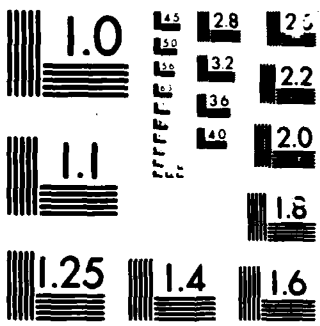
AD-A167 230 EXTREME POINT METHODS IN THE DETERMINATION OF THE STRUCTURE OF A CLASS OF..(U) PITTSBURGH UNIV PA CENTER FOR MULTIVARIATE ANALYSIS M B RAO ET AL. JAN 86 1/1

UNCLASSIFIED TR-86-01 AFOSR-TR-86-0254 F49620-85-C-0008 F/G 12/1 NL



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EXTREME POINT METHODS IN THE DETERMINATION OF THE STRUCTURE OF A CLASS OF BIVARIATE DISTRIBUTIONS AND SOME APPLICATIONS TO CONTINGENCY TABLES*

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Technical Report No. 86-01

January 1986

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Abstract. A decomposition of the class of all bivariate positive quadrant dependent distribution into compact convex subsets is obtained. Extreme points of these compact convex subsets are investigated. A similar decomposition works out for the class of all bivariate distributions. Extreme points of the compact convex sets figuring in this decomposition are analyzed. An application to contingency tables is presented.

A PAPER PRESENTED AT THE INTERNATIONAL CONFERENCE ON THE ADVANCES IN MULTIVARIATE ANALYSIS, INDIAN STATISTICAL INSTITUTE, CALCUTTA, DECEMBER 16-20, 1985.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 86-0254
6a. NAME OF PERFORMING ORGANIZATION University of Pittsburgh; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) Dept Mathmatics and Statistics Pittsburgh, PA 15260	7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR; 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620-85-C-0008
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A5; WORK UNIT NO.

11. TITLE (Include Security Classification) EXTREME POINT METHODS IN THE DETERMINATION OF THE STRUCTURE OF A CLASS OF BIVARIATE DISTRIBUTIONS AND SOME APPLICATIONS TO CONTINGENCY TABLES*

12. PERSONAL AUTHOR(S) M. BHASKARA RAO - P.K. KRISHNAIAH - K. SUBRAMANYAM

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical	FROM ______ TO ______	86 January	28

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Major Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5027; 22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE. UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

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1. Introduction

The class of bivariate distributions in which we are interested in this paper are those which are positive quadrant dependent. To be more specific, let $\mathcal{B}$ be the Borel σ-field on the real line, R and $\mathcal{B}\times\mathcal{B}$ be the Borel σ-field on the two-dimensional euclidean space, R^2. If μ is a probability measure on $\mathcal{B}\times\mathcal{B}$, we denote the first and second marginals of μ by μ_1 and μ_2 respectively. The probability measure μ is said to be <u>positive quadrant dependent</u> if

$$\mu\{[c,\infty) \times [d,\infty)\} \geq \mu_1\{[c,\infty)\}\mu_2\{[d,\infty)\}$$

for every c,d in R. The above notion can be restated as follows. Let X and Y be two real random variables with some joint distribution function F. X and Y are said to be positive quadrant dependent if

$$\Pr(X \geq x,\ Y \geq y) \geq \Pr(X \geq x)\Pr(Y \geq y)$$

for all x,y in R. See Lehmann (1966) for this and related notions of dependence.

We want to look at the above notion of dependence from a global point of view. Choquet's theorem is in the background of the ensuing discussion. Suppose M is a compact convex subset of a locally convex linear topological space. Then by Krein-Milman theorem, there is atleast one extreme point of M. Choquet's theorem represents every member of M as a mixture (in some sense) of the extreme points of M. For a full discussion on Choquet's theorem, see Phelps (1966). We want to pursue the same line of tack in studying the notion of positive quadrant dependence. Accordingly, let M_{PQD} denote the collection of all bivariate positive quadrant dependent distribtuions. We equip M_{PQD} with the topology of weak convergence. Suppose (i) M_{PQD} is compact and (ii) M_{PQD} is convex. Then in accordance with Choquet's theorem we can write every member of M_{PQD} as a mixture of extreme points of M_{PQD}. There are certain common properties of extreme point distributions which are preserved under mixtures and a good understanding of the set of all extreme point distributions would provide an insight into the structure of a general bivariate positive quadrant dependent distribution.

But, the set M_{PQD} is neither compact nor convex. Non-compactness of the set M_{PQD} is not surprising. The malaise stems from the non-compactness of the class of all bivariate distributions. The following bivariate distributions are positive quadrant dependent.

$$\mu = \begin{array}{c|cc|c} X \backslash Y & 1 & 2 & \\ \hline 1 & 1/4 & 1/4 & \\ 2 & 1/4 & 1/4 & \\ \hline & & & 1 \end{array}, \qquad \lambda = \begin{array}{c|cc|c} X \backslash Y & 1 & 2 & \\ \hline 1 & 2/9 & 1/9 & \\ 2 & 4/9 & 2/9 & \\ \hline & & & 1 \end{array}.$$

But $(1/2)\mu + (1/2)\lambda$ is not positive quadrant dependent.

In the next section, we identify some natural subsets of M_{PQD} which are compact and convex. In Section 3, we discuss a method of enumerating the extreme points of these compact convex sets. In Section 4, an application of the results of Sections 3 and 4 is discussed in detail in the context of contingency tables.

2. Compact convex subsets of M_{PQD}.

Let λ and ν be two fixed probability measures on the Borel σ-field $\mathcal{B}$ of the real line. Let $M(\lambda,\nu)$ be the set of all bivariate distributions μ with the first and second marginals λ and ν respectively, i.e.,

$$M(\lambda,\nu) = \{\mu;\ \mu \text{ is a probability measure on } \mathcal{B}\text{x}\mathcal{B}\ ,\ \mu_1 = \lambda \text{ and } \mu_2 = \nu\}\ .$$

Let $M_{PQD}(\lambda,\nu)$ be the set of all bivariate positive quadrant dependent distributions with the first and second marginals λ and ν respectively, i.e

$$M_{PQD}(\lambda,\nu) = \{\mu\ ;\ \mu \text{ is positive quadrant dependent, } \mu_1 = \lambda \text{ and } \mu_2 = \nu\}.$$

These sets are topologically and functional analytically are nice as the following result exemplifies.

