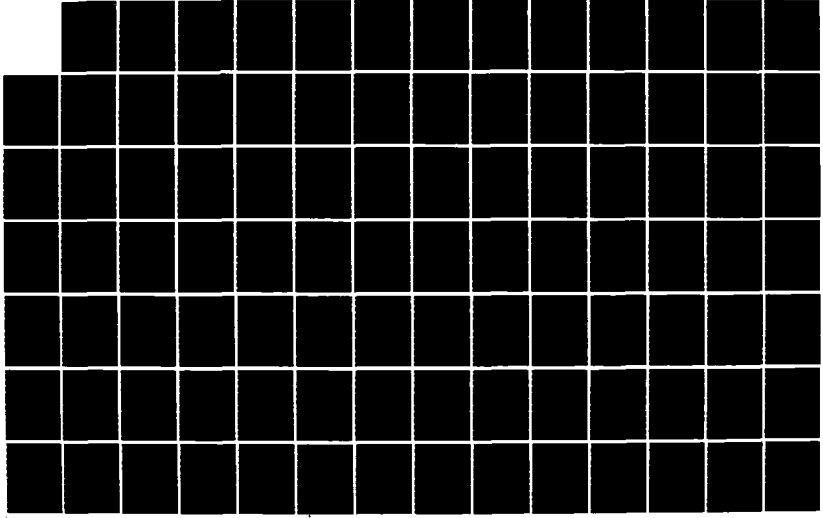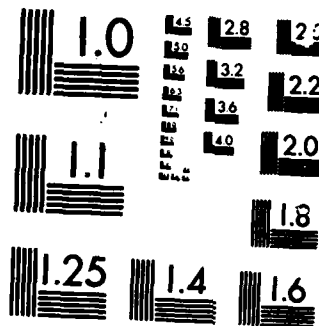AD-A165 975   NONLINEAR SIGNAL PROCESSING AND TEAM DIFFERENTIAL GAMES   1/2
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NONLINEAR SIGNAL PROCESSING

AND

TEAM DIFFERENTIAL GAMES

by

Francois J. Bugnon

R.R. Mohler
Principal Investigator and Dissertation Supervisor

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ABSTRACT

Team pursuit-evasion games are studied here with one performance index for the team as a unit in competition with one common opponent. Particular structures of team games are discussed after a brief introduciton of the two-player differential games. The classical calculus of variations is used to derive the feedback strategies for team linear, quadratic pursuit-evasion games. Several definitions of the performance index that correspond to different levels of cooperation and hierarchical organization in the team are investigated. The game of kind analysis partitions the players and the space according to their role in the team. Practical solutions to these complex problems rely best on suboptimal schemes. Thus a structural analysis is presented with the intent to simplify the computation of optimal decision and communication processes. Then approximated solutions as well as suboptimal hierarchies for linear quadratic team games are derived. Two-player games provide a great deal of information concerning the solution team games, allowing to compute an approximate solution of a three-player game using a composition method and to derive exactly the solution of a complex linear quadratic team game from a controllability study by providing terminal-time criteria of selection of unknowns. Hierarchical structures naturally arise; in particular, different filtering structures for a stochastic team game are compared. Detection and localization of the opponent players requires processing from several sources. In the underwater case, direction finding techniques may fail because of the environment (multipath propagation) or, in competitive situations, because of jamming signals. The nonlinear processing method developed to alleviate these difficulties also increases the class of problems solved by a given aperture, and is based on the eigenstructure method applied to Mth-order multiplicative signals.

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TEAM DIFFERENTIAL GAMES AND NON-LINEAR SIGNAL PROCESSING

I. INTRODUCTION

Many control problems involve the modelling of some unknown,
possibly deterministic, processes, as the signal processing of
measurements from a target, performed by one or more channels. In
the absence of knowledge of the target motion, most models assume
either a stochastic type of dynamics or a given simple motion, such
as a constant course. Then various techniques are used to recover
from unexpected target maneuvers, including input estimation, variable
dimension filtering approaches, white noise models with adjustable
level or with several levels (Reference [1] provides further biblio-
graphy). The signal processing task could be made easier if some
knowledge of the behavior of targets were known. But, usually, in a
hostile environment, the target expects to be tracked or chased, thus,
a competitive situation arises between the various observing channels
processing the data and the unwilling target, requiring the study of
a team pursuit-evasion game problem.

Formerly introduced by the pioneering work of Von Neuman and
Morgenstern [2] the discrete game analysis embodied most of the prin-
ciples of the game theory yet recognized as important. But it is not
until the concurrent development of the optimal control theory and the
introduction of the differential games by Isaacs [3] that the game
theory came of age.

After a ten year initial surge triggered by the publication of Isaacs' work, the study of the games was mainly captured by mathematicians rather than engineers. A more mathematical approach of the games emerges; the early results were given a more solid theoretical support and were extended, as in Friedman [4]. In particular, bargaining games, in which N players individually attempt to achieve some optimal performance by contracting alliances, were advanced to a better understanding. Nevertheless, the last few years experienced a renewed interest from the engineering community for differential games, a typical problem being the stochastic missile versus airplane control problem viewed as a game (Jarmark [5], Shinar [6]). Together with the progresses made in multi-target tracking (Bar-Shalom [7]), the 1-vs.-N game where a single missile faces several possible targets was also considered (Breakwell [8]).

This brief history of the games is herein schematically partitioned into four different "eras"; though questionable, this classification has the merit of enhancing the various N-player games considered. First came, as a generalization, the games in which the N players compete individually, then a few particular games were solved relying on geometrical or intuitive remarks, as in the game where two cutters attempt to prevent the escape of an evader (Isaacs [3]). Then came the bargaining games and coalition formation problems, both with a clear economical or political ulterior motive, and last came the one versus many games where a single pursuer must choose between several possible evaders as a first target.

Team differential pursuit-evasion games studied here involve coplayers who strive against a common opponent, such as an athletic

team with a common goal. The individual performance is superseded by that of the team as a whole. Clearly, this is not a typical N-player game of the more common variety. Nor can it be treated as a second phase of a bargaining game with coalitions already formed, such as studied by Von Neuman and Morgenstern [2], since the principle of optimality does not apply to the individual players but only to the team as a unit.

In the literature, team decision theory refers to the games where a group of agents, acquiring different information, work to-gether in a coordinated effort to achieve a common goal, as in Bagchi and Basar [9], but the competitive aspect is missing in these games.

Hereafter, the common opponent game is referred to as the team game or the N-versus-one game. The players of the game are called "pursuers" and the common opponent, "evader", in keeping with the spirit of "sink the Bismark".

Solving even the simplest team game is a very difficult task, and to cope with the curse of dimensionality, analysis of the simpler 1-vs.-1 game will be taken advantage of. Long before the battle of Salamis (480 BC), the need for a strict organization in obtaining optimal results from individual elements was recognized. Therefore, hierarchical structures are introduced and sometimes simplified, as early as possible; decentralized and suboptimal structures are looked for because they usually yield tractable solutions or more robust schemes.

In Chapter II, matrix games and the well known homicidal chauffeur game are used as a guideline to present a few concepts about

differential games whose understanding is required. Chapter III dis-
cusses various pitfalls and underlying assumptions in the formulation
of a team game; the calculus of variations is applied to yield the
necessary conditions of optimality for team games. The game of kind,
partitioning the pursuers and early structural game solutions is the
object of Chapter IV. The next chapter focuses on the linear quad-
ratic team differential games, expressing the solution in a compact
form and deriving C^3 (command, control, communication) suboptimal
structures. A composition approximation to reduce the computations
required for linear quadratic team games is presented in Chapter VI.
Chapter VII studies a fixed terminal time quadratic game from a con-
trollability point of view to provide criteria to select the terminal
time unknowns, and shows how it yields the solution to a complex
problem of optimal location of a pursuer in a team. Stochastic team
games are envisioned in Chapter VIII, for which hierarchical options
prove fundamental. The next chapter features a non-linear signal
processing technique to detect and locate the various players in an
unfriendly environment, where multipath propagation and jamming
either force the classical, linear methods to fail or, at best, to
considerably reduce the effective array aperture.

II. TWO-PLAYER DIFFERENTIAL GAMES

1. INTRODUCTION

Let S be a system including two variables u and v controlled by two distinct parties P1 and P2 who strive to maximize two corresponding performance indices J1 and J2. Then, a game siutation arises whenever the control policies of either player at the present time is not known by both parties, hereafter named "players".

When each player has all information about the system to control, the form of the performance indices and the other player's choice of strategies, the game is a perfect information game.

Particular games for which only one player is aware of the other player's strategy are called hierarchical or Stackelberg games. One player, the leader, announces his strategy first and the other player, the follower, reacts accordingly. When both maximizing players have identical performance indices and goals, a Pareto game is defined. Otherwise, whenever J1 and J2 differ, conflicting interests create a competitive situation. The game is a zero-sum game if J1 + J2 = 0, that is, when the interests of the players are opposite; otherwise, a non-zero-sum game is defined. Competitive games are usually defined in terms of a miximizing and a minimizing player.

When the number of strategies available is countable, the game is said to be discrete. If there is a finite number of possible strategies, a payoff can immediately be associated with each playable control pair (u,v) and the game can be put under a convenient matrix

form. On the other hand, an uncountable number of possible strategies characterizes continuous games. A differential game is a continuous game for which the system on which the controls apply is defined in terms of a set of lumped differential equations. Classical examples are Lanchester's equations and the predator-prey equations.

The study of gambling gave game theory its name, but the most obvious application of game theory is to clear competitive situations such as pursuit-evasion games, combat models and macro-economic behavior and strategies. Yet, a very important domain of application of game theory is in optimization problems with unpredictable parameters or forcing functions. The classical method solves that problem as a stochastic control problem, modelling the unknown as a random parameter, provided that the statistics of the unknown be a priori specified. Otherwise, a worst case study might be necessary. This conservative approach assumes that the unknown parameter or forcing function is controlled by an intelligent adversary, in a zero-sum game formulation. The method can be used to replace a stochastic problem by a deterministic one. Problems such as ship collision avoidance or games against nature are treated this way.

A few of the important concepts concerning two-player game theory are introduced below, rather informally, in order to provide a basis to the study of team games. A more complete study of discrete games can be found in [2]; [3] introduces differential games, and pursuit-evasion games are focused upon in [10].

2. DISCRETE GAMES

In the following perfect information, zero-sum competitive

discrete game, the maximizing player P1 has two available strategies: u_1 and u_2 that select a row in the matrix, when P2, the minimizing player, can choose either column 1 or 2.

Table 1. Matrix game with a pure strategy.

Player P1

		v_1	v_2
	u_1	2	3
Player P2	u_2	1	4

The payoff corresponding to the play (u_i, v_j) is the matrix co-efficient L_{ij}. Since both players must announce their strategy at the same time, it seems natural for P1 to choose the row with the smallest maximum and for P2 to choose the column with the largest minimum. Then:

$$\max_{v_i} \min_{u_i} L_{ij} = 3 \text{ , P1 plays first}$$

and

$$\min_{u_i} \max_{v_j} L_{ij} = 3 \text{ , P2 plays first.}$$

When both expected payoffs match, the result is called "value of the game". The strategy $(u^* = u_1, v^* = v_2)$ is an equilibrium solution, named "minimax stragegy".

The main problem with the minimax strategy is that it requires an exhaustive search over all the possible strategies. Clearly, this is unsuitable to large dimensioned matrices or to continuous

games. A quicker converging strategy is the Nash equilibrium strategy. It is defined as:

$$u^* = \arg(\max_{u_i} L(u_i, v^*)) ,$$

$$v^* = \arg(\min_{v_j} L(u^*, v_j)) .$$

The two above equations must hold simultaneously, and the search is made for a fixed assumed optimal strategy for the other player. It can immediately be understood that a Nash strategy corresponds to a local equilibrium concept, when the minimax strategy corresponds to a global equilibrium. Consequently, uniqueness is not ensured, as the following example shows, in which both minimizing players P1 and P2 face two possible Nash equilibria, namely (u_1, v_1) and (u_2, v_2), and a minimax (u_2, v_1).

Table 2. Matrix game with a mixed strategy.

		Player P1	
		v_1	v_2
	u_1	0	2
Player P2	u_2	1	0

Moreover, games for which no unique choice, named pure strategy, can satisfy $\min \max L_{ij} = \max \min L_{ij}$, do not have a Nash strategy in pure strategies. Then, as the game repeats itself, the controls must be chosen in a probabilistic way, defining a "mixed strategy". The corresponding averaged payoff then satisfies $\min \max L_{ij} = \max \min L_{ij}$.

A well known theorem states that the value of a game always exists in pure or mixed strategies for matrix games.

Though simpler in their definition, discrete games show many of the important characteristics of continuous games, such as the information structure, the various equilibrium strategies, the definition of a unique quantity called the value of the game, and the possible occurrence of a mixed strategy.

3. A PURSUIT-EVASION DIFFERENTIAL GAME: THE HOMICIDAL CHAUFFEUR GAME

3.1 Presentation of the game

A general form for the performance index of a zero-sum game is

$$J(u,v) = g(x(t_f),t_f) + \int_{t_0}^{t_f} h(x,u,v,t)dt . \qquad (1)$$

If $g(x(t_f),t_f) = 0$ and $\max_{v} \min_{u} h(x,u,v,t) \geq 0$, the game is said to be a generalized pursuit-evasion game.

It is assumed that the differential equations describing the motion or policy of the various players as well as the initial positions, are given. Then, as the game proceeds, the evader attempts to avoid the various capture zones related to the pursuers. The classical method divides the game initially at $t = t_0$, into two distinct phases, named the game of kind and the game of degree.

The game of kind can be summarized as finding the answer to the question: is capture possible? Once capture is assumed to be possible, the game of degree attempts to find the optimal way to conclude the game. Three approaches are used here to solve the game of degree;

i.e. classical optimal control theory, heuristic approaches relying on simple geometrical considerations and approximation methods tailored to specific games.

To illustrate further definitions and remarks, a specific example is treated. The homicidal chauffeur game, also known as "the two-car problem", is one of the best examples of pursuit-evasion games. Both its simplicity and its versatility are remarkable. The derivation of the classical one-pursuer-one-evader solution borrows heavily from Isaacs [3].

A pursuer P, of speed p and control u, attempts to capture, in minimum time, an evader E of speed e and control v. The heading angle rates are subject to the controls. The dynamics obey the following lumped differential equations

$$\dot{X}_{1p} = p\sin(\theta_p) \ ,$$
$$\dot{X}_{2p} = p\cos(\theta_p) \ , \tag{2}$$
$$\dot{\theta}_p = u \ ,$$

for the pursuer, and the dynamics of the evader are

$$\dot{X}_{1e} = e\sin(\theta_e) \ ,$$
$$\dot{X}_{2e} = e\cos(\theta_e) \ , \tag{3}$$
$$\dot{\theta}_e = v \ .$$

In an unconstrained space, only the relative position of the two players matter. The classical approach reduces the game by fixing P at the origin, defining the new coordinates, according to Figure 1, as

$$x_1 = (X_{1e} - X_{1p})\sin(\theta_p) - (X_{2e} - X_{2p})\cos(\theta_p) \ , \tag{4}$$

$$x_2 = (X_{1e} - X_{1p})\cos(\theta_p) + (X_{2e} - X_{2p})\sin(\theta_p) \ .$$
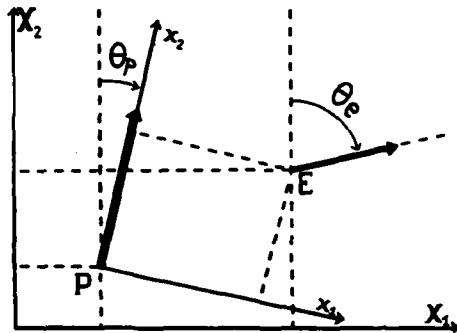
Figure 1. Relative coordinates.

The reduced state differential equations take the form

$$\dot{x}_1 = -ux_2 + e\sin(\theta_e - \theta_p) \ ,$$

$$\dot{x}_2 = ux_1 + e\cos(\theta_e - \theta_p) - p \ , \tag{5}$$

$$(\dot{\theta}_e - \dot{\theta}_p) = v - u \ .$$

The control of the pursuer, u, is constrained by $|u| \leq U$, defining a sharpest turn rate of U rd/sec. The evader is assumed to have complete control of its angle rate v. Therefore, E controls the angle $\theta_e - \theta_p$ completely. That angle is now defined as the new variable, hereby reducing the state equations to

$$\dot{x}_1 = -ux_2 + e\sin(v) \ , \tag{6}$$

$$\dot{x}_2 = ux_1 + e\cos(v) - p \ .$$

In general, $p > e$ is enforced, thus the pursuit-evasion game represents a study of the speed versus maneuverability type.

Capture is achieved whenever E is forced within the terminal manifold, or lethal area, of P, described by a matrix M and a scalar r as

$$x^T M x \leq r^2 \ . \tag{7}$$

In the sequel, $M = I$, and the lethal area is a circle of radius r; the terminal manifold is described by

$$x_1^2 + x_2^2 - r^2 \leq 0 \ . \tag{8}$$

3.2 *Solution of the game by the calculus of variations*

The performance index

$$J(u,v) = \int_{t_o}^{t_f} dt = t_f - t_o \ , \tag{9}$$

is associated with the state (6) in the Hamiltonian

$$H(u,v) = 1 + \lambda_1(-ux_2 + e\sin(v)) + \lambda_2(ux_1 + e\cos(v) - p) \ , \tag{10}$$

by adjoining the costate vector $\lambda = (\lambda_1, \lambda_2)$. The costate variables propagate according to

$$\dot{\lambda}_1 = \frac{-\partial H}{\partial x_1} = -\lambda_2 u \ ,$$

$$\dot{\lambda}_2 = \frac{-\partial H}{\partial x_2} = \lambda_1 u \ , \tag{11}$$

and the transversality condition provides a terminal time relation-
ship as

$$\lambda_1(t_f) = \nu \frac{-\partial (x_1^2 + x_2^2 - r^2)}{\partial x_1} = \nu x_1(t_f) \ ,$$

$$\lambda_2(t_f) = \nu \frac{-\partial (x_1^2 + x_2^2 - r^2)}{\partial x_2} = \nu x_2(t_f) \ ,$$

$$(12)$$

expressing the orthogonality of the costate vector with the terminal

manifold at the terminal time; ν is a Lagrange multiplier whose value

must be such that

$$H(t_f) = 0 \tag{13}$$

holds.

The optimal controls corresponding to a Nash equilibrium strategy,

or defining a game theoretic saddle-point, are computed by applying

the minimum (and maximum) principle as

$$u^* = \arg(\min_{|u| \leq U} H(u,v^*)) \ ,$$

$$v^* = \arg(\max_{v} H(u^*,v)) \ . \tag{14}$$

Generally, a game theoretic saddle-point does not exist unless the

Hamiltonian is separable, that is, unless Isaacs' condition

$$H(u,v) = H_u(u) + H_v(v) \ , \tag{15}$$

is satisfied.

The knowledge of the initial state together with (12) allows

classification of the one-pursuer-one-evader homicidal chauffeur game

as a classical two-point boundary-value problem.

Developing the above equations, introducing a reverse time $\tau = t_f - t$ and a terminal hit angle α, the optimal policies can be derived as

$$u^* = U\text{sign}(x_2 \sin(U\tau+\alpha) - x_1 \cos(U\tau+\alpha)) ,$$

$$v^* = U\tau+\alpha , \tag{16}$$

where sign is the signum function.

The retrograde path equations, integrated from a terminal hit point in the reduced coordinate system are

$$x_1(\tau) = p-p\cos(U\tau) + e(r-\tau)\sin(U\tau+\alpha) ,$$

$$x_2(\tau) = p\sin(U\tau) + e(r-\tau)\cos(U\tau+\alpha) . \tag{17}$$

Equations (17) are valid up to the first switch in u, then, according to the new value of u, a new set of differential equations are to be integrated until the next switch. A trial and error procedure must be used to find the proper hit angle that corresponds to the initial condition $(x_1(t_o), x_2(t_o))$.

Due to the particular example, the exact value of the Lagrange multiplier ν is irrelevant. In most games, finding a suitable value of ν that satisfies (13) is an acute problem since the multiplication of the costate vector by the controls yields, even in the simplest case, a second-order equation in ν, often including non-linear elements such as absolute values, which may produce zero, one or two possible solutions. Though a global study, even for the simplest linear game is nearly impossible, only one value of ν at a time seems

to be the rule since possible values of ν are checked against two important conditions: the playability and capturability conditions.

The playability condition states that, if pursuer P captures E at $t = t_f$, and if $g(x(t_f)) \leq 0$ describes the terminal manifold, then the scalar product

$$\overrightarrow{grad}(g(x(t_f))) \cdot \overrightarrow{\dot{x}}(t_f) \leq 0 \tag{18}$$

is negative. The playability condition expresses the simple fact that, in order for capture to be performed, the evader must penetrate into the terminal manifold. When the terminal manifold is a circle, $x(t_f)$ is a normal vector to the terminal manifold, parallel to the costate vector since the transversality condition holds. Then, an equivalent form is

$$\overrightarrow{\lambda}(t_f) \cdot \overrightarrow{\dot{x}}(t_f) \leq 0 \ . \tag{19}$$

The capturability condition is defined as

$$H(t_f) - k \leq 0 \ , \tag{20}$$

with k, a positive constant. Compared with (13), the capturability condition appears trivial. For minimum-time problems, it is convenient to choose $k = 1$. Then, (20) and (19) are equivalent conditions for one-pursuer-one-evader games.

Nevertheless, the capturability is a global condition on the game, when the playability condition must be respected by every single pursuer actively participating in the capture, thus, for team games, these two conditions are clearly different. The distinction between capturability and playability has been overlooked so far in the study of the one-versus-one games.

For the two-car problem, the condition expressed by (19) gives $e \leq p$, the obvious requirement that the pursuer be faster than the evader.

When the capture area is defined by a time-invariant equation, it is convenient to apply the playability condition to the terminal manifold. In some instances, it defines the portion of the terminal manifold that can be used in order to capture the evader, named "the usable part of the terminal manifold". The usable part is not a characteristic of the game studied but depends rather on the way the game is studied. For the homicidal chauffeur game, the usable part can easily be computed as the part of the circle for which

$$x_2(t_f) \geq re/p . \tag{21}$$

The optimal trajectories conducted from the two points $x_2(t_f) = re/p$ are referred to as "semi-permeable lines". These lines separate capture from escape and are best characterized by the fact that the payoff is discontinuous across them. Permeability ensures that only under a non-optimal play can the trajectory cross that border line. If the semi-permeable lines intersect, the capture zone is finite. Solving $\dot{x}_1 = 0$ with (21) and (17) gives the condition $r \leq r_o$ under which the semi-permeable lines intersect.

Figures 2 and 3 show two possible sets of trajectories. In the more interesting case of Figure 2, singular behaviors are numerous. The semi-permeable lines stop at point C and C´, consequently, if the evader is located around point E, a swerve motion is adopted by P, turning left, away from the evader at first, in order to go around C,
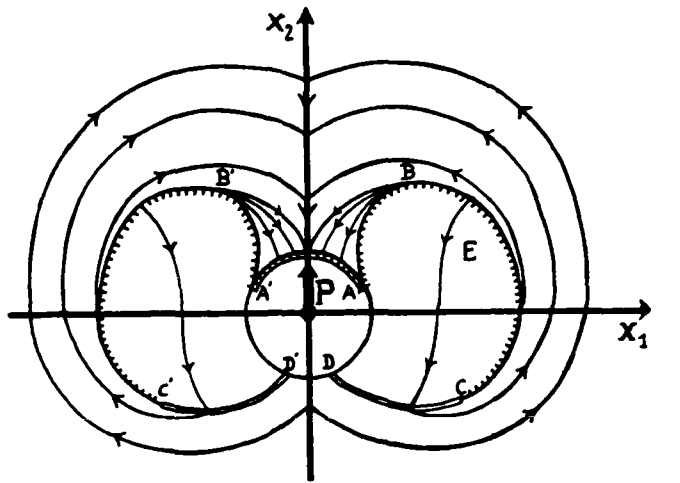
≡≡≡	Switching line in u.
▬▬	Usable part of the terminal manifold.
⊓⊓⊓⊓	Semi-permeable line.
→►—	Trajectory.

Figure 2. Homicidal chauffeur game with infinite capture zone.
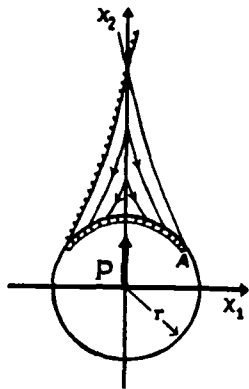
≡≡≡	Usable part of the terminal manifold.
⊓⊓⊓⊓	Semi-permeable line.
→►—	Trajectory.

Figure 3. Homicidal chauffeur game with restricted capture zone.

and then right to face the evader and achieve capture in a linear course. At points B and B´, an infinite number of optimal strategies are available. Along the lower half of axis x_2, the optimal value for u is U or -U, whereas the upper half corresponds to a more classical singular solution $u^* = 0$. Thus, the study of the singularities nearly represents the whole effort in solving the two-car problem.

3.3 Further studies of the homicidal chauffeur game

Unlike the choice made so far, an obvious reduction choice to study the N-versus-one homicidal chauffeur game is to position the evader at the origin. If all the capture sets are circles of identical radius, an equivalent "safety set" surrounding the evader can be defined. Then, the state equations are

$$\dot{x}_1 = x_2 v + p\sin(u) \ ,$$

$$\dot{x}_2 = -x_1 v + p\cos(u) - e \ . \tag{22}$$

When $p > e$, the usable part of the terminal manifold is not restricted, and consequently, the part of the study involving singular behavior such as the semi-permeable lines cannot be conducted. Trajectories, from a hit angle α, are integrated as

$$x_1 = -e + e\cos(V\tau) + p(r+\tau)\sin(-V\tau+\alpha) \ ,$$

$$x_2 = -e\sin(V\tau) + p(r+\tau)\cos(-V\tau+\alpha) \ , \tag{23}$$

and are plotted in Figure 4.
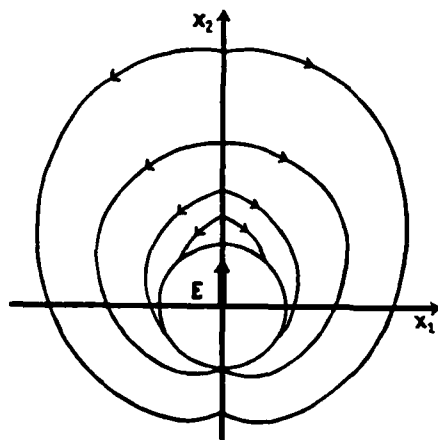
Figure 4. Reversed homicidal chauffeur game.

Trigonometric functions as in (6) are quite cumbersome to deal with, a linear or bilinear set of differential equations to approximate the exact game would simplify the study of the team game. The homicidal chauffeur game is equivalent to an extended-state bilinear model if the control v is constrained as

$$|v| \leq V = 0.3 . \qquad (24)$$

Then, the state equations for that game are

$$\dot{x} = Ax + Bxu + Cv ,$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e\text{-}p \\ 0 & 0 & 0 \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \qquad C = \begin{bmatrix} e \\ 0 \\ 0 \end{bmatrix} . \qquad (25)$$

A deviation of the solution can be conducted in a similar fashion as previously. The trajectories, shown in Figure 5, are sections of circles whose centers and radii change at every switchings of the control variables u and v.

validity band
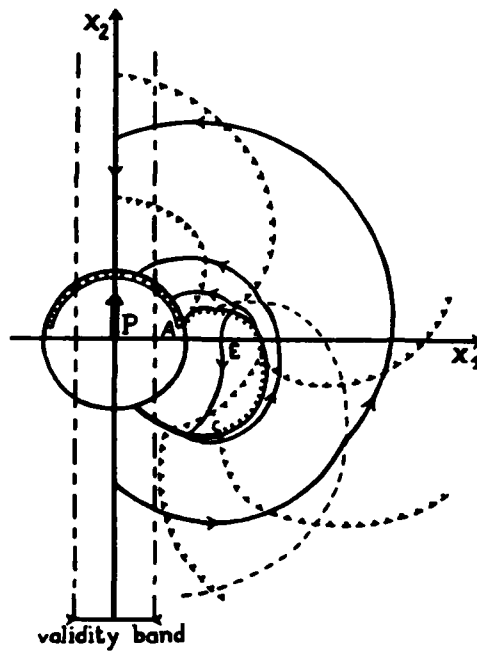
▬▬▬	Usable part.
▲▲▲▲	Constant terminal time.
➤—	Optimal trajectory.
▼▼▼▼	Semi-permeable line.
▬▬▬	Switching line of u.
- - - -	Switching line of v.

Figure 5. Bilinear approximation of the homicidal chauffeur game.

