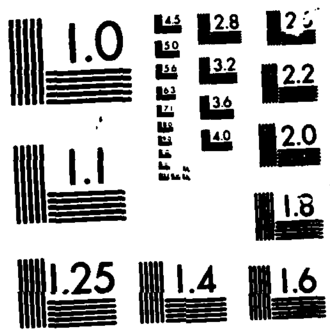
AD-A165 841 CONFIDENCE REGIONS FOR VARIANCE COMPONENTS IN UNBALANCED MIXED LINEAR MODELS(U) IOWA STATE UNIV AMES STATISTICAL LAB A P FENECH ET AL. JAN 86 1/1

UNCLASSIFIED PREPRINT-SER-86-12 N00014-85-K-0418 F/G 12/1 NL




MICROCOPY RESOLUTION TEST CHART

# CONFIDENCE REGIONS FOR VARIANCE COMPONENTS IN UNBALANCED MIXED LINEAR MODELS

by

Alan P. Fenech and David A. Harville[(1)]
University of California-Davis and Iowa State University

Preprint Number 86-12

# Statistical Laboratory Preprint Series



IOWA STATE UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCIENCE with PRACTICE



Ames, Iowa

5

(A)

# CONFIDENCE REGIONS FOR VARIANCE COMPONENTS

# IN UNBALANCED MIXED LINEAR MODELS

by

Alan P. Fenech and David A. Harville[1]
University of California-Davis and Iowa State University

Preprint Number 86-12

January 1986

Department of Statistics
Iowa State University
Ames, Iowa 50011

(1) Research supported in part by the Office of Naval Research, Contract N0014-85-K-0418.

American Mathematical Society 1980 subject classifications.
Primary 62J10; Secondary 62F25.

# CONFIDENCE REGIONS FOR VARIANCE COMPONENTS IN UNBALANCED MIXED LINEAR MODELS

by

Alan P. Fenech and David A. Harville
University of California-Davis and Iowa State University

## ABSTRACT

~~We~~ This document presents a general procedure for obtaining exact confidence regions for the variance components in unbalanced mixed linear models. The procedure utilizes, as pivotal quantities, quadratic forms that may depend on the variance components in a complicated way and that are distributed independently as chi-square variates. In the special case of balanced classificatory models, the pivotal quantities simplify to scalar multiples of sums of squares from the usual analysis of variance. The procedure can be easily modified so as to obtain an exact confidence region for ratios of variance components and can be regarded as a generalization of Wald's procedure for obtaining a confidence interval for a single variance ratio.

# 1. INTRODUCTION

Suppose that y is an n x 1 observable vector that follows the general mixed linear model

$$y = X_0\beta_0 + X_1b_1 + \dots + X_kb_k + b_{k+1} \tag{1.1}$$

where $X_i$ is an n x $m_i$ known matrix (i = 0,...,k), and $\beta_0$ is an $m_0$ x 1 vector of unknown parameters, $b_i$ is an $m_i$ x 1 unobservable random vector whose distribution is $N(0, \sigma_i^2 I)$, that is, multivariate normal with mean vector 0 and variance-covariance matrix $\sigma_i^2 I$ (i=1,...,k+1), and $\sigma_1^2$, ...,$\sigma_{k+1}^2$ are unknown parameters. Assume that $b_1,\dots,b_{k+1}$ are independently and that $\sigma_i^2 > 0$ (i=1,...,k) and $\sigma_{k+1}^2 > 0$. Define $\sigma^2 = (\sigma_1^2, \dots, \sigma_k^2, \sigma_{k+1}^2)^T$, $\gamma_i = \sigma_i^2/\sigma_{k+1}^2$ (i = 1,...,k) and $\gamma = (\gamma_1,\dots,\gamma_k)^T$.

We devise an exact 100 (1-α)% confidence region for the vector $\sigma^2$ of variance components. Essentially the same approach can be used to obtain an exact 100 (1-α)% confidence region for the vector γ of variance ratios. Confidence regions for $\sigma^2$ and γ can be of direct interest. They can also be used to obtain (generally conservative) confidence intervals for functions of $\sigma^2$ or γ (Spjotvoll, 1972; Khuri, 1981) and for linear combinations of the fixed and random effects, that is, linear combinations of the elements of the vectors $\beta_0$, $b_1$, ..., $b_k$ (Jeske, 1985).

