{j+1} + \frac{1}{\alpha_2 - 1} \alpha_2^{j+1}\right] ,$$

where

$$\alpha_1 = 1 + \frac{h}{2} - \sqrt{h} \cdot \left(1 + \frac{h}{4}\right)^{1/2} ,$$

$$\alpha_2 = 1 + \frac{h}{2} + \sqrt{h} \cdot \left(1 + \frac{h}{4}\right)^{1/2} ,$$

and furthermore, since $u_j > 0$, we have that

$$|u^\Delta - u|_\infty > u_J > \frac{h}{\alpha_2 - 1} .$$

Finally, calculating the right hand side above, we arrive at

$$|u^\Delta - u|_\infty > \sqrt{h} \left(\frac{\sqrt{h}}{2} + \left(1 + \frac{h}{4}\right)^{1/2}\right)^{-1} = \sqrt{h} + O(h) ,$$

which is exactly the rate of Theorem 3.

We shall prove Theorem 3 via three lemmas.

Lemma 2. Assume that $\bar{H}(x,u,p_1,p_2)$ satisfies properties 1, 2 and 3 above. Then, if the difference scheme (4.3) had a solution, say u^Δ, u^Δ is bounded and has a bounded Lipschitz constant, uniformly in $\Delta > 0$.

Proof: We first prove that u^Δ must be uniformly bounded. Suppose that $\max\limits_{0\le j\le J} u_j \ge 0$ is attained for some $1 \le j_0 \le J-1$. Since at an interior maximum $D^+u_{j_0} \le 0 \le D^-u_{j_0}$, (4.3) and Property 2 imply that

$$\bar{H}(x_{j_0},u_{j_0},0,0) \le \bar{H}(x_{j_0},u_{j_0},D^+u_{j_0},D^-u_{j_0}) = 0 \ .$$

Therefore, we have from Property 3 that

$$\mu_0 u_{j_0} \le (\max(u_{j_0},1))^{\eta_1}|\bar{H}(x_{j_0},0,0,0)| \ .$$

Similarly, if $\max\limits_{0\le j\le J} u_j \ge 0$ is attained at $j = 0$ or $j = J$, we would have that

$$\mu|\gamma_0|u_0 \le (\max(u_0,1))^{\eta_1}|\bar{H}(0,0,0,-\gamma_0)| \ ,$$

or

$$\mu|\gamma_1|u_J \le (\max(u_J,1))^{\eta_1}|\bar{H}(1,0,\gamma_1,0)| \ ,$$

which proves that $u^\Delta(x)$ must be bounded above independent of $\Delta > 0$. An identical argument would show that $\min\limits_{0\le j\le J} u_j$ must be bounded below independent of $\Delta > 0$.

Next, we show that $|D^+u_j|$ must be uniformly bounded. Suppose that $\max\limits_{0\le j\le J-1} D^+u_j \ge \max(-\gamma_0,\gamma_1,0)$ is attained at j_0. Again using (4.3), we must have that

$$\begin{aligned} 0 &= \bar{H}(x_{j_0+1},u_{j_0+1},D^+u_{j_0+1},D^+u_{j_0}) - \bar{H}(x_{j_0},u_{j_0},D^+u_{j_0},D^+u_{j_0-1}) \\ &\ge \bar{H}(x_{j_0+1},u_{j_0+1},D^+u_{j_0},D^+u_{j_0}) - \bar{H}(x_{j_0},u_{j_0},D^+u_{j_0},D^+u_{j_0}) \\ &= H(x_{j_0+1},u_{j_0+1},D^+u_{j_0}) - H(x_{j_0},u_{j_0},D^+u_{j_0}). \end{aligned}$$

Setting $K = D^+u_{j_0} \ge 0$, we have from above and Property 3 that

$$\mu_K K \le \left|\frac{\partial}{\partial x} H(\xi,u_{j_0},K)\right|(\max(|u_{j_0}|,1))^{\eta_1} \ ,$$

and this inequality combined with Assumption B' implies that

$$K \le (\max(K,1))^{n_2} C(|u_{j_0}|)(\max(|u_{j_0}|,1))^{n_1} .$$

Therefore, D^+u_j is bounded above, again independent of $\Delta > 0$. A similar argument would show that D^+u_j is bounded below independent of $\Delta > 0$. This proves the lemma.

Lemma 3. Assume that $\bar{H}(x,u,p_1,p_2)$ satisfies properties 1, 2 and 3. Then, the difference scheme (4.3) has a unique solution.