The non-reduced trajectories would show that the evader adopts a nearly parallel course to the pursuer, because of the constraint on v. Nevertheless, it is remarkable that the same type of trajectories and policies are found for the bilinear approximation of the homicidal chauffeur game even outside the area defined by $|v| \leq 0.3$, in which the approximation is valid. On the other hand, the validity band only includes the terminal end game maneuvers, thereby excluding the interesting phases of the pursuit.

Actually, the vectogram corresponding to the original homicidal chauffeur game is given by the vector $(\sin(v),\cos(v))$, when for the bilinear approximation, it is given by the vector $(1,v)$ for $|v| \leq 0.3$. As Figure 6 shows, it is a rather poor approximation. A better approximation is the square vectogram approximation. To give a more convenient set of state equations, the circle is approximated as closely as possible with the simplest form possible. The new equations are

$$\dot{x}_1 = -ux_2 + av_1 \, ,$$

$$\dot{x}_2 = ux_1 + av_2 - p \, , \tag{26}$$

where the controls are constrained as

$$|u| \leq U \, ,$$

$$|v_1| + |v_2| \leq a \, . \tag{27}$$

Actually, only the four vectors corresponding to the corners of the square $(v_1 \cdot v_2 = 0)$ are used, the evader switching from one to the other, as Figure 7 shows. The results show the same general behavior as the exact solution. The discretization performed by the approximation on the control v, causes some troubles in the stabilizations of the trajectories about the semi-permeable line. In particular, the use of the Euler formula to integrate the differential equations produces cleaner switches than the fourth order Runge-Kutta method, because of the important number of switches required. In order for this vectogram approximation to be an optimal fit, the parameter a can be optimized. Dolezal [11] addresses this type of parameter
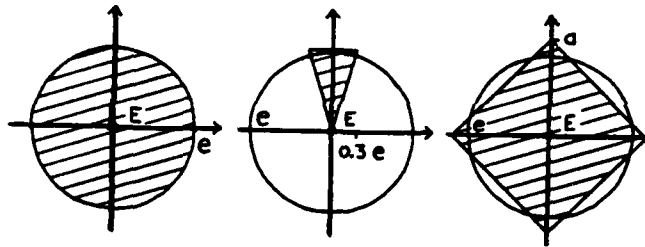
A. Original game.   B. Bilinear approx.   C. Square vectogram approx.

Figure 6.   Vectogram approximations.
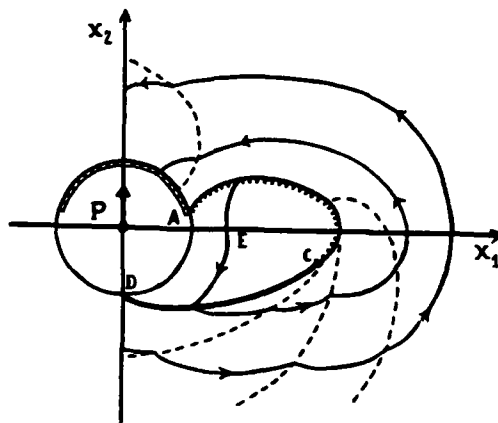
▬▬▬	Usable part.
——➤—	Optimal trajectory.
⊓⊓⊓⊓	Semi-permeable line.
═══	Switching line for u.
- - - -	Switching line for v.

Figure 7.   Square vectogram approximation of the homicidal chauffeur
           game.

optimization problem prior to the completion of a game. The perfor-
mance index to be minimized may become complex. By changing the value
of a, simulation shows that it seems to be always possible to approxi-
mate fairly closely parts of the exact solution, but significant
deviations always occured at some point.

The difficulties to approximate the equations of the homicidal
chauffeur game are quite typical to the differential games. Lineari-
zations around open-loop trajectories are hazardous since the other
player may choose a tailored (closed-loop) strategy such that the
result will significantly deviate. Local minimax properties cannot
be claimed safely, due to this generic unstable behavior of differ-
ential games. Open-loop strategies are not interesting from a
practical point of view. Closed-loop, or feedback strategies are
more useful, not only because they allow to cope with uncertainties
in the position, parameters, etc., but in order to recover from a
deliberate non-optimal play of the opponent or to take advantage of
an error committed by the other party. Minimum time, one-versus-one
pursuit-evasion games, that have a convex terminal manifold can
easily be studied globally, the trajectories being, in essence, closed
loop. This is not the case of games involving a fixed terminal time
or integral constraints on the controls or states. In this latter
case, a particular trajectory corresponds to each point, a complete
state-plane representation of the trajectories is impossible, and,
in the event of a detected non-optimal play, a new open-loop solution
must be recomputed.

The classical method uses the calculus of variations to solve
the game of degree, thus, the game of kind must have been solved first.

Hence the weakness of the method proposed so far, that requires a qualitative study beforehand. A global study of the problem has been suggested, applying the Lyapunov theory operating together with dynamical systems. Unfortunately, quoting Skowronski [12], the qualitative study is still at an infancy stage.

Classical results are extended by Simaan and Cruz [13] to the continuous games in which the state is available only at discrete instant of time, when Regade and Sarma [14] show that the optimal payoff of linear differential games under partial observation is not altered from the complete observation case, an obvious property of open-loop policies.

3.4 Two-pursuer-one-evader homicidal chauffeur game

Two-pursuer-one-evader homicidal chauffeur games ameanable to a solution by ways of simple heuristic geometrical considerations are studied here, in an attempt to demonstrate what can be done without any thorough or even partial study of team games.

The reversed one-versus-one homicidal chauffeur game is solved classically as above; cases where the two pursuers are symetrically disposed according to the orientation of the evader, are investigated. A direct pursuit occurs when E is located on the median of P1P2, and properly oriented, according to Figure 8. Then, the game is symmetrical and the optimal play for E is to move straight along x_2. This articular two-versus-one homicidal chauffeur game is equivalent to the one-versus-one case of the "wall pursuit game" in Isaacs [3], where E, constrained to x_2, is chased by P_1 alone. Trivial geometrical considerations are used to solve that game according to Figure 9.
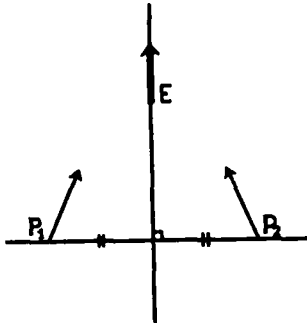
Figure 8. The direct pursuit case.
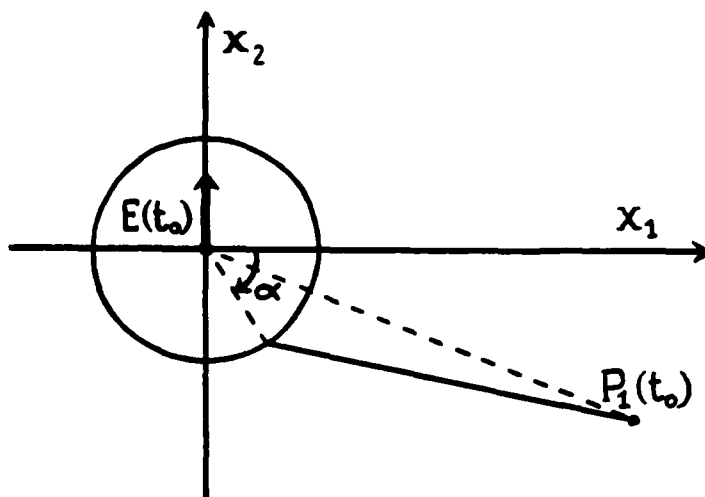
Figure 9. The wall pursuit game.

A collision dilemma arises whenever E and the symmetrical pair face each other. Then, E must compare its payoff for two strategies: going straight ahead or turning either right of left, as sharp as possible. If E decides to turn, say to the right, he will favor one of the pursuers, namely P_1, and the game will be concluded by that very pursuer. Though inactive, P_2 is responsible for having forced E to turn towards P_1 and not away from him. That optimal trajectory is given by the one-versus-one game in which the regressive path trajectories from the terminal hit points, are not stopped from symmetry reasons, along the axis x_2. Two examples of the collision dilemma to be solved by E are shown in Figure 10.

→»—	P_1 alone . . . . : $t_f = 2.67$	
→»—	$P_1 \& P_2$, E turns . . . . . : $t_f = 2.09$	E must turn.
→—	$P_1 \& P_2$, E moves straight: $t_f = 1.74$	
→»—	P_1 alone . . . . : $t_f = 1.34$	
→»—	$P_1 \& P_2$, E turns . . . . . : $t_f = 1.02$	E must move straight.
→—	$P_1 \& P_2$, E moves straight: $t_f = 1.07$	
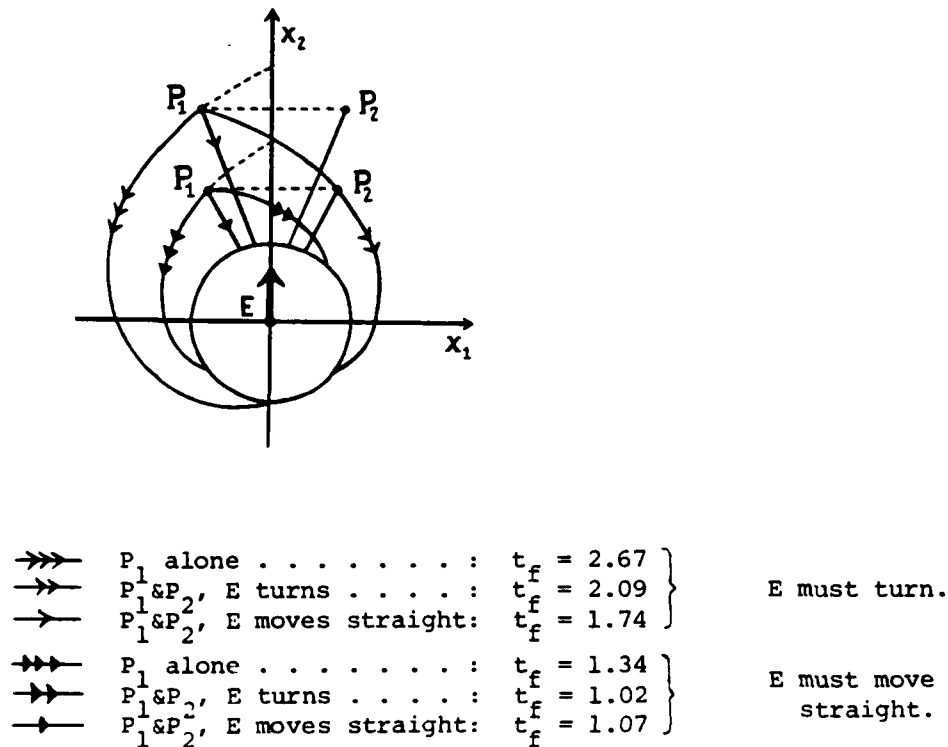
Figure 10. The collision dilemma.

When the pursuers are not symmetrically disposed, a straight
optimal trajectory of E is possible if P_1 and P_2 are located on the
circle of center $(0, et_f)$ and radius $(r+t_f)$. Then, E oriented away
from the line P_1P_2 is an obvious condition to ensure the straight line
play. For P_1 fixed, three possible situations can arise, depending
on the location of P_2 along the circle, two of which are solved,
i.e. the straight-line play and the sharp-turn play, when the third
situation corresponds to a "moderate" turn of E. Figure 11 depicts
these three possibilities.

Even though a wide variety of cases can be solved using simple
considerations as above, a method to solve general situations for
team games is wanting.
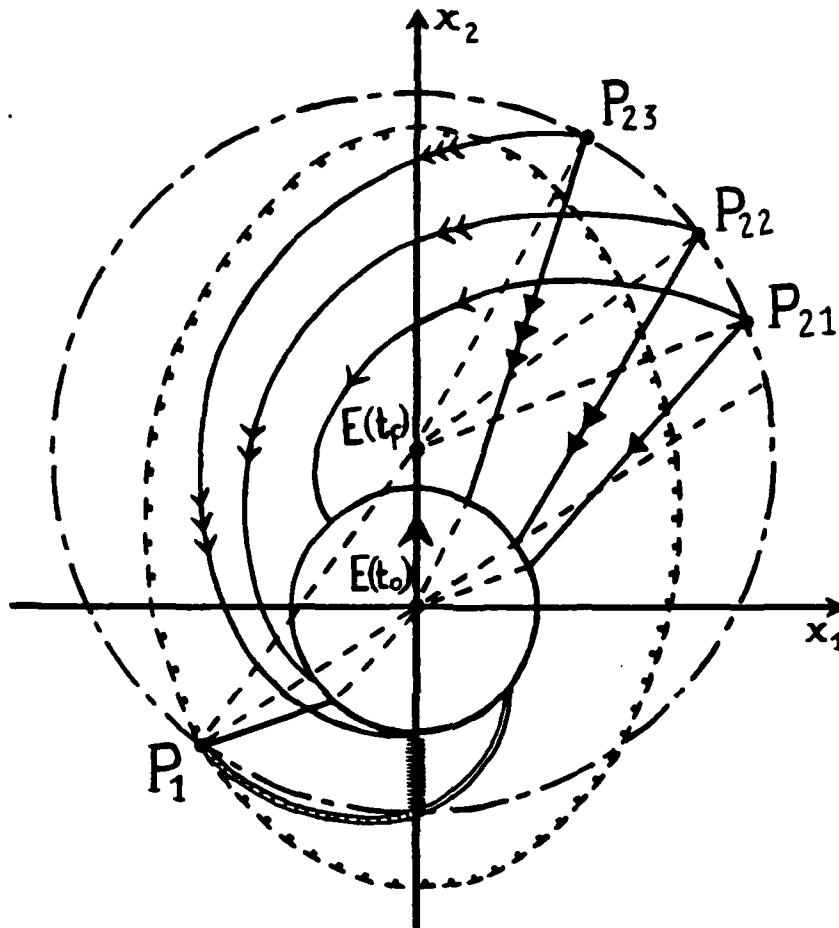
▬▬▬	P_1 alone . . . . : t_{f1} = 3.05	(1)
▬▬▬	P_1 if E always turns right: t_{f2} = 2.70	
→	$P_1 \& P_{21}$, E turns right. . . : t_{f3} = 1.97	(2)
→	$P_1 \& P_{21}$, E moves straight . : t_{f4} = 2.25	
→→	$P_1 \& P_{22}$, E turns right. . . : t_{f5} = 2.47	(3)
→→	$P_1 \& P_{22}$, E moves straight . : t_{f4} = 2.25	
→→→	$P_1 \& P_{23}$, E turns right. . . : t_{f6} = 3.17	(4)
→→→	$P_1 \& P_{23}$, E moves straight . : t_{f4} = 2.25	

(1) one-versus-one game.

(2) E moves straight, caught by P_1 and P_{21}.

(3) E turns right, caught by P_{22}.

(4) E turns slightly to the right ($t_{f6} > t_{f2}$), caught by P_1
     and P_{23}.

Figure 11.   Three particular asymmetrical games.

III.   INTRODUCTION TO TEAM DIFFERENTIAL GAMES

1.   INTRODUCTION

Not very many team games are analyzed in the literature; to name a few, Isaacs [3] derives sufficient conditions in order for a team of patrollers to block a channel, Hagedorn and Breakwell [15] study the game in which a fast evader attempts to pass between two pursuers, dividing the game into two distinct phases, one in which the three players move in a straight course, and one in which the evader stays at a constant distance from its closest pursuer.

The same partitioning of the game into distinct phases allowed Foley and Schmitendorf [16] to solve a differential game with two pursuers and one evader, but in a non-zero sum game formulation, with a performance index for each pursuer. On the other hand, linear one-pursuer, one-evader games are studied by Pshenichnyi, Chikrii and Rappoport [17], with a unique performance index, but where all pursuers must capture the evader individually, and thus the evader strives against the slowest pursuer. These team games are solved because a convenient mapping from the one-versus-one game is possible, not unlike the example of the two-pursuer one-evader homicidal chauffeur game in the previous chapter.

Together with a discussion of the major difficulties in the statement of team differential games, this chapter formally applies the calculus of variations to derive the necessary conditions of optimality for team differential games.

2. FACTORS TO CONSIDER TO STATE A TEAM GAME

In most practical situations, the controls are physically bounded, notable exceptions being systems without inertia that are controlled directly through their heading angle as in the homicidal chauffeur game. Unbounded controls are more suitable to derive the feedback strategies and to avoid the switching functions that come together with bang-bang controls, hence the popularity of quadratic performance indices that provide a self-limitation in the controls. However, not every type of quadratic game is suitable to a deterministic pursuit-evasion game study. As an example, the following bilinear quadratic game has a relative state

$$\dot{x} = Ax + Bxu + Cv , \tag{28}$$

a quadratic performance index

$$J = \frac{1}{2} \int_{t_o}^{t_f} (||u||_R^2 - ||v||_S^2)dt , \tag{29}$$

and capture is achieved whenever the evader is forced within the circle of radius r. A complete study of this two player game would show that the capture zone is finite and that the Nash equilibrium strategies within that zone are u = v = 0. Thus, either capture is impossible, or it is bound to happen, such that no action is to be taken by either player. It is quite a paradox to find such a trivial solution to such a seemingly complex game. The difficulty is in the merging of the pursuit-evasion concept with the quadratic performance index. If capture only matters, the value of the control used by the evader becomes infinite, but so would immediately be the pursuer's

control. It would achieve the same result as choosing u = v = 0 but

for a possibly different value of J. This is an example of a game in

which the perfect information structure prevents any evolution and

hampers the interpretation of the results.

It seems that a better approach is to apply integral constraints

on the controls. The first effect is to allow only open-loop solu-

tions since, in most cases, the players try to spend their whole con-

trol capabilities and then, capture zones, singular behaviors etc,

become very fuzzy. If the bilinear quadratic game above is used as a

representation of the homicidal chauffeur game, according to the

previous chapter, then, the controls are, in fact, angles, and the

very definition of an integral constraint of an angle is rather

peculiar. On the other hand, the previous chapter shows that sub-

stitution of the original controls by a vectogam approximation is an

unsafe manipulation for differential games.

An easy generalization from the one-versus-one games can be

done if the performance indices of the team games include summations

of the controls, states, etc.- merely adding the various energies etc.

spent by the team. While no particular problems arise for minimum-

time team games, and even quadratic team games, fixed terminal-time

games that include a terminal miss distance are more difficult to

generalize since the distance between the evader and the closest

pursuer is to be considered. Thus, most fixed terminal-time team

games include a cumbersome minimum operator in their performance

index as

$$J = \int_{t_0}^{t_f} (\min_i g(x_i(t)) + L(u_i,v,t))dt \; , \tag{30}$$

where the reduced state equation for that game is

$$\dot{x}_i = f(x_i, u_i, v, t) \tag{31}$$

for pursuer P_i, and x_i is the relative distance between the evader E

and P_i. An equivalent form for a two-versus-one game is

$$J = \int_{t_o}^{t_f} (wg(x_1(t)) + (1-w)g(x_2(t_f)) + L(u_i, v, t))dt \tag{32}$$

by introducing the switching variable w, such that

$$w = 1 \text{ for } g(x_1(t)) < g(x_2(t)) \quad \text{and} \tag{33}$$

$$w = 0 \text{ for } g(x_1(t)) \geq g(x_2(t)) .$$

w can be viewed as a control variable belonging to a mythical mini-

mizing third party, such that (33) is a Nash strategy. It can be

shown that the equivalent game requires the adjunction of a state y

such that

$$y = w(g(x_1(t)) - g(x_2(t))) , \tag{34}$$

and the new performance index is

$$J = y(t_f) + \int_{t_o}^{t_f} (g(x_2(t)) + L(x_i, u_i, v, t))dt . \tag{35}$$

Here w is constrained to the interval [0,1]. Thus, the two-versus-

one team game with the non-linear minimum operator in the performance

index, is equivalent to a three-versus-one augmented team game with

the simplified performance index (35).

Actually, it can be proved that N-versus-one team games in which

a term such as $\min(a_1, a_2, \ldots, a_m)$ is included in the performance index,

where $a_1, a_2, \ldots, a_m$ are m independent linear functions of the states, are equivalent to simpler forms of (N + m - 1)-versus-one team differential games. If $a_1, a_2, \ldots, a_m$ are not linearly independent, then it is possible to find a set $(b_1, b_2 \ldots, b_e)$ of independent vectors such that for 1 < m,

$$\min (a_1, a_2, \ldots, a_m) = \min (b_1, b_2 \ldots, b_e) \ . \tag{36}$$

On the other hand, particular games for which the control of the evader affects independently (m-1) of the state components of the functions $(a_1, a_2, \ldots, a_m)$, proceed according to m distinct phases: first E plays according to the "closest" pursuer in the sense of the minimum function, until a second pursuer is just as close, then, E plays to keep both pursuers equally distant until a third pursuer becomes equally distant and so on. At the terminal time, $a_1 = a_2 = \ldots = a_m$. The games studied by Hagedorn and Breakwell [15] and Foley and Schmitendorf [16] belong to this category. If the above is not met, then chances are that, for some initial conditions, $a_i < a_j$ holds for some pair i,j, thereby reducing the number of arguments of the minimum operator.

The proof of this theorem is rather easy and will not be presented here, it provides a technique to put team games under more classically tractable forms, at the expense of an increase in the dimensionality. In the sequel, team games are assumed to have been put under this form, thus, cumbersome minimum operators are not considered in further studies.

Another problem is the definition of the differential equations describing the game. In a pursuit-evasion game, only the relative

state between the evader and the pursuers is important in an unconstrained space. Therefore, there is no need to carry the equations of motion of every single player, when a so-called reduced state is simpler. The reduction is long recognized as one of the major steps in studying such a differential game as a pursuit-evasion game, as, for example, Pontryagin [18] shows clearly.

Nevertheless, various approaches are possible. First is the brute force manner whereby the equations describing the motion of each player are kept, at the expense of an unwelcomed increase in the dimensions of the game. Moreover, keeping track of the positions of the players and of their associated terminal manifolds is a lot more difficult, as the guessing of the terminal state, so important in the solution of the N-point boundary-value problems, using optimal control theory. An excellent way to avoid any trouble is to start from a reduced set of equations. Sometimes, a proper choice of coordinate systems can simplify the problem or the controls, but this depends on the type of study conducted.

3. NECESSARY CONDITIONS OF OPTIMALITY FOR GENERALIZED TEAM GAMES

3.1 Presentation

In the following, the notations and the general method adopted are borrowed from Bryson and Ho [19]. The free terminal time game obeys the differential equations

$$\dot{x} = f(x,u,v,t) \ , \tag{37}$$

and $x(t_o) = x_o$ is given. The unconstrained controls are u for the pursuer and v for the evader. State constraints

$$\psi(x(t_f), t_f) = 0 , \tag{38}$$

are adjoined to the performance index by the set of Lagrange multipliers ν_p and ν_e. The more general non-zero sum game is studied; then the performance index for the pursuer takes the form

$$J_p = \phi_p(x(t_f), t_f) + \nu_p^T \psi(x(t_f), t_f) + \int_{t_o}^{t_f} (L_p(x, u, v, t) + \lambda_p^T (f(x, u, c, t) - \dot{x})) dt \tag{39}$$

and for the evader

$$J_e = -\phi_e(x(t_f), t_f) - \nu_e^T \psi(x(t_f), t_f) - \int_{t_o}^{t_f} (L_e(x, u, v, t) + \lambda_e^T (f(x, u, v, t) - \dot{x})) dt , \tag{40}$$

where ϕ_e and ϕ_p are terminal payoffs and L_e, L_p are integral functions. The Lagrange multipliers ν_p and ν_e have the dimensions of the vector ψ. From now on, the various arguments are omitted for brevity.

For minimum-time linear team games where only the pursuers that perform capture are present, the results derived by Pshenichnyi, Chikrii and Rappoport [17], further generalized by Satimov, Azamov and Khaidarov [20], can be referred to, stating sufficient conditions that ensure the existence of a solution in finite time.

3.2 Stationarity for the one-versus-one game

Stationarity of dJ_e and dJ_p is expressed taking into account differential changes in the terminal time.

For the pursuer, it yields

$$dJ_p = (\frac{\partial \phi_p}{\partial t_f} dt + \nu_p^T \frac{\partial \psi}{\partial t} dt + \frac{\partial \phi_p}{\partial x} dx + \nu_p^T \frac{\partial \psi}{\partial x} dx) + (L_p)_{t_f} dt_f - (L_p)_{t_o} dt_o +$$

$$(\lambda_p^T(f-\dot{x}))_{t_f} dt_f - (\lambda_p^T(f-\dot{x}))_{t_o} dt_o + \int_{t_o}^{t_f} (\frac{\partial L_p}{\partial x} \delta x + \lambda_p^T \frac{\partial f}{\partial x} \delta x -$$

$$\lambda_p^T \delta \dot{x} + (\frac{\partial L_p}{\partial u} \delta u + \lambda_p^T \frac{\partial f}{\partial u} \delta u) + (\frac{\partial L_p}{\partial v} \delta v + \lambda_p^T \frac{\partial f}{\partial v} \delta v)) dt \tag{41}$$

At time $t = t_o$ and $t = t_f$,

$$\lambda_p^T(f(x,u,v,t) - \dot{x}) = 0 , \tag{42}$$

and, introducing the Hamiltonian

$$H_p = L_p + \lambda_p^T f , \tag{43}$$

together with the function

$$\Phi_p = \phi_p + \nu_p^T \psi , \tag{44}$$

(41) simplifies into

$$dJ_p = [(\frac{\partial \Phi_p}{\partial t} + L_p) dt_f + \frac{\partial \Phi_p}{\partial x} dx]_{t_f} - (L_p)_{t_o} dt_o +$$

$$\int_{t_o}^{t_f} (\frac{\partial H_p}{\partial x} \delta x + \frac{\partial H_p}{\partial u} \delta u + \frac{\partial H_p}{\partial v} \delta v - \lambda_p^T \dot{\delta x}) dt . \tag{45}$$

From Figure 12,

$$dx(t_f) = \delta x(t_f) + \dot{x}(t_f) dt_f ,$$

$$\delta x = dx - \dot{x} dt \tag{46}$$
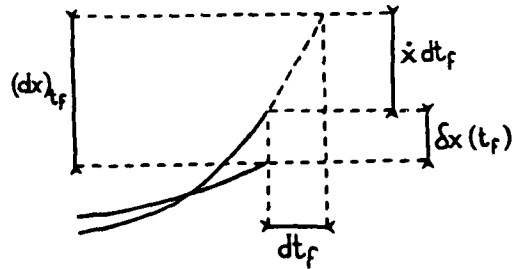
Figure 12. Terminal time differentials.

Then, integrating by parts, the following equality holds:

$$\int_{t_o}^{t_f} -\lambda_p^T \delta \dot{x} = -(\lambda_p^T dx)_{tf} + (\lambda_p^T \dot{x} dt)_{tf} + (\lambda_p^T \delta x) +$$

$$\int_{t_o}^{t_f} \dot{\lambda}_p^T \delta x \quad , \tag{47}$$

together with (45), it yields

$$dJ_p = (\frac{\partial \Phi_p}{\partial t} + L_p + \lambda_p^T \dot{x})_{tf} dt_f + ((\frac{\partial \Phi_p}{\partial x} - \lambda_p^T) dx)_{tf} + (\lambda_p^T \delta x)_{to} -$$

$$(L_p)_{to} dt_o + \int_{t_o}^{t_f} ((\frac{\partial H_p}{\partial x} + \dot{\lambda}_p^T) \delta x + \frac{\partial H_p}{\partial u} \delta u + \frac{\partial H_p}{\partial v} \delta v) dt \quad , \tag{48}$$

when a similar derivation yields

$$-dJ_e = (\frac{\partial \Phi_e}{\partial t} + L_e - \lambda_e^T \dot{x})_{tf} dt_f + ((\frac{\partial \Phi_e}{\partial x} + \lambda_e^T) dx)_{tf} - (\lambda_e^T \delta x)_{to} -$$

$$(L_e)_{to} dt_o - \int_{t_o}^{t_f} ((\frac{\partial H_e}{\partial x} + \dot{\lambda}_e^T) \delta x + \frac{\partial H_e}{\partial u} \delta u + \frac{\partial H_e}{\partial v} \delta v) dt. \tag{49}$$

In order for the game to have a stationary value, the coefficients of dt_f and dx must vanish for both dJ_p and dJ_e, regardless of t. Therefore, these terms must be individually set to zero, that is

$$-\frac{\partial H_p}{\partial x} = \dot\lambda_p^T \quad , \quad -\frac{\partial H_e}{\partial x} = \dot\lambda_e^T \quad ,$$

$$\left[\frac{\partial \Phi_p}{\partial x}\right]_{tf} = (\lambda_p^T)_{tf} \quad , \quad \left[\frac{\partial \Phi_e}{\partial x}\right]_{tf} = (\lambda_e^T)_{tf} \quad , \tag{50}$$

$$(\frac{\partial \Phi_p}{\partial t} + L_p + \lambda_p^T \dot{x})_{tf} = (\frac{\partial \Phi_p}{\partial t} + H_p)_{tf} = 0 \quad , \quad (\frac{\partial \Phi_e}{\partial t} + L_e - \lambda_e^T \dot{x})_{tf} =$$

$$(\frac{\partial \Phi_e}{\partial t} + H_e)_{tf} = 0$$

or, since

$$\frac{\partial \Phi_p}{\partial t} = \frac{d\Phi_p}{dt} - \frac{\partial \Phi_p}{\partial x} \dot{x} \quad , \tag{51}$$

an equivalent condition is

$$(\frac{\partial \Phi_p}{\partial t} + L_p)_{tf} = 0 \quad , \quad (\frac{\partial \Phi_e}{\partial t} + L_e)_{tf} = 0 \quad . \tag{52}$$

Then, if the above holds, dJ_p and dJ_e are left to be

$$dJ_p = \int_{t_o}^{t_f} (\frac{\partial H_p}{\partial u} \delta u + \frac{\partial H_p}{\partial v} \delta v) dt + (\lambda_p^T dx)_{to} - (H_p)_{to} dto \quad ,$$

$$\tag{53}$$

$$dJ_e = \int_{t_o}^{t_f} (\frac{\partial H_e}{\partial u} \delta u + \frac{\partial H_e}{\partial v} \delta v) dt + (\lambda_e^T dx)_{to} + (H_e)_{to} dto \quad .$$

A stationary value for both J_p and J_e requires that

$$\frac{\partial H_p}{\partial u} \delta u + \frac{\partial H_p}{\partial v} \delta v = 0$$

and (54)

$$\frac{\partial H_e}{\partial u} \delta u + \frac{\partial H_e}{\partial v} \delta v = 0 .$$

The Nash equilibrium strategy is defined by

$$\frac{\partial H}{\partial u} = 0 \text{ with } v = v^* \text{ given}$$

and (55)

$$\frac{\partial H}{\partial v} = 0 \text{ with } u = u^* \text{ given} .$$

It achieves the desired stationarity, but it must be understood that the controls u^*, v^* that satisfy the above are candidates that must also verify the Nash inequality:

$$\forall u \in U \quad H_p(u^*,v^*) \leq H (u,v^*) ,$$

and (56)

$$\forall v \in V \quad H_e(u^*,v) \leq H (u^*,v^*) .$$

The necessary conditions applied to the one-versus-one differential game define a two-point boundary-value problem. For a generalized pursuit-evasion game, $\phi_p = \phi_e = \phi$, $L_p = L_e = L$ hold and the equations describing the terminal manifold are usually one-dimensioned; then, ν_p and ν_e are scalars, moreover $\lambda_p = -\lambda_e$, $\nu_p = -\nu_e = \nu$ and $H_p = -H_e = H$.