Let $\chi^2(f)$ represent a chi-square distribution with f degrees of freedom, and take $\chi^*_{\alpha,f}$ to be the upper-α point of this distribution. Under certain circumstances, there exist k+1 quadratic forms

$y^T A_1 y, \ldots, y^T A_{k+1} y$, where $A_1, \ldots, A_{k+1}$ are $n \times n$ symmetric matrices of known constants, such that (i) $y^T A_1 y, \ldots, y^T A_{k+1} y$ are distributed independently, (ii) $y^T A_i y/(\sigma^2_{k+1} c_i) \sim \chi^2(f_i)$ for a positive integer $f_i$ and a scalar $c_i$ that are necessarily given by $f_i = \text{rank}\,(A_i)$ and and $c_i = f_i^{-1}(d_{i,k+1} + \Sigma_j d_{ij}\gamma_j)$ with $d_{ij} = \text{tr}(X_j^T A_i X_j)$ $(j=1,\ldots,k)$ and $d_{i,k+1} = \text{tr}(A_i)$ $(i=1,\ldots,k+1)$, and (iii) the $(k+1) \times (k+1)$ matrix with $ij^{th}$ element $d_{ij}$ is nonsingular. If the $k+1$ quadratic forms $y^T A_1 y, \ldots,$ $y^T A_{k+1}\, y$ exist, then clearly, for $0 < \alpha < 1$, a 100 $(1-\alpha)$% confidence region for the vector $\sigma^2$ is given by the set $S(y)$ consisting of those values of the vector $\sigma^2$ that simultaneously satisfy the $k+1$ inequalities

$$\chi^*_{1-\alpha_{i2}, f_i} < y^T A_i y/(\sigma^2_{k+1} c_i) < \chi^*_{\alpha_{i1}, f_i} \quad (i=1,\ldots,k+1), \tag{1.2}$$

where $\alpha_{i1}$ and $\alpha_{i2}$ represent nonnegative constants such that $\alpha_{i1} + \alpha_{i2} < 1$ $(i=1,\ldots,k+1)$ and $\Pi_i(1-\alpha_{i1} - \alpha_{i2}) = 1-\alpha$.

It is well known that, for balanced classificatory models, the requisite quadratic forms exist and, in fact, can be taken to be those sums of squares in the customary analysis of variance that correspond to the random effects and errors, in which case $f_1 + \ldots + f_k = n -$ $\text{rank}(X_0)$ (e.g., Broemeling, 1969). Other special cases for which such quadratic forms have been found are treated by, for example, Broemeling and Bee (1976).

Unfortunately, in many cases, quadratic forms $y^T A_1 y, \ldots, y^T A_{k+1} y$, whose matrices $A_1, \ldots, A_{k+1}$ are matrices of known constants and that satisfy the three desired conditions, may, if they exist at all, be hard to find and/or may be such that $\Sigma f_i < n - \text{rank}(X_0)$, in which case the confidence region S(y) may be overly large. In what follows, we present a general procedure for forming an exact 100(1-α)% confidence region for the vector $\sigma^2$. Our approach differs from the aforementioned approach in that we allow the elements of the matrix $A_i$ of the $i^{th}$ quadratic form $y^T A_i y$ to be functionally dependent on the last k-i+1 variance ratios $\gamma_i = \sigma_i^2/\sigma_{k+1}^2, \ldots, \gamma_k = \sigma_k^2/\sigma_{k+1}^2$ $(i=1,\ldots,k)$. By allowing this dependence, we are able to construct quadratic forms $y^T A_1 y, \ldots, y^T A_{k+1} y$ that satisfy the three desired properties and which, in addition, are such that $\Sigma f_i = n - \text{rank}(X_0)$. Then, as in the aforementioned approach, an exact 100(1-α)% confidence region consists of the set S(y) of all values of the vector $\sigma^2$ that simultaneously satisfy the k+1 inequalities (1.2).

Our procedure can be regarded as an extension of Wald's (1940 and 1947) procedure. Wald's procedure, when extended along the lines discussed by Thompson (1955), Spjotvoll (1968), Seely and El-Bassiouni (1983), and Harville and Fenech (1984), covers the special case k=1. Hartley and Rao (1967,Sec. 9) proposed a general procedure for obtaining an exact 100 (1-α)% confidence region which, like ours, can be viewed as a generalization of Wald's procedure. However, as can easily be shown, it produces confidence regions of a seemingly unappealing form and, in fact, can with high probability produce confidence regions of infinite volume.

In principle, the likelihood ratio could be used to generate a confidence region (Hartley and Rao, 1967, Sec. 8). However, the percentiles of its distribution are approximated on the basis of asymptotic results. The accuracy of these approximations is questionable and is not easily investigated. (What few asymptotic results are available for mixed linear models--see, for example, Miller (1977)--seem unassuring, and, except in special cases, Monte Carlo studies are computationally unfeasible.)