Proof: Consider the map $F_\nu : \mathbb{R}^{J+1} \to \mathbb{R}^{J+1}$, defined by

$$(F_\nu(u))_j = u_j - \nu\bar{H}(x_j,u_j,D^+u_j,D^-u_j) , \tag{4.5}$$

for $0 \le j \le J$, where $-D^-u_0 = \gamma_0$ and $D^+u_J = \gamma_1$. We show below that F_ν has a unique fixed point and obviously this fixed point is the desired solution of difference scheme (4.3). We may assume that $\bar{H}(x,u,p_1,p_2)$ above is globally Lipschitz continuous, since $\bar{H}$ could be modified in a smooth way outside the bounded a priori domain established by the previous lemma.

We now claim that $(F_\nu(u))_j$ is a nondecreasing function in u_{j-1}, u_j and u_{j+1}, provided that ν is chosen sufficiently small. Assume for simplicity that $\bar{H}$ is smooth. We then find upon differentiating

$$\frac{\partial}{\partial u_{j-1}}(F_\nu(u))_j = \nu\bar{H}_{p_2}/(x_j - x_{j-1}) \quad \text{for} \quad 1 \le j \le J ,$$

$$\frac{\partial}{\partial u_{j+1}}(F_\nu(u))_j = -\nu\bar{H}_{p_1}/(x_{j+1} - x_j) \quad \text{for} \quad 0 \le j \le J-1 ,$$

and Property 2 implies that these quantities are nonnegative. Furthermore,

$$\frac{\partial}{\partial u_j}(F_\nu(u))_j = 1 - \nu\{\bar{H}_u - \bar{H}_{p_1}/(x_{j+1} - x_j) + \bar{H}_{p_2}/(x_j - x_{j-1})\} , \tag{4.6}$$

for $1 \leq j \leq J-1$, and $\bar{H}_{p_2} = 0$ for $j = 0$ and $\bar{H}_{p_1} = 0$ for $j = J$. Therefore, since $\bar{H}$ is assumed to be globally Lipschitz continuous, we can choose ν small enough so that these derivatives are nonnegative as well.

Next, we show that F_ν has the fixed-point property for ν as above; (ν should be thought of as an artificial time parameter and the restriction on ν imposed in (4.6) as a CFL condition). Let $u \in \mathbb{R}^{J+1}$ and $v \in \mathbb{R}^{J+1}$ and define $\tau = v - u$. Now consider

$$F_\nu(v) - F_\nu(u) = F_\nu(u + \tau) - F_\nu(u) .$$

Setting $\tau_M = \max\left(\max_{0 \leq j \leq J} \tau, 0\right)$, we have by the claim above, that

$$(F_\nu(u + \tau) - F_\nu(u))_j \leq (F_\nu(u + \tau_M \dagger) - F_\nu(u))_j . \tag{4.8}$$

Now, recalling the definition of F_ν in (4.5), we see that the right hand side of (4.8) is equal to

$$\tau_M - \nu(\bar{H}(x_j, u_j + \tau_M, D^+u_j, D^-u_j) - \bar{H}(x_j, u_j, D^+u_j, D^-u_j)) ,$$

which by Property 3 is bounded above by

$$\tau_M(1 - \nu\hat{\mu}) ,$$

where $\hat{\mu}$ is the appropriate positive constant of Property 3 governed by the a priori domain of Lemma 2. Setting $\tau_m = \min\left(\min_{0 \leq j \leq J} \tau, 0\right)$ and repeating the argument above, we find that

$$\tau_m(1 - \nu\hat{\mu}) \leq (F_\nu(v) - F_\nu(u))_j \leq \tau_M(1 - \nu\hat{\mu}) . \tag{4.9}$$

Therefore, the Banach fixed-point theorem guarantees a unique fixed point of F_ν, for ν sufficiently small, which is the desired result.

<u>Remark 4.1</u>. Inequality (4.9) tells us that implementing an artificial time method, $(u^{n+1} = F_\nu(u^n))$, to obtain a solution of difference scheme (4.3),

converges at an ℓ^∞ rate of $e^{-\hat{\mu}t}$. This of course, is computationally slow in light of the increment restriction imposed by (4.6). We recommend a few iterations of artificial time to pull the initial approximation into the ℓ^∞ domain of attraction for Newton's method, which with some "smoothness", converges at a much faster quadratic rate.

The next lemma is crucial to establish the fact that u^Δ satisfies the approximate viscosity inequalities.