Then, together with the Nash inequality, the necessary conditions of optimality are

$$(\frac{\partial \phi}{\partial t} + \nu \frac{\psi}{\partial t} + L + \lambda^T \dot{x})_{tf} = (\frac{\partial \phi}{\partial t} + \nu \frac{\partial \psi}{\partial t} + H)_{tf} = 0$$

$$- \frac{\partial H}{\partial x} = \dot{\lambda}^T \quad ,$$

(57)

$$(\frac{\partial \phi}{\partial x} + \nu \frac{\partial \psi}{\partial x})_{tf} = (\lambda^T)_{tf} \quad ,$$

$$\frac{\partial H}{\partial u} = 0 \quad , \quad \frac{\partial H}{\partial v} = 0 \quad .$$

3.3 Stationarity for the N-versus-one team game

For the team game opposing the pursuers P_i to E according to the state differential equations

$$\dot{x}_i = f_i (x_i, u_i, v, t)$$

(58)

J_p and J_e are defined as

$$J_p = \sum_i \phi_{p_i} (x(t_f), t_f) + \sum_i \nu_{p_i}^T \psi_i + \int_{t_o}^{t_f} (L_p + \sum_i \lambda_{p_i}^T (f_i - \dot{x}_i)) dt \quad ,$$

(59)

$$J_e = -\sum_i \phi_{e_i} (x(t_f), t_f) - \sum_i \nu_{e_i}^T \psi_i + \int_{t_o}^{t_f} (-L_e + \sum_i \lambda_{e_i}^T (f_i - \dot{x}_i)) dt \quad .$$

and the Hamiltonians are

$$H_p = L_p + \sum_i \lambda_{p_i}^T f_i \quad ,$$

(60)

$$H = -L_e + \sum_i \lambda_{e_i}^T f_i \quad ,$$

when the relation between the one-versus-one and the N-versus-one game is

$$L_p = \sum_i (L_{p_i} - L_{p\nu_i}) + L_{p\nu} \, ,$$

(61)

$$L_e = \sum_i (L_{e_1} - L_{e\nu_i}) + L_{e\nu} \, ,$$

with, in the one-versus-one case of P_i-vs.-E:

$$J_{p_i} = \phi_{p_i}(x_i(t_f), t_f) + \nu_{p_i}^T \psi_{p_i}(x_i(t_f), t_f) + \int_{t_o}^{t_f}(L_{p_i u_i}(x_i, u_i, t) +$$

$$L_{p\nu_i}(x_i, v, t) + \lambda_{e_i}^T(f_i - \dot{x}_i)) dt \, ,$$

(62)

$$J_e = \phi_{e_i}(x_i(t_f), t_f) + \nu_{e_i}^T \psi_{e_i}(x_i(t_f), t_f) + \int_{t_o}^{t_f}(L_{e_i u_i}(x_i, u_i, t) -$$

$$-L_{e\nu_i}(x_i, v, t) + \lambda_{e_i}^T(f_i - \dot{x}_i)) dt.$$

A step by step derivation identical to the one conducted in the two player case, yields the necessary conditions of optimality for the team games, as

$$(\sum_i \frac{\partial \phi_{p_i}}{\partial t_f} + L_p + \sum_i \lambda_{p_i}^T x_i)_{t_f} = 0 \, , \quad (\sum_i \frac{\partial \phi_{e_i}}{\partial t_f} + L_e - \sum_i \lambda_{e_i}^T x_i)_{t_f} = 0$$

$$-\frac{\partial H_{p_i}}{\partial x_i} = \dot{\lambda}_{p_i}^T \, , \quad -\frac{\partial H_{e_i}}{\partial x_i} = \dot{\lambda}_{e_i}^T \, ,$$

(63)

$$(\frac{\partial \phi_{p_i}}{\partial x_i})_{t_f} = (\lambda_{p_i}^T)_{t_f} \, , \quad (\frac{\partial \phi_{e_i}}{\partial x_i}) = (\lambda_{e_i}^T)_{t_f} \, ,$$

$$\frac{\partial H_{p_i}}{\partial u_i} = 0 \text{ if } v^* \text{ given }, \quad \sum_i \frac{\partial H_{ei}}{\partial v} + \frac{\partial L_{ev}}{\partial v} - \sum_i \frac{\partial L_{ev_i}}{\partial v} = 0 \text{ if } u^* \text{ given },$$

in which some equations are expressed N times, one for each pursuer.

For this team game, the Nash strategy is expressed as

$$J_p(u_1^*, u_2^*, \ldots, u_N^*, v^*) \leq J_p(u_1, u_2, \ldots, u_N, v^*)$$

$$\forall (u_1, u_2, \ldots, u_N) \in U_1 \cdot U_2 \ldots U_N ,$$

(64)

$$J_e(u_1^*, u_2^*, \ldots, u_N^*, v) \leq J_e(u_1^*, u_2^*, \ldots, u_N^*, v^*) \quad \forall v \in V ,$$

when, in the general N-player case, it would have been

$$(\forall i \in 1, \ldots, N) \quad J(u_1^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^* \ldots, u_N^*) \leq$$

(65)

$$J(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*), \quad (\forall u_i \in U_i)$$

The distinction between a team game and an N-player game appears clearly.

For a generalized team pursuit-evasion game, $\phi_{pi} = \phi_{ei} = \phi_i$, $L_p = L_e = L$ and ψ has dimension one. Then $\lambda_{pi} = -\lambda_{ei} = \lambda_i$ and $-\nu_{ei} = \nu_{pi} = \nu_i$.

The necessary conditions of optimality are summarized below:

$$(\sum_i (\frac{\partial \phi_i}{\partial t} + \nu_i \frac{\partial \psi_i}{\partial t}) + H)_{t_f} = 0 , \quad -\frac{\partial H_i}{\partial x_i} = \lambda_i^T ,$$

(66)

$$(\frac{\partial \phi_i}{\partial x_i} + \nu_i^T \frac{\partial \psi_i}{\partial x_i})_{t_f} = (\lambda_i^T)_{t_f} , \quad \frac{\partial H}{\partial u_i} = 0 , \quad \frac{\partial H}{\partial v} = 0 .$$

The first equation in (66) can only determine one of the N scalars ν_i. The remaining scalars must be chosen so that the solution to the equations matches the initial conditions of the game. The (N-1) unknowns, called strategic variables are defined as

$$\nu \, z_i = \nu_i, \; z_1 = 1 \; . \tag{67}$$

Utilizing this formulation, the first equation in (66) is used to determine the unique Lagrange multiplier ν.

The strategic variable z_i, is such that, when z_i is small, P_i has nearly no effect on the game as seen from (43) and because λ_i has a module proportional to z_i. On the other hand, if z_i is large, then, P_i is more important than other players. Thus, there is a clear relation between the importance of a pursuer and the value of its strategic variable. Conversely, the value of z_i can be used a-posteriori as a selecting device to choose the relevant players to simplify a game and to compute approximate solutions.

The strategic variable is not constant when the evader is surrounded by the pursuers. That possibility exists whenever there are at least N pursuers for an evader moving on a space of dimensions (N-1). Then z_i adopts a value such that the control of the evader disappears from the Hamiltonian.

The main problem in solving the team games, with this approach, is to guess properly the strategic variables. The capturability and playability conditions can be used to restrict the domain of definition of z_i. When a given level of cooperation is required from a pursuer, the value of its associated strategic variable is necessarily

restricted. However, finding the exact value of the strategic

variable z_i usually requires a painstaking trial and error procedure.

Thus a formula to give a first approximation of z_i is desired. The

correspondence between z_i and the effect of the pursuers suggests

formulae such as

$$z_i = \frac{(P_1 E)}{(P_i E)} \tag{68}$$

or

$$z_i = \frac{(P_1 E - r_1)/p_1}{(P_i E - r_i)/p_i} \; , \tag{69}$$

where $P_i E$ is the distance separating the evader from pursuer P_i,

of maximum speed p_i.

Also, it can be shown easily that, under favorable initial con-

ditions, two slow pursuers can catch a faster evader. Also, if the

one-versus-one game (P_1, E) admits a solution from $x_1(t_f)$, then, the

two-versus-one game with $x_1(t_f)$ is possible under any value of the

strategic variable. Conversely, for any pair $(x_1(t_f), x_2(t_f))$, there

exists a value of the strategic variable such that the necessary

conditions of optimality are satisfied. Then, it is clear that

restricted usable parts of terminal manifolds can no longer be found

and, consequently, the convenient use of the singularities to study

the boundary of the usable part, barriers, capture zones and par-

ticular trajectories do not apply to team pursuit-evasion games.

Feedback solutions can only be given relatively to the other

pursuer, thus, the added dimension due to the strategic variable

prevents the global solution to be represented by a unique state

portrait, since trajectories corresponding to different values of z_i do intersect.

The most crucial point in this study is that an N-pursuer team game is not a classical (N+1) point boundary value problem because another (N-1) unknowns, modelled as strategic variables, are to be found. The team of pursuers minimizes a common performance index; it defines a single Hamiltonian for the team, and then, whenever a pursuer is added to the team, one equation is missing. Nevertheless, the general structure of the equations enables one to take advantage of the individual 1-vs.-1 games to derive structures of team games.

IV.   THE GAME OF KIND

1.   INTRODUCTION

By analogy with the 1-vs.-1 game, the game of kind study for team differential pursuit-evasion games could be summarized by finding the answer to the question "can the pursuit team catch the evader in finite time?" Sufficient conditions ensuring the existence of a control to achieve capture in finite time are stated by Satimov, Azamov and Khaidarov [20]. But another dimension to the game of kind has to be added, and the next question is "which are the relevant pursuers?" In other words, how to distinguish the pursuers whose presence is required from those whose effect on the optimal solution is null.

A pursuer P_i, attempting to capture an evader E, is expecting some cooperation from other pursuers. Fixing P_i, it is convenient to partition the space into various zones in which a copursuer would be expected to behave in a particular way. As the game proceeds, the zones evolve dynamically. The possible zones of interest are numerous, therefore the pursuers will be assumed to play optimally, reducing the number of the zones of interest.

After the various definitions of the zones for the minimum time problem, a parallel with the game of degree allows computation of the most useful zones, the help zones. Approximated formulae are also of interest since the computation of the exact zones is, at best, difficult. Finally, an example is treated to illustrate the procedure.

2. PURSUER CLASSES

Generally, three classes of pursuers can be distinguished as follows:

i) The first is made of the pursuers that are irrelevant to the game because of their initial position, their dynamical characteristics (speed, maneuverability, etc.) or because other pursuers are located in positions that completely cover the possibilities of these pursuers.

ii) The second class includes the pursuers that have a temporary effect on the game. The typical example is a very slow pursuer located on the course of the evader. This pursuer will deny a direct course, but once passed, its effect will be null. State constraints like islands can be modeled as a pursuer of speed equal to zero, with a lethal area covering the island.

iii) The third class consists of the pursuers that actually perform the capture. It is often implicitly assumed that the pursuers belong to this class.

The three classes must be studied in both events: an optimal as well as a non-optimal play of the evader. The actual control that a pursuer should adopt in the (unlikely) event that the evader would not play optimally, when such an optimal play of the evader would not allow this pursuer to be useful, is difficult to find since the very definition of the criteria to optimize is problematic.

The study of class two partitions the game into several sequences divided by time t_s from which the pursuer of class two has no

effect any more. Very often, the evader is on the border of the lethal area of this pursuer at time t_s.

The global study of the game of kind helps in finding the conditions under which capture is possible. These conditions on the relative speeds and maneuverability, ensure capture independently of the initial state (capture cond'tion) or not (playability condition). Under particular initial conditions, it is possible for two slow pursuers to catch a faster evader.

3. TWO-VERSUS-ONE PURSUIT-EVASION ZONES

The zones represent a parametric study of the pursuit-evasion game. Many parameters are candidates but the one used is the initial position of an eventual copursuer. Thus, the other parameters such as the position of the evader at $t = t_o$ and the characteristics of the various players are assumed to be known. Another assumption is the optimality of the strategies of the pursuers.

The specific pair (P_i, P_j) is separately studied. To a pursuer P_i, six zones, corresponding to six possible cases, are relevant. The following notations are used:

$$x_i^T M_i x_i \leq r_i^2 \text{ is the capture condition corresponding to pursuer } P_i.$$

The control vector of pursuer P_i is u_i, defined on the set of admissible control functions U_i. The control of the evader is v, defined on V. Usual definitions of U_i and V are the set of control vectors bounded by a given maximum norm.

$t_{P_i}^*(v)$ is the capture time of the one-versus-one game (P_i, E),

where E applies the control v, and P_i, initially located at $x_i(t_o)$, plays optimally. v^* is the optimal strategy of E. To each pair $(v, x_i(t_o))$ thus corresponds a time $t_{P_i}^*(v)$.

$t_{P_j}^*(v^*) > t_{P_i}^*(v^*)$ means that the optimal game (P_j, E) finishes in a longer time than the optimal game (P_i, E). Then P_i is a more dangerous player than P_j and $z_i < z_j$ is likely in the 2-vs.-1 (P_i, P_j, E) game. Note that one of the times $t_{P_j}^*(v^*)$ or $t_{P_i}^*(v^*)$ may be infinite if the corresponding pursuer is unable to catch, alone, the evader.

$Z1P_{jo}(\overline{P}_i/P_j^*)$ is to be interpreted as zone number 1 where the parameter of study is the position of pursuer P_j at time $t = t_o$, in which P_j is located such that P_i has no effect on the game, given that P_j plays optimally.

In the following, $x_i(t_0)$ is given and the parameter describing the zones is the initial position $x_j(t_o)$ of an eventual copursuer.

i) $Z1P_{jo}(\overline{P}_i/P_j^*) = \{x_j(t_o) \,|\, x_i^T(t)M_i x_i(t) > r_i^2, \forall\, v \varepsilon V, \forall\, t \varepsilon$

$$[t_o, t_{P_j}^*(v)], \forall\, u_i \varepsilon U_i\} .$$

If P_j belongs to this zone at $t = t_o$, and the game (P_j, E) is finished in $t_{P_j}^*(v)$, then P_i will not play any role in any game involving P_j, provided that P_j plays optimally. In this game, a strategic variable associated with P_i is $z_i = 0$, and P_i is of class one as defined earlier.

ii) $Z2P_{jo}(\tilde{P}_i/P_j^*) = \{x_j(t_o) \mid x_i^T(t)M_ix_i(t) > r_i^2, \forall u_i \epsilon U_i,$

$\forall t \epsilon [t_o, t_{P_j}^*(v^*)]$, and $\exists u_i \epsilon U_i, \exists v \epsilon V,$

$\exists t \epsilon [t_o, t_{P_j}^*(v)]; x_i^T(t)M_ix_i(t) \leq r_i^2\}$.

P_i cannot catch an evader playing optimally according to P_j but there exists at least one play of the evader that would enable P_i to capture before P_j does. P_i can play a role in a game involving P_j only if E does not play optimally or if the presence of other pursuers forces E to deviate from the (P_j, E) optimal strategy. If pursuer P_i is of class two as defined earlier, then P_i is able to intercept the trajectory of the 1-vs.-1 (P_j, E) game in the prescribed time otherwise P_i would not be able to deny the evader a course. The fact that P_i might not achieve capture is here irrelevant: the other member of the team, namely P_j, will do it; the question is to know if P_i plays a role in the team effort. A pursuer P_i of class two belongs to the next zone.

iii) $Z3P_{jo}(P_i/P_j^*) = \{x_j(t_o) \mid \exists u_i \epsilon U_i, \exists t \epsilon [t_o, t_{P_j}^*(v^*)];$

$x_i^T(t)M_ix_i(t) \leq r_i^2$ and $t_{P_j}^*(v^*) < t_{P_i}^*(v^*)\}$.

If P_j belongs to this zone, then P_i will play a role in the (P_i, P_j, E) game, in which P_j is the primary threat.

iv) $Z4P_{jo}(P_j/P_i^*) = \{x_j(t_o) \mid \exists u_j \epsilon U_j, \exists t \epsilon [t_o, t_{P_i}^*(v^*)] ;$

$x_j^T(t)M_jx_j(t) \leq r_j^2$ and $t_{P_i}^*(v^*) \leq t_{P_j}^*(v^*)\}$.

If P_j belongs to this zone, then P_j will play a role in the (P_i, P_j, E) game, in which P_i is the primary threat. A pursuer of class three would belong to Z3 or Z4.

$$v) \quad Z5P_{jo}(\tilde{P}_j/P_i^*) = \{x_j(t_o) \mid x_j^T(t)M_j x_j(t) > r_j^2, \ \forall \, u_j \epsilon U_j, \ \forall \, t\epsilon$$

$$[t_o, t_{P_i}^*(v^*)] \text{ and } \exists \, u_j \, U_j, \ \exists \, v\epsilon V, \ \exists \, t\epsilon$$

$$[t_o, t_{P_i}^*(v)]; \ x_j^T(t)M_j x_j(t) \leq r_j^2\} \ .$$

P_j cannot catch an evader playing optimally according to P_i but there exists at least one play of the evader that would enable P_j to capture before P_i does. If P_j belongs to this zone, then P_j can play a role in a game involving P_i only if E does not play optimally or if the presence of other pursuers forces E to deviate from the (P_i, E) optimal strategy.

$$vi) \quad Z6P_{jo}(\overline{P}_j/P_i^*) = \{x_j^T(t)M_j x_j(t) > r_j^2, \ \forall \, u_j \epsilon U_j, \ \forall \, v\epsilon V,$$

$$\forall \, t\epsilon[t_o, t_{P_i}^*(v)]\} \ .$$

If P_j belongs to this zone, then P_j will not play any role in any game involving P_i, provided that P_i plays optimally. In this zone, the strategic variable z_j associated with P_j is zero, and P_j is of class one.

Section 6. shows an example of the derivation of these zones for the problem of the cutters vs. railroad.

The definitions above concern the 2-vs.-1 game (P_i, P_j, E). The zones are the same for every pursuer identical to P_j but depend on

the position $x_i(t_o)$ of P_i at $t = t_o$, i.e. to a new initial position $x_i(t_o)$ corresponds a completely new set of zones. The same kind of definitions can be derived for a three-vs.-one game as:

$Z1P_{ko}(\overline{P}_k/P_i^*, P_j^*)$ etc. The derivation of these zones is, of course, more complex than in the two-pursuer-versus-one evader game.

In solving a N-pursuer game, it is hoped that the simple case by case study of the zones associated with the possible pairs allows selection of the relevant players.

4. THE HELP ZONE

The help zone $HP_{jo}(P_i^*, E) = Z1 \cup Z2 \cup Z3 \cup Z4$ is the zone in which P_j plays a role in the (P_i, P_j, E) game where P_i and E play optimally.

The first way to compute the boundary of the help zone is by remarking that this boundary separates the optimal 2-vs.-1 game (P_i, P_j, E) in which P_j plays a role with the 2-vs.-1 game in which P_j does not play a role; this latter game is, thus, equivalent to the 1-vs.-1 game (P_i, E), since P_j is irrelevant. Therefore, if pursuer P_j, at $t = t_o$, is located on this boundary, then the equations of the necessary conditions of optimality describing the 2-vs.-1 (P_i, P_j, E) game and the equations describing the 1-vs.-1 (P_j, E) game both hold, and consequently, the boundary of the help zone is determined by all the possible $x_j(t_o)$ that satisfy these conditions. Then all the equations must be jointly solved; this control approach seems heavy, but is easily shown to be equivalent to the solution of the equations of the 2-vs.-1 game and letting the strategic variable z_j approach zero, since P_j has less and less influence while reaching the boundary

of the help zone, beyond which P_j becomes useless. This is a relatively easy task. That smooth transition between the 1-vs.-1 and the 2-vs.-1 games is ensured by z_j. The absence of the strategic variable prevents any attempt to define a help zone for fixed terminal-time games.

The second way to derive the help zone is a gaming approach. The Hamiltonian of the 2-vs.-1 game is H, and the Hamiltonians of the 1-vs.-1 games (P_i,E) and (P_j,E) are H_i, H_j. Then it is always possible to set $H = H_i + H_j + C_{ij}$ where C_{ij} is a correction term such that the equality holds. If P_j collaborates with P_i in the capture, then $H \neq H_j$ must hold. Thus, $H_j + C_{ij} \neq 0$ is the cooperation condition, and conversely, $H_j + C_{ij} = 0$, with the controls u_i^* from the 1-vs.-1 game, gives the boundary of the help zone. Though different in their approaches, both methods are, in fact, equivalent.

5. APPROXIMATIONS OF THE NO-HELP ZONE

Zone Z6 is important because a pursuer located in zone Z6 of another pursuer can immediately be eliminated. Unfortunately, if the derivation of the help zone is more or less possible, Z6 is more difficult to compute since a non-optimal play of the evader (optimal in the 2-vs.-1 sense) must be accounted for.

From the definition of Z6 given above, three simplifications that yield approximated results can be made.

$$i) \quad Z63P_{jo}(\overline{P}_j/P_i^*) = \{ \; x_j(t_o) \,|\, y_j^T(t) M_j y_j(t) > r_j^2, \; \forall u_j \varepsilon U_j,$$

$$\forall t \varepsilon [t_o, t_{P_i}^*(v^*)] \} \; ,$$

where y_j is the state vector corresponding to the Pareto game (P_j,E) in which E is willing to be caught (rendez-vous game). $y_j(t)$ obeys the same dynamics as $x_j(t)$ and $y_j(t_o) = x_j(t_o)$, but during the development of the game, the evader uses its control v to help P_j in the capture. If P_j is not in the area of constant terminal time $t_{P_j}^*(v^*)$ in the Pareto game (P_j,E), then P_j belongs to Z6. The idea is that (P_i,E) lasts at most $t_{P_i}^*(v^*)$; if the game (P_j,E), with the cooperation of E, cannot be ended within this time, then, for sure, P_j cannot help P_i in catching E. Z63 is, of course, a coarse approximation of Z6; also Z63 $\subset$ Z6.

ii) $Z62P_{jo}(\overline{P}_j/P_i^*) = \{x_j(t_o)\,|\,y_j^T(t)M_jy_j(t)>r_j^2,\ \forall\,u_j\epsilon U_j,$

$$\forall\,v\epsilon V,\ \forall\,t\epsilon[t_o,t_{P_i}^*(v^*)]\}\ ,$$

where y_j is the state vector corresponding to the Pareto game (P_j,E). Z62 uses the same approach as Z63, but now, E is constrained to the reachable zone of the evader allowed by P_i. In the 1-vs.-1 game (P_i,E) where P_i plays optimally, E, playing all the possible policies v, can cover, before capture, a zone named reachable zone of the evader allowed by P_i, and the largest time to capture is $t_{P_j}^*(v^*)$. In Z62, E is constrained to this zone that P_j attempts to reach. Z62 is a better approximation that Z63 at the expense of some more computation; Z63 $\subset$ Z62 $\subset$ Z6.

iii) $Z61P_{jo}(\overline{P}_j/P_i^*) = \{x_j(t_o)\,|\,y_j^T(t)M_jy_j(t)>r_j^2,\ \forall\,u_j\epsilon U_j,$

$$\forall\,v\epsilon V,\ \forall\,t\epsilon[t_o,t_{P_i}^*(v)]\}\ ,$$

where y_j is the state vector corresponding to the Pareto (P_j, E) game. In Z61, E is also constrained to its reachable zone but the exact time $t_{P_i}^*(v)$ allowed by P_i, associated with every policy v of E, is computed and fixes the maximum duration of the Pareto game (P_j, E) during which P_j attempts to reach E.

For those game in which the 1-vs.-1 game (P_i, E) lasts long enough compares with the inertia of P_j, a major simplification can be made if the control u_j considered is fixed to be a trajectory orthogonal to the trajectory of the evader at time $t_{P_i}^*(v^*)$.

Z63 is to be used when $t_{P_i}^*(v^*)$ only is available, Z62 when the reachable zone of E according to the 1-vs.-1 game (P_i, E) is known, and Z61 when the exact timing of this reachable zone is known.

6. TWO-CUTTERS VS. RAILROAD EXAMPLE

6.1 Time-optimal conditions

The cutters P_i attempt to intercept in minimum time a train E, constrained on a railway (axis x_2). The lethal area of the cutters is a circle of radius r_i; the cutters of speed P_i control their heading angle u_i, and the evader controls its speed v, bounded by e. The 1-vs.-1 version of this game has been referred to as "the wall pursuit game" by Isaacs [3], or sometimes as "ICBM vs. railroad". Due to its simplicity, this game can be solved geometrically or using control theory. The important case where E is between or surrounded by its two opponents as well as its implications on the strategic variable z_2 ($z_1 = 1$) are investigated.

The game of kind will be studied through two examples.

The Hamiltonian, the optimal policies (u_i^*, v^*), the costate vectors $\lambda_i = (\lambda_{i1}, \lambda_{i2})$ and the states (x_i) are given by

$$H = -\lambda_{11} p_1 \cos(u_1) + \lambda_{12}(v - p_1 \sin(u_1)) - \lambda_{21} p_2 \cos(u_2) +$$

$$\lambda_{22}(v - p_2 \sin(u_2)) + 1 , \tag{70}$$

$$\cos(u_i^*) = \lambda_{i1}/\sqrt{\lambda_{i1}^2 + \lambda_{i2}^2} , \tag{71}$$

$$\sin(u_i^*) = \lambda_{i2}/\sqrt{\lambda_{i1}^2 + \lambda_{i2}^2} ,$$

$$v^* = e \cdot \text{sign}(\lambda_{12} + \lambda_{22}) ,$$

where "sign" is the signum function,

$$\dot{\lambda}_{i1} = 0 ,$$

$$\dot{\lambda}_{i2} = 0 , \tag{72}$$

at $t = t_f$ $\lambda_{11} = \nu_1 r_1 \cos(\alpha_1)$, $\lambda_{21} = \nu_2 r_2 \cos(\alpha_2)$,

$$\lambda_{12} = \nu_1 r_1 \sin(\alpha_1) , \quad \lambda_{22} = \nu_2 r_2 \sin(\alpha_2) ,$$

and $z_2 \nu_1 r_1 = \nu_2 r_2$,

$$x_{i1} = r_i \cos(\alpha_i) + \tau p_i \cos(\alpha_i) , \tag{73}$$

$$x_{i2} = r_i \sin(\alpha_i) + \tau p_i \sin(\alpha_i) - \tau e \cdot \text{sign}(\sin(\alpha_i)) + \tau e C_i ,$$

with $C_i = 1 \cdot \text{sign}(\sin(\alpha_i)) - 1 \cdot \text{sign}(\sin(\alpha_1) + z_2 \sin(\alpha_2))$;

α_i are line of sight angles relative to x_2 axis at the terminal time; τ is defined as $t_f - t$. The strategic variable z_2 appears in (73) only in a signum function, therefore has a switching effect.