## 2. PRELIMINARIES

Define $V_i = I + \Sigma_{s=i+1}^{k}\gamma_s X_s X_s^T$ $(i=0,\ldots,k-1)$ and $V_{k+1} = V_k = I$, and let $V = V_0 = I + \Sigma_{s=1}^{k}\gamma_s X_s X_s^T$. Under the assumed model (1.1), $y \sim N(X_0\beta_0,\ \sigma_{k+1}^2 V)$, and the parameter space for the vector $\sigma^2$ is

$$\Omega_0 = \{\sigma^2 : \sigma_{k+1}^2 > 0,\ \gamma_i > 0\ (i=1,\ldots,k)\}.$$

Note, however, that the matrix V is positive definite for some values of the vector $\gamma$ that include one or more negative elements.

Subsequently, we take the model to be the generalization of model (1.1) that results from disregarding our original definitions of $\sigma_1^2,\ldots,\sigma_{k+1}^2$ as variances and from assuming only that $y \sim N(X_0\beta_0,\ \sigma_{k+1}^2 V)$ and that the parameter space for the vector $\sigma^2$ is

$$\Omega = \{\sigma^2 : \sigma_{k+1}^2 > 0,\ (\gamma_{i+1},\ldots,\gamma_k)^T \in \Gamma_i\ (i=0,\ldots,k-1)\},$$

where $\Gamma_i$ is the set of all values of $(\gamma_{i+1},\ldots,\gamma_k)^T$ such that $V_i$ is positive definite.

We shall have occasion to refer to a model, to be called Model i, in which

$$y = X_0\beta_0 + X_1\beta_1 + \ldots + X_i\beta_i + e_i,$$

where $\beta_1,\ldots,\beta_i$ are vectors which, like $\beta_0$, are composed of unknown parameters, where $e_i$ is an n x 1 unobservable random vector with $E(e_i) = 0$ and $\mathrm{var}(e_i) = \sigma^2_{k+1}V_i$, and where $\gamma_{i+1},\ldots,\gamma_k$ and hence $V_i$ are assumed to be known ($i=1,\ldots,k$). Model i is essentially the same as model (1.1) except that the parameter vectors $\beta_1,\ldots,\beta_i$ appear in place of the random vectors $b_1,\ldots,b_i$ and the parameters $\gamma_{i+1},\ldots,\gamma_k$ are taken to be known instead of unknown.

For $i = 0,1,\ldots,k$, define $X_i^* = (X_0, X_1,\ldots,X_i)$. Note that $X_i^* = (X_{i-1}^*, X_i)$ ($i=1,\ldots,k$). Let $r_0 = \mathrm{rank}(X_0)$, $r_i = \mathrm{rank}(X_i^*) - \mathrm{rank}(X_{i-1}^*)$ ($i=1,\ldots,k$), and $r_{k+1} = n - \mathrm{rank}(X_k^*)$. Subsequently, we assume that $r_i > 0$ ($i=1,\ldots,k+1$).

We write $A^-$ for an arbitrary generalized inverse of a matrix A, that is, $A^-$ is any matrix satisfying $AA^-A = A$. Define

$$P_{i-1} = X_{i-1}^*(X_{i-1}^{*T}V_i^{-1}X_{i-1}^*)^-X_{i-1}^{*T}V_i^{-1} \qquad (i=1,\ldots,k+1).$$

Note that

$$P_{i-1}^2 = P_{i-1},\ P_{i-1}^TV_i^{-1} = (V_i^{-1}P_{i-1})^T = V_i^{-1}P_{i-1}$$

and that $P_{i-1}X_{i-1}^* = X_{i-1}^*$ and $X_{i-1}^{*T}V_i^{-1}P_{i-1} = (P_{i-1}X_{i-1}^*)^TV_i^{-1} = X_{i-1}^{*T}V_i^{-1}$, implying that, for $j = 0,\ldots,i-1$,

$$P_{i-1}X_j = X_j,\ X_j^TV_i^{-1}P_{i-1} = X_j^TV_i^{-1},\ P_{i-1}X_j^* = X_j^*,\ X_j^{*T}V_i^{-1}P_{i-1} = X_j^{*T}V_i^{-1}$$

$(i=1,\ldots,k+1)$. Note also that, since the matrix $I-P_k = I - X_k^*(X_k^{*T}X_k^*)^-X_k^*$ is symmetric and idempotent, there exists an $n \times r_{k+1}$ matrix $F$ such that $I-P_k = FF^T$ and $F^TF = I$ and that $FF^TX_i = 0$, implying that $F^TX_i = 0$ $(i=0,\ldots,k)$.

## 3. QUADRATIC FORMS

### 3.1 Definition

We now introduce the quadratic forms on which the proposed confidence region is to be based.