<u>Theorem 2.1</u> $M(\lambda,\nu)$ and $M_{PQD}(\lambda,\nu)$ are compact convex sets.

<u>Proof</u>. See Bhaskara Rao. Krishnaiah and Subramanyam (1985).

Thus we obtain a decomposition of M_{PQD} into compact convex subsets as follows.

$$M_{PQD} = \bigcup_{\lambda} \bigcup_{\nu} M_{PQD}(\lambda,\nu)$$

3. Extreme points

Extreme points of the compact convex set $M(\lambda,\nu)$ have been characterized by Lindenstrauss (1965) and Douglas (1964). The following is their result.

Theorem 3.1 A probability measure μ in $M(\lambda,\nu)$ is an extreme point of $M(\lambda,\nu)$ if and only if the collection of all functions h from $R\times R$ to R which are of the form $h(x,y) = f(x) + g(y)$, x,y in R for some f and g real-valued integrable functions on the real line integrable with respect to λ and ν respectively is dense in $L_1(R\times R, \mathcal{B}\times\mathcal{B}, \mu)$.

Proof. See Douglas (1964, Theorem 1, p.243). See also Lindenstrauss (1965, Theorem 1, p.379).

We generalize the above result. We operate in the more general realm of probability triplets. Let $(X,\mathcal{C},\lambda)$ and $(Y,\mathcal{D},\nu)$ be two probability spaces. Let $M(\lambda,\nu)$ be the collection of all probability measures μ on the product σ-field $\mathcal{C}\times\mathcal{D}$ such that the first marginal of μ is λ and the second marginal of μ is ν. This notation agrees with the notation introduced above in the context of probability measures on the real line. Let

$$F_o = \{I_{B\times Y}\ ;\ B \in \mathcal{C}\}\ \{I_{X\times C}\ ;\ C \in \mathcal{D}\},$$

where I_A denotes the indicator function of the set A. Let F be the linear manifold spanned by F_o.

<u>Theorem 3.2</u> A probability measure μ on $C\times\mathcal{D}$ is an extreme point of $M(\lambda,\nu)$ if and only if F is dense in $L_1(X\times Y,\ C\times\mathcal{D},\mu)$.

<u>Proof</u>. The proof here is modelled on the proof of Theorem 1 of Douglas (1964, p.243). Suppose μ is not an extreme point of $M(\lambda,\nu)$. Let $\mu = (\zeta + \eta)/2$ for some ζ and η in $M(\lambda,\nu)$. This implies that $2\mu \geq \eta \geq 0$, and by Radon-Nikodym theorem, there exists a function h in $L_\infty(X\times Y,\ C\times\mathcal{D},\mu)$ such that

$$\frac{d\eta}{d\mu} = h \quad \text{a.e. } [\mu] \quad , \text{ and}$$

$$1 - h \neq 0 \quad \text{a.e. } [\mu] \quad .$$

The function $1 - h$ is orthogonal to F, i.e.,

$$\begin{aligned} \int f(1 - h)\, d\mu &= \int f\, d\mu - \int fh\, d\mu \\ &= \int f\, d\mu - \int f\, d\eta = \int f\, d\mu - \int f\, d\mu \\ &= 0 \text{ for every } f \text{ in } F. \end{aligned}$$

This clearly demonstrates that F is not dense in $L_1(X\times Y,\ C\times\mathcal{D},\mu)$. For, if F were to be dense, since the dual $L_1^*(X\times Y,\ C\times\mathcal{D},\mu)$ of $L_1(X\times Y,\ C\times\mathcal{D},\mu)$ is $L_\infty(X\times Y,\ C\times\mathcal{D},\mu)$, the linear functional induced by $1 - h$ on L_1 vanishes identically on F would imply that $1 - h = 0$ a.e.$[\mu]$ .This contradiction proves the assertion made above.

Conversely, suppose F is not dense in $L_1(X\times Y, C\times\mathcal{D}, \mu)$. Then there exists an essentially non-zero function h in $L_\infty(X\times Y, C\times\mathcal{D}, \mu)$ orthogonal to F. Set

$$\eta(E) = \int_E h\, d\mu \qquad \text{for } E \text{ in } C\times\mathcal{D}.$$

Since $\int hf\, d\mu = 0$ for every f in F, we have, in particular, $\int h\, d\mu = 0$. This implies that $\eta(X\times Y) = 0$. Define

$$\zeta(E) = \int_E (1 + h/||h||_\infty)\, d\mu \quad , \quad \text{and}$$

$$\upsilon(E) = \int_E (1 - h/||h||_\infty)\, d\mu \qquad \text{for } E \text{ in } C\times\mathcal{D},$$

where $||h||_\infty$ is the L_∞-norm of h. Since $1 \pm (h/||h||_\infty) \geq 0$ a.e. $[\mu]$, the set functions ζ and υ are non-negative. In fact, $\zeta, \upsilon \in M(\lambda,\nu)$. We can now write $\mu = (\zeta + \upsilon)/2$. This completes the proof.