The capture condition is expressed by

$$P_1 + z_2 p_2 - e|\sin(\alpha_1) + z_2 \sin(\alpha_2)| \geq 0 . \tag{74}$$

Three inequalities restrict the strategic variable: the capture condition, expressed at the terminal time and the playability conditions applied to each pursuer, which state that the trajectories, as τ increases from zero, must go away from the terminal manifold. Defining $\vec{d}_i$ as the tangent vector to the trajectory at time $\tau = t_f - t$, this geometrical condition is expressed by

$$\vec{d}_i = (x_{i1} - r_i \cos(\alpha_i) ; x_{i2} - r_i \sin(\alpha_i)) ,$$

$$\tag{75}$$

$$(x_{i1} - r_i \cos(\alpha_i))\cos(\alpha_i) + (x_{i2} - r_i \sin(\alpha_i))\sin(\alpha_i) \geq 0 .$$

The strategic variable is not constant when, for example, the common opponent is surrounded by the pursuers. That possibility exists whenever a team of at least N members faces an opponent moving on a space of dimension $N - 1$. Then z_i adopts a value such that the control of the evader, v, disappears from the Hamiltonian at the last moments of the game.

6.2 Game of kind analysis

The trajectories are given by (73), the limit of the help zone for P_1 is obtained by letting z_2 approach zero in the trajectory (73)

for P_2. The result is given below; the variable is α_2, $\tau = t_f$ and α_1 are given by the 1-vs.-1 game (P_1, E).

$$x_1 = r_1\cos(\alpha_2) + \tau p_2\cos(\alpha_2) \quad ,$$

<div align="right">(77)</div>

$$x_2 = r_2\sin(\alpha_2) + \tau p_2\sin(\alpha_2) - \tau e\cdot\text{sign}(\sin(\alpha_1)) \quad ,$$

From the above computations, various significant zones of capture and help for location of P_2 at $t = t_o$, may be computed, relative to $x_1(t_o)$. For example, with $p_1 = e$, $e = 1$, $r_1 = 1$, $r_2 = 0.5$ and $x_1(t_o) = (4 ; -4)$. Figure 13 presents the zones for $p_2 = 1.5$ and Figure 14, for $p_2 = 0.5$, is an example where P_2 is unable to capture E in the game (P_2, E) and $t_{P_2}^*(v^*)$ is infinite. Z6, the zone of capture by P_1 alone, is the most difficult to calculate. The three approximations are computed according to their definitions in Section 5.

The interval given by $x_2 \epsilon [3.95, -1.79]$ in Figures 13 and 14 provides the segment that E can reach when P_1 plays optimally.
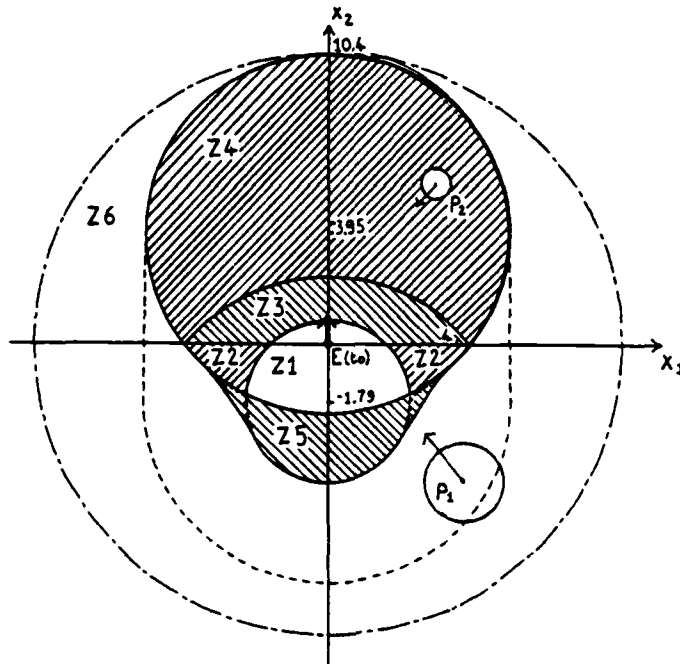
$e < p_2 < p_1$

Z1: P_2 alone will capture E.
Z2: P_2 alone will capture an optimal E.
Z3: P_2 will be helped by P_1 in capturing E.
Z4: P_1 will be helped by P_2 in capturing E.
Z5: P_1 alone will capture an optimal E.
Z6: P_1 alone will capture E.

Approximations of Z6:

—··— Z61.
—————— Z62.
·—·— Z63.

Figure 13. Cooperative zones for 2-vs.-1 example with pursuer speed
$p_2 = 1.5$.
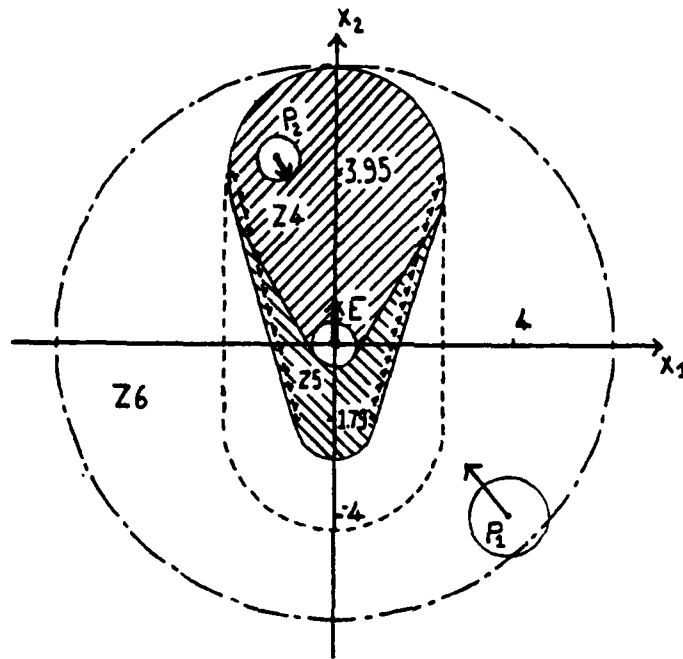
$p_2 < e < p_1$

Z1, Z2, Z3 are identical to the capture area of P_2.
Z4: P_1 will be helped by P_2 in capturing E.
Z5: P_1 alone will capture an optimal E.
Z6: P_1 alone will capture E.

Approximations of Z6:

— ▼•▼•— Z61.

- - - - - Z62.

-•——•— Z63.

Figure 14. Cooperative zones for 2-vs-1 example with pursuer speed
$p_2 = 0.5$.

V. THE LINEAR QUADRATIC GAME OF DEGREE

1. THE QUADRATIC TEAM GAMES

The hierarchical command and communication structure for a linear
quadratic game of N pursuers and 1 evader is derived. The optimal
solution and a simpler form of this solution are given yielding the
general solution to the N-pursuer vs. one-evader game. A sensitivity
study leads to a simple hierarchical structure which greatly reduces
the amount of computation required.

Two-player, linear, quadratic games have been extensively studied
in the literature. According to Clemhaut and Wan [21], conceptually,
a linear quadratic model may be regarded as a Taylor approximation
to more general models. Another advantage is that, computationally,
the closed-loop control can, in principle, be numerically determined.

On the other hand, two main disadvantages must be overcome. A
quadratic objective function implies satiability at some finite state
vector. Unbounded controls make any pursuit-evasion, game-of-kind
type of analysis impossible. More importantly, even for a two-person,
two-state problem, the solution involves a Ricatti system including
six second-order differential equations.

Team games emphasize these disadvantages; the very definition
of the performance index actually defines the type of cooperation
between teammates; this issue is addressed in Chapter VII. The
game becomes an N+1 point, boundary-value problem involving heavily

interrelated Ricatti equations and has a complex hierarchical structure.

2. 1-vs.-1 LINEAR QUADRATIC GAME OF DEGREE

The game studied is a linear, quadratic, generalized N-pursuer, one-evader game with the reduced state given by

$$\dot{x}_i = A_i x_i + B_i u_i + C_i v \ , \tag{78}$$

where $i = 1, 2, .., N$; $x_i \in R^n$, $u_i \in R^n$ is the control of pursuer P_i and $v \in R^n$ is the control of the evader; A_i, B_i, C_i are matrices of appropriate dimension.

The terminal condition is given by a manifold $g(x_i(t_f)) \leq 0$ or again more specifically

$$x_i^T(t_f) M_i x_i(t_f) \leq r_i^2 \ , \tag{79}$$

and the performance index, to be minimized by the pursuers and maximized by the evader, is

$$J = 0.5 \int_{t_o}^{t_f} (\sum_i (x_i^T Q_i x_i + u_i^T R_i u_i) - v^T S v) dt \ , \tag{80}$$

where M_i, Q_i, R_i, S are appropriate, positive definite, symmetric matrices.

Let $\lambda_i \in R^n$ be the costate vector required by the game solution with the minimum principle such that the transversality condition is given by the relation

$$\lambda_i(t_f) = \nu z_i M_i x_i(t_f) \ , \tag{81}$$

where ν, a Lagrange multiplier, satisfies the terminal Hamiltonian

$$H(t_f) = [\sum_i (A_i x_i + B_i u_i + Cv)^T \lambda_i - \frac{1}{2}\sum_i (x_i^T Q_i x_i + u_i^T R_i u_i) +$$

$$\frac{1}{2}v^T Sv)]_{t_f} = 0 \tag{82}$$

The strategic variable, $z_1 = 1$ for a 1-vs.-1 game, and it can be assumed that $z_1 = 1$ (most effective pursuer) and $z_j < 1$ for $j = 2, \ldots, N$ in general.

The open-loop Nash solution according to the minimum principle for the one-pursuer case (i=1) is given by

$$u_i^* = R_i^{-1} B_i^T T_i \Phi_i(t, t_o) x_i(t_o) \;,$$

$$\tag{83}$$

$$v^* = S^{-1} C_i^T T_i \Phi_i(t, t_o) x_i(t_o) \;,$$

where the state transition matrix,

$$\dot{\Phi}_i(t, t_o) = (A_i + N_i T_i + L_i T_i)\Phi_i(t, t_o) \;,$$

$$\Phi_i(t, t) = I \;, \tag{84}$$

$$N_i = B_i R_i^{-1} B_i^T \;, \quad L_i = C_i S^{-1} C_i^T \;;$$

T_i is the solution to the matrix Ricatti equation,

$$\dot{T}_i = -A_i^T T_i - T_i A_i - T_i(N_i + L_i)T_i + Q_i \;,$$

$$\tag{85}$$

$$T_i(t_f) = \nu z_i M_i \;.$$

The solution is classical. The non-zero-sum game formulation, in which P and E do not have a similar performance index is treated by Mohler, Kolodziej and Bugnon [22]. The vector λ_i and the matrix T_i are duplicated; (85) becomes a system of coupled Ricatti equations. A procedure to decouple the Ricatti equations is given by Simaan and Cruz [23], taking advantage of a preliminary solution common to several initial conditions, reducing the problem to the computation of successive linear equations.

A complete derivation of the two-player, linear, quadratic game is given by Ichikawa [24] and Hämäläinen [25].

3. TWO-vs.-ONE-GAME SOLUTION

The two-pursuer, one-evader game requires two costate vectors with transversality condition (81) for $i = 1,2$. u_i^*, $i = 1,2$, is given by (83) and

$$v^* = S^{-1} [\sum_{i=1}^{2} C_i^T T_i \Phi_i(t,t_o) x_i(t_o)] , \tag{86}$$

The state transition matrices are given by (84), and $H(t_f)$ by (82).

T_i on the other hand, is computed from

$$\dot{T}_1 x_1 = [-T_1 A_1 - A_1^T T_1 - T_1 (N_1 + L_1) T_1 + Q_1] x_1 - T_1 C_1 S^{-1} C_2^T T_2 x_2 ,$$

$$\dot{T}_2 x_2 = [-T_2 A_2 - A_2^T T_2 - T_2 (N_2 + L_2) T_2 + Q_2] x_2 - T_2 C_2 S^{-1} C_1^T T_1 x_1 , \tag{87}$$

where $T_1(t_f) = \nu z_1 M_1$, $T_2(t_f) = \nu z_2 M_2$.

The solution is very complex compared to the 1-vs.-1 game. The differential systems, (84) and (87), are tightly coupled, requiring parallel computation whereas the solution to the 1-vs.-1 game could proceed in several steps.

z_2 must be selected properly to correspond to the initial conditions of the game. It might require a trial and error procedure whose computational burden would be somewhat reduced using the method described by Simaan and Cruz [23].

By analogy with the strategic variable, a strategic matrix Z_i is defined as a symmetric, non-singular matrix satisfying the relation

$$Z_i T_1 x_1 = T_i x_i \; , \tag{88}$$

and $Z_1 = I$.

The solution, while (85) and (86) provide u^*, v^*, is given by

$$\dot{T}_1 = -T_1 A_1 - A_1^T T_1 - T_1 (N_1 + L_1) T_1 - T_1 C_1 S^{-1} C_2^T Z_2 T_1 + Q_1 \; ,$$

$$\dot{T}_2 = -T_2 A_2 - A_2^T T_2 - T_2 (N_2 + L_2) T_2 - T_2 C_2 S^{-1} C_1^T Z_2^{-1} T_2 + Q_2 \; , \tag{89}$$

with $\quad T_1(t_f) = \nu z_1 M_1 \; ,$

$$T_2(t_f) = \nu z_2 M_2 \; ,$$

$$\dot{Z}_2 = Z_2 (A_1^T + Q_1 T_1^{-1}) - (A_2^T + Q_2 T_2^{-1}) Z_2 \; , \tag{90}$$

$$\dot{Z}_2^{-1} = Z_2^{-1} (A_2^T + Q_2 T_2^{-1}) - (A_1^T + Q_1 T_1^{-1}) Z_2^{-1} \; ,$$

and $\qquad Z_2(t_f) M_1 x_1(t_f) = (z_2/z_1) M_2 x_2(t_f)$,

$$Z_2^{-1}(t_f) = [Z_2(t_f)]^{-1}$$

The similarities with the 1-vs.-1 game are now obvious; the equations are decoupled and can be solved in turn. However, the problem of guessing the strategic variable remains. Note the simplification that $Q_1 = Q_2 = 0$ would introduce.

4. STRUCTURE OF AN N-IDENTICAL-PURSUER, ONE-EVADER LINEAR QUADRATIC GAME

The solution for Q=0 takes the form given by

$$u_i^* = B^T T_i x_i \ , \tag{91}$$

$$v^* = C \sum_{i=1}^{N} T_i x_i \ ,$$

$$\dot{T}_i = -T_i A - A^T T_i - T_i (N+L) T_i - T_i L \left(\sum_{\substack{j=1,N \\ j \neq i}} Z_j \right) Z_i^{-1} T_i \ , \tag{92}$$

and $T_i(t_f) = \nu z_i M$, ν given by $H(t_f) = 0$,

$$\dot{Z}_i = Z_i A^T - A^T Z_i \ ,$$

$$\dot{Z}_i^{-1} = Z_i^{-1} A^T - A^T Z_i^{-1} \ , \tag{93}$$

with $\qquad Z_i(t_f) M x_1(t_f) = (z_i/z_1) M x_i(t_f)$

$$Z_i^{-1}(t_f) = [Z_i(t_f)]^{-1} \ .$$

The summation sign (Σ) on the control of the evader v^*, and the independence of the controls u_i^* as well as the form of the solution show that each pursuer P_i needs to receive information about

the optimal control, v^*, about the strategic matrices Z_j and the strategic variables z_j in order to carry out its own optimization algorithm. This results in u_i while providing information about $T_i x_i$ (to compute v^*), $x_i(t_f)$ (to compute Z_i), and $J_i = \frac{1}{2} \int_{t_o}^{t_f} (u_i^T R_i u_i) dt$ (to compute z_i and z_j).

It can be viewed as the nesting of three hierarchical structures that must iteratively be optimized, going from the lowest one to the highest one. Figure 15 shows that structure.

First, according to a given pair (Z_i, z_i), the loop involving AL1 is taken into account, each pursuer producing $x_i(t_f)$. Then, according to these $x_i(t_f)$, the new Z_i are to be computed by AL2 according to (93), and so on until the point where the new values of Z_i match the old ones. As a last step, J_i are analyzed in AL3 to produce the new strategic vector $z = [z_1, z_2, \ldots, z_N]$. Consequently,

i) AL1 is purely deterministic: $v^* = C \sum_1^N T_i x_i$.

ii) AL2 is tactical: the main strategic option being defined by the strategic vector z, AL2 merely computes the trajectories, etc. to perform the capture. $\sum_{\substack{j \neq i}}^{1,N} Z_j$ must be computed according to the differential equations given for Z_j which depend on $x_j(t_f)$. An algorithm that ensures the convergence of Z_j to yield the values of $x_j(t_f)$ produced by the pursuers, must be added.

iii) AL3 is strategical. The values in the strategic vector define early strategical choices viz. which is the most important player representing the main threat to be considered by the evader, which are the irrelevant pursuers (of class one), etc... z_i must be determined according to both the a priori information and the a posteriori information based on J_i for the previous vector z.
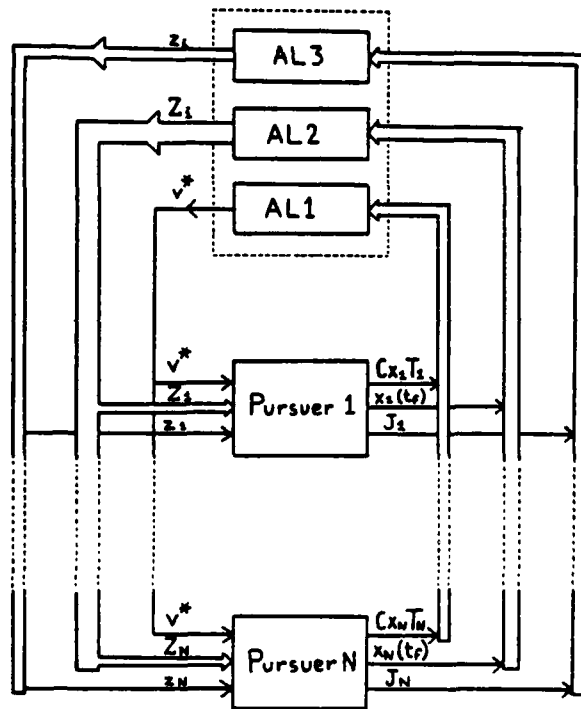
Figure 15.   Nesting hierarchical structure for N-vs.-1 linear quadratic game.

ALl is known; AL2 is a simple converging algorithm, but so far, little is developed to avoid an exhaustive search for AL3.

A few fules in finding AL3 are as follows:

i) Study the game of kind, trying to select the relevant pursuers to the game.

ii) Attempt to guess the value of z_i according to the initial positions and some simple geometric rules.

iii) The capturability and playability conditions that must be satisfied by z_i, delimit the hyper-volume of definition of the vector z.

iv) A heuristic decomposition approach to the solution that will be introduced in the next chapter.

5. A SUB-OPTIMAL STRUCTURE

Equation (92) can be redefined as

$$\dot{T}_i + -T_i A - A^T T_i - T_i (N + L) T_i - \beta_i - \omega_i , \qquad (94)$$

if

$$\beta_i = T_i L (\sum_{j=1}^{i-1} z_j) z_i^{-1} T_i ,$$

$$\omega_i = T_i L (\sum_{j=i+1}^{N} z_j) z_i^{-1} T_i . \qquad (95)$$

Except for the addition of β_i and ω_i in (94) and the summation sign in (91), the N-vs.-1 game solution is identical to the 1-vs.-1 solution.

This suggests a sensitivity study of both β_i and ω_i. In some instances the β_i are nearly equal and the ω_i are negligible when the pursuers represent equal threats to the evader. In this case, the difference between the 1-vs.-1 game and n-vs.-1 game is seen to

come hardly from the controls adopted by the pursuers but from the controls adopted by the evader. β_i is an important term if P_i plays a minor role in the capture. On the other hand, for all cases considered, ω_i is negligible when the playability condition is not violated, provided that the pursuers be classified in order of importance, P_1 being the most important one (the closest one to E for the minimum-time problems).

Consequently, the result is shown in Figure 16, where each pursuer solves (91) - (93), finding the couple $x_i(t_f)$, z_i corresponding to $x_i(t_o)$, and passes this information forward.
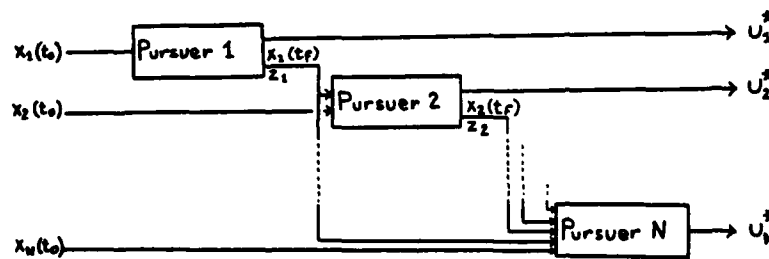
Figure 16.   Sub-optimal simple field hierarchical structure for the N-vs.-1 linear quadratic game.

This structure takes advantage of the autonomy of the pursuers, breaks the complex solution into simpler steps, and, to an individual pursuer, the number and class of the co-pursuers is completely irrelevant.

The previous structure of Figure 15, though exact, is inferior in that the search of the optimal solution is quite tedious, AL3, in particular, is vague. The information passed (matrices Z_i, vectors z and x_i, Cx_iT_i) is substantial compared to the new structure.

A parallel structure is derived from this "ripple" structure in Figure 17, enhancing the independence of the individual pursuers with respect to the team, but at the expense of an increased number of equations to solve. A "minor" pursuer can be added but the gain produced by this pursuer must be weighted against the amount of delay or computation that this very pursuer will have to cope with.

The structure depicted in Figure 17, and implemented for the examples considered does not show any variation in state, control or performance superior to 0.1% of the rigorous solution.
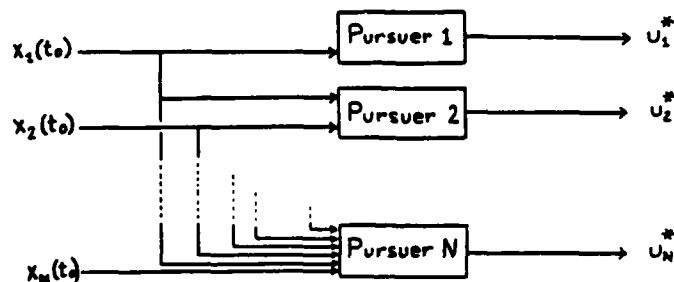
Figure 17. Independent hierarchical structure for the N-vs.-1 linear quadratic game.

VI. A COMPOSITION APPROXIMATING ALGORITHM

1. INTRODUCTION

The main difficulty in solving an N-vs.-1 pursuit-evasion game, using the necessary conditions of optimality, is to guess properly various variables such as the terminal hit points and the strategic variables. This problem gets more and more difficult as the number of pursuers increases. The 1-vs.-1 problem, though complex, is often solvable whereas the 2-vs.-1 problem is nearly impossible to derive globally.

In this section, the solution to the 2-vs.-1 linear, minimum-time, team game is simplified using a composition approach. The two individual 1-vs.-1 games are assumed to be previously solved. The approach consists of the computing of an intermediate stage in which an "equivalent" 1-vs.-1 game is defined. The solution to this equivalent 1-vs.-1 game is used to find, in a similar way, the solution to the 2-vs.-1 game.

The study of the 2-vs.-1 game of kind done so far corresponds to a direct optimal approach. The direct composition is defined first; i.e., the problem addressed is how to compute the equivalent 1-vs.-1 game solution from the two 1-vs.-1 games.

The first step is to prove that such an equivalent game exists; therefore the composition is first derived analytically from the results of both the 1-vs.-1 and the 2-vs.-1 games.

2. ANALYTIC COMPOSITION

The computation of the initial position, dynamics and terminal manifold of pursuer P whose effect on the game studied (P,E) is identical to the effect of the pair (P_1, P_2), is named composition.

Three major properties of the ideal composition must be relaxed:

i) P must be equivalent to P_1 and P_2 for every optimal and non-optimal play of the pursuers. So far, in every problem studied, the optimality of the pursuers was assumed since the main goal is to study the team and not the escape maneuvers. Thus, optimality will be assumed for the pursuers, even though a problem is to get rid of a non-optimal play of a (P_1, P_2, E) game that might, in fact, correspond to an optimal play of a (P_1, P_2, P_3, E) game.

ii) P must be equivalent to P_1 and P_2 for every optimal and non-optimal play of the evader. This is a reasonable request, but for the sake of simplicity, the evader will be assumed to behave optimally.

iii) To every position of P_1 and P_2 there corresponds a position of the equivalent pursuer P. Enforcing this property would solve the 2-vs.-1 game globally. The study of this functional will not be undertaken.

By composition, all the above simplifications are assumed. The scope of interest of the composition operator is considerably reduced, but, as shown, existence is the main obstacle to a broader definition.

In the following equations, the 2-vs.-1 game is written with indices when the equivalent 1-vs.-1 game has none. The pursuers are identical. The minimum-time problem, where v, the control of the evader, is constrained by $|v| \leq V$ and u_i, the control of pursuer P_i, by $|v_i| \leq U_i$, is analyzed.

The reduced states, the Hamiltonians and the optimal controls are given by

$$\dot{x} = Ax + Bu + Cv,$$

$$\dot{x}_1 = Ax_1 + Bu_1 + Cv, \tag{96}$$

$$\dot{x}_2 = Ax_2 + Bu_2 + Cv,$$

$$H = \lambda^T(Ax + Bu + Cv) + 1,$$

$$H_{1,2} = \lambda_1^T(Ax_1 + Bu_1 + Cv) + \lambda_2^T(Ax_2 + Bu_2 + Cv) + 1 \tag{97}$$

$$v_{1,2}^* = V \cdot \text{sign}((\lambda_1^T + \lambda_2^T)C),$$

$$v^* = V \cdot \text{sign}(\lambda^T C), \tag{98}$$

and the capture condition by

$$x^T(t_f)Mx(t_f) \leq r^2. \tag{99}$$

By definition of the composition, in (99), both lines are identical. Thus, $\lambda^T C$ and $(\lambda_1^T + \lambda_2^T)C$ must be identical switching functions in order for the composition to exist.

A simple possible choice is

$$\lambda^T = k(\lambda_1^T + \lambda_2^T) \ , \tag{100}$$

in which k is a positive constant.

Both the 2-vs.-1 and the 1-vs.-1 game must verify the necessary conditions of optimality when (100) is verified.

The maximum principle, that gives the control policies, is obviously verified, the costate dynamics given by

$$\dot{\lambda}^T = -\lambda^T A \ ,$$

$$\lambda_1^T = -\lambda_1^T A \ , \tag{101}$$

$$\dot{\lambda}_2^T = -\lambda_2^T A \ ,$$

do not conflict with the time derivative of (100) expressed as

$$\dot{\lambda}^T = k(\dot{\lambda}_1^T + \dot{\lambda}_2^T) \ . \tag{102}$$

At $t = t_f$, the transversality conditions

$$\lambda^T(t_f) = \nu x^T M \ ,$$

$$\lambda_1^T(t_f) = \nu_{1,2} z_1 x_1^T M, \tag{103}$$

$$\lambda_2^T(t_f) = \nu_{1,2} z_2 x_2^T M \ ,$$

must be verified; (100) and (103) yield

$$k(\lambda_1^T(t_f) + \lambda_2^T(t_f)) = \nu x^T M = \nu_{1,2}(z_1 x_1^T + z_2 x_2^T) kM \ , \tag{104}$$

or, defining

$$x_e(t) = z_1 x_1(t) + z_2 x_2(t) \ , \tag{105}$$

(104) becomes

$$\nu x^T(t_f)M = \nu_{1,2} x_e^T(t_f)kM \ , \tag{106}$$

where the Lagrange multipliers $\nu_{1,2}$ and ν must verify the terminal time conditions

$$H(t_f) = 0 \ ,$$
$$\tag{107}$$
$$H_{1,2}(t_f) = 0 \ .$$

It can be shown that the choice made in (100) is a valid choice, summarized by

$$x(t_f) = x_e(t_f)/a \ ,$$

$$-1/k = 2\nu_{1,2}(x_e^T M A x_e/a + x_e^T C v - x_e^T M B U sign(x_e^T M B)) \ , \tag{108}$$

$$a = \pm \sqrt{x_e^T(t_f) M x_e(t_f)} \ /r \ .$$

a is chosen so that k is positive, and then, $\nu = ak\nu_{1,2}$.

Note that, in finding $x(t_o)$, the position of the equivalent pursuer at time $t - t_o$, only "a" really matters. Thus, provided that the 2-vs.-1 game be solved first, finding the equivalent pursuer (i.e., composing) is an easy procedure for the linear system (96), (100) is, in particular, a valid choice.