Lemma 4. Suppose $\phi(x) = \kappa|x - y|^2 + \psi(x)$, where $y \in [0,1]$ is fixed, κ is a constant and $\psi(x)$ is an affine function with $-\psi'(0) = \gamma_0$ and $\psi'(1) = \gamma_1$. Then:

(a) If $\kappa > 0$ and $\max\limits_{x\in[0,1]} (u^\Delta(x) - \phi(x))$ is attained for some $\xi \neq x_j$, $0 < j < J$, we have $D^+u_{j_0} - D^-u_{j_0} < \Delta\cdot\phi_{xx}(\xi)$, where x_{j_0} is the nearest grid point to ξ.

(b) If $\kappa < 0$ and $\min\limits_{x\in[0,1]} (u^\Delta(x) - \phi(x))$ is attained for some $\xi \neq x_j$, $0 < j < J$, we have $D^+u_{j_0} - D^-u_{j_0} > \Delta\cdot\phi_{xx}(\xi)$, where x_{j_0} is the nearest grid point to ξ.

Proof: We prove (a) only since the proof of (b) is identical. Let x_{j_0} be the nearest grid point to ξ. We have three basic cases to examine: They are: $x_{j_0} = 0$, $x_{j_0} = 1$ and $0 < x_{j_0} < 1$.

Case 1: When $x_{j_0} = 0$, we must have $D^+u_0 = \phi_x(\xi)$ since $\xi \in (0,x_1)$ is where the maximum of $u^\Delta - \phi$ occurs. However, because ϕ is quadratic, $\phi_x(\xi) = \phi_x(0) + \xi\phi_{xx}(\xi)$. Therefore, $D^+u_0 = \xi\phi_{xx}(\xi) + \phi_x(0) < \xi\phi_{xx}(\xi) - \gamma_0$

Case 2: The case when $x_{j_0} = 1$ is identical to case 1 above.

Case 3: Suppose now that $\xi \in (x_{j_0-1},x_{j_0})$ and choose an arbitrary $\tau \in (x_{j_0},x_{j_0+1})$, (if on the otherhand, $\xi \in (x_{j_0},x_{j_0+1})$, the argument below is essentially the same). Using the definition of u^Δ and the fact that

$(u^\Delta - \phi)(\xi)$ is maximum, we have that

$$(4.10) \qquad \begin{aligned} & u_{j_0} + D^- u_{j_0}(\xi - x_{j_0}) - \phi(\xi) \\ & \quad \geq u_{j_0} + D^+ u_{j_0}(\tau - x_{j_0}) - \phi(\tau), \\ & D^- u_{j_0} = \phi_x(\xi) . \end{aligned}$$

Therefore, a simple calculation will show that (4.10) implies

$$(4.11) \qquad D^+ u_{j_0} - D^- u_{j_0} \leq \frac{\phi_x(\xi)(\xi - \tau) + \phi(\tau) - \phi(\xi)}{\tau - x_{j_0}} .$$

Taylor's theorem allows us to write the right hand side of (4.11) as

$$\frac{1}{2} \left[\frac{(\tau - \xi)^2}{\tau - x_{j_0}}\right] \phi_{xx}(\xi) .$$

Recall that we have assumed our grid satisfies the constraint $(x_{j+1} - x_j) \geq \frac{1}{2}(x_j - x_{j-1})$. This allows us to minimize the bracketed term above by choosing $\tau = 2x_j - \xi$. Doing this, we have

$$D^+ u_{j_0} - D^- u_{j_0} \leq 2(x_{j_0} - \xi)\phi_{xx}(\xi) ,$$

and since x_{j_0} is the <u>nearest</u> grid point to ξ, the proof is complete.

<u>Proof of Theorem 3</u>: The proof of the theorem is complete, (Lemmas 2 and 3), except for showing that u^Δ satisfies the approximate viscosity inequalities of Definition 2. With this in mind, set $\phi_1(x) = |x - y_0|^2/\delta + \psi(x)$, where $y_0 \in [0,1]$, is fixed and $\psi(x)$ is affine, with $\psi'(0) = -\gamma_0$ and $\psi'(1) = \gamma_1$, (as in Definition 2a). Suppose now that $u^\Delta - \phi_1$ is maximum at $\xi \in [0,1]$. To show that u^Δ is an approximate viscosity subsolution, we

must verify that

$$H(\xi,u^{\Delta}(\xi),\phi_{1x}(\xi)) \leq (K\cdot\Delta)\cdot\phi_{1xx}(\xi) + \underline{C}\Delta ,$$

where K is some constant, independent of Δ, and as always in this section, $\Delta = \max\limits_{0\leq j\leq J-1} (x_{j+1} - x_j)$.