We now exhibit some extreme points of $M(\lambda,\nu)$ using measure preserving transformations. Let T be a measure preserving transformation from X to Y preserving the measures λ and ν, i.e., $T^{-1}(D) \in C$ and $\lambda(T^{-1}(D)) = \nu(D)$ for every D in $\mathcal{D}$. Let G be the graph of T, i.e.,

$$G = \{(x, Tx) \; ; \; x \in X\}.$$

Let P_1 be the projection map from $X\times Y$ onto X, i.e., $P_1(x,y) = x$ for every (x,y) in $X\times Y$. We claim that for every E in x, $P_1(E \cap G)$ is available in C. For, let $E = \{E \in C\times\mathcal{D} \; ; \; P_1(E \cap G) \in C\}$. One can check that the rectangle sets in $\mathcal{C}\times\mathcal{D}$ are available in $\mathcal{E}$, $\mathcal{E}$ is closed under complementation and countable unions. Hence $\mathcal{E} = \mathcal{C}\times\mathcal{D}$. Define a set function μ on $\mathcal{C}\times\mathcal{D}$ as follows.

$$\mu(E) = \lambda(P_1(E \cap G)) \quad \text{for } E \text{ in } \mathcal{C}\times\mathcal{D}.$$

We now check that μ is a probability measure on $\mathcal{C}\times\mathcal{D}$ with marginals λ and ν.

$$\begin{aligned} \mu(C\times Y) &= \lambda(P_1((C\times Y) \cap G)) \\ &= \lambda(C \cap T^{-1}Y) = \lambda(C) \quad \text{for } C \text{ in } \mathcal{C}. \\ \mu(X\times D) &= \lambda(P_1((X\times D) \cap G)) \quad \text{for } D \text{ in } \mathcal{D}. \\ &= \lambda(X \cap T^{-1}D) = \lambda(T^{-1}D) \\ &= \nu(D), \text{ since } T \text{ is measure preserving.} \end{aligned}$$

The assertion that μ is a probability measure is obvious.

We now claim that G is a thick subset of $X\times Y$ under μ, i.e., the outer measure of G, $\mu^*(G) = 1$. For, if E is any set in $\mathcal{C}\times\mathcal{D}$ containing G, then $P_1(E \cap G) = P_1(G) = X$. Loosely speaking, the measure μ is concentrated on the graph of T.

Finally, we assert that μ is an extreme point of $M(\lambda,\nu)$. Suppose $\mu = (\zeta + \upsilon)/2$ for some ζ and υ in $M(\lambda,\nu)$. It is obvious that $\zeta^*(G) = 1 = \upsilon^*(G)$. Further, for any E in $\mathcal{C}\times\mathcal{D}$, $\zeta^*(E \cap G) = \zeta(E)$ and $\upsilon^*(E\cap G) = \upsilon(E)$. If $C\times D \in \mathcal{C}\times\mathcal{D}$, $(C\times D) \cap G \subset (C \cap T^{-1}D) \times Y$. Consequently,

$$\zeta(C \times D) \leq \zeta((C \cap T^{-1}D) \times Y) = \lambda(C \cap T^{-1}D) = \mu(C \times D),$$

and

$$\upsilon(C \times D) \leq \upsilon(C \cap T^{-1}D) \times Y) = \lambda(C \cap T^{-1}D) = \mu(C \times D).$$

Hence $\mu(C \times D) = \zeta(C \times D) = \upsilon(C \times D)$ for every C in $\mathcal{C}$ and D in $\mathcal{D}$. Therefore, $\mu = \zeta = \upsilon$. This completes the proof.

We now want to discuss characterizing the extreme points of $\mathcal{M}_{PQD}(\lambda, \nu)$, where λ and ν are fixed probability measures on the Borel σ-field $\mathcal{B}$ of the real line. One can prove that the product measure $\mu \times \nu$ is always an extreme point of $\mathcal{M}_{PQD}(\lambda, \nu)$. See Subramanyam and Bhaskara Rao (1985). But this is in total contrast with the case of $\mathcal{M}(\lambda, \nu)$. The product measure $\mu \times \nu$ is an extreme point of $\mathcal{M}(\lambda, \nu)$ if and only if either λ is 0-1 valued or ν is 0-1 valued. See Kemp (1968, p.1356). We do not have any characterization of the extreme points of $\mathcal{M}_{PQD}(\lambda, \nu)$ in the general case. Note that $\mathcal{M}_{PQD}(\lambda, \nu)$ is a compact convex subset of $\mathcal{M}(\lambda, \nu)$. It is easy to see that if μ is an extreme point of $\mathcal{M}(\lambda, \nu)$ and μ belongs to $\mathcal{M}_{PQD}(\lambda, \nu)$, then μ is an extreme point of $\mathcal{M}_{PQD}(\lambda, \nu)$. If μ is an extreme point of $\mathcal{M}_{PQD}(\lambda, \nu)$, μ need not be an extreme point of $\mathcal{M}(\lambda, \nu)$. The numbers also do not exhibit a pattern. If λ and ν have finite support, there are cases when the number of extreme points of $\mathcal{M}_{PQD}(\lambda, \nu)$ is strictly less than the number of extreme points of $\mathcal{M}(\lambda, \nu)$, and cases the opposite happens.

When the probability measures λ and ν have finite support, the subject of the determination of the extreme points of $M(\lambda,\nu)$ and of $M_{PQD}(\lambda,\nu)$ have a distinct flavour. We now proceed to describe a method of determining the extreme points of these compact convex sets. Without loss of generality, assume that the support of λ is $\{1,2,\ldots,m\}$ and that of ν is $\{1,2,\ldots,n\}$. Let $p_i = \lambda(\{i\})$, $i = 1,2,\ldots,m$ and $q_j = \nu(\{j\})$, $j = 1,2,\ldots,n$. From now on, we use the suggestive notation :

$$M(\lambda,\nu) = M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$$

and

$$M_{PQD}(\lambda,\nu) = M_{PQD}(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n).$$

<u>Extreme points of</u> $\underline{M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)}$

Any bivariate distribution in $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$ can be described in the form of a matrix $P = (p_{ij})$ of order $m \times n$ with the following properties.

(i) $p_{ij} \geq 0$ for all i and j.

(ii) $\sum_{i=1}^{m} p_{ij} = q_j$ for $j = 1,2,\ldots,n$.

(iii) $\sum_{j=1}^{n} p_{ij} = p_i$ for $i = 1,2,\ldots,m$.