3. AN APPROXIMATION PROCEDURE

The game is solved in two steps, according to Figure 18. The
first step is to compute the 1-vs.-1 equivalent game, namely $x(t_o)$
and z_2 to find t_f, v^* and then, as a second step, the complex 2-vs.-1
game is transformed into a two-dimensional, time-varying, one-sided
control problem with fixed terminal time, much simpler to solve than
the original game.

The procedure is mainly concerned with the so-called direct
composition. The two main phases of the problem are to find $x(t_o)$
and to estimate z_2.
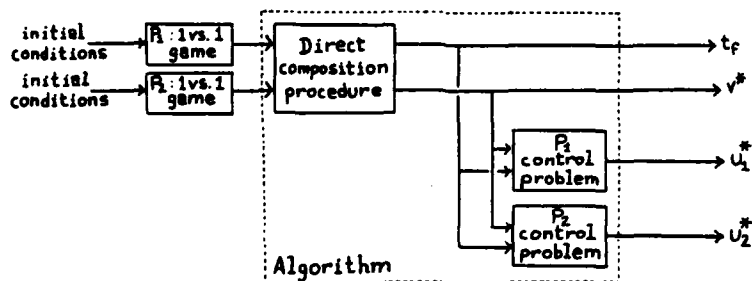
Figure 18. Composition approximating algorithm for 2-vs.-1 game.

The composition is performed using backward integration from a terminal hit point given by

$$x(t_f) = x_e(t_f) k\nu/\nu_{1,2} \ , \tag{109}$$

where x_e is defined in (105). This relation merely expresses the colinearity between x and x_e at $t = t_f$, the constant term $k\nu/\nu_{1,2}$ simply adjusts the modules in order to satisfy the terminal condition (99). If $r = 1$ then it is readily seen that $||x_e(t_f)|| = \nu_{1,2}/(k\nu)$.

Then, integrating backwards in time, the state of the 2-vs.-1 and the equivalent 1-vs.-1 games is given by

$$x(t) = \Phi(t,t_f)x(t_f) + \int_{t_f}^{t}\Phi(t,\tau)Bu(\tau)d\tau + \int_{t_f}^{t}\Phi(t,\tau)Cv(\tau)d\tau \ ,$$

$$x_1(t) = \Phi_1(t,t_f)x_1(t_f) + \int_{t_f}^{t}\Phi_1(t,\tau)Bu(\tau)d\tau +$$

$$\int_{t_f}^{t}\Phi_1(t,\tau)Cv(\tau)d\tau \ , \tag{110}$$

$$x_2(t) = \Phi_2(t,t_f)x_2(t_f) + \int_{t_f}^{t}\Phi_2(t,\tau)Bu(\tau)d\tau +$$

$$\int_{t_f}^{t}\Phi_2(t,\tau)Cv(\tau)d\tau \ ,$$

where Φ, Φ_1, Φ_2 are the corresponding state transition matrices, and A,B,C are time-invariant matrices.

P, P_1 and P_2 have the same homogeneous part in their differential equation (96), therefore

$$\Phi(t,t_f) = \Phi_1(t,t_f) = \Phi_2(t,t_f) = \exp(A(t - t_f)) \ . \tag{111}$$

The optimal controls are

$$u^* = -U \cdot \text{sign}(\lambda^T B) \ ,$$

$$u_1^* = -U_1 \cdot \text{sign}(\lambda_1^T B) \ ,$$

$$u_2^* = -U_2 \cdot \text{sign}(\lambda_2^T B) \ , \tag{112}$$

$$v^* = V \cdot \text{sign}(\lambda_1^T C + \lambda_2^T C) \ ,$$

$$v^* = V \cdot \text{sign}(\lambda^T C) \ .$$

Using

$$\lambda(t_f) = (x_1(t_f) + z_2 x_2(t_f))/a = x_e(t_f)/a \ , \tag{113}$$

with (100) and (110), $U = U_1 = U_2 = 1$, $V = 1$, then

$$x(t) = \exp(A(t-t_f)) x(t_f) + \int_t^{t_f} \exp(A(t-\tau)) B \cdot \text{sign}((\lambda_1^T + \lambda_2^T) B) d\tau$$

$$\tag{114}$$

$$-\int_t^{t_f} \exp(A(t-\tau)) C \cdot \text{sign}((\lambda_1^T + \lambda_2^T) C) d\tau \ ,$$

is derived. (114) can be expressed as

$$x(t) = \exp(A(t-t_f)) x_1(t_f)/a + \int_t^{t_f} \exp(A(t-\tau)) B \cdot \text{sign}(\lambda_1^T B) d\tau/a$$

$$-\int_t^{t_f} \exp(A(t-\tau)) C \cdot \text{sign}((\lambda_1^T + \lambda_2^T) C) d\tau/a$$

$$\tag{115}$$

$$- z_2 \int_t^{t_f} \exp(A(t-\tau)) C \cdot \text{sign}((\lambda_1^T + \lambda_2^T) C) d\tau/a$$

$$+ z_2 \exp(A(t-t_f)) x_2(t_f)/a$$

$$+ z_2 \int_t^{t_f} \exp(A(t-\tau)) B \cdot \text{sign}(\lambda_2^T B)) d\tau/a$$

$$+ \int_t^{t_f} \exp(A(t-\tau)) B(\text{sign}((\lambda_1^T + \lambda_2^T)B) - \text{sign}(\lambda_1^T B)/a$$

$$- z_2 \, \text{sign}(\lambda_2^T B)) d\tau/a - \int_t^{t_f} \exp(A(t-\tau)) C(\text{sign}((\lambda_1^T + \lambda_2^T)C)$$

$$- \text{sign}((\lambda_1^T + \lambda_2^T)C)/a - z_2 \text{sign}((\lambda_1^T + \lambda_2^T)C/a) d\tau \ ,$$

In (115), terms $x_1(t)/a$ and $z_2 x_2(t)/a$ can be recognized. Thus (115) equates

$$x(t) = x_e(t)/a + ER \ ,$$

$$ER = \int_t^{t_f} \exp(A(t-\tau)) [B(\text{sign}((\lambda_1^T + \lambda_2^T)B - \text{sign}(\lambda_1^T B)/a$$

$$- z_2 \, \text{sign}(\lambda_2^T B)/a) - C(1 - (1$$

$$- (1+z_2)/a)\text{sign}((\lambda_1^T + \lambda_2^T)C)] d\tau \quad .$$

(116)

Now, (116) is an important result, from which the procedure is derived. ER, in particular must be carefully studied: similar terms subtract from each other and the result, a switching function, is integrated against the exponential matrix. Thus, a small value of ER is likely, especially when the number of switches is important.

In the sequel, ER will be assumed to be negligible compared to $x_e(t)/a$, thus, an estimate on the upper bound of ER is of interest. Such a bound can easily be obtained by squaring ER and using Schwartz's inequality. Unfortulately, such an inequality has the drawback of

totally suppressing the compensations in ER, resulting in a very bad upper bound.

Since (116) holds at any time, it holds for both $t = t_o$ and $t = t_f$ where ER equals zero, thereby verifying hypothesis (113).

As a conclusion, the approximated method performs the composition from the 2-vs.-1 game, therefore requiring the estimation of z_2 and then, according to (116) the initial position of the equivalent pursuer is $x(t_o) = x_e(t_o)/a$. Then, the 1-vs.-1 game is solved, v^* the optimal control of the evader and t_f the terminal time, are computed. As a last step, the control problem of finding the trajectories and controls of the two pursuers, knowing v^* and t_f is solved.

The estimation of z_2 is the most delicate problem in the method; the information available consists of the 1-vs.-1 games. The estimate of the strategic variable will take full advantage of the whole prior information of the game: the 1-vs.-1 Nash and Pareto (i.e., cooperative) game solutions, the study of the game of kind and, particularly, the computation of the help zones.

Let t_1 and t_2 be the terminal times of the individual 1-vs.-1 (P_1, E) and (P_2, E) games. Let t_{1m} and t_{1M} be two limits such that, because of the very position of P_2, cooperation is possible with P_1 only if $t_{1m} \leq t_1 \leq t_{1M}$. Then, if $t_1 \leq t_{1m}$, $z_2 = 0$ (P_1 is so close to E, that P_2 is useless) and if $t_1 \geq t_{1M}$, $z_2 = +\infty$ (P_1 is so far away from E that P_2 does need any such help).

The simplest formula verifying the above requirements is

$$\hat{z}_2 = k(t_1 - t_{1m})/(t_{1M}-t_1) , \tag{117}$$

where k is a constant.

Using the same approach, if t_{2m} and t_{2M} can be found such that cooperation is possible with P_2 only if $t_{2m} \leq t_2 \leq t_{2M}$, then

$$\hat{z}_2 = k(t_1-t_{1m})(t_{2M}-t_2)/(t_{1M}-t_1)(t_2-t_{2m}), \qquad (118)$$

respects the limit conditions, but the problem is to fix k. When $t_1 = t_2$, both pursuers are likely to be equally dangerous to the evader, therefore $z_2 = 1$ should be enforced. This remark yields the formula that will be used for z_2. I. e.,

$$\hat{z}_{21} = (t_1-t_{1m})(t_{1M}-t_2)(t_{2M}-t_2)(t_1-t_{2m})/(t_{1M}-t_1)(t_2-t_{1m})$$

$$(t_2-t_{2m})(t_{2M}-t_1) \qquad (119)$$

If none of the four limits t_{1m}, t_{2m}, t_{1M} and t_{2M} can be computed, then $t_{1m} = t_{2m} = 0$ and $t_{1M} = t_{2M} = \infty$ is assumed, the estimate becomes

$$\hat{z}_{22} = (t_1/t_2)^2 , \qquad (120)$$

whereas the simplest estimate for z_2 is

$$\hat{z}_{23} = t_1/t_2 . \qquad (121)$$

The four time limits are computed as follows:

i) When $t_1 \leq t_{1M}$, then P_2 does not play any role ($z_2 = 0$). The best case for P_2 is when E is willing to be caught: the corresponding time is the terminal time of the Pareto (P_2,E) game. If t_1 is even smaller than this time, then, for sure, P_2 is useless.

Thus: t_{1m} is the terminal time of the Pareto (P_2, E) game.

t_{2m} is the terminal time of the Pareto (P_1, E) game.

ii) When $t_1 \geq t_{1M}$, then P_1 does not play any role, and z_2 becomes infinite. When $t_1 = t_{1M}$, then P_1 is located on the limit of the help zone of P_2, limiting, by definition, the cooperation zone. Thus a possible bound is to take the maximum of the terminal times of the (P_1, E) games with P_1 located on the help zone. The help zone is usually assymetrical, since, for a given distance $x_1(t_o)$, the cooperation is best when the evader is surrounded by the pursuers. Thus, every 1-vs.-1 trajectory intersects with the boundary of the help zone at a very different time, depending on the position of the intersection. Thus, t_{1M} will be taken as the terminal time on the limit of the help zone, along the trajectory going through $x_1(t_o)$ (an example will be shown). $t_{1M} - t_1$ represents the time separating $x_1(t_o)$ from the help zone. A similar definition is adopted for t_{2M}.

4. TIME OPTIMAL EXAMPLE

The game studied has the reduced state equations

$$\dot{x}_i = Ax_i + Bu_i + Cv , \tag{122}$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} , \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad |u_i| \leq 1, \quad |v| \leq 1 ,$$

and capture is achieved whenever

$$x_i^T(t_f) M x_i(t_f) \leq r_i^2 \qquad (123)$$

with $M = I$, $r_i = 1$.

The minimum-time problem with identical pursuers is considered. Applying the necessary conditions of optimality yield the optimal controls, the costate dynamics and the formula for the Lagrange multiplier as

$$u_i^* = -1 \cdot \text{sign}(\lambda_i^T B) \quad ,$$

$$v^* = 1 \cdot \text{sign}(\sum_i \lambda_i^T C) \quad ,$$

$$\dot{\lambda}_i^T = -\lambda_i^T A \quad , \qquad (124)$$

$$\lambda_i^T(t_f) = 2 \nu z_i x_i^T(t_f) M.$$

$$\nu \geq 0 \rightarrow -\sum_i z_i x_i^T A x_i + \sum_i |z_i x_i^T B| - |\sum_i z_i x_i^T C| = \nu/2 \quad ,$$

$$\nu \leq 0 \rightarrow \sum_i z_i x_i^T A x_i + \sum_i |z_i x_i^T B| - |\sum_i z_i x_i^T C| = \nu/2 \quad .$$

For a fixed $x_1(t_o)$, Figure 19 shows the possible locations of $x_2(t_o)$ according to the amount of cooperation allowed by z_2.

It is remarkable that the trajectory of P_1 is not at all affected by z_2. Since this trajectory is fixed, and differs from the equivalent 1-vs.-1 trajectory, the cooperation of P_2 is required, defining a minimum for z_2 equal to 0.38 in this case.

Figure 20 shows that, when P_1 is chosen very close to the equivalent 1-vs.-1 trajectory, then $P_2(t_o)$ is constrained to a very limited arc, because the requested help is very specific.
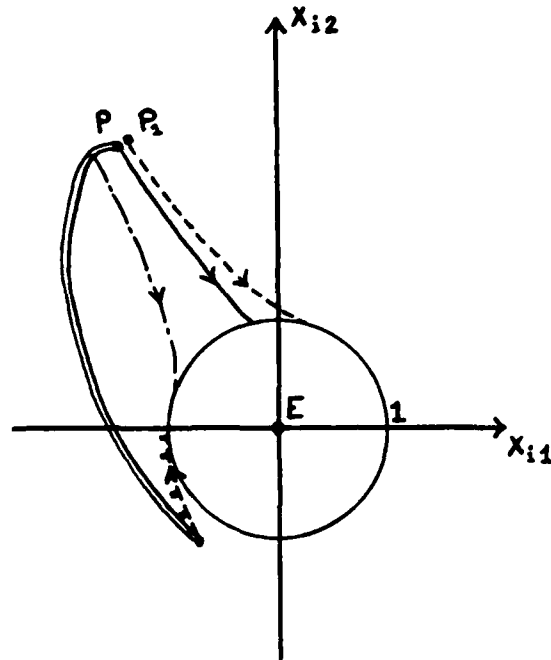
→── 1-vs.-1 game. $x_1(t_f) = -0.27$.

-→-- 2-vs.-1 game. Trajectory of P_1 for $x_{11}(t_f) = 0.1$.

▼▼▼▼ 2-vs.-1 game. Trajectory of P_2 for $x_{11}(t_f) = 0.1$, $z_2 = 0.38$.

-→-- 2-vs.-1 game. Trajectory of P_2 for $x_{11}(t_f) = 0.1$, $z_2 = 1$.

→── 2-vs.-1 game. Trajectory of P_2 for $x_{11}(t_f) = 0.1$, $z_2 = 100$.

═══ Location of $P_2(t_o)$ as a function of z_2, for $t_f = 0.5$.

Figure 19. Time-optimal solution sensitivity to cooperation by z_2.
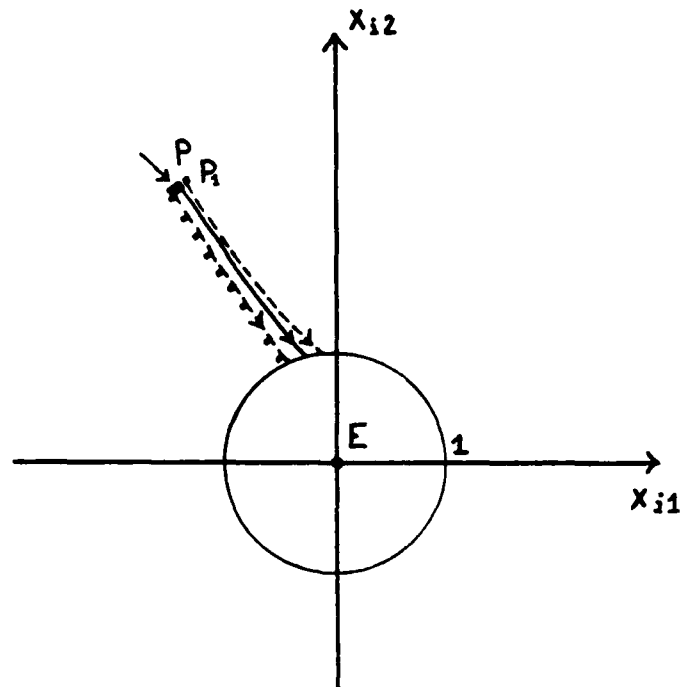
——→— 1-vs.-1 game. $x_1(t_f) = -0.27$.

-→-- 2-vs.-1 game. Trajectory of P_1 for $x_{11}(t_f) = -0.25$.

▼▼▼· 2-vs.-1 game. Trajectory of P_2 for $x_{11}(t_f) = -0.25$, $z_2 = 0.38$.

━━━ Location of $P_2(t_o)$ as a function of z_2 for $t_f = 0.5$.

Figure 20. Sensitivity relative to second pursuer close to 1-vs.-1 game.

The solution to the 1-vs.-1 game is given in Figure 21. The area in which capture can be avoided is very small. This is due to the stable eigenvector of the matrix A added to the advantage in control of the pursuer with respect to the evader, illustrated by the matrices B and C. The lines of constant terminal time tend to be disformed along the unstable eigenvector $x_{i2} = (-3-\sqrt{17})x_{i1}/2$, more favorable to the escape.

- →--   Semi-permeable line.
- ▭▭▭   Usable part of the terminal manifold.
- →—   Trajectory.
- ——   Constant t_f.
-   Switching line.
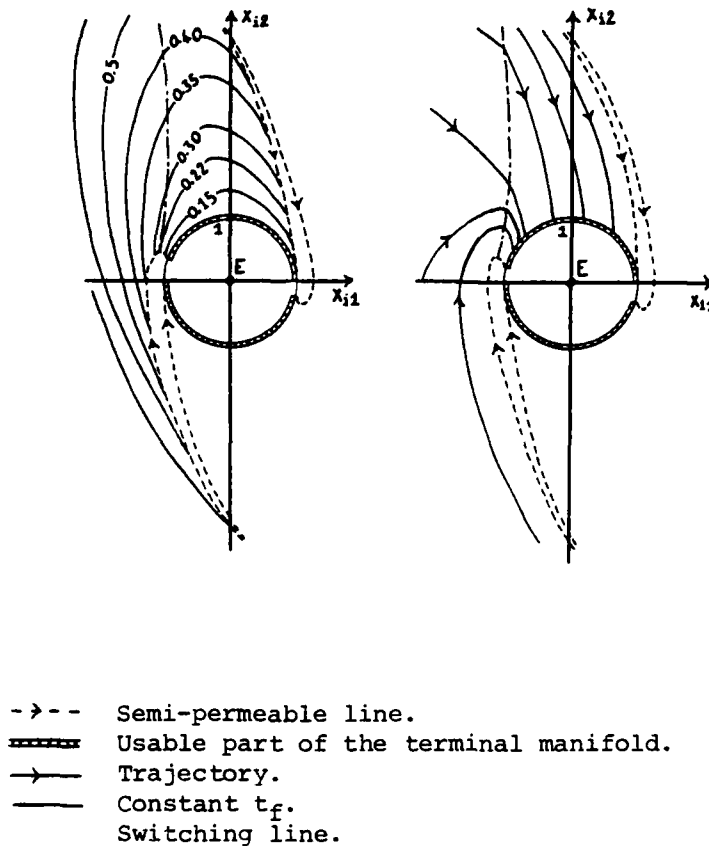
Figure 21. Time-optimal solution for 1-vs.1 game.

The game is symmetrical, the trajectories and the constant
terminal time lines are plotted only over half of the space.

The solution to the cooperative (Pareto) game is given in
Figure 22.
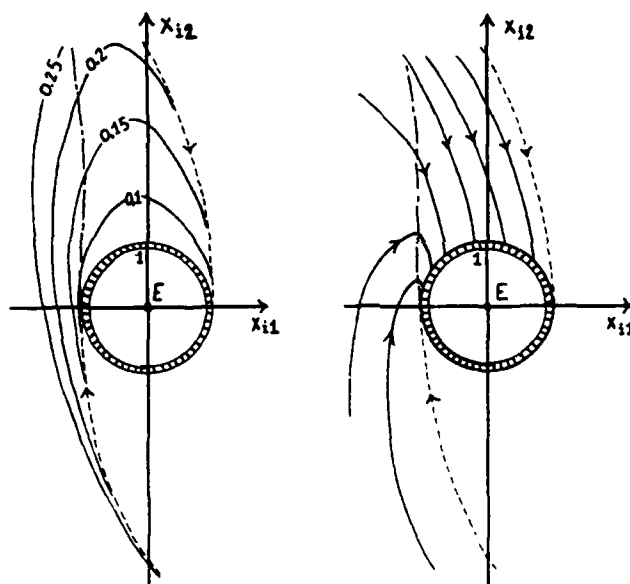
→ - - Semi-permeable line.
▭▭▭▭ Usable part of the terminal manifold.
→── Trajectory.
──── Constant t_f.
Switching line.

Figure 22. Time-optimal solution for 1-vs-1 Pareto game.
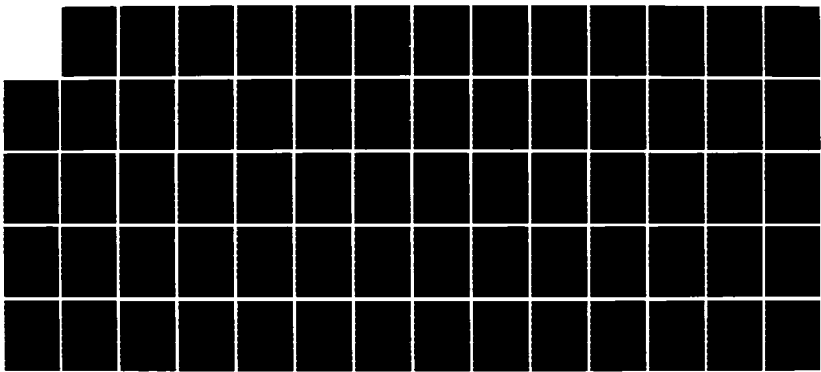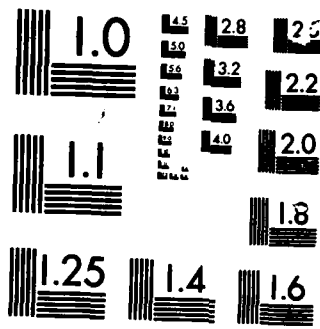
MICROCOPY RESOLUTION TEST CHART

The lines of constant terminal time are wider apart and there is no place where capture is impossible; this is expected since, in the Pareto game, E collaborates with P.

The computations for three cases are summarized in Table 1. Cases 1 and 3 correspond to a low and a high value of z_2, quantifying the cooperation. Case 2 corresponds to the worst possible case, both $z_2 = 1$ and $a = 0$ force a bad result. But the expected failure of the method in that case is not a problem since it can be computed that $v^* = 0$, this corresponds to a case of maximum advantage of the pursuit team, and this case is treated in Chapter VII.

The controls corresponding to the three cases are plotted in Figure 23. In Case 1, the equivalent pursuer has the same control as P_2; in case 3, the same as P_1, when case 2 corresponds to an equal threat from both pursuers. Of course, both v^* and t_f are identical since it is the goal of the procedure to estimate those variables. P_1 and P_2, trying to collaborate as much as possible, adopt opposite controls, so that E, playing a control opposing one of the pursuers, then plays in favor of the second pursuer.

The three different estimates of z_2 are given in Table 3. $\hat{z}_{21}$ gives excellent results in cases 1 and 3, but fails in case 2, where $\hat{z}_{23}$ is superior. $\hat{z}_{22}$ corresponds to $\hat{z}_{21}$ when the various limiting times are not available. The results are centered around $z_2 = 1$; in case of uncertainty, the best estimate is a rather equal cooperation between the pursuers.

Figure 24 shows, as an example, how t_{1M} is computed for case 1.
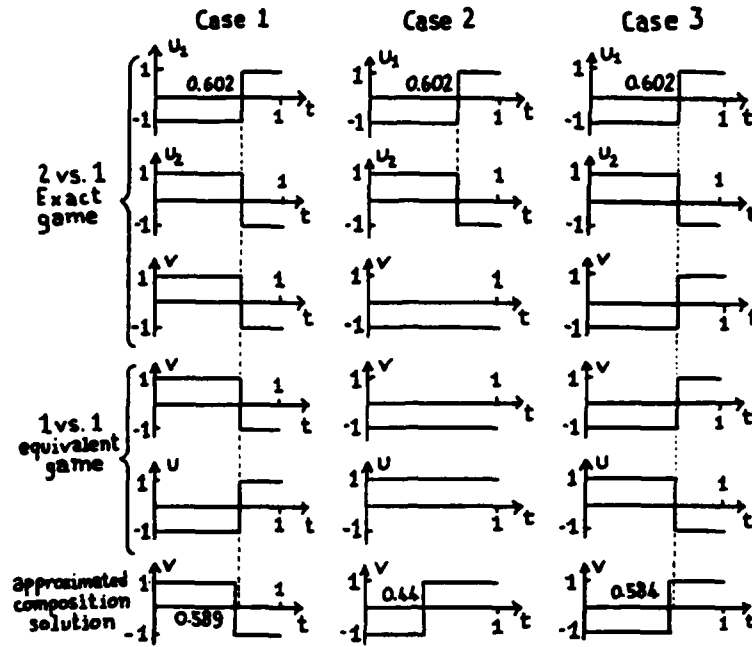
Figure 23. Time-optimal control approximations for Table 1.

——— 1-vs.-1 trajectory.
- ——- Limit of the help zone.
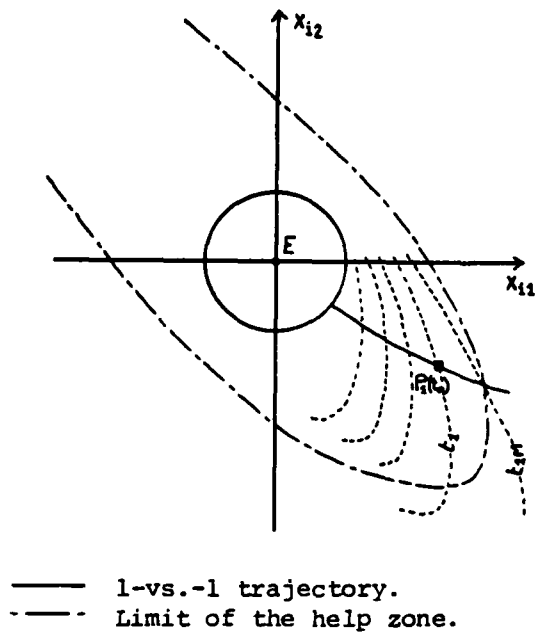
Figure 24. Computation of cooperation-time limit.

Table 3. Time-optimal approximate solution comparisons for three different degrees of cooperation.

		Case 1	Case 2	Case 3
Examples	$P_1(t_o)$	(-5.36,13.0)	(-7.67,19.2)	(-7.79,19.4)
	$P_2(t_o)$	(7.78,-19.4)	(5.47,-12.5)	(5.35,-13.0)
2-vs.-1 exact solution	$P_1(t_f)$	(-0.2,0.98)	(-0.20,0.98)	(-0.20,0.98)
	$P_2(t_f)$	(0.2,-0.98)	(0.20,-0.98)	(0.20,-0.98)
	z_2	0.2	1.	5.
	t_f	1.00	1.00	1.00
Exact composition	$P(t_o)$	(-5.38,13.08)	(4.27,-16.16)	(5.38,-13.08)
	$P(t_f)$	(-0.20,0.98)	(-0.98,-0.20)	(0.20,-0.98)
	t_1	1.0025	1.100	1.1075
	t_2	1.1075	1.0425	1.0025
	t_{1m}	0.860	0.780	0.7525
	t_{2m}	0.7525	0.850	0.860
Estimating z_2	t_{1M}	1.40	1.27	1.32
	t_{2M}	1.32	1.40	1.40
	$\hat{z}_{21}$	0.199	1.85	5.007
	$\hat{z}_{23}$	0.905	1.05	1.105
	$\hat{z}_{22}$	0.811	1.103	1.22
Estimated $\|x_e\|_{t_f}=a$		0.787	0.943	4.03
Exact a		0.8	0.	4.
Approximated composition:	$P(t_o)$	(-4.81,11.61)	(2.618,-4.412)	(4.71,-11.34)
Switch in v:	t_ν	0.589	0.44	0.584
Exact switch:	t_ν	0.602	--	0.602

VII. A CONTROLLABILITY STUDY FOR LINEAR QUADRATIC TEAM GAMES

1. INTRODUCTION

The fixed terminal time aspect of the linear, quadratic team
games is studied in this chapter. A controllability study of the
1-vs.-1 version of the problem was made by Behn and Ho [26]. These
results are adapted to suit the N-vs.-1 team game under study, and
the controllability of the team is used as a measure of the efficiency
of the individual pursuers, providing a sufficient condition for a
pursuer to be useful to a team.