Define $C_i = X_i^TV_i^{-1}(I-P_{i-1})X_i$ and $q_i = X_i^TV_i^{-1}(I-P_{i-1})y$ $(i=1,\ldots,k)$. It is known that rank $(C_i) = r_i$ and that there exists an $m_i \times r_i$ matrix $\Lambda_i$ of rank $r_i$ such that, under Model i, $\tau_i = \Lambda_i^T\beta_i$ would be an estimable parametric function. It is further known that $\Lambda_i^T = L_i^TC_i$ for some matrix $L_i$, which is necessarily of rank $r_i$, and that, under Model i, the minimum-variance linear unbiased estimator of $\tau_i$ would be

$$\tilde{\tau}_i = (\tilde{\tau}_{i1},\ldots,\tilde{\tau}_{ir_i})^T = L_i^Tq_i.$$

The proposed confidence region is based on the k+1 quadratic forms

$$Q_i = Q_i(\gamma_i,\ldots,\gamma_k;y) = \tilde{\tau}_i^T[L_i^T(C_i + \gamma_iC_i^2)L_i]^{-1}\tilde{\tau}_i \quad (i=1,\ldots,k),$$

$$Q_{k+1} = Q_{k+1}(y) = y^T(I-P_k)y = z^Tz,$$

where $z = F^T y$. As shown in Section 3.2, $Q_1/\sigma^2_{k+1},\ldots, Q_{k+1}/\sigma^2_{k+1}$ are distributed independently as chi-square random variables with degrees of freedom $r_1,\ldots,r_{k+1}$, respectively. The quadratic forms $Q_1,\ldots,Q_{k+1}$ are invariant to the choice of the matrices $\Lambda_1,\ldots,\Lambda_k$, as is easily verified.

### 3.2 Joint distribution

Under the assumed distribution for y, which is $N(X_0\beta_0,\sigma^2_{k+1}V)$, we find that, for $j < i = 1,\ldots,k$, $\mathrm{cov}(q_i,q_j)$

$$= \sigma^2_{k+1}X_i^TV_i^{-1}(I-P_{i-1})VV_j^{-1}(I-P_{j-1})X_j$$

$$= \sigma^2_{k+1}X_i^TV_i^{-1}(I-P_{i-1})(I+\Sigma^k_{s=j}\gamma_sX_sX_s^T)V_j^{-1}(I-P_{j-1})X_j$$

$$= \sigma^2_{k+1}X_i^TV_i^{-1}(I-P_{i-1})(V_j+\gamma_jX_jX_j^T)V_j^{-1}(I-P_{j-1})X_j$$

$$= \sigma^2_{k+1}X_i^TV_i^{-1}(I-P_{i-1})V_jV_j^{-1}(I-P_{j-1})X_j$$

$$= \sigma^2_{k+1}[X_i^TV_i^{-1}(I-P_{i-1})X_j - X_i^TV_i^{-1}(I-P_{i-1})X^*_{j-1}(X^{*T}_{j-1}V_j^{-1}X^*_{j-1})^-X^{*T}_{j-1}V_j^{-1}X_j]$$

$$= 0 - 0 = 0. \tag{3.1}$$

It can be shown, in similar fashion, that $\mathrm{cov}(z,q_j) = 0$ $(j=1,\ldots,k)$. Thus, $q_1,\ldots,q_k,z$ are distributed independently, implying that $\tilde{\tau}_1,\ldots,\tilde{\tau}_k,z$ are distributed independently and hence that $Q_1,\ldots,Q_k,Q_{k+1}$ are distributed independently.

Further, for $i=1,\ldots,k$, we find that

$$E(q_i) = X_i^T V_i^{-1}(I-P_{i-1})X_0\beta_0 = 0, \tag{3.2}$$

$$\mathrm{var}(q_i) = \sigma_{k+1}^2 X_i^T V_i^{-1}(I-P_{i-1})VV_i^{-1}(I-P_{i-1})X_i$$

$$= \sigma_{k+1}^2 X_i^T V_i^{-1}(I-P_{i-1})(V_i+ \gamma_i X_i X_i^T)V_i^{-1}(I-P_{i-1})X_i$$

$$= \sigma_{k+1}^2(C_i+ \gamma_i C_i^2), \tag{3.3}$$

implying that $\tilde{\tau}_i \sim N[0,\ \sigma_{k+1}^2 L_i^T(C_i + \gamma_i C_i^2)L_i]$ and hence that $Q_i/\sigma_{k+1}^2 \sim \chi^2(r_i)$. Also, $z \sim N(0,\sigma_{k+1}^2 I)$, and, consequently, $Q_{k+1}/\sigma_{k+1}^2 \sim \chi^2(r_{k+1})$.