<u>Definition 3.1</u> Let $P = (p_{ij})$ be any matrix in $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$. An ordered sequence $[p_{i_1j_1}, p_{i_2j_2}, \ldots, p_{i_kj_k}]$ in P is said to be a <u>loop</u> in P if

(i) k is even,

(ii) $i_1 = i_2$, $i_3 = i_4$, $\ldots$, $i_{k-1} = i_k$,

(iii) $j_2 = j_3$, $j_4 = j_5$, $\cdots$, $j_k = j_1$,

(iv) the pairs (i_r, j_r) for $r = 1,2,\ldots,k$ are distinct.

A loop in P is said to be positive if every member of the loop is positive. Support of P is the collection of all pairs (i,j) for which p_{ij} is positive.

The following lemmas are helpful in characterizing the extreme points of $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$.

Lemma 3.1 Every row (column) of a matrix P in $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots q_n)$ contains an even number of elements of any loop in P.

Proof. Obvious.

Lemma 3.2 If $p_{i_1j_1}, p_{i_1j_2}; p_{i_2j_2}, p_{i_2j_3}; \cdots ; p_{i_rj_r}, p_{i_rj_{r+1}}; \cdots$ is an infinite sequence of elements from a a given matrix P with the property that any two consecutive suffixes are distinct, then we can find a loop in P consisting of elements from the given sequence.

Proof. Let b be the first element in the sequence whose suffix agrees with the suffix of one of the elements a preceding b.

Case (i). $a = p_{i_kj_k}$ and $b = p_{i_rj_r}$. Then $[p_{i_kj_k}, p_{i_kj_{k+1}}; \cdots ;$ $p_{i_{r-1}j_{r-1}}, p_{i_{r-1}j_r}]$ is a loop in P.

Case (ii). $a = p_{i_ki_{k+1}}$ and $b = p_{i_ri_{r+1}}$. Then $[p_{i_{k+1}j_{k+1}}, p_{i_{k+1}j_{k+2}}; \cdots$ $p_{i_rj_r}, p_{i_rj_{r+1}}]$ is a loop in P.

Case (iii). $a = p_{i_kj_k}$ and $b = p_{i_rj_{r+1}}$. Then $[p_{i_{k+1}j_{k+1}}, p_{i_{k+1}j_{k+2}}; \cdots ;$

$p_{i_r j_r}, p_{i_r j_{r+1}}]$ is a loop in P.

Case (iv). $a = p_{i_k j_{k+1}}$ and $b = p_{i_r j_r}$. This case can be disposed of as in Case (iii).

Lemma 3.3 If every row and column of P contains atleast two positive elements, then P contains a positive loop.

Proof. It is easy to construct an infinite sequence of positive elements in P satisfying the hypothesis of Lemma 3.2.

We now give five different ways of characterizing the extreme points of $M(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n)$.

Theorem 3.3 Let P be any matrix in $M(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n)$. The following statements are equivalent.

(i) P is not an extreme point of $M(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n)$.

(ii) There is a positive loop in P.

(iii) There exists a submatrix E of P having the property that every row and column of E has atleast two positive elements.

(iv) There exists a square submatrix D of P having the property that every row and column of D contains atleast two positive elements.

(v) There exists a square submatrix F of P, say, of order k, having the property that the number of positive elements in F is atleast $2k$.

Proof. The equivalence of (i) and (v) was proved by Lindenstrauss (1965, p.382). The equivalence of (i) and (ii) was proved by Klee and Witzgall (1968, Theorem 4, p.265). We prove the equivalence of these five statements directly as follows. (Lemma 3.2 plays a crucial role in our proofs.)

(i) ===> (ii). Suppose P is not an extreme point of $M(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n)$. For the remainder of this proof, we denote the above compact convex set simply by M. Then we can write $P = \alpha B + (1-\alpha)C$ for some distinct $B = (b_{ij})$ and $C = (c_{ij})$ in M and $0 < \alpha < 1$. If $p_{ij} = 0$, then $b_{ij} = 0 = c_{ij}$. Since P and B are distinct, there exists a pair (i_1, j_1) such that $p_{i_1j_1} \neq b_{i_1j_1}$. Clearly, $p_{i_1j_1}$ is positive. Assume, without loss of generality, that $p_{i_1j_1} > b_{i_1j_1}$. There exists $j_2 \neq j_1$ such that $p_{i_1j_2} < b_{i_1j_2}$. Note that $p_{i_1j_2}$ is positive. There exists $i_2 \neq i_1$ such that $p_{i_2j_2} > b_{i_2j_2}$, and $j_3 \neq j_2$ such that $p_{i_2j_3} < b_{i_2j_3}$ and so on. Thus we can construct a sequence $p_{i_1j_1}, p_{i_1j_2}; p_{i_2j_2}, p_{i_2j_3}; \ldots$ of positive elements in P with the property that any two consecutive suffixes are distinct. By Lemma 3.2, we can find a positive loop in P.

(ii) ===> (v). Suppose P contains a positive loop $p_{i_1j_1}, p_{i_2j_2}, \ldots, p_{i_rj_r}$. Let s_1 = the number of distinct elements among $i_1, i_2, \ldots, i_r$ and s_2 = the number of distinct elements among $j_1, j_2, \ldots, j_r$. Assume, without loss of generality, that $s_1 \leq s_2$. Let E be the submatrix of order $s_1 \times s_2$ determined by $i_1, i_2, \ldots, i_r$ - th rows and $j_1, j_2, \ldots, j_r$ -th columns. Let D be a submatrix of E obtained by deleting $s_2 - s_1$ columns from E. Since every column of D contains atleast two positive elements, the number of positive elments in D is atleast $2s_1$, and the order of the matrix D is $s_1 \times s_1$.

(v) ===> (iv). This is clear.

(iv) ===> (iii). This is also clear.

(iii) ===> (ii). One can easily extract a positive loop from E and hence from P.