The evader is shown to be neutralized by the maximum control-
lability strategies. In cases where these strategies are game-
optimal, it yields a simple method of solving the difficult problem
of optimal disposition of a team of pursuers by providing a choice of
optimal conditions to simplify the search for the optimal solution.

2. FORMULATION OF THE GAME

The state of pursuer P_i, controlling u_i, is defined by the
linear dynamic system

$$\dot{x}_i = F_i(t)x_i + \overline{G}_i(t)u_i(t) ,$$

$$x_i(t_o) = x_{io} .$$

(125)

The state of the evader, controlling v, takes a similar form:

$$\dot{x}_e = F_e(t)x_e + \overline{G}_e(t)v(t) \quad,$$

$$x_e(t_o) = x_{eo} \quad - \tag{126}$$

where $i = 1, 2, \ldots, N$; x_i, $x_e \epsilon R^n$, $u_i \epsilon R^m$, $v \epsilon R^p$, and $F_i(t)$, $\overline{G}_i(t)$, $F_e(t)$, $\overline{G}_e(t)$ are matrices of appropriate dimensions.

The performance index is given in terms of a miss distance at the fixed terminal time t_f and an energy integral component as

$$J = \frac{1}{2}a^2\sum_i||x_i(t_f) - x_e(t_f)||^2_{A^TA} + \frac{1}{2}\int_{t_o}^{t_f}\sum_i||u_i(t)||^2_{R_i}$$

$$- ||v(t)||^2_{R_e})dt \quad, \tag{127}$$

where A^TA selects the relevant components of the state; R_i, R_e are appropriate positive-definite matrices, and a is a weighting factor such that, if $a^2 \to \infty$, the problem is to capture the evader with minimum energy. Here $a^2 \to \infty$ is used in the sense $\frac{1}{2}a^2\sum_i||x_i(t_f) - x_e(t_f)||^2_{A^TA} = 0$ if $\sum_i||x_i(t_f) - x_e(t_f)||^2_{A^TA} = 0$ and ∞ otherwise. Such a performance index, including a summation of the pursuer's achievements does not favor as much cooperation between teammates as a performance involving a minimum operator on the miss distances would, as discussed in Chapter VIII.

A reduced state vector is defined, for each pursuer, as

$$y_i(t) = A [\Phi_i(t_f,t)x_i - \Phi_e(t_f,t)x_e(t)] \quad,$$

$$i = 1, 2, \ldots, N \quad, \tag{128}$$

where Φ_i and Φ_e are the state transition matrices corresponding to (125) and (126). $y_i(t)$ is the terminal miss $A(x_i(t_f) - x_e(t_f))$ predicted at time t, on the basis that no control will be applied during the interval $[t, t_f]$. Consequently, a new set of state differential equations is defined by

$$\dot{y}_i(t) = G_i(t_f, t) u_i(t) - G_e(t_f, t) v(t) \quad ,$$

$$y_i(t_o) = y_{io} \quad , \quad i = 1, 2, \ldots, N \quad , \tag{129}$$

where G_i and G are time-varying matrices satisfying

$$G_i(t_f, t) = A\Phi_i(t_f, t)\overline{G}_i(t) \quad ,$$

$$G_e(t_f, t) = A\Phi_e(t_f, t)\overline{G}_e(t) \quad , \tag{130}$$

and the p·· ·rmance index is restated as

$$J = \frac{1}{2} a^2 \sum_i ||y_i(t_f)||^2 + \frac{1}{2} \int_{t_o}^{t_f} (\sum_i ||u_i(t)||^2_{R_i}$$

$$- ||v(t)||^2_{R_e}) \, dt \quad . \tag{131}$$

Then, applying the classical calculus of variations, and without lumping the controls of the pursuers into a single control vector, the open-loop optimal controls can be shown to be

$$u_i^*(t) = - a^2 R_i^{-1}(t) G_i^T(t_f, t) y_i(t_f) \quad ,$$

$$v^*(t) = - a^2 R_e^{-1}(t) G_e^T(t_f, t) (\sum_i y_i(t_f)) \quad . \tag{132}$$

The optimal controls are a function of the terminal misses, multiplied by a gain which varies according to a position estimate. The form of the control of the pursuers is irrelevant to the number of the team-mates, enhancing the relative independence of each pursuer; on the other hand, the evader must compound the various threats represented by the pursuers.

To derive the feedback solutions, two approaches are possible. The first one defines the controls, for $a = 1$, as

$$u_i^* = - R_i^{-1} G_i^T K_i^{-1} y_i \ ,$$

(133)

$$v^* = - R_e^{-1} G_e^T (\sum_i K_i^{-1} y_i) \ ,$$

where matrix K_i obeys

$$\dot{K}_i(t_f, t) K_i^{-1}(t_f, t) y_i(t) = -G_i(t_f, t) R_i^{-1}(t) G_i^T(t_f, t) K_i^{-1}(t_f, t) y_i(t)$$

$$+ G_e(t_f, t) R_e^{-1}(t) G_e^T(t_f, t) \sum_j (K_j^{-1}(t_f, t) y_j(t)) \ , \quad (134)$$

$$K_i(t_f, t_f) = I \ .$$

I is the identity matrix. (134) can immediately be simplified for a two-player game, $i = j = 1$, but for the team game under study, sim-plification of (134) requires the introduction of a strategic matrix $Z_i(t)$ according to Chapter V, defined as a symmetric, non-singular matrix which satisfies

$$K_i^{-1}(t_f, t) y_i(t) = Z_i(t) K_1^{-1}(t_f, t) y_1(t) \ . \tag{135}$$

Then the propagation of K_i simplifies into

$$\dot{K}_i(t_f, t) = - G_i(t_f, t) R_i^{-1}(t) G_i^T(t_f, t)$$

$$+ G_e(t_f, t) R_e^{-1}(t) G_e^T(t_f, t) (\sum_j Z_j(t)) Z_i^{-1}(t) \quad . \tag{136}$$

By differentiation of (135), $\dot{Z}_i(t)$ can be shown to be null--a direct consequence of the reduction chosen in (128). Then $Z_i(t)$ is a constant matrix, which might be a bit surprising, but is true only for the game-optimal strategies.

This approach to the solution is used by Behn and Ho [26] for the simpler two-player case; its advantage is to lend itself to nice controllability interpretations. The major drawback is the inversion of K_i required to compute the controls. A better computational form was adopted in Chapter V, where

$$u_i^* = - R_i^{-1} G_i^T T_i y_i \quad ,$$

$$v^* = - R_e^{-1} G_e^T (\sum_i T_i y_i) \tag{137}$$

and T_i propagates according to Riccati equations that are readily simplified, using the same strategic matrix approach, into

$$\dot{T}_i(t) = T_i(t) [G_i(t_f, t) R_i^{-1}(t) G_i^T(t_f, t)$$

$$- G_e(t_f, t) R_e^{-1}(t) G_e^T(t_f, t) (\sum_j Z_j(t)) Z_i^{-1}(t)] T_i(t) \tag{138}$$

$$T_i(t_f) = a^2 I \quad .$$

This form is computationally superior, and a strict derivation of the simpler two-player, linear, quadratic games of this kind can be found in Ichikawa [24].

Using Z_i, the optimal control of the evader can be restated as

$$v^*(t) = -a^2 R_e^{-1}(t) G_e^T(t_f, t) (\sum_i Z_i(t)) K_1(t_f, t) y_1(t) . \qquad (139)$$

The computation of the optimal controls requires that the N matrices K_i be inverted. According to Ho, Bryson and Baron [27], the non-singularity of K_i is equivalent to the non-existence of a conjugate point, in the one-vs.-one case. Existence of K_i^{-1} is studied in terms of the team controllability.

3. CONTROLLABILITY STUDY OF THE TEAM DIFFERENTIAL GAME

From (136), the N matrices K_i are integrated, from the terminal time, as

$$K_i(t_f, t) = I/a^2 + \int_t^{t_f} G_i(t_f, \tau) R_i^{-1}(\tau) G_i^T(t_f, \tau) d\tau$$

$$\qquad (140)$$

$$- \sum_j \int_t^{t_f} G_e(t_f, \tau) R_e^{-1}(\tau) G_e^T(t_f, \tau) Z_j(\tau) Z_i^{-1}(\tau) d\tau .$$

Matrices $Z_i(t)$ being constant, a more compact notation is

$$K_i(t_f, t) = I/a^2 + M_{ri}(t_f, t) - M_e(t_f, t) (\sum_{j \neq i} Z_j) Z_i^{-1} , \qquad (141)$$

where

$$M_{ri}(t_f, t) = M_i(t_f, t) - M_e(t_f, t) \quad,$$

$$M_i(t_f, t) = \int_t^{t_f} G_i(t_f, \tau) R_i^{-1}(\tau) G_i^T(t_f, \tau) d\tau \quad, \tag{142}$$

$$M_e(t_f, t) = \int_t^{t_f} G_e(t_f, \tau) R_e^{-1}(\tau) G_e^T(t_f, \tau) d\tau \quad.$$

From [26], M_i and M_e are the reduced controllability grammians of the pursuer and of the evader, and, for the 1-vs.-1 game, where P_i faces E, M_{ri} is called the relative controllability grammian of pursuer P_i. It expresses the control superiority of pursuer P_i over the evader, in the 1-vs.-1 game.

A sufficient condition to ensure existence of matrices K_i^{-1} is the semi-positive definiteness of the N matrices

$$M_{ri}(t_f, t) - M_e(t_f, t) \, (\sum_{j \neq i} Z_j) Z_i^{-1} \geq 0 \quad, \tag{143}$$

where, for the 1-vs.-1 case, a sufficient condition is

$$M_{ri}(t_f, t) \geq 0 \quad. \tag{144}$$

If all the pursuers are playing a "positive" role in the game, pursuer P_i must be better off as a team member than playing alone. Therefore a sufficient condition for pursuer P_i to be efficient is that

$$- M_e(t_f, t) \, (\sum_{j \neq i} Z_j) Z_i^{-1} \geq 0 \quad, \tag{145}$$

holds. Since M_e is positive definite, another sufficient condition is

$$(\sum_{j \neq i} Z_j) Z_i^{-1} \leq 0 \quad. \tag{146}$$

Both conditions provide criteria to select the relevant pursuers to form a team.

(143) can also be expressed as

$$M_i(t_f, t) - M_e(t_f, t) (\sum_j Z_j) Z_i^{-1} \geq 0 \quad . \tag{147}$$

Then a possible strategy for P_i, to which corresponds a strategic matrix Z_i, is to try and mazimize (147), in order to be as "efficient" as possible. This strategy is named the maximum controllability strategy since it is related to the relative controllability grammians. If all the pursuers play a maximum controllability strategy, then the sum of the terms as (147), i.e.

$$\sum_{i=1}^{N} ((M_i(t_f, t) - M_e(t_f, t) (\sum_j Z_j) Z_i^{-1})) Z_i^2 \quad , \tag{148}$$

is also maximized, since Z_i^2 is a non-singular, positive-definite matrix. (148) can be reorganized as

$$\sum_i M_i(t_f, t) Z_i^2 - M_e(t_f, t) (\sum_j Z_j)(\sum_i Z_i) \quad . \tag{149}$$

(149) is named the team controllability grammian. Due to the symmetry of the strategic matrices, the product

$$(\sum_j Z_j)(\sum_i Z_i) = (\sum_i Z_i)^2 \quad , \tag{150}$$

is positive definite, and so are M_i, M_e and Z_i^2. Then, in order to play a maximum controllability strategy, (150) must be maximized and $(\sum_i Z_i) = 0$ must hold or, for all $i = 1, \ldots, N$, the pursuers must play a strategy such that

$$Z_i = - \sum_{j \neq i} Z_j \quad , \tag{151}$$

From (151) and (139), the optimal control of the evader corresponding to the maximum controllability strategy of the team, is identically null. If the game is to be played, the best strategy of the evader is to do...nothing! The evader is denied any incentive to move and is said to be neutralized by this (possibly non-game optimal) strategy played by the team.

4. MOST CONTROLLABLE DISPOSITION OF PURSUIT TEAM

The most controllable disposition of a pursuit team is such that the maximum controllability strategy is also the game-optimal strategy for the team of pursuers. And, as shown, the evader is neutralized by that disposition. Then the team is said to be optimal.

If the matrices defined in (145) are positive definite, then multiplying each end by the vector $K_i^{-1}(t_f,t)y_i(t)$ and using definition (135), yields the scalar inequality

$$-y_i^T(t)K_i^{-T}(t_f,t)M_e(t_f,t)\sum_{j\neq i}K_j^{-1}(t_f,t)y_j(t) \geq 0 \quad , \qquad (152)$$

and, as $t \to t_f$, $M_e(t_f,t)$ is proportional to

$$C(t_f) = A\overline{G}_e(t_f)R_e^{-1}(t_f)\overline{G}_e^T(t_f)A^T \quad . \qquad (153)$$

Then, since $K_j(t_f,t) = I/a^2$ at $t = t_f$, (153) becomes

$$\alpha y_i^T(t_f)C(t_f)\sum_{j\neq i}y_j(t_f) \leq 0 \quad , \qquad (154)$$

where α is a positive constant.

In order for pursuer P_i to play a maximum controllability strategy, and for a given miss distance, $y_i(t_f)$ must minimize the

vector product (154); it expresses the fact that an efficient team-mate P_i must, at the terminal time, face the vectorial sum of the miss vectors of the remaining pursuers, modified by the constant matrix $C(t_f)$, expressing the evader's interest and capabilities, to cut the evader's main exit direction.

If a pursuer P_i is to be joined to the team (P_j), $i \neq j$, to provide a maximum controllability strategy over the evader, then (154) must be minimized. Or, in order for a team of pursuers to be optimal facing an evader, then the set of equations (154) must be jointly minimized. (154) provides $N - 1$ independent terminal time equations for N pursuers, greatly simplifying the formidable task of finding the optimal controls of this $(N + 1)$ point, boundary-value problem.

One important consequence for two-identical pursuer games is that (151) becomes $Z_1 = -Z_2 = I$, because of the very definition of the strategic matrices (135). Then, for a fixed $x_1(t_o)$, $x_2(t_f)$ is chosen as to minimize (154). The trajectories corresponding to the two-player game (P_1, E) and the trajectory of P_1 in the three-player game (P_1, P_2, E) must match at $t = t_o$. Due to the symmetry of the differential game, to locate "optimally" P_2 as to obtain the most controllable of the pursuit team, the two-pursuer game does not need to be studied at all i.e., the one-pursuer game (P_2, E) equations are merely integrated backwards in time from the terminal state $y_2(t_f)$ given by (154), to find, at $t = t_o$, the position of P_2 which is optimal.

The above procedure applies, in fact, whenever N-1 pursuers are already located, then the study of the (N-1)-pursuer game gives the solution. Then the trajectories and controls of the N-pursuer game are computed from a simple one-sided, control problem, since the game-optimal strategy is a maximum controllability strategy for the optimal team, and, as shown, it implies that the evader is neutralized, i.e., $v^* = 0$ and N-1 independent terminal-time relations minimizing (154) are known to hold.

5. EXAMPLE

A second-order, two-pursuer game is studied, in which both pursuers are identical, with $A = I$, $R_e = R_i = I$, $\overline{G}_i = I$, $\overline{G}_e = \frac{1}{2} I$, $t_f = 0.75$ and

$$F_i = F_e = \begin{matrix} -0.1 & 0.1 \\ 0.1 & -0.2 \end{matrix} . \qquad (155)$$

Figure 25 shows the real trajectories, with $x_e(t_o) = (0, 0)$. The line is traced which corresponds to every possible position $x_2(t_o)$ for a fixed $x_1(t_o)$ and such that the terminal optimal condition (154) is minimized. This line of perfect help passes exactly through the point $-x_1(t_o)'$, as expected. This line is limited because the miss distances of P_1 and P_2 must be positive. Considering $x_2(t_o)$ as a parameter, the line of perfect help can be thought of as separating the area in which P_2 strives to decrease the terminal miss from the area in which t_f is large enough to allow P_2 to primarily think of reducing its energy spending.
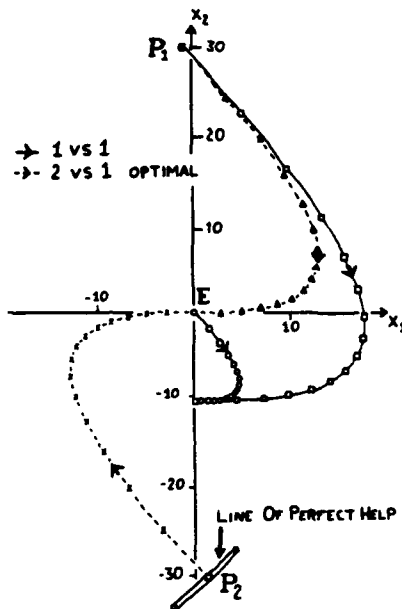
Figure 25. Most controllable two-pursuer team.

A more painstaking study, to compute the lines of constant terminal payoff, is required to use the 1-vs.-1 trajectory to locate optimally pursuer P_2, constrained to a zone at $t = t_o$, in order to minimize the terminal payoff. This task is more feasible concerning minimum time problems.

VIII. STRUCTURAL CHOICES OF A STOCHASTIC DIFFERENTIAL TEAM GAME

1. INTRODUCTION

Stochastic differential games in which both parties make noisy measurements of the state have a closure problem due to the fact that an optimal control should take advantage of the estimate of the error made by the opponent in its own estimate, but this estimate, in turn, is not exact and must be estimated by the former player, etc.

On the other hand, games in which only one party makes perfect measurements do not have such problems. Moreover, the separation principle, dividing the stochastic game into an estimation problem followed by a game analysis, can easily be studied. In the problem studied below, the evader makes perfect measurements.

The team stochastic differential game arises between two sonars (or radar) systems and a target. The target uses a mixed strategy by adding white noise into its controller to hamper the tracking of the sonars. The study, relying on the classical calculus of variations, provides insight into the solutions of difficult problems specific to team games. In particular, various cooperative modes between the two sonar systems are shown to produce different results. The game is studied according to a centralized or a decentralized organization which are equivalent for passive targets that do not use their white noise control capabilities. Conditions are derived under which the Nash-optimal choices of a structure are demonstrated.

To illustrate further team-game problems, a non-optimal, generated, scalar case is presented in which the decentralized structure is proven superior, and where singularities are studied. Due to the open-loop nature of the problem, a Stackelberg (hierarchical) game can be defined in the same context.

2. GAME STATEMENT

Two ships, equipped with sonar (or radar) systems are tracking a target whose dynamics are assumed to have the linear form

$$\dot{x} = F(t)x + G(t)v , \tag{156}$$

where $F(t)$ and $G(t)$ are known matrices of dimension n.n and n.p; x is an n-vector of state variables, and v is a p-control vector.

The action of the ships as a team is not the a posteriori result of a coalition dictated by a common interest but an a priori assumption whereby the overall performance of the team supersedes the individual payoffs.

The target, detected at time t_o, is tracked up to time t_f. In the mean time, it uses its control capabilities to achieve both good accuracy in reaching its destination and a maximum in the tracking error estimate made by the ships.

Ho [28] suggests a control law of the form $u = K(t)x + \zeta$. $K(t)$ is a p.n feedback matrix and ζ a p-vector of Gaussian white noise components with statistical parameters $E(\zeta(t)) = 0$ and $E(\zeta(t)\zeta(\tau)) = T(t)\delta(t-\tau)$, where $T(t)$ is a p.p matrix used by the target as a control variable. Through the control of the statistical

parameters in the matrix T, the target is able to play a mixed strategy to complicate the estimation algorithm of the tracking team, but at the expense of self-inflicted perturbations.

The ships simultaneously, but on an independent basis, record continuous noisy measurements of the state x of the target, in the following form:

$$z_i = H_i(t)x + s_i, \quad i = 1,2. \tag{157}$$

H_i is a known q.n matrix, and s_i is a q-vector containing Gaussian white noise components such that $E(s_i) = o$ and $E(s_i(t)s_i^T(\tau)) = S_i(t)\delta(t-\tau)$ where $S_i(t)$ is given. For convenience, s_1 and s_2 are assumed to be statistically independent, i.e., $S_{12} = E(s_1(t)s_2^T(\tau)) = 0$.

From these two measurements, and knowing the target dynamics (156), the tracking team is able to design a (Kalman) filter by optimizing its gain according to the performance index. If T(t) were known by the team, then ζ could be treated as a mere corrupting noise of given characteristics, to be averaged out by the filter. Unfortunately, the actual characteristics of the corrupting noise ζ are unknown to the pursuers since it is controlled by the target. At this point, two approaches are possible. The brute-force method assumes in (156) a value T_{max} of maximum corrupting-noise covariance, covering the worst possible case, and performs the state estimation on that basis. This method has the obvious advantage of great simplicity but might be overwhelmingly penalizing if the target decides not to use a mixed control at all.

Consequently, the second method, namely the gaming approach, is considered here. The overall problem might be decomposed into two

steps: first, the game is solved by the tracking team which computes the game-optimal controls played by the target. Then, on line, when the target is actually detected, the tracking is performed according to these assumptions. Of course, the actual target might not adopt the controls predicted by the game analysis, but the tracking team is guaranteed a minimum payoff by playing according to the game-optimal controls. The same type of approach is used by the target to predict the optimal filter gains on which to base its optimal control strategy.

A remark of importance is that neither the tracking team, nor the target is able to compute the exact performance. Actually, the error estimate covariance computed on line by the tracking team corresponds to the truth only in the physically improbable event of an optimal play made by both parties. Throughout, it will be assumed that a referee has access to both sides to compute the real performance.

Therefore, the problem as introduced is a game and not a classical estimation problem. In the sequel, the optimality of the filters does not refer to the control actually played by the target but the filters are to be understood as game-optimal filters.

Team game analysis, as opposed to the 1-vs.-1 game problem, such as studied by Speyer [23] is the main objective. In particular, the various communication structures that define cooperation levels reflected by the very choice of the performance index, together with structural choices that supersede control strategies, are examined.

3. COMMUNICATION STRUCTURE

Perfect information is assumed; in particular, the form of the target dynamics and control, the measurement equations, the form of the estimator filters as well as the statistical parameters of the various random variables and the initial states are known by both parties.

Several filtering structures to estimate the state of the target from the measurements are discussed but, throughout, the various filters have the linear form

$$\dot{\overline{x}}_i = (F + GK)\overline{x}_i + L_i(z_i - \overline{z}_i) , \tag{158}$$

where $\overline{z}_i = H_i\overline{x}_i$ and L_i is the filter gain (control) to be optimized.

The linear form of the equations and the Gaussian assumptions allow a restatement of the game in terms of a differential equation governing the propagation of the covariance of the state variables by

$$\dot{X} = (F + GK)X + X(F + GK)^T + GTG^T ,$$

$$\tag{159}$$

$$X = E(x(t)x^T(\tau)) .$$

The target makes perfect measurements of its own state; the separation principle holds, and the stochastic game is reformulated as a deterministic game of perfect information, belonging to the class of problems investigated by Isaacs [3].

Similarly, a linear estimator based on measurements $z_i = H_i(t)x + s_i$ has an error-covariance matrix propagating as

$$\dot{P}_i = (F+GK-L_iH_i)P + (F+GK-L_iH_i)^T P + L_iS_iL_i^T + GTG^T , \quad (160)$$

$$P_i = E((x - \bar{x}_i)(x - \bar{x}_i)^T) .$$

The cooperation levels between the members of the team are related to the density of their communication network. For example, when the two ships are not allowed, or do not possess the ability, to improve their estimate by comparing their results, then the target strives to increase the uncertainty of the best performer according to the performance index

$$J = tr\{0.5 \ AA^TX(t_f) - 0.5 \int_{t_o}^{t_f} \min_{1,2}(P_1(t),P_2(t))dt\} , \quad (161)$$

with AA^T an n.n weighting matrix and tr the trace operator. Though playing independently, it can be shown that teammates cannot solve the game unless they can compare P_1 with P_2 to compute the target control that is used. The operator "min" can be eliminated by defining an equivalent 3-vs.-1 game, somewhat as suggested by Gourishankar and Salama [30], as shown in Section III.2.

On the other hand, a fairly easy generalization of a result due to Speyer [29] can be derived by defining J as

$$J = tr\{0.5 \ AA^TX(t_f) - 0.5 \int_{t_o}^{t_f}\sum_i P_i(t)dt\} . \quad (162)$$

In that case, it can be shown that no communication is required between teammates; therefore, the structure of the game does not possess feedback links. Though in a simpler form, this performance index is ill defined since, from the target's point of view, a large P_1 does not compensate for a small P_2.

When a permanent link is established between the pursuers,
then cooperation can be total. The tracking team comes up with a
unique estimate and associated covariance $P(t)$ according to the per-
formance index, adopted from now on:

$$J = tr\{0.5 \ AA^T X(t_f) - 0.5 \int_{t_o}^{t_f} P(t)dt\} \ . \tag{163}$$

Computing directly the overall optimal tracking strategy will
not be attempted, but rather, structures are defined a priori in
order to compare the classical 1-vs.-1 game with the more peculiar
team games. Two filtering structures are developed in detail in the
next section. In the first one, referred to henceforth as the cen-
tralized structure, the two input measurements are merged together
using a zero-memory filter to produce a unique early estimate; this
is conceptually equivalent to a measurement which is the object of
the study. Therefore, the game becomes a 1-vs.-1 game of the
classical type, such as studied by Speyer [29].

In the second case, named decentralized structure, each team
member optimizes its very own filter gain, producing an estimate
which is, in a later stage, combined with that of its copursuer to
produce the overall team estimate. This 2-vs.-1 game structure is
compared to the previous one.

The estimation counterpart to this game problem, i.e., the
brute-force method proposed earlier, is also interesting. Both
structures can be compared with the overall optimal linear filter
based on the optimization of the gains L_1 and L_2 as

$$\dot{\bar{x}} = (F + GK)\,\bar{x} + L_1(z_1 - \bar{z}_1) + L_2(z_2 - \bar{z}_2) \ , \tag{164}$$

$$\bar{z}_i = H_i \bar{x} \ .$$

On the other hand, the decentralized structure, because of the common information it is based upon, is not to be compared with a decomposition scheme, such as arises when the optimal smoothing solution for linear dynamic systems is derived as the combining of two Kalman filters working on intervals (t_o, t) and (t, t_f), for non overlapping measurement subsets.

4. THE TWO STRUCTURES

4.1 The centralized structure

Figure 26 shows a block-diagram representation of the centralized structure. The two measurements z_1 and z_2 collected from the sonar channels are merged in a zero-memory, linear filter (LF) of the form

$$\bar{z} = z_1 + C(z_2 - z_1) \ . \tag{165}$$

C is to be computed as to minimize the mean square error:

$$S = E((\bar{z}-z)(\bar{z}-z)^T) \ , \tag{166}$$

where z=Hx. For $H_i = H$, S can be computed as

$$S = S_1 + S_{12}C^T - S_1C^T + CS_{12} - CS_1 + C(S_1 + S_2 - 2S_{12})C^T \ , \tag{167}$$

or alternatively as
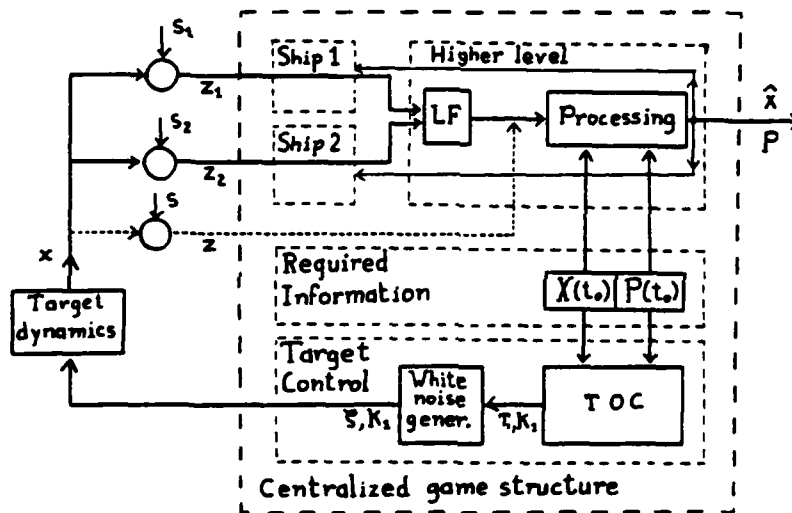
Figure 26. Block diagram of the centralized game structure. TOC is the target optimal controller.

$$S = S_1 - M_1(S_1-S_{12}) + (C-M_1)(S_1+S_2-2S_{12})(C-M_1)^T ,$$

<div align="right">(168)</div>

$$M_i = (S_i-S_{12})(S_1+S_2-2S_{12})^{-1} .$$

Now, since M_1 and $(S_1+S_2-2S_{12})$ are positive definite, (168) is minimized for $C = M_1$, and then, using (157), (165) can be expressed as

$$\bar{z} = Hx + (M_1 s_2 + M_2 s_1) .$$

<div align="right">(169)</div>

An equivalent noise s is defined as

$$s = M_1 s_2 + M_2 s_1 .$$

<div align="right">(170)</div>

It can be verified that $E(s) = 0$ and the covariance of s, i.e., $S = E(s(t)s^T(\tau))$, is

$$S = S_1(S_1 + S_2)^{-1}S_2 ,$$

<div align="right">(171)</div>

$$\text{or} \quad S^{-1} = S_1^{-1} + S_2^{-1} ,$$

since $S_{12} = 0$ is assumed. Thus, the result of the merging of the two measurements is conceptually equivalent to a single measurement vector z, corrupted by the Gaussian white noise s. By using the zero-memory linear filter, the problem is reduced to a simple 1-vs.-1 game. Thus 1-vs.-1 game is then solved using calculus of variations. The performance index and the variational Hamiltonian are defined as

$$J(K_1,T_1) = tr\{X_1(t_f) - 0.5\int_{t_o}^{t_f}P(t)dt\} ,$$

<div align="right">(172)</div>

$$H = \text{tr}\{-0.5\ P + \Lambda_x((F+GK_1)X_1+X_1(F+GK_1)^T + GT_1G^T)$$

(173)

$$+ \Lambda_p((F+GK_1-I\cdot)P+P(F+GK_1-LH)^T + LSL^T + GT_1\ G^T)\}\ .$$

Λ_x and Λ_p are the Lagrange-multiplier matrices associated with the covariances X and P. T_1 is the target variance control and P is the estimator error variance. K is the feedback-control matrix.