Using difference scheme (4.3) and Property 1, we have that for every $0 \leq j \leq J$

$$H(\xi,u^{\Delta}(\xi),\phi_{1x}(\xi)) = \bar{H}(\xi,u^{\Delta}(\xi),\phi_{1x}(\xi),\phi_{1x}(\xi)) - \bar{H}(x_j,u_j,D^+u_j,D^-u_j) ,$$

and we rewrite this identity as

$$(4.13) \quad H(\xi,u^{\Delta}(\xi),\phi_{1x}(\xi))$$

$$= [\bar{H}(x_j,u_j,\phi_{1x}(\xi),\phi_{1x}(\xi)) - \bar{H}(x_j,u_j,D^+u_j,D^-u_j)]$$

$$+ [\bar{H}(\xi,u^{\Delta}(\xi),\phi_{1x}(\xi),\phi_{1x}(\xi)) - \bar{H}(x_j,u_j,\phi_{1x}(\xi),\phi_{1x}(\xi))] .$$

The second term on the right hand side of (4.13) is bounded above by

$$L_x|\xi - x_j| + L_u L|\xi - x_j| ,$$

where L_x and L_u are the Lipschitz constants of $\bar{H}$ in the x and u arguments respectively, and L is the Lipschitz constant of u^{Δ}. The first term on the right hand side of (4.13) can be written as

$$(4.15) \quad \bar{H}_{p_1}\cdot(\phi_{1x}(\xi) - D^+u_j) + \bar{H}_{p_2}\cdot(\phi_{1x}(\xi) - D^-u_j) ,$$

where again, we have assumed that $\bar{H}$ is smooth for simplicity.

If $\xi = x_{j_0}$ for some $0 \leq j_0 \leq J$, we have nothing to prove since it's an easy exercise to determine that in this case $D^+u_{j_0} \leq \phi_{1x}(\xi) \leq D^-u_{j_0}$ when x_{j_0}, $(=\xi)$, is a maximizer of $u^{\Delta} - \phi_1$. (Recall by the definition of ϕ_1 that $\phi_{1x}(0) \leq D^-u_0$ and $\phi_{1x}(1) \geq D^+u_J$ in the event that $\xi = 0$ or 1.) Therefore, setting $j = j_0$ in (4.15) and recalling Property 2, (which says that $\bar{H}_{p_1} \leq 0 \leq \bar{H}_{p_2}$), verifies the approximate viscosity inequality 2a here in a trivial way.

If on the other hand, $\xi \neq x_j$ for all $0 \leq j \leq J$, take x_{j_0} to be the nearest grid point to ξ. Set $j = j_0$ in (4.15) and insert the identity $\phi_{1x}(\xi) = D^- u_{j_0}$, (or $\phi_{1x}(\xi) = D^+ u_{j_0}$) into it. Using the result of Lemma 4, allows us to combine (4.15) with (4.13), to arrive at

$$H(\xi, u^{\Delta}(\xi), \phi_{1x}(\xi)) \leq (K \cdot \Delta) \cdot \phi_{1xx}(\xi) + \underline{C}\Delta ,$$

where $K = \max(-H_{p_1}, H_{p_2})$ and $\underline{C}$ is given by $\frac{1}{2}(L_x + L_u L)$.

An identical argument will show that u^{Δ} is an approximate viscosity supersolution, (see Definition 2b), and so by applying the abstract result of Theorem 2, the proof of Theorem 3 is complete.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
[illegible]2905

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
THE NEUMANN PROBLEM FOR NONLINEAR SECOND ORDER SINGULAR PERTURBATION PROBLEMS

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Benoit Perthame and Richard Sanders

8. CONTRACT OR GRANT NUMBER(s)
DAAG29-80-C-0041,
NSF- MCS 82-00676

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53705

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Work Unit Number 1 - Applied Analysis

11. CONTROLLING OFFICE NAME AND ADDRESS
See Item 18 below.

12. REPORT DATE
January 1986

13. NUMBER OF PAGES
32

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
singular perturbation
viscosity solution
viscosity inequalities

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Singularly perturbed second order elliptic partial differential equations with Neumann boundary conditions arise in many areas of application. These problems rarely have smooth limit solutions. In this paper, we characterize the limit solution for a wide class of such problems. We also give an abstract rate of convergence theorem and apply the abstract theorem to certain finite difference approximations.

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