### 3.3 Canonical representation

We now consider a particular choice for the parameter vectors $\tau_1,\ldots,\tau_k$ or, equivalently, for the coefficient matrices $\Lambda_1^T,\ldots,\Lambda_k^T$. This choice produces representations for the quadratic forms $Q_1,\ldots,Q_k$ that are, as we discuss in Section 4, informative about the nature of the proposed confidence region, and that can be useful computationally.

Let $\Delta_{i1},\ldots,\Delta_{ir_i}$ represent the nonzero, and hence positive, characteristic values of the matrix $C_i$, define $D_i = \mathrm{diag}(\Delta_{i1},\ldots,\Delta_{ir_i})$, and take $R_i$ to be an $m_i \times r_i$ matrix whose columns are orthonormal characteristic vectors of $C_i$ corresponding to the values $\Delta_{i1},\ldots,\Delta_{ir_i}$, respectively. Thus, by definition,

$$C_iR_i = R_iD_i, \qquad R_i^TR_i = I,$$

implying that

$$R_i^TC_iR_i = D_i, \quad R_i^TC_i^2R_i = (C_iR_i)^TC_iR_i = D_iR_i^TR_iD_i = D_i^2.$$

Consider the choice

$$\Lambda_i^T = D_i^{-1/2}R_i^TC_i = L_i^TC_i$$

with $L_i^T = D_i^{-1/2}R_i^T$. Note that this choice varies with $\gamma_{i+1},\ldots,\gamma_k$. It leads to the representation

$$Q_i = \tilde{\tau}_i^T(I+\gamma_iD_i)^{-1}\tilde{\tau}_i = \sum_{j=1}^{r_i} \frac{\tilde{\tau}_{ij}^2}{1+\gamma_i\Delta_{ij}} \tag{3.4}$$

with $\tilde{\tau}_i = D_i^{-1/2}R_i^Tq_i$.

### 3.4 An alternative approach

The quadratic forms $Q_1,\ldots,Q_{k+1}$ can be defined, and their properties established, via a vector-space approach, as we now demonstrate. Denote by $R^n$ the vector space consisting of all n-dimensional real vectors, and let $W^\perp$ represent the orthogonal complement of a subspace W of $R^n$ with respect to the usual inner product, that is, the inner product that assigns the value $y_1^Ty_2$ to any two n-dimensional real vectors $y_1$ and $y_2$. Further, let $W^{\perp i}$ represent the orthogonal complement of W with respect to the inner product $y_1^TV_iy_2$. Denote by C(A) the column space of a matrix A.

Define $U_{k+1} = C(X_k^*)^\perp$ and

$$U_i = C(X_{i-1}^*)^\perp \cap [C(X_i^*)^\perp]^{\perp i} \qquad (i=1,\ldots,k).$$

Then, $\dim(U_{k+1}) = r_{k+1}$ and

$$\dim (U_i) = \dim[C(X_{i-1}^*)^\perp]-\dim[C(X_i^*)^\perp]$$

$$= n-\text{rank}(X_{i-1}^*)-[n-\text{rank}(X_i^*)] = r_i \quad (i=1,\ldots,k).$$

Take $H_i$ to be an $n \times r_i$ matrix whose columns form a basis for $U_i$ $(i=1,\ldots,k+1)$. Derivations paralleling those of results (3.1), (3.2), and (3.3) reveal that $H_i^T V H_j = 0$ $(j<i=1,\ldots,k+1)$, that $H_i^T X_0 = 0$ $(i=1,\ldots,k+1)$, and that $H_i^T V H_i = H_i^T(V_i + \gamma_i X_i X_i^T)H_i$ $(i=1,\ldots,k)$ and $H_{k+1}^T V H_{k+1} = H_{k+1}^T H_{k+1}$. Note that $U_1,\ldots,U_{k+1}$ are orthogonal with respect to the inner product $y_1^T V y_2$ and that their direct sum is $C(X_0)^\perp$.

We have, in effect, established that the vectors $H_1^T y,\ldots,H_{k+1}^T y$ are distributed independently, with $H_i^T y \sim N[0, \sigma_{k+1}^2 H_i^T(V_i + \gamma_i X_i X_i^T)H_i]$ $(i=1,\ldots,k)$ and $H_{k+1}^T y \sim N(0, \sigma_{k+1}^2 H_{k+1}^T H_{k+1})$. It follows that the k+1 quadratic forms

$$Q_i^* = y^T H_i [H_i^T(V_i + \gamma_i X_i X_i^T)H_i]^{-1} H_i^T y \qquad (i=1,\ldots,k)$$

$$Q_{k+1}^* = y^T H_{k+1}(H_{k+1}^T H_{k+1})^{-1} H_{k+1}^T y$$

are distributed independently, with $Q_i^*/\sigma_{k+1}^2 \sim \chi^2(r_i)$ $(i=1,\ldots,k+1)$. These quadratic forms are invariant to the choice of the basis matrices $H_1,\ldots,H_{k+1}$, as is easily verified.