The costate variables propagate as

$$\dot{\Lambda}_x = -\Lambda_x(F+GK_1) - (F+GK_1)^T\Lambda_x\ ,$$

$$\Lambda_x(t_f) = I\ ,$$

$$\dot{\Lambda}_p = 0.5\ I - \Lambda_p(F+GK_1) - (F+GK_1)^T\Lambda_p\ ,$$

(174)

$$\Lambda_p(t_f) = 0\ .$$

The necessary conditions of optimality yield the optimal control gain L as

$$\frac{\partial H}{\partial L} = (-2HP + 2SL^T)\ \Lambda_p = 0\ ,$$

(175)

and since it can be checked from (174) that Λ_p is non-singular except at $t = t_f$, the optimal control L for this 1-vs.-1 case is

$$L = PH^TS^{-1}\ .$$

(176)

Substituted back into (160), this yields the familiar Kalman-filter formulation,

$$\dot{P} = (F+GK_1)P + P(F+GK_1)^T\quad PH^TS^{-1}HP + GT_1G^T\ ,$$

(177)

optimized according to the given couple (K_1,T_1).

The target control T_1 takes values in the interval $[0, T_{max}]$, and the feedback matrix K_1 is bounded by the matrices K_{min} and K_{max}. The optimal control law K_1 is computed as

$$\frac{\partial H}{\partial K_{1ij}} = \{(2X\Lambda_x + 2P\Lambda_p)G\}_{ji} \quad \begin{array}{l} > 0 \rightarrow K_{1ij} = K_{ijmin} , \\ \\ < 0 \rightarrow K_{1ij} = K_{ijmax} \end{array} \quad (178)$$

and T_1 is

$$\frac{\partial H}{\partial T_{1ij}} = \{G^T(\Lambda_x + \Lambda_p)G\}_{ji} \quad \begin{array}{l} > 0 \rightarrow T_{1ij} = 0 , \\ \\ < 0 \rightarrow T_{1ij} = T_{ijmax} , \end{array} \quad (179)$$

The above generalizes the results obtained in [29] where, for the scalar case, it is proven that no singular arc can occur in (178). However, singularities play a major role for T_1 in the derivation of the solution of the 1-vs.-1 game. This point is illustrated by an example in [29] which is unfortunately in error.

4.2 The decentralized structure

In the 2-vs.-1 team game depicted in Figure 27, each ship computes its own estimate $\overline{x}_i$ and associated error covariance P_i which are then combined by the same type of zero memory linear filter as in the previous structure. Each ship is given more processing power but the task of the higher hierarchical level is a lot simpler than for the centralized structure.

The processing algorithms P1 and P2 associated with the ships include the sonar channel, a Kalman filter and a model of the target dynamics. Feedback, under the form of the output covariance Q is
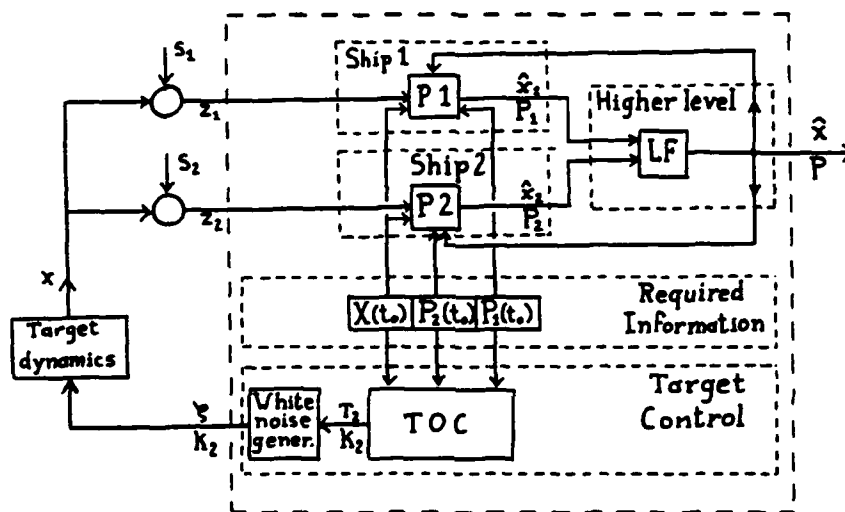
Figure 27. Block diagram of the decentralized game structure.

required by the individual ships to compute their optimal controls. Elaborated information is passed on to the higher level whereas, previously, raw measurements are communicated.

According to section 4.1, the output covariance is

$$Q = P_1 - (P_1 - P_{12}) (P_1 + P_2 - 2P_{12})^{-1} (P_1 - P_{12}) , \qquad (180)$$

or
$$Q = 0.5 (P_1 M_2^T + P_2 M_1^T + P_{12}) , \qquad (181)$$

where
$$M_i = (P_i - P_{12}) (P_1 + P_2 - 2P_{12}) . \qquad (182)$$

P_{12} is the cross covariance defined as $P_{12} = E((x - \overline{x}_1)(x - \overline{x}_2)^T)$ and can be shown to propagate as

$$\dot{P}_{12} = GT_2 G^T , \qquad (183)$$

where

$$P_{12}(t_o) = E(x(t_o) - \overline{x}_1(t_o))(x(t_o) - \overline{x}_2(t_o))^T)$$

$$\qquad (184)$$

$$= E(s_1(t_o) s_2^T(t_o)) = 0 .$$

Thus, it can be seen that, through the use of the mixed strategy T_2, the target is able to control directly the amount of cross covariance or redundancy in the computation performed by the teammates.

Then, the game is solved using the calculus of variations approach. The performance index is

$$I(K_2, T_2) = tr\{X_2(t_f) - 0.5 \int_{t_o}^{t_f} Q(t) dt\} , \qquad (185)$$

where Q is given by (181). The state equations are

$$\dot{X}_2 = (F+GK_2)X_2+X_2(F+GK_2)^T + GT_2G^T ,$$

$$\dot{P}_i = (F+GK_2)P_i+P_i(F+GK_2)^T - L_iH_iP_i - P_iH_i^TL_i^T + L_iS_iL_i^T + GT_2G^T,$$

$$\dot{P}_{12} = GT_2G^T . \tag{186}$$

The Hamiltonian is

$$H = tr\{-0.5\ Q+\Lambda_x((F+GK_2)X_2+X_2(F+GK_2)^T+GT_2G^T)+\Lambda_{p12}GT_2G^T$$

$$+\Lambda_{p1}((F+GK_2)X_2+X_2(F+GK_2)^T-L_1H_1P_1-P_1H_1^TL_1^T+L_1S_1L_1^T+GT_2G^T)$$

$$+\Lambda_{p2}((F+GK_2)X_2+X_2(F+GK_2)^T-L_2H_2P_2-P_2H_2^TL_2^T+L_2S_2L_2^T+GT_2G^T)\} ,$$

$$\tag{187}$$

where the Lagrange matrices Λ_x, Λ_{p1}, Λ_{p2} and Λ_{p12} obey

$$\dot{\Lambda}_x = -\Lambda_x(F+GK_2)-(F+GK_2)^T\Lambda_x;\ \Lambda_x(t_f) = I ,$$

$$\dot{\Lambda}_{p12} = -0.5\ (M_1M_2^T + M_2M_1^T);\ \Lambda_{p12}(t_f) = 0 ,$$

$$\dot{\Lambda}_{p1} = 0.5\ (M_1M_1^T+M_2M_1^T+M_1M_2^T)-\Lambda_{p1}(F+GK_2)-(F+GK_2)^T\Lambda_{p1} \tag{188}$$

$$+ \Lambda_{p1}^TP_1H_1^TL_1^T + L_1H_1P_1\Lambda_{p1}^T;\ \Lambda_{p1}(t_f) = 0 ,$$

$$\dot{\Lambda}_{p2} = 0.5\ (M_2M_2^T+M_2M_1^TM+M_1M_2^T) - \Lambda_{p2}(F+GK_2)-(F+GK_2)^T\Lambda_{p2}$$

$$+\Lambda_{p2}^TP_2H_2^TL_2^T + L_2H_2P_2\Lambda_{p2}^T ;\ \Lambda_{p2}(t_f) = 0 ,$$

and

$$M_i = (P_i-P_{12})(P_1+P_2-2P_{12})^{-1} . \tag{182}$$

The necessary conditions of optimality yield the optimal controls L_1 and L_2 as

$$\frac{\partial H}{\partial L_i} = (-2H_i P_i + 2S_i L_i^T)\Lambda_{pi} = 0 \,, \tag{189}$$

or

$$L_i = P_i H_i^T S_i^{-1} \,. \tag{190}$$

The target controls are given by

$$\frac{\partial H}{\partial K_{2ij}} = \{(2X_2\Lambda_x + 2P_1\Lambda_{p1} + 2P_2\Lambda_{p2})G\}_{ji} \quad \begin{array}{l} > 0 \rightarrow K_{2ij} = K_{ijmin} \\ \\ < 0 \rightarrow K_{2ij} = K_{ijmax} \end{array} \tag{191}$$

and

$$\frac{\partial H}{\partial T_{2ij}} = \{G^T(\Lambda_x + \Lambda_{p1} + \Lambda_{p2} + \Lambda_{p12})G\}_{ji} \quad \begin{array}{l} > 0 \rightarrow T_{2ij} = 0 \,, \\ \\ < 0 \rightarrow T_{2ij} = T_{ijmax} \end{array} \tag{192}$$

4.3 Comparison of the structures

In order to make a decision concerning the choice of structure, $I(K_2,T_2)$ is to be compared with $J(K_1,T_1)$. A possible way involves extensive simulation for the very example under study. Here, an analytical method aimed at deriving sufficient conditions is chosen instead. For all (K,T), the Nash optimal inequalities,

$$I(K_2,T_2) \leq I(K,T) \,,$$

$$\tag{193}$$

$$J(K_1,T_1) \leq J(K,T) \,,$$

hold since the target, controlling K and T, is the minimizing

player; in particular

$$I(K_2,T_2) \leq I(K_1,T_1) \ ,$$

$$\text{(194)}$$

$$J(K_1,T_1) \leq J(K_2,T_2) \ ,$$

and the above problem is solved if $I(K_1,T_1)$ and $J(K_1,T_1)$ or $I(K_2,T_2)$

and $J(K_2,T_2)$ can be compared. The four payoffs of (194) can be

rearranged according to the matrix form in Table 4. If $I(K_1,T_1) \leq$

$J(K_1,T_1)$, then, considering (194), the above matrix game admits a

Nash equalibrium in pure strategies corresponding to the centralized

choice of structure. On discrete games, among others, Luce and

Raiffa [31] can be referred to. Conversely, if $J(K_2,T_2) \leq I(K_2,T_2)$,

then the decentralized choice made by both parties is the Nash

optimal.

Table 4. Payoff matrix.

		Tracking structure:	
		centralized	decentralized
Structure of the (minimizing) target.	centralized	$J(K_1,T_1)$	$I(K_1,T_1)$
	decentralized	$J(K_2,T_2)$	$I(K_2,T_2)$

Consequently, by studying the perfect information game in which

both players are bound to jointly choose either structure, and using

the above approach, the solution of the non-perfect information game

in which neither player is sure of the other's choice of structure

is also solved.

$I(K_1,T_1)$ is the performance achieved when the target plays the strategies K_1, T_1, optimal if the tracking team adopts the centralized structure when, actually, the tracking team adopts the decentralized structure, based on the assumption that the target plays accordingly. Therefore, the tracking filters are not fitted to the real control policy K_1, T_1. Let $I^*(K,T)$ be the payoff of the filter optimized with respect to the very pair (K,T). Then, due to optimality,

$$I(K,T) \leq I^*(K,T) \tag{195}$$

holds for the maximizing tracking team. And, in particular,

$$I(K_1,T_1) \leq I^*(K_1,T_1) , \tag{196}$$

which, together with (194) yields

$$I(K_2,T_2) \leq I^*(K_1,T_1) . \tag{197}$$

(K_2,T_2) is the control pair assumed by the tracking team to optimize I. Therefore,

$$I^*(K_2,T_2) = I(K_2,T_2) \tag{198}$$

and, eventually

$$I^*(K_2,T_2) \leq I^*(K_1,T_1) . \tag{199}$$

Similar demonstration performed on $J(K_2,T_2)$ gives

$$J^*(K_1,T_1) \leq J^*(K_2,T_2) . \tag{200}$$

(199) and (200) form a set equivalent to (194). Therefore,

$$I^*(K_1, T_1) \leq J^*(K_1, T_1) \rightarrow I(K_1, T_1) \leq J(J_1, T_1) \ ,$$

<div align="right">(201)</div>

$$J^*(K_2, T_2) \leq I^*(K_2, T_2) \rightarrow J(K_2, T_2) \leq I(K_2, T_2) \ .$$

Though more restrictive in the conclusions, working with I^* and J^* has the advantage that the performance computed by the tracking team and the real performance do coincide, thereby simplifying the computation task attempted in the next section.

As pointed out previously, neither the target nor the tracking team has access to the real value of $I(K_1, T_1)$ or $J(K_2, T_2)$ during the actual tracking phase; only a referee, having free access to both sides could compute these values. The performances computed by either side differ since they are based upon different assumptions. If a decision had to be taken according to these quantities, it would define a non-zero sum game which translates into a bimatrix game. This approach, a little more involved, though more satisfying for the players, is the one adopted in the example of Section 7.

5. COMPARING $I(K_1, T_1)$ WITH $J(K_1, T_1)$

The target plays according to the centralized structure, corresponding to equations (159), (160), (172) to (179). It thinks it achieves a payoff computed on the basis of $X_1(t_f)$ and $P(t)$.

The tracking team, on the other hand, plays according to the decentralized structure, described by equations (180) to (191), adopting the control policy $L_1 = P_1 H_1^T S_1^{-1}$ and $L_2 = P_2 H_2^T S_2^{-1}$ defined in (190), and assuming an achievement $I(K_2, T_2)$ computed after $X_2(t_f)$ and $Q(t)$ as in (185).

Nevertheless, the true performance achieved is

$$I(K_1, T_1) = \text{tr}\{\overline{X}(t_f) - 0.5 \int_{t_0}^{t_f} \overline{Q}(t)dt\} \ , \tag{202}$$

where the over bars denote the true values, such that

$$\dot{\overline{X}} = (F+GK_1)\overline{X} + \overline{X}(F+GK_1)^T + GT_1G^T \ , \tag{203}$$

$$\dot{\overline{Q}} = 0.5 \ (\overline{P}_1\overline{M}_2^T + \overline{P}_2\overline{M}_1^T + \overline{P}_{12}) \ ,$$

where $\overline{P}_1$ and $\overline{P}_2$ propagate as

$$\dot{\overline{P}}_1 = (F+GK_1)\overline{P}_1 + \overline{P}_1(F+GK_1)^T - P_1H_1^TS_1^{-1}H_1\overline{P}_1 - \overline{P}_1H_1^TS_1^{-1}H_1P_1$$

$$+ P_1H_1^TS_1H_1P_1 + GT_1G^T \ , \tag{204}$$

$$\dot{\overline{P}}_2 = (F+GK_2)\overline{P}_2 + \overline{P}_2(F+GK_2)^T - P_2H_2^TS_2^{-1}H_2\overline{P}_2 - \overline{P}_2H_2^TS_2^{-1}H_2P_2$$

$$+ P_2H_2^TS_2^{-1}H_2P_2 + GT_1G^T \ ,$$

$$\dot{\overline{P}}_{12} = GT_1G^T \ .$$

The initial conditions are identical. Computing $I(K_1, T_1)$ amounts to solving in parallel three differential games, some equations being interdependent as (204) shows. When $I^*(K_1, T_1)$ is studied, the performance computed by the tracking team is the real performance; therefore equations (202) to (204) are irrelevant, resulting in considerable simplifications.

$I(K_1, T_1)$ is to be compared with $J(K_1, T_1)$ which actually corresponds to the performance assumed by the target and expressed by (172). Simulation of the above, for even a two dimensional case, amounts to 68 scalar differential equations, 16 switching functions and 24 parameters from trial and error (the 2-vs.-1 game is a 4-point boundary-value problem). This task must be somewhat duplicated to compute $J(K_2, T_2)$. The complexity is a characteristic of team games.

Since $\bar{X}(t_o) = X(t_o)$, (203) and (159) are identical propagation functions, therefore, $\bar{X}(t_f) = X(t_f)$. In other words, since, in both cases, the target plays the same strategy, it alters its strategy the same way. Thus, the two integral terms in (202) and (172) are to be compared. As a sufficient condition, their differential elements $\bar{Q}(t)$ and $P(t)$ are compared. If $\bar{Q}(t) - P(t)$ is positive definite for all t in the interval considered (i.e., $\bar{Q}(t) \geq P(t)$), then $I(K_1, T_1) \leq J(K_1, T_1)$, and the centralized structure is to be chosen. If $\bar{Q}(t) - P(t)$ is negative definite, then the decentralized structure is best. Since $\bar{Q}(t_o) = P(t_o)$, the study focuses on $\dot{P}$ and $\dot{\bar{Q}}$.

Next, the identities

$$\bar{M}_1 + \bar{M}_2 = \bar{M}_1^T + \bar{M}_2^T = I ,$$

$$(\bar{P}_i - \bar{P}_{12})\dot{\bar{M}}_i^T = -\bar{M}_i(\dot{\bar{P}}_1 + \dot{\bar{P}}_2 - 2\dot{\bar{P}}_{12})\bar{M}_i^T + \bar{M}_i(\dot{\bar{P}}_i - \dot{\bar{P}}_{12}) ,$$

$$(205)$$

are recognized, and $\Delta P_i = P_i - \bar{P}_i$ is introduced as the difference between the covariance computed by the trackers and the real one. Then, after some cumbersone algebra,

$$\dot{\overline{Q}} = (F+GK_1)\overline{Q} + \overline{Q}(F+GK_1)^T - \overline{Q}H^T S^{-1} H \ Q + GT_1 G^T$$

(206)

$$- \ \overline{M}_1 \overline{B}_1 \overline{P}_{12} \overline{M}_2^T - (\overline{M}_1 \overline{B}_1 \overline{P}_{12} \overline{M}_2^T)^T - \overline{M}_2 \overline{B}_2 \overline{P}_{12} \overline{M}_1^T - (\overline{M}_2 \overline{B}_2 \overline{P}_{12} \overline{M}_1^T)^T \ ,$$

where

$$\overline{B}_1 = F + GK_1 - \overline{P}_2 R_2 - 0.5 \ \overline{M}_1^{-1} \overline{M}_2 (\overline{P}_{12} R_2 + \Delta P_1 R_1 \Delta P_1 P_{12}^{-1}) \ ,$$

(207)

$$\overline{B}_2 = F + GK_1 - \overline{P}_1 R_1 - 0.5 \ \overline{M}_2^{-1} \overline{M}_1 (\overline{P}_{12} R_1 + \Delta P_2 R_2 \Delta P_2 P_{12}^{-1}) \ ,$$

and $R_i = H_i^T S_i^{-1} H_i$.

(206) is to be compared with

$$\dot{P} = (F+GK_1)P + P(F+GK_1)^T - PH^T S^{-1} HP + GT_1 G^T \ .$$

(177)

Two remarks are in order at this point. First, the difference between both equations is seen to come from the feedback control and from the quadratic term. Then, when the target decides not to use its mixed strategy, P_{12} remains zero and $\dot{\overline{Q}}$ is identical to $\dot{P}$; thus, both structures perform equally.

Matrices $\overline{M}_i$, $\overline{P}_{12}$ are positive definite, therefore, if $\overline{B}_1$ and $\overline{B}_2$ are negative definite then $\overline{Q}(t) \geq P(t)$ is ensured and the centralized structure chosen.

Except for the term $(F+GK_1)$, both $\overline{B}_1$ and $\overline{B}_2$ are negative definite terms; thus, in order for the centralized structure to be chosen, $F+GK_{max}$ must not be so positive definite as to force the integral of $\overline{Q}$ to be larger than the integral of P. Since it is difficult to estimate bounds on $\overline{B}_i - (F+GK_1)$, unless performing the actual simulation, a sufficient condition to ensure the choice of the centralized structure is $F+GK_{max} \leq 0$ or $GK_{max} \leq - F$. As an

example, if F is negative definite and little positive feedback is used, then the centralized structure is chosen.

When $I^*(K_1,T_1)$ is compared to $J^*(K_1,T_1)$ then $P_i = \bar{P}_i$ and $\Delta P_i = 0$; thus it is more difficult for $\bar{B}_1$ and $\bar{B}_2$ to be negative definite.

6. COMPARING $J(K_2,T_2)$ with $I(K_2,T_2)$

This is the dual case to the one developed previously. The actual covariance is $\bar{P}$, where

$$\dot{\bar{P}} = (F+GK_2)\bar{P} + \bar{P}(F+GK_2)^T - PH^TS^{-1}H\bar{P} - \bar{P}H^TS^{-1}HP$$

$$+ PH^TS^{-1}HP + GT_2G^T . \tag{208}$$

The target assumes a decentralized structure, computing K_2, T_2 and a covariance Q according to equations (180) to (192) when the tracking team computes its control $L = PR$ according to the centralized structure described by equations (159), (160), (172) to (179).

$\dot{Q}$ is compared to $\dot{\bar{P}}$, using a similar derivation; i.e.,

$$\dot{Q} = (F+GK_2)Q + Q(F+GK_2)^T - GH^TS^{-1}HQ + GT_2G^T - M_1B_1P_{12}M_2^T$$

$$\tag{209}$$

$$-(M_1B_1P_{12}M_2)^T - M_2B_2P_{12}M_1^T - (M_2B_2P_{12}M_1^T)^T ,$$

where

$$B_1 = (F+GK_2) - P_2R_2 - 0.5\, M_1^{-1}M_2(P_{12}R_2 - \Delta PR_1\Delta PP_{12}^{-1}) ,$$

$$B_2 = (F+GK_2) - P_1R_1 - 0.5\, M_2^{-1}M_1(P_{12}R_1 - \Delta PR_2\Delta PP_{12}^{-1}) , \tag{210}$$

$$\Delta P = P - \bar{P} .$$

A sufficient condition for the decentralized structure to be chosen is that B_1 and B_2 be positive definite. Confronting (210) and (207), the term involving ΔP plays, in both cases, a favorable role in forcing a definite conclusion, when, if $J^*(K_2,T_2)$ and $I^*(K_2,T_2)$ are computed, then $\Delta P = 0$ and that factor disappears. Again, it is difficult to evaluate (210). The decentralized structure is adopted, for example, when F and GK_{min} are positive definite matrices of fairly large norm.

As a last remark, at $t = t_f$, $T = 0$. Thus, for games of short total duration $t_f - t_o$, no switch in T can occur; then $P_{12} = 0$ and, consequently, both structures are identical. Otherwise, depending on the very game studied, in particular the target capabilities and the initial state, either structure might be chosen.

7. A MODIFIED SCALAR CASE

A simplified scalar, and slightly modified example is developed here to illustrate further some particularities of team games.

To focus on the sole study of the effect of the mixed strategy, the feedback control K is forced to zero; also F = 0, G = 1, $H_1 = H_2 = 1$, and T is constrained to [0,1], where the maximum tolerable self-added noise in the target's dynamics is 1.

Then, the state and observation equations for the centralized case are

$$\dot{x} = \zeta$$

$$z_1 = x + s_1 \ , \tag{211}$$

$$z_2 = x + s_2 \ .$$

The two measurements z_1 and z_2 are combined into the equivalent measurement z:

$$z = x + s, \tag{212}$$

where the equivalent noise s has covariance $S = S_1 S_2 / (S_1 + S_2)$. The covariances associated with x and the error estimate propagate as

$$\dot{X} = T ,$$

$$\dot{P} = -2LP + L^2 S + T . \tag{213}$$

The performance index and Hamiltonian are

$$J(T_1) = X(t_f) - 0.5 \int_{t_o}^{t_f} P(t) \, dt ,$$

$$H = -0.5 \, P + \Lambda_x T_1 + \Lambda_p (-2LP + L^2 S + T_1) , \tag{214}$$

and the necessary conditions of optimality yield the optimal control variable L as

$$\frac{\partial H}{\partial L} = \Lambda_p (-2P + 2SL) = 0, \quad \text{or} \quad L = P/S . \tag{215}$$

Substitution back into (213) yields

$$\dot{P} = -P^2/S + T_1 , \tag{216}$$

when the costate variables propagate as

$$\dot{\Lambda}_x = 0, \quad \Lambda_x(t_f) = 1 ,$$

$$\dot{\Lambda}_p = P\Lambda_p + 0.5, \quad \Lambda_p(t_f) = 0 . \tag{217}$$

The Pontryagin maximum principle is applied to find the optimal control T_1 as

$$\Lambda_x + \Lambda_p < 0 \rightarrow T = 1 ,$$

<div align="right">(218)</div>

$$\Lambda_x + \Lambda_p > 0 \rightarrow T = 0 ,$$

Singularities play an important role whenever bang-bang policies are present for differential games. A reference on the subject is provided by Forhouar and Leondes [32].

A singularity of order n at the switching point arises whenever the generalized Legendre-Clebsch condition is met, i.e.,

$$\frac{\partial^n}{\partial T^n} \left[\frac{\partial^2}{\partial t^2} \left[\frac{\partial H}{\partial T} \right] \right] < 0 .$$

<div align="right">(219)</div>

Furthermore, stationarity along the singular arc implies that

$$\frac{\partial}{\partial t} \left[\frac{\partial H}{\partial T} \right] = 0 ,$$

<div align="right">(220)</div>

and

$$\frac{\partial^2}{\partial t^2} \left[\frac{\partial H}{\partial T} \right] = 0 .$$

<div align="right">(221)</div>

Together with the switching condition, this yields three equations to fix the three unknowns P, Λ_p and T. Then, the equations are solved forward in time and backwards in time on either side of the singular arc, yielding the solution.

The decentralized structure has the performance index

$$J(T_2) = X(t_f) - 0.5 \int_{t_o}^{t_f} P_1(t)P_2(t)/(P_1(t) + P_2(t))dt. \quad (222)$$

It differs somewhat from (185) and (180). This modification of the decentralized structure is assumed by both parties. Such an assumption over a non-optimal structure might be questioned as far as the target is concerned, but the purpose here is to study the team game peculiarities rather than choosing a given tracking structure. A study similar to the previous one yields the equations for the two Kalman filters and target policy as

$$\dot{x} = \zeta \ ,$$

$$z_1 = x + s_1 \ , \qquad\qquad\qquad\qquad (211)$$

$$z_2 = x + s_2 \ ,$$

$$\dot{X} = T_2 \ , \qquad\qquad X(t_o) = 1 \ ,$$

$$\dot{P}_1 = -P_1^2/S_1 + T_2 \ , \quad P_1(t_o) = P_{10} \ , \qquad (223)$$

$$\dot{P}_2 = -P_2^2/S_2 + T_2 \ , \quad P_2(t_o) = P_{20} \ ,$$

$$\dot{\Lambda}_x = 0 \ , \qquad\qquad\qquad\qquad \Lambda_x(t_f) = 1 \ ,$$

$$\Lambda_{P1} = 2P_1\Lambda_{P1}/S + 0.5 \ P_2^2/(P_1 + P_2)^2 \ , \ \Lambda_{p1}(t_f) = 0 \ , \quad (224)$$

$$\Lambda_{P2} = 2P_2\Lambda_{p2}/S + 0.5 \ P_1^2/(P_1 + P_2)^2 \ , \ \Lambda_{p2}(t_f) = 0 \ ,$$

$$\Lambda_x + \Lambda_{P1} + \Lambda_{P2} < 0 \rightarrow T_2 = 1 \ ,$$
$$> 0 \rightarrow T_2 = 0 \ . \qquad\qquad (225)$$

A possibility for a singular arc arises again whenever the switching function is equal to zero. Stationarity along the singular arc implies that the first two time derivatives of the switching function be zero. These three equations cannot fix the five unknowns, namely P_1, P_2, Λ_{P1}, Λ_{P2} and T_2. In the five-dimensional space spanned by the unknowns, a manifold is defined of two dimensions, constrained further by the restrictions on time, $t \in [t_o, t_f]$ and on $T_2 \in [0,1]$. Reaching such a manifold happens under rarely met initial conditions, as the simulation proved. The conclusion is that the singularity does not play a major role in the decentralized structure, unlike the centralized structure.