(ii) ===> (i). Suppose $[p_{i_1j_1}, p_{i_2j_2}, \ldots, p_{i_kj_k}]$ is a positive loop in P. Let $0 < \varepsilon <$ minimum$\{p_{i_1j_1}, p_{i_2j_2}, \ldots, p_{i_kj_k}\}$. Define two matrices $B = (b_{ij})$ and $C = (c_{ij})$ as follows.

$$
\begin{aligned}
b_{i_sj_s} &= p_{i_sj_s} + \varepsilon \ , \ \text{if } s \text{ is odd,} \\
&= p_{i_sj_s} - \varepsilon \ , \ \text{if } s \text{ is even,} \\
b_{ij} &= p_{ij} \ , \ \text{if } (i,j) \neq (i_s,j_s) \text{ for any } s = 1,2,\ldots,k, \\
c_{i_sj_s} &= p_{i_sj_s} + \varepsilon \ , \ \text{if } s \text{ is even,} \\
&= p_{i_sj_s} - \varepsilon \ ; \ \text{if } s \text{ is odd,} \\
c_{ij} &= p_{ij}, \ \text{if } (i,j) \neq (i_s,j_s) \text{ for any } s = 1,2,\ldots,k.
\end{aligned}
$$

Note that B and C are in $\mathcal{M}$, distinct and $P = \frac{1}{2}(B + C)$. Consequently, P is not an extreme point of $\mathcal{M}$.

This completes the proof of the theorem.

The above theorem provides some information on the supports of the extreme point bivariate distributions in $\mathcal{M}(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$.

<u>Corollary 3.1</u> Let P and Q be two distinct extreme points of $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$. Then the support of P neither contains nor is contained in the support of Q properly.

<u>Proof</u>. Suppose the support of P is contained properly in the support of Q. Let (i_1,j_1) be such that $q_{i_1j_1} > 0 = p_{i_1j_1}$, where $P = (p_{ij})$ and $Q = (q_{ij})$. There exists $j_2 \neq j_1$ such that $q_{i_1j_2} < p_{i_1j_2}$. Clearly, $q_{i_1j_2}$ is positive. For, otherwise, $p_{i_1j_2}$ will be equal to zero. There exists $i_2 \neq i_1$ such that $q_{i_2j_2} > p_{i_2j_2}$ which implies that $q_{i_2j_2}$ is positive. Proceeding as above, we can construct an infinite sequence $q_{i_1j_1}, q_{i_1j_2}; q_{i_2j_2}, q_{i_2j_3}; \ldots$ of positive elements in Q with the property that any two consecutive suffixes are distinct. By Lemma 3.2, there exists a positive loop giving rise to a contradiction to the veracity of Theorem 3.3. A similar argument shows that the support of Q is not contained in the support of P properly.

<u>Corollary 3.2</u> Let $P = (p_{ij})$ and $Q = (q_{ij})$ be two distinct extreme points of $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$. Then the supports of P and Q are distinct.

<u>Proof.</u> Suppose that the supports of P and Q are equal. Since P and Q are distinct, there exists a pair (i_1,j_1) such that $p_{i_1j_1} \neq q_{i_1j_1}$. Assume, without loss of generality, that $p_{i_1j_1} > q_{i_1j_1}$ which obviously

implies that that $p_{i_1j_1}$ is positive. There exists $j_2 \neq j_1$ such that $p_{i_1j_2} < q_{i_1j_2}$. Since the supports are equal, $p_{i_1j_2}$ is positive. There exists $i_2 \neq i_1$ such that $p_{i_2j_2} > q_{i_2j_2}$, and so on. Thus we can construct an infinite sequence $p_{i_1j_1}, p_{i_1j_2}; p_{i_2j_2}, p_{i_2j_3}; \ldots$ of positive elements in P with the property that any two consecutive suffixes are distinct. By Lemma 3.2, there exists a positive loop in P negating the validity of Theorem 3.3. This contradiction proves the result.

<u>Corollary 3.3</u> The number of extreme points of $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$ is finite.

<u>Proof</u>. The set of all subsets of $\{(i,j)\ ;\ i = 1 \text{ to } m,\ j = 1 \text{ to } n\}$ is finite.

<u>Corollary 3.4</u> Let $S = (s_{ij})$ be the matrix in $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$, where $s_{ij} = p_iq_j$ for all i and j. Then S is an extreme point of $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$ if and only if $p_i = 1$ for some i or $q_j = 1$ for some j.

<u>Proof</u>. If either $p_i = 1$ for some i or $q_j = 1$ for some j, then $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$ contains only one matrix S and consequently, it is an extreme point of $M(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$. On the other hand, if $p_i \neq 1$ for every i and $q_j \neq 1$ for every j, then we can find p_u, p_v, q_r, q_t all positive with $u \neq v$ and $r \neq t$. Then $[s_{ur}, s_{ut}, s_{vt}, s_{vr}]$ is a positive loop in S. Hence S is not an extreme point.

Remarks. The matrix S defined above is the product measure of the distributio $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_n$. The above corollary is a discrete analogue of Theorem 3 of Kemp (1968, p.1356).

Examples.

1. $m = 2$, $n = 2$; $p_1 = p_2 = \frac{1}{2} = q_1 = q_2$.

   Extreme points of $M(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ are

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

2. $m = 2$, $n = 2$; $p_1 = 1/4$, $p_2 = 3/4$; $q_1 = 2/3$ and $q_2 = 1/3$.

   Extreme points of $M(1/4,3/4;\ 2/3,1/3)$ are

$$\begin{bmatrix} 0 & 3/12 \\ 8/12 & 1/12 \end{bmatrix} \text{ and } \begin{bmatrix} 3/12 & 0 \\ 5/12 & 4/12 \end{bmatrix}.$$

3. $m = 2$, $n = 3$; $p_1 = p_2 = 1/2$; $q_1 = q_2 = q_3 = 1/3$.