It is easy to show that one choice for the matrices $H_1^T,\ldots,H_{k+1}^T$ is $H_i^T = L_i^T X_i^T V_i^{-1}(I-P_{i-1})$ $(i=1,\ldots,k)$, $H_{k+1}^T = F^T$. This result can, in turn, be used to show that the quadratic forms $Q_1^*,\ldots,Q_{k+1}^*$ are identical to the quadratic forms $Q_1,\ldots,Q_{k+1}$, introduced in Section 3.1. The representations $Q_1,\ldots,Q_{k+1}$ are informative about the nature of the computations required to evaluate the k+1 quadratic forms for specified values of $\gamma_1,\ldots,\gamma_k$.

## 4. NATURE OF CONFIDENCE REGION

### 4.1 General case

The set S(y), consisting of those values of the vector $\sigma^2$ ($\sigma^2 \in \Omega$) that simultaneously satisfy the k+1 inequalities

$$\chi^*_{1-\alpha_{i2},r_i} < Q_i/\sigma^2_{k+1} < \chi^*_{\alpha_{i1},r_i} \quad (i=1,\ldots,k+1),$$

is a $100(1-\alpha)\%$ confidence region for $\sigma^2$. We now present an alternative description of this set, one which provides more insight into the nature of the set and which is more useful computationally.

Let $\lambda_i^*$ represent the maximum characteristic value of the matrix $X_i^T V_i^{-1} X_i$, and define $\Delta_i^* = \max(\Delta_{i1},\ldots,\Delta_{ir_i})$ $(i=1,\ldots,k)$. It is easy to show that $\Delta_i^* < \lambda_i^*$. For any fixed value of $(\gamma_{i+1},\ldots,\gamma_k) \in \Gamma_i$, the matrix $V_{i-1} = V_i + \gamma_i X_i X_i^T$ is positive definite for those values of $\gamma_i$ belonging to the interval $-1/\lambda_i^* < \gamma_i < \infty$, but the quadratic form $Q_i$ is a well-defined function of $\gamma_i$ over the more extensive interval $-1/\Delta_i^* < \gamma_i < \infty$, as is evident from representation (3.4). Further, as a function of $\gamma_i$, $Q_i$ is strictly decreasing and strictly convex over the interval $-1/\Delta_i^* < \gamma_i < \infty$, and

$$\lim_{\gamma_i \to -1/\Delta_i^*} Q_i = \infty, \qquad \lim_{\gamma_i \to \infty} Q_i = 0$$

(Harville and Fenech, 1984, Section 3). For convenience, define $Q_i(-1/\Delta_i^*, \gamma_{i+1},\ldots,\gamma_k;y) = \infty$ and $Q_i(\infty,\gamma_{i+1},\ldots,\gamma_k;y) = 0$.

Let $\ell_{k+1} = \ell_{k+1}(y) = Q_{k+1}/\chi^*_{\alpha_{k+1,1},r_{k+1}}$ and $u_{k+1} = u_{k+1}(y) = Q_{k+1}/\chi^*_{1-\alpha_{k+1,2},r_{k+1}}$. Further, for any fixed value of $(\gamma_{i+1},\ldots,\gamma_k) \in \Gamma_i$ and for any fixed positive value of $\sigma^2_{k+1}$, define $\ell_i = \ell_i(\gamma_{i+1},\ldots,\gamma_k, \sigma^2_{k+1};y)$ to be the unique value of $\gamma_i$ that satisfies $Q_i = \sigma^2_{k+1}\chi^*_{\alpha_{i1},r_i}$ and $u_i = u_i(\gamma_{i+1},\ldots,\gamma_k,\sigma^2_{k+1};y)$ to be the unique value of $\gamma_i$ that satisfies $Q_i = \sigma^2_{k+1}\chi^*_{1-\alpha_{i2},r_i}$. Then, an alternative description of the $100(1-\alpha)\%$ confidence region $S(y)$ is $S(y) =$

$$\{\sigma^2 : \sigma^2 \in \Omega,\ \ell_{k+1} < \sigma^2_{k+1} < u_{k+1},\ \sigma^2_{k+1}\ell_i < \sigma^2_i < \sigma^2_{k+1}u_i\ (i=1,\ldots,k)\}.$$

## 4.2 Special case

The upper and lower bounds $u_i$ and $\ell_i$ on $\gamma_i$ were defined as solutions to equations in $\gamma_i$ which are, in general, inherently nonlinear and not amenable to explicit solution. We now consider a special case where these equations can be solved explicitly.