To compare both structures, the differential equation governing the propagation of the error covariance of the decentralized structure can be computed as

$$\dot{Q} = -Q^2/S + T_2 - 2T_2 P_1 P_2 / (P_1 + P_2)^2 , \qquad (226)$$

and is compared with (213).

If the target is passive, then $T_2 = 0$ and both structures are equivalent. Otherwise, for $T_2 = T_1$, $P(t) \geq Q(t)$ due to the fact that the term $P_1 P_2 / (P_1 + P_2)^2$ is bounded by 0 and 0.25. Thus, $I^*(T) \geq J^*(T)$ for any given policy T, and, as shown previously, it implies that the decentralized structure is to be chosen.

Simulation was performed for noise levels $S_1 = 8$ and $S_2 = 4$. Figures 28 and 29 show the optimal solutions corresponding to two different initial conditions. The resulting control policies differ widely but a higher covariance in the centralized case is compensated
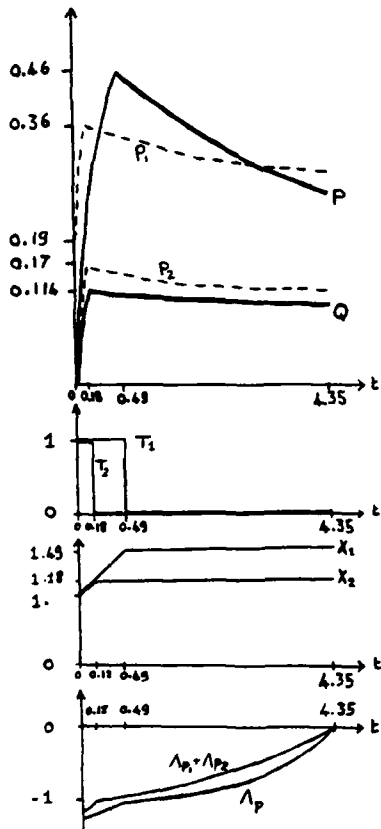
$J(T_1) = 0.731$ (One Switch) $J(T_1) = 0.223$ (Singular arc)
$I(T_2) = 0.843$ (One Switch) $I(T_2) = 0.385$ ($T_2=0$ uniformly)

Figure 28. Optimal trajectories.   Figure 29. Optimal trajectories
            for case 1.   for case 2.

to some extent by an increase in $X(t_f)$. As expected, $I(K_2,T_2) \geq J(K_1,T_1)$ is verified.

More cooperation, resulting in a larger difference in $J(K_1,T_1)$ and $I(K_2,T_2)$ would have happened if both measurements had identical noise levels as previous results by Mohler, Kolodziej and Bugnon [22] show.

Thus far, the target and the ships were assumed to play according to either the centralized scheme or the decentralized scheme. When the choice of the opponent is unknown, both parties must choose a structure and play accordingly. As explained earlier, in that non-perfect information case, the performance indices differ and the game is of the non-zero sum type. It defines a bimatrix game which, for the case of Figure 28, is given by Table 5.

Table 5. Performances for non-perfect information game.

| | | Target strategies | | | | Target strategies | |
		a	b			a	b
Ships	a	0.94	0.96	Ships	a	0.94	1.04
strategies	b	0.84	0.73	strategies	b	0.76	0.73
		Ships performance				Target performance	

A Nash equilibrium in pure strategies exists for this example. It is unique and corresponds to the assumption of a decentralized structure made by both parties.

Actually, the open-loop nature of the structures depicted in Figures 26 and 27 allows an implementation of both strategies in parallel by the tracking team, a simple device could compare the

performance indices obtained from both schemes in order to select the best choice.

Viewed from such a perspective, the above bimatrix game becomes a Stackelberg or leader-follower game in which the target (the leader) announces its strategy first, and the ships (the followers) react accordingly. The Stackelberg equilibrium and the Nash equilibrium coincide for all five examples run in simulation. The conclusion is that both the target and the ships should play according to the decentralized structure, even in the event of non-perfect information.

8. CONCLUSION

A stochastic team differential game is presented, in which the target, by the use of a mixed strategy, has a direct control over the cross correlation between the members of the tracking team. The study is conducted in the general case and two important features of team games are demonstrated; they are as follows:

i) Depending on the kind of cooperation allowed between the team members, various games can be defined. It ranges from independent players up to totally collaborating members. In the latter case, certain choices must be made in the game structure which are not readily apparent in the 1-vs.-1 case.

ii) Hierarchical structures naturally arise with the classical trade off between performance and computational burden.

Two particular game structures, i.e., a centralized and a de-centralized one, are compared, yielding a matrix-game study in a

non-perfect information frame. The choice of structure can be
expressed as a (hierarchical) Stackelberg game due to the open-loop
nature of the problem. It is remarkable that the same problem can
be studied as a zero-sum matrix game, a non-zero sum bimatrix game
and a Stackelberg game.

Complexity and dimensionality are major issues in team games.
For example, to use the same approach as in this chapter, there are
15 ways to combine the measurements of a 4-vs.-1 team game, resulting
in a 15 by 15 matrix game!

Both structures yield equivalent results whenever the target
does not use mixed strategy, or is passive. Otherwise, sufficient
conditions, that depend on the game dynamics, are derived under
which one or the other structure is to be chosen.

The centralized structure might be viewed as a convenient way
to alleviate difficulties by rejecting the game study up to a unique
higher player in the hierarchy. Nevertheless, the advantages of
the decentralized structure are numerous. Among others, the structure
is more practical, by distributing the computational burden among
the players. Thus it can be adapted to several team configurations
or can recover better from individual failures. Also, the example
shows that it is a lot less likely that a singularity will be encount-
ered in a team game.

IX. A NONLINEAR APPROACH TO THE
DIRECTION-FINDING PROBLEM

1. INTRODUCTION

Deterministic team games presented in the first chapters are of
perfect information. That is, the number of players is common know-
ledge, the state of the pursuers and evader is perfectly known by
both parties. In stochastic team games as the one studied in
Chapter VIII, the second requirement is relaxed somewhat, since the
players are measuring the opponent's state corrupted by additive
noise. Nevertheless, at time $t = t_o$, the game is clearly defined in
terms of the number of the players. In a practical, competitive
situation, both the pursuers and the evaders, before the theoretical
beginning of the game and as the game evolves, are searching for in-
formation about the other party. Then the problem is one of detection,
localization and identification. In the underwater case, from a set
of spatially distributed measurements, high resolution techniques
such as the direction-finding techniques allow retrieval of both the
number and the direction of arrival of incident waves. These waves
are unwillingly emitted by the pursuers (machineries, communications,
etc.), or, in a competitive situation, by potential jammers. Obviously
high-power jamming can be used, to hide a given source. But even a
low power jammer, which emits an identical signature away from the
source, is efficient since classical, i.e., linear, direction-finding

methods fail in the case where coherent signals are received, as shown below.

Eigenstructure based methods have been successfully applied to the harmonic retrieval problems (Pisarenko [33]). Likelihood methods are used to derive detection tests for the number of sources, as studied by several authors: Bienvenu and Kopp [34], Rissanen [35], Wax and Kailath [36]. On the other hand, the related problem of estimators for the location of the sources has been studied by Bienvenu and Kopp [34] and Henderson [37].

These direction-finding methods based on the eigenstructure usually rely on the hypothesis that the signals are not coherent, i.e. not fully correlated. Unfortunately, coherent sources occur frequently in practical problems, in case of jamming or in case of multipath propagation. When this hypothesis is no longer valid, the spectral density matrix of the sources becomes singular. This results in an inconsistency in the eigenstructure method: though u sources are detected, the associated direction vectors are not proper. Then, for example, linear processing techniques performed by Henderson [37] propagate the singularity of the source spectral density matrix, and the methods described fail.

Shan, Wax and Kailath [38] propose a method to recover from coherent sources by averaging spectral densities computed from different linear subarrays, taking advantage of the regular spacing of the sensors. This spacial smoothing approach trades off, in fact, half the effective aperture to recover from the possible coherence of the sources.

The nonlinear approach developed below solves the case where all the sources are coherent. It can be applied to any type of array and shows drastic improvements in terms of minimal aperture when the number of coherent sources is high. Multiplicative nonlinear signals are built by convolution operations on the data to provide a sufficient number of Mth-order direction vectors to solve the problems. In general, the minimum number of sensors required to solve a particular problem is theoretically independent of the total number of sources, but rather depends on the number of uncoherent sources. The nonlinear method also provides the coherence coefficients between the sources, unlike the method by Shan, Wax and Kailath [38]. Formulae relating the number of sources to the order of the nonlinear method required are also provided. A brief discussion of the practical limitations of the method concludes the chapter.

2. PROBLEM STATEMENT

n wideband sources s_k are impinging on a linear array from directions θ_k. The linear array consists of d sensors located at distances D_i of sensor 1, such that $0 = D_1 < \ldots < D_d$. Then the signals received at sensor r_i are

$$r_i(t) = \sum_{k=1}^{n} s_k(t - \frac{D_i}{c} \sin\theta_k) + n_i(t) \quad , \qquad (227)$$

where c is the speed of propagation, and the additive noises at the ith sensor, $n_i(t)$ are independent, identically distributed, Gaussian noises. When the noise field is unknown, techniques described by Paulraj and Kailath [39] can be applied.

For a finite observation time, the vectors R, S, N are defined as

$$R^T(\omega) = [R_1(\omega),\ldots,R_d(\omega)] \ ,$$

$$S^T(\omega) = [S_1(\omega),\ldots,S_n(\omega)] \ , \tag{228}$$

$$N^T(\omega) = [N_1(\omega),\ldots,N_d(\omega)] \ ,$$

where $R_i(\omega)$, $S_k(\omega)$ and $N_i(\omega)$ are the Fourier transforms of $r_i(t)$, $s_k(t)$ and $n_i(t)$. Then the Fourier transform of (227) yields

$$R_i(\omega) = \sum_{k=1}^{n} e^{-j\omega\tau_{ik}} s_k(\omega) + N_i(\omega) \ , \tag{229}$$

where $\tau_{ik} = \dfrac{D_i}{c} \sin \theta_k$. More generally,

$$R(\omega) = A_1(\omega) \ S(\omega) + N(\omega) \ , \tag{230}$$

and matrix $A_1(\omega)$ is the (first-order) direction matrix

$$A_1(\omega) = \begin{bmatrix} 1 & \cdots & 1 \\ e^{-j\omega\tau_{11}} & \cdots & e^{-j\omega\tau_{1n}} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ e^{-j\omega\tau_{d1}} & \cdots & e^{-j\omega\tau_{dn}} \end{bmatrix} \tag{231}$$

whose columns are the direction vectors of sources s_k:

$$\vec{d}_k^T(\omega) = [1, \ e^{-j\omega\tau_{1k}},\ldots, \ e^{-j\omega\tau_{nk}}] \ . \tag{232}$$

Multiplying (230) by its conjugate transpose, and taking expectations, for a sufficiently long observation time, the result converges in the mean to the spectral density matrix with the usual

covariance problems associated with the periodogram estimate for finite-length samples. Now this estimate of the (first-order) spectral density $L_1(\omega)$ is given by

$$L_1(\omega) = A_1(\omega) \ E(S(\omega)\overline{S}(\omega)) \ \overline{A}_1(\omega) + E(N(\omega)\overline{N}(\omega)) \ , \qquad (233)$$

where the overbar denotes the conjugate transpose. Or, dropping the arguments for short,

$$L_1 = A_1 P_1 \overline{A}_1 + \sigma^2 I \ , \qquad (234)$$

with

$$P_1 = E(S(\omega)\overline{S}(\omega)) \qquad (235)$$

and σ^2 is the Gaussian noise spectral density coefficient.

Equation (234) has the desired structure to apply the eigenstructure method. If P_1 has rank n, i.e., if no two sources are coherent, the d by d spectral density matrix L_1 has n eigenvalues larger than σ^2 and $d - n$ identical eigenvalues equal to σ^2. The associated eigenvectors are orthogonal to the columns of the direction matrix A_1, i.e., to the direction vectors.

If two sources are coherent, e.g., $s_2 = \alpha s_1$, then P_1 has rank $n - 1$ and the $d - n + 1$ eigenvectors are not orthogonal to the direction vectors as defined in (232) but are orthogonal to the d by (n - 1) direction matrix

$$A_1 = \begin{bmatrix} 1 + \alpha & 1 & \dots & 1 \\ e^{-j\omega\tau_{11}} + \alpha e^{-j\omega\tau_{12}} & e^{-j\omega\tau_{13}} & \dots & e^{-j\omega\tau_{1n}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ e^{-j\omega\tau_{d1}} + \alpha e^{-j\omega\tau_{d2}} & e^{-j\omega\tau_{d3}} & \dots & e^{-j\omega\tau_{dn}} \end{bmatrix} \qquad (236)$$

with a compound direction vector in the first column, as a function of three unknowns, namely θ_1, θ_2 and α. A_1 has one less column compared to (231).

If there are only $d = n + 1$ sensors, then there are only two eigenvalues equal to σ^2 with eigenvectors that are orthogonal to the compound direction vector. This provides two equations that cannot fix the three unknowns and the problem cannot be solved using this method, as pointed out in [38].

3. THE EIGENSTRUCTURE GENERATING FUNCTION

Mth-order signals, from $r_{i,1}(t) = r_i(t)$ down to $r_{i,M}(t)$ are constructed from the definition

$$r_{i,M}(t) = r_{i,M-1}(t) \overset{*}{} r_i(t) , \tag{237}$$

where * denotes the convolution operator. Then, the corresponding Mth-order vector R_M is

$$R_M^T = [R_1^M, R_2^M, \ldots, R_d^M] , \tag{238}$$

and it follows that the Mth-order spectral density matrix is defined as

$$L_M(\omega) = E[R_M(\omega) \overline{R}_M(\omega)] , \tag{239}$$

or

$$L_M = E \begin{bmatrix} (R_1\overline{R}_1)^M & (R_1\overline{R}_2)^M & \cdots & \cdot \\ (R_2\overline{R}_1)^M & (R_2\overline{R}_2)^M & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{bmatrix} \tag{240}$$

Introducing a new matrix operator Δ, R_M is defined as the integer Mth power of R,

$$R_M(\omega) = R(\omega)^{\Delta M} , \tag{241}$$

in terms of Δ products. E.G. when $M = 2$, each matrix element is simply squared.

Δ performs a component to component mulitplication, i.e.

$$A \ \Delta \ B = \ \leftrightarrow c_{ij} = a_{ij} \cdot b_{ij} , \tag{242}$$

where it is implicit that matrices A, B and, of course, C are of identical dimensions.

Applied to p by q matrices, useful identities are

$$A \ \Delta \ B = B \ \Delta \ A ,$$

$$A \ \Delta \ (B \ \Delta \ C) = (A \ \Delta \ B) \ \Delta \ C ,$$

$$\overline{A \ \Delta \ B} = \overline{A} \ \Delta \ \overline{B} , \tag{243}$$

$$(A + B) \ \Delta \ C = A \ \Delta \ C + B \ \Delta \ C ,$$

$$I \ \Delta \ I = I ,$$

and for n by 1 vectors,

$$(A \ \Delta \ B) \ \overline{C \ \Delta \ D} = (A\overline{C}) \ \Delta \ (B\overline{D}) = (A\overline{D}) \ \Delta \ (B\overline{C}) , \tag{244}$$

$$(AB)^{\Delta M} = A^{\Delta M} \overline{B}^{\Delta M} ,$$

and, by definition, it is assumed that

$$(AD) \ \Delta \ (BC) \ = \ AD \ \Delta \ BC \ . \tag{245}$$

Applied to complex scalars, Δ is the regular scalar multiplication, i.e.

$$t^{\Delta M} = t^M \ . \tag{246}$$

The Δ-exponential of matrix A takes a form analogous to the classical matrix exponential:

$$e\Delta(A) \ = \ \sum_{M=0}^{\infty} A^{\Delta M}/M! \ , \tag{247}$$

where the regular matrix product has been replaced by the Δ-product.

Let an Mth-order spectral-density generating function be given by

$$\Phi_{\Delta R\overline{R}}(t) \ = \ E[e\Delta(R\overline{R}t)] \ . \tag{248}$$

Then, using definition (248) and properties (244), (248) yields

$$\Phi_{\Delta R\overline{R}}(t) \ = \ \sum_{M=0}^{\infty} E((R\overline{R}^{\Delta M})t^M/M! \ , \tag{249}$$

where

$$L_M \ = \ E((R\overline{R})^{\Delta M}) \tag{250}$$

is immediately identified. Thus

$$\Phi_{\Delta R\overline{R}}(t) \ = \ \sum_{M=0}^{\infty} L_M t^M/M! \ , \tag{251}$$

and hence the name of Mth-order spectral-density generating function.

As an example, L_2 is computed as

$$L_2 = E((R\overline{R})^{\Delta 2}) = E((A_1 S\overline{S}\overline{A}_1 + A_1 S\overline{N} + N\overline{S}\overline{A}_1 + N\overline{N})^{\Delta 2}) \quad , \quad (252)$$

Then, using the independence of the noises with respect to the sources, for Gaussian noises, odd powers of N and $\overline{N}$ are averaged out. Moreover, terms as $E(N \, \Delta \, N)$ and $E(\overline{N} \, \Delta \, \overline{N})$ go to zero since

$$E(N_i \Delta N_i) = \mathcal{F}(E(n_i(t) * n_i(t))) \quad , \quad (253)$$

where $\mathcal{F}$ is the Fourier transform operator. $E(n_i(t) * n_i(t))$ is the correlation estimate between $n_i(t)$ and $n_i(-t)$ and is expected to be zero since, for any time shift τ, at most one point is correlated. Thus L_2 is equal to

$$L_2 = B_2 + 4 B_1 \Delta\sigma^2 I + E((N\overline{N})^{\Delta 2}) \quad , \quad (254)$$

where

$$B_M = E((A_1 S\overline{S}\overline{A}_1)^{\Delta M}) \quad , \quad (255)$$

and, for Gaussian noises,

$$E((N\overline{N})^{\Delta M}) = \frac{(2M)!}{M! 2^M} \sigma^{2M} I \quad . \quad (256)$$

More generally, it can be shown that

$$L_M = \sum_{p=0}^{M} B_p \binom{p}{M}^2 \Delta E((N\overline{N})^{\Delta (M-p)}) \quad , \quad (257)$$

with $B = I$, and $\binom{p}{M}$ is the binomial coefficient, or

$$L_M = B_M + \sum_{p=0}^{M-1} B_p \Delta \frac{M!^2 (2(M-p))!}{(M-p)!^3 p!^2 2^{M-p}} \sigma^{2(M-p)} I \quad . \quad (258)$$

Let K_M be the Mth-order eigenstructure matrix defined as

$$K_M = B_M + \sigma^{2M} I \quad . \quad (259)$$

Then K_M can be computed from L_M by the recurrent relation

$$K_1 = L_1$$

$$K_M = L_M - \sum_{p=1}^{M-1} K_p \sigma^{2(M-p)} \frac{M!^2 (2(M-p))!}{(M-p)!^3 p!^2 2^{M-p}} \, \Delta \, I +$$

<div align="right">(260)</div>

$$\sum_{p=1}^{M-1} \sigma^{2(M-p)} \frac{M!^2 (2(M-p))!}{(M-p)!^3 p!^2 2^{M-p}} \, I - \sigma^{2M} \frac{(2M)!}{M! 2^M} I + \sigma^{2M} I \, .$$

K_M and L_M differ only by their diagonal components σ^2 is estimated as the smallest eigenvalue of L_1.

Then, the eigenstructure generating function is defined as

$$\Phi_{\square R\bar{R}}(t) = \sum_{M=0}^{\infty} K_M t^M / M = E(e \,\square(R\bar{R}t)) \, ,$$

<div align="right">(261)</div>

or

$$K_M = E((R\bar{R})^{\square M}) \, ,$$

<div align="right">(262)</div>

where the $\square$-operator is defined a posteriori from relations (260) and (262).

When all the sources are coherent, A_1 is composed of a unique compound vector and

$$B_M = E((A_1 S\bar{S}\bar{A}_1)^{\Delta M}) = A_M P_M \bar{A}_M \, ,$$

<div align="right">(263)</div>

where

$$A_M = A_1^{\Delta M}$$

<div align="right">(264)</div>

is the Mth-order compound vector and

$$P_M = P_1^{\Delta M} \tag{265}$$

is now a scalar (dimension 1).

Then the d by d Mth-order eigenstructure matrices K have the prescribed structure

$$K_M = A_M P_M \overline{A}_M + \sigma^{2M} I . \tag{266}$$

Thus 1 of the d eigenvalues is larger than σ^{2M}, the remaining d − 1 eigenvalues are equal to σ^{2M} and the associated eigenvectors are orthogonal to the columns of matrices A_M, i.e. to the Mth-order direction vectors, thereby yielding, for each other M = 1, 2,... another set of equations which are functions of the unknown parameters.

Moreover, the Mth-order direction vector has for components the components of the corresponding first-order direction vector raised to power M. Thus they are, in general, independent vectors and functions of the very same parameters.

The Mth-order eigenstructure method is applied to matrices K_1, K_2,...,K_M. Consequently there are no theoretical limitations to M, nor to the number of equations provided by the method. The only constraint is that there must be at least one eigenvalue equal to σ^{2M}, and thus d − n $\geq$ 1 is enforced, i.e. d $\geq$ 2 where n = 1 is now the order of the source spectral-density matrix, which is the number of non-coherent sources, 1 in the case where all sources are coherent.

4. MINIMUM SENSOR CONFIGURATIONS

The incident source signals impinging on the array are classified

by $(n:u,c,k)$ where

n is the total number of sources,

u is the number of non-coherent sources,

c is the number of coherent sources,

k is the maximum number of coherent sources present in uncorrelated groups of sources, and k is smaller than n.

Then it can be shown that

$$n = u + c ,$$

$$k + 1 \in [\lfloor(n/u)\rfloor, c + 1] \quad \text{for } u > 1 , \tag{267}$$

$$k + 1 = c + 1 \quad \text{for } u = 1 ,$$

and $\lfloor$ is such that $\lfloor(n/u)\rfloor$ is the greatest integer smaller or equal to n/u.

For an array of d sensors, receiving signals from n directions, the classical, i.e. first-order, method is applied first. If coherent sources are present, the u non-coherent sources are detected and only $q \leq u$ orthogonal proper direction vectors are computed. Then there are $n - u = c$ coherent sources that are a function of the $u - q$ non-coherent ones. It must be emphasized that, at this point, c is to be guessed by the experimenter. Nevertheless, a large value for c solves all problems where c is smaller but at the expense of more computation. A trial and error procedure, choosing first $c = 1$ and increasing to $c = 2$ if no match is found is also possible.

The nonlinear method if applied when all the sources are coherent. In that case $u = 1$ and $k = n - 1$. The sources are

$$s_i(t) = \alpha_i s_1(t) , \tag{268}$$

$i = 1, 2, \ldots, n,$ α_i are complex constants, $\alpha_1 = 1$. The compound direction vector is

$$A_1 = \vec{d}_1 = \sum_{i=1}^{n} \alpha_i \vec{d}_i^* \ , \tag{269}$$

where $\vec{d}_i^*$ are the uncompounded direction vectors as defined in equation (231). $\vec{d}_1$ is a function of $2n - 1$ unknowns: $n - 1$ coherence co-efficients α_i and n direction parameters θ_i.

In the minimum configuration case, there is only one eigenvalue equal to σ^{2M} and thus, each order produces one more equation from the orthogonality of the corresponding eigenvector with the compound eigenvector $\vec{d}_1^{\Delta M}$.

Consequently, the minimum configuration to solve the problem is

$$d = u + 1 = 2$$
$$M = 2n - 1 \ . \tag{270}$$

When $d = n + 1$, then there are at least n minimum eigenvalues equal to σ^{2M}, as a consequence, n new equations are produced from each order. Thus the second-order nonlinear method produces $2n$ equations that solve the problem in its entirety.

More generally if d_1 is the minimum number of sensors required by the method described in [38], d_2 by the second-order nonlinear method and d_M by the Mth-order nonlinear method, then

$$d_1 = n + k + 1 = 2n \ ,$$
$$d_2 = n + 1 \ , \tag{271}$$
$$d_M = u + 1 = 2 \ .$$

Figure 30 compares minimum sensor configurations that allow to solve the problem for these three methods. Gains in terms of sensors are substantial for high number of coherent sources. The required value of M in the Mth-order method is indicated in parenthesis.
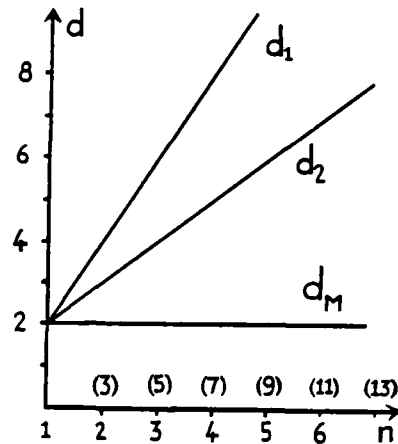

Figure 30. Minimum sensor configuration for n coherent sources.

5. CONCLUSION

The Mth-order nonlinear method yields, theoretically, the receiving angles and the coherence coefficients in the case of coherent sources. For a constant number of sensors, an increased number of coherent sources is usually followed by an increase in the order of the non-linear method requested to solve the problem.

Practically, for a finite observation time T, the Mth-order signals have lengths M.T, but even though they are longer, they do not converge any better than the first-order signals to the Mth-order spectral-density matrices. Actually, for a short observation

time, the repeated convolutions have the drawback of enhancing the irregularities in the noise. This clearly sets a limitation on the practical order of the method, for a given observation time, added to the computational requirements to derive the Mth-order matrices.

Nevertheless, it must be underlined that, for each order, the smallest eigenvalue is σ^{2M}; which provides a convenient way of checking degree of convergence for the corresponding estimated Mth-order matrix compared to the first order.

The most serious limitation of the method is that, so far, it can only be applied to the particular case where all the sources are coherent.

X.   CONCLUSIONS

Team differential games feature several particular problems. Classical N-player games are, generally, N point boundary value problems but an (N-1)-pursuer-1 evader team differential game includes N-2 extra unknowns due to the fact that optimality applies to the team as a unit and not to individuals. For minimum-time problems, these unknowns are modeled as strategic variables whose values quantify the activity of the pursuers. Singularities that play a major role in the solution of a two-player differential game do not appear nearly as important for team games. On the other hand, the study of the game of kind is a lot more complex and crucial.

Early choices on the kind of cooperation allowed between team-mates are reflected in the form of the performance index, simpler choices yield formulae easily generalized at the expense of a reduced interaction level in the team, hampering the result. More powerful team structures can be treated classically only by increasing the dimensions of the game. Through a suitable definition of the reduced coordinates, convenient studies of team games can be conducted but depend on the type of analysis conducted for the very type of game.

Due to its relative simplicity, the one-versus-one game can often be studied quite extensively. In most instances, the addition of an extra pursuer to a team is not directly reflected on the form of the controls of the pursuers as it is on the control of the evader.

Moreover, useful time and performance limitations to the team game can be derived from the 1-versus-1 game. Then, it allows a study of the team game from geometrical analogies or the computation of approximate solutions which are tailored to a given game as the composition approximation or the simple suboptimal solution to the linear quadratic team game. The same approach can even yield the exact solution to the otherwise untractable problem of optimal location of a pursuer in a team. Maximum team controllability criteria though generally non-game optimal, still provide with handy relationships to approximate terminal time state unknowns.

Because of the complexity of team differential games, structures and hierarchies arise naturally. Analysis or computational burden is the reason for introducing structures that break the solution into easier steps, as in the games involving a minimum operator in the performance index that show two distinct phases or as the composition approximation. When the hierarchy that corresponds to the complete solution is prohibitively complex, a careful sensitivity study can yield suboptimal hierarchies to reduce both the information structure and the computations required.

The stochastic team differential game investigated shows that hierarchical choices must be made beforehand that reflect a team philosophy or an early strategic option. Decentralized structures are probably more adequately adapted to team games but optimality is usually not achieved.

The nonlinear approach to the direction finding problem addresses the problems of jamming and multipath propagation. Although

computationally costly, it shows great improvements when the coherence level of the received signals is high. For a given number of sensors, it enables the solution of a wider class of problems, but the observation time sets a limitation on the order of the method, thereby limiting the possibilities.

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