   Extreme points of $M(1/2,1/2;\ 1/3,1/3,1/3)$ are

$$\begin{bmatrix} 2/6 & 1/6 & 0 \\ 0 & 1/6 & 2/6 \end{bmatrix}, \begin{bmatrix} 2/6 & 0 & 1/6 \\ 0 & 2/6 & 1/6 \end{bmatrix},$$

$$\begin{bmatrix} 1/6 & 2/6 & 0 \\ 1/6 & 0 & 2/6 \end{bmatrix}, \begin{bmatrix} 0 & 2/6 & 1/6 \\ 2/6 & 0 & 1/6 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1/6 & 2/6 \\ 2/6 & 1/6 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1/6 & 0 & 2/6 \\ 1/6 & 2/6 & 0 \end{bmatrix}.$$

<u>Extreme points of $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$</u>

We discuss a method of enumerating the extreme points of the above compact convex set with the help of some examples. From this discussion, a general strategy can be devised to enumerate the extreme points in the general case.

1. <u>The 2 x 2 case</u>.

One can show that any bivariate distribution in $M_{PQD}(p_1,p_2;\ q_1,q_2)$ given in the form of a matrix $P = (p_{ij})$ must satisfy the inequality

$$p_2q_2 \leq p_{22} \leq p_2 \wedge q_2,$$

where $a \wedge b$ denotes the minimum of the two numbers a and b. Conversely, if a number p_{22} satisfies the above inequalities, one can find a matrix in $M_{PQD}(p_1,p_2;\ q_1,q_2)$ whose (2,2)-th element is p_{22}. More precisely, this matrix is given by

$$\begin{bmatrix} p_1-q_2+p_{22} & q_2-p_{22} \\ p_2-p_{22} & p_{22} \end{bmatrix}.$$

Consequently, the extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2)$ are obtained by setting $p_{22} = p_2q_2$ and $p_{22} = p_2 \wedge q_2$ separately and fill the rest of the entries of the matrix using marginality conditions. There are only two extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2)$ and these are given by

$$\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} q_1 & q_2-p_2 \\ 0 & p_2 \end{bmatrix} \qquad \text{if } p_2 \wedge q_2 = p_2,$$

$$\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_1 & 0 \\ p_2-q_2 & q_2 \end{bmatrix} \qquad \text{if } p_2 \wedge q_2 = q_2.$$

Every member of $M_{PQD}(p_1,p_2;\ q_1,q_2)$ is a convex combination of these two extreme points. The following are two concrete examples.

<u>Example 1</u>. $p_1 = p_2 = 1/2 = q_1 = q_2$.

The extreme points are

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}.$$

<u>Example 2</u>. $p_1 = 1/4$, $p_2 = 3/4$ and $q_1 = 2/3$, $q_2 = 1/3$.

The extreme points are

$$\begin{bmatrix} 3/12 & 0 \\ 5/12 & 4/12 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2/12 & 1/12 \\ 6/12 & 3/12 \end{bmatrix}.$$

2. <u>The 2×3 case</u>.

In this case, the determination of the extreme points of

$M_{PQD}(p_1,p_2;\ q_1,q_2,q_3)$ can be achieved graphically. If the matrix (p_{ij}) belongs to $M_{PQD}(p_1,p_2;\ q_1,q_2,q_3)$, then one can check that the following inequalities must be satisfied.

$$p_2q_3 \le p_{23} \le p_2 \wedge q_3 \quad \text{and}$$

$$(p_2q_2 + p_2q_3) \vee p_{23} \le p_{22} + p_{23} \le p_2 \wedge (q_2 + p_{23}).$$

($a \vee b$ means the maximum of the two numbers a and b.) Conversely, if p_{22} and p_{23} are two numbers satisfying the above inequalities, then the matrix

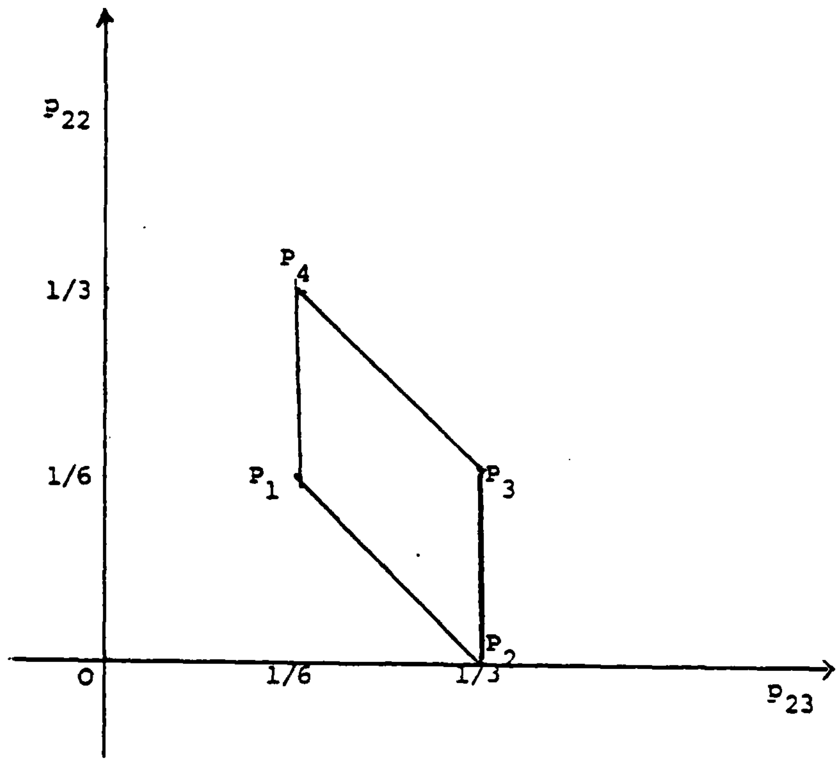
$$\begin{bmatrix} q_1-p_2+p_{22}+p_{23} & q_2-p_{22} & q_3-p_{23} \\ p_2-p_{22}-p_{23} & p_{22} & p_{23} \end{bmatrix}$$

belongs to $M_{PQD}(p_1,p_2;\ q_1,q_2,q_3)$. The above two inequalities determine a simplex in the two dimensional $p_{22} - p_{23}$ plane. The extreme points of this simplex give the extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2,q_3)$. As an illustration, let $p_1 = p_2 = 1/2$ and $q_1 = q_2 = q_3 = 1/3$. The determining inequalities are

$$1/6 \le p_{23} \le 1/3 \quad \text{and}$$

$$(1/3) \vee p_{23} = 1/3 \le p_{22} + p_{23} \le 1/2 \wedge (1/3 + p_{23}) = 1/2.$$

These inequalities determine the following simplex in the $p_{22} - p_{23}$ plane. There are four extreme points of the set $M_{PQD}(1/2,1/2;\ 1/3,1/3,1/3)$ which are determined by the four extreme points of the simplex. Note that the set $M(1/2,1/2;\ 1/3,1/3,1/3)$ has six extreme points.