Let $P^{(0)}_{i-1}$ represent the value of the matrix $P_{i-1}$ when $\gamma_{i+1} = \cdots = \gamma_k = 0$; that is, $P^{(0)}_{i-1} = X^*_{i-1}(X^{*T}_{i-1}X^*_{i-1})^- X^*_{i-1}$ $(i=1,\ldots,k+1)$. Note that the quadratic form $y^T(I-P^{(0)}_0)y$ represents the residual sum of squares obtained from a least squares fit of the submodel $y = X_0\beta_0 + b_{k+1}$.

Following Brown (1984), we define an analysis of variance for the variance components to be a partitioning $y^T(I-P_0^{(0)})y = y^TA_1y + \dots + y^TA_sy$, where $A_1,\dots,A_s$ are n x n symmetric matrices of known constants, such that (i) $y^TA_1y,\dots,y^TA_sy$ are distributed independently, (ii) $y^TA_iy/(\sigma_{k+1}^2c_i) \sim \chi^2(f_i)$ for a positive integer $f_i$ and a scalar $c_i$ $(i=1,\dots,s)$, and (iii) the scalars $c_1,\dots,c_s$, which are necessarily linear functions of $\gamma_1,\dots,\gamma_k$, are distinct. Brown showed, in effect, that an analysis of variance for the variance components exists if and only if the matrices $(I-P_0^{(0)})X_1X_1^T(I-P_0^{(0)}),\dots,(I-P_0^{(0)})X_kX_k^T(I-P_0^{(0)})$ commute in pairs, in which case the sums of squares $y^TA_1y,\dots,y^TA_sy$ are unique up to order.

In the Appendix, we show, by construction, that, if an analysis of variance for the variance components exists, then

$$Q_i = \Sigma_{j\epsilon I_i} y^TA_jy/c_j \qquad (i=1,\dots,k+1),$$

where $I_1,\dots,I_{k+1}$ represents a partitioning of the integers $1,\dots,s$ into k+1 sets. We show further that, if $s = k+1$, then, for $i=1,\dots,k$,

$$Q_i = \frac{y^T(P_i^{(0)} - P_{i-1}^{(0)})y}{1 + \Sigma_{j=i}^k \lambda_{ji}\gamma_j}$$

where $\lambda_{ji} = r_i^{-1}\mathrm{tr}[X_j^T(P_i^{(0)} - P_{i-1}^{(0)})X_j]$ $(j=1,\dots,k)$.

We see that, in the special case where there exists an analysis of variance for the variance components and where, in addition, $s = k+1$,

$$(4.1) \qquad \ell_i = \lambda_{ii}^{-1}[y^T(P_i^{(0)}-P_{i-1}^{(0)})y/(\sigma_{k+1}^2\chi^*_{\alpha_{i1},r_i}) - (1 + \Sigma_{j=i+1}^k \lambda_{ji}\gamma_j)],$$

(4.2) $u_i = \lambda_{ii}^{-1}[y^T(P_i^{(0)}-P_{i-1}^{(0)})y/(\sigma_{k+1}^2\chi^*_{1-\alpha_{i2},r_i}) - (1 + \Sigma_{j=i+1}^k \lambda_{ji}\gamma_j)]$,

and that, for balanced classificatory models, our procedure for forming a confidence region reduces to the traditional procedure. Note that, in this special case, $\ell_i$ and $u_i$ are linear functions of $\gamma_{i+1},\ldots,\gamma_k$ and $1/\sigma_{k+1}^2$ or, equivalently, $\sigma_{k+1}^2\ell_i$ and $\sigma_{k+1}^2 u_i$ are linear functions of $\sigma_{i+1}^2,\ldots,\sigma_k^2$, $\sigma_{k+1}^2$.

Even if an analysis of variance for all k+1 variance components does not exist, there may exist a less extensive partitioning $y^T(I-P_{k'}^{(0)})y = y^TA_1y + \ldots + y^TA_sy$ such that $y^TA_1y,\ldots,y^TA_sy$ are distributed independently as distinct scalar multiples of chi-square random variables, in which case formulas (4.1) and (4.2) are still applicable, at least for $i = k'+1,\ldots,k$.

## 5. DISCUSSION

Some modifications of the proposed procedure for obtaining a confidence region for the variance components $\sigma_1^2,\ldots,\sigma_{k+1}^2$ and some considerations in its implementation are as follows:

1. The proposed procedure can be modified so as to obtain a confidence region for the variance ratios $\gamma_1,\ldots,\gamma_k$. Define $F_i = (r_{k+1}/r_i)(\chi_i^2/\chi_{k+1}^2)$ $(i=1,\ldots,k)$, where $\chi_1^2,\ldots,\chi_{k+1}^2$ represent independently distributed chi-square random variables with degrees of freedom $r_1,\ldots,r_{k+1}$, respectively. Take $F_i^{(\ell)}$, $F_i^{(u)}$ $(i=1,\ldots,k)$ to be any constants such that

$$\mathrm{pr}[F_i^{(\ell)} < F_i < F_i^{(u)}\ (i=1,\ldots,k)] = 1-\alpha.$$

Then, a $100(1-\alpha)\%$ confidence region for $\gamma_1,\ldots,\gamma_k$ is the set

$$S^*(y) = \{\gamma : (\gamma_1,\ldots,\gamma_k)^T \in \Gamma_{i-1},\ \ell_i^* < \gamma_i < u_i^*\ (i=1,\ldots,k)\}$$

where $\ell_i^* = \ell_i^*(\gamma_{i+1},\ldots,\gamma_k;y)$ is the unique value of $\gamma_i$ that satisfies $(r_{k+1}/r_i)(Q_i/Q_{k+1}) = F_i^{(u)}$ and $u_i^* = u_i^*(\gamma_{i+1},\ldots,\gamma_k;y)$ is the unique value of $\gamma_i$ that satisfies $(r_{k+1}/r_i)(Q_i/Q_{k+1}) = F_i^{(\ell)}$. In the special case of a balanced classificatory model and for $\alpha_{11} = \ldots = \alpha_{k1} = 0$, the confidence region $S^*(y)$ simplifies to that discussed by Sahai and Anderson (1973).

If k is sufficiently large, the determination of the constants $F_i^{(\ell)}$, $F_i^{(u)}$ $(i=1,\ldots,k)$ may be computationally unfeasible, in which case our procedure for obtaining an exact $100(1-\alpha)\%$ confidence region can not be implemented. However, an approximate $100(1-\alpha)\%$ confidence region can be obtained by replacing $F_i^{(\ell)}$ and $F_i^{(u)}$ by the upper-$(1-\alpha_{i2})$ and upper-$\alpha_{i1}$ points, respectively, of the marginal distribution of $F_i$. It follows from Kimball's (1951) inequality that, for $\alpha_{12} = \ldots = \alpha_{k2} = 0$, this region is conservative; that is, its probability of coverage equals or exceeds $1-\alpha$. In the special case of a balanced classificatory model and for $\alpha_{11} = \ldots = \alpha_{k1} = 0$, the approximate confidence region simplifies to that proposed by Broemeling (1969).

2. The proposed procedure for using the quantities $Q_1,\ldots,Q_{k+1}$ to form a confidence region for the variance components $\sigma_1^2,\ldots,\sigma_{k+1}^2$ or the variance ratios $\gamma_1,\ldots,\gamma_k$ can be modified, in an obvious way, to obtain a confidence region for the last $k-k^*+2$ variance components $\sigma_{k^*}^2,\ldots,\sigma_{k+1}^2$ or the last $k-k^*+1$ variance ratios $\gamma_{k^*},\ldots,\gamma_k$, based on the quantities $Q_{k^*},\ldots,Q_{k+1}$.

3. By following the approach described, for example, by Spjotvoll (1972) and Khuri (1981), the proposed confidence region for the variance components $\sigma_1^2,\ldots,\sigma_{k+1}^2$ or variance ratios $\gamma_1,\ldots,\gamma_k$ can be transformed into a generally conservative confidence region for a function of the variance components or variance ratios or, more generally, for a family of such functions. For instance, let $R_f(y)$ represent the range of a vector $f = f(\sigma_1^2,\ldots,\sigma_{k+1}^2)$ of functions of variance components when the domain of f is restricted to the set S(y). Then, clearly, $pr[f \varepsilon R_f(y)] \geq 1-\alpha$; that is, $R_f(y)$ is a generally conservative $100(1-\alpha)\%$ confidence region for f.

4. Often, the parameter space for the vector $\sigma^2$ is a proper subset $\Omega'$ of the set $\Omega$, rather than $\Omega$ itself. In particular, the parameter space for $\sigma^2$ may be the set $\Omega_0$.

The confidence region S(y) may include values of $\sigma^2$ not belonging to $\Omega'$. Let S'(y) represent the subset of S(y) obtained by