The four extreme points corresponding to P_1, P_2, P_3, P_4 respectively are

$$\begin{bmatrix} 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 \end{bmatrix}, \quad \begin{bmatrix} 1/6 & 1/3 & 0 \\ 1/6 & 0 & 1/3 \end{bmatrix},$$

$$\begin{bmatrix} 1/3 & 1/6 & 0 \\ 0 & 1/6 & 1/3 \end{bmatrix}, \quad \begin{bmatrix} 1/3 & 0 & 1/6 \\ 0 & 1/3 & 1/6 \end{bmatrix}.$$

Every member of $M_{PQD}(1/2,1/2;\ 1/3,1/3,1/3)$ is a convex combination of these four extreme points.

3. <u>The</u> <u>2 x 4</u> <u>case.</u>

One can check that if a matrix (p_{ij}) belongs to $M_{PQD}(p_1,p_2;$

$q_1, q_2, q_3)$, then

$$p_2 q_4 \leq p_{24} \leq p_2 \wedge q_4,$$

$$(p_2 q_3 + p_2 q_4) \vee p_{24} \leq p_{23} + p_{24} \leq p_2 \wedge (q_3 + p_{24}),$$

$$(p_2 q_2 + p_2 q_3 + p_2 q_4) \vee (p_{23} + p_{24}) \leq p_{22} + p_{23} + p_{24} \leq$$

$$p_2 \wedge (q_2 + p_{23} + p_{24}).$$

Conversely, if p_{22}, p_{23} and p_{24} are three numbers satisfying the above inequalities, then they determine a member of $M_{PQD}(p_1, p_2; q_1, q_2, q_3)$ in an obvious way. For details, see Subramanyam and Bhaskara Rao (1985). We illustrate the determination of the extreme points of this set with the help of a concrete example.

Let $p_1 = p_2 = 1/2$ and $q_1 = q_2 = q_3 = q_4 = 1/4$. The determining inequalities are

$$1/8 \leq p_{24} \leq 2/8,$$

$$2/8 \leq p_{23} + p_{24} \leq 2/8 + p_{24} \quad \text{and}$$

$$3/8 \vee (p_{23} + p_{24}) \leq p_{22} + p_{23} + p_{24} \leq 4/8 \wedge (2/8 + p_{23} + p_{24}).$$

The first step in the determination of all extreme points consists of eliminating the symbols $\vee$ and $\wedge$ from the above set of inequalities by splitting, if necessary, some of the inequalities. For example, the inequality $2/8 \leq p_{23} + p_{24} \leq 2/8 + p_{24}$ is equivalent to the two inequalities $2/8 \leq p_{23} + p_{24} \leq 3/8$ and $3/8 \leq p_{23} + p_{24} \leq 2/8 + p_{24}$.

The splitting certainly helps to get rid of the symbols $\vee$ and $\wedge$. The above set of determining inequalities is equivalent to the following two sets of inequalities.

A. $1/8 \leq p_{24} \leq 2/8$

$$2/8 \leq p_{23} + p_{24} \leq 3/8$$

$$3/8 \leq p_{22} + p_{23} + p_{24} \leq 4/8$$

B. $1/8 \leq p_{24} \leq 2/8$

$$3/8 \leq p_{23} + p_{24} \leq 2/8 + p_{24}$$

$$p_{23} + p_{24} \leq p_{22} + p_{23} + p_{24} \leq 4/8$$

Now, the second step consists of the following mancevres. In each set of inequalities, set the central expression equal to the expression either on the right or on the left. This will result in three equations in three unknowns. These linear equations are very easy to solve. From the solution so obtained, one can build a matrix in $M_{PQD}(1/2,1/2;\ 1/4,1/4,1/4,1/4)$ using the marginality restrictions. The set of all extreme points of $M_{PQD}(1/2,1/2;\ 1/4,1/4,1/4,1/4)$ is contained in the collection of all matrices so obtained above. After eliminating the duplicates, the following
the
is collection of all extreme points of $M_{PQD}(1/2,1/2;\ 1/4,1/4,1/4,1/4)$.

$$\begin{bmatrix} 1/8 & 1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/8 & 1/8 \end{bmatrix}, \quad \begin{bmatrix} 2/8 & 0 & 1/8 & 1/8 \\ 0 & 2/8 & 1/8 & 1/8 \end{bmatrix},$$

$$\begin{bmatrix} 1/8 & 2/8 & 0 & 1/8 \\ 1/8 & 0 & 2/8 & 1/8 \end{bmatrix}, \quad \begin{bmatrix} 2/8 & 1/8 & 0 & 1/8 \\ 0 & 1/8 & 2/8 & 1/8 \end{bmatrix},$$

$$\begin{bmatrix} 1/8 & 1/8 & 2/8 & 0 \\ 1/8 & 1/8 & 0 & 2/8 \end{bmatrix}, \quad \begin{bmatrix} 2/8 & 0 & 2/8 & 0 \\ 0 & 2/8 & 0 & 2/8 \end{bmatrix},$$

$$\begin{bmatrix} 1/8 & 2/8 & 1/8 & 0 \\ 1/8 & 0 & 1/8 & 2/8 \end{bmatrix}, \quad \begin{bmatrix} 2/8 & 1/8 & 1/8 & 0 \\ 0 & 1/8 & 1/8 & 2/8 \end{bmatrix},$$

and

$$\begin{bmatrix} 2/8 & 2/8 & 0 & 0 \\ 0 & 0 & 2/8 & 2/8 \end{bmatrix}.$$

The determination of the set of all extreme points in the general case can be achieved by following basically the above two steps. For further details, see Subramanyam and Bhaskara Rao (1986).

4. Applications to Contingency Tables

In this section, we are interested in the following problem. Let X and Y be random variables with known marginal distributions λ and ν respectively but with unknown joint distribution function. We want to test the hypothesis that X and Y are independent against the alternative that X and Y are strictly positive quadrant dependent. For simplicity, assume that the support of λ is $\{1,2,\ldots,m\}$ and that of ν is $\{1,2,\ldots,n\}$. Let the data consist of N independent realizations of (X,Y). Let n_{ij} = Number of (X,Y)'s with $X = i$ and $Y = j$, $i = 1$ to m and $j = 1$ to n. The data can be arranged in the form of a contingency table as follows.

$$
\begin{array}{|cccc|c}
\hline
n_{11} & n_{12} & \cdots & n_{1n} & \\
n_{21} & n_{22} & \cdots & n_{2n} & \\
 & \cdots\cdots & \cdots\cdots & & \\
n_{m1} & n_{m2} & \cdots & n_{mn} & \\
\hline
\multicolumn{4}{c|}{} & N \\
\end{array}
$$

There are a plethora of tests available to test the hypothesis of independence. A problem of choice arizes to test the above hypothesis of independence against the above specific alternative. One way to resolve the dilemma is to compare the power functions of the tests. The domain of the power function is $\mathcal{M}_{PQD}(\lambda,\nu)$. Computation of the power of a given test at every point in the domain is not feasible practically. The following theorem asserts that it is enough to compute the power at all extreme points of $\mathcal{M}_{PQD}(\lambda,\nu)$.

<u>Theorem 4.1</u> Let the extreme points of $\mathcal{M}_{PQD}(\lambda,\nu)$ be $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}$. Let T be a test proposed to test the hypothesis of independence and $\beta_T(\ )$ its power function. Let μ be an arbitrary distribution in $\mathcal{M}_{PQD}(\lambda,\nu)$. (Then we can write

$$\mu = \alpha_1 \mu^{(1)} + \alpha_2 \mu^{(2)} + \ldots + \alpha_k \mu^{(k)}$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 0$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$.)

Then

$$\beta_T(\mu) = \alpha_1 \beta_T(\mu^{(1)}) + \alpha_2 \beta_T(\mu^{(2)}) + \ldots + \alpha_k \beta_T(\mu^{(k)}).$$

For a discussion of this theorem, see Bhaskara Rao, Krishnaiah and Subramanyam (1985). In view of this theorem, if we wish to compare the performance of two tests, we merely compute the powers of these tests at the extreme point distributions and then compare these powers point by point. To illustrate the mechanism of this theorem, we consider the case $m = 2 = n$.

Assume, for simplicity, that $p_2 \leq q_2$. Then the extreme points of $\mathcal{M}_{PQD}(p_1, p_2; q_1, q_2)$ are

$$P_1 = \begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \text{ and } \begin{bmatrix} q_1 & q_2-p_2 \\ 0 & p_2 \end{bmatrix} = P_2 .$$

There are two popular measures of association between X and Y defined by

$$\text{Gamma Ratio} = \gamma = \frac{p_{11}p_{22} - p_{12}p_{21}}{p_{11}p_{22} + p_{12}p_{21}}$$

(see Goodman and Kruskal)

and

$$\text{Spearman's rho} = \rho = \frac{p_{11}p_{22} - p_{12}p_{21}}{(p_1p_2q_1q_2)^{\frac{1}{2}}} .$$

If X and Y are independent, then $\gamma = 0 = \rho$. Further, one can show that $\gamma > 0$ if and only if X and Y are strictly positive quadrant dependent and that $\rho > 0$ if and only if X and Y are strictly positive quadrant dependent.

<u>Test based on the gamma ratio</u> : $\underline{T_1}$

Reject the null hypothesis if and only if

$$\hat{\gamma} = \frac{n_{11}n_{22} - n_{12}n_{21}}{n_{11}n_{22} + n_{12}n_{21}} \geq a.$$

<u>Test based on the rho criterion</u> : $\underline{T_2}$

Reject the null hypothesis if and only if

$$\hat{\rho} = \frac{n_{11}n_{22} - n_{12}n_{21}}{(p_1p_2q_1q_2)^{\frac{1}{2}}} \geq a.$$

Let us calculate the power of the tests T_1 and T_2 at the extreme points of $M_{PQD}(p_1,p_2; q_1,q_2)$.

$$\beta_{T_1}(P_1) = \beta_{T_2}(P_1) = \text{Size of the tests } T_1 \text{ and } T_2.$$

Under P_2, $n_{21} = 0$ almost surely and consequently, $\hat{\gamma} = 1$ with probability unity. Therefore,

$$\beta_{T_1}(P_2) = \Pr(\text{Rejecting the null hypothesis}/P_2)$$

$$= \Pr(\hat{\gamma} \geq a/ P_2) = 1$$

It is now obvious that $\beta_{T_2}(P_2) \leq \beta_{T_1}(P_2)$. Hence the power function of T_1 dominates the power function of T_2. As a matter of fact, the power function of T_1 will always dominate the power function of any test proposed for testing the hypothesis of independence. In a nutshell, what this means is that the test based on Gamma Ratio is the uniformly most powerful test for testing the hypothesis of independence against the alternative of strict positive quadrant dependence in the context of 2 x 2 contingency tables.

Comparisons of the power functions of various tests have been carried out more elaborately in Bhaskara Rao , Krisnnaiah and Subramanyam (1985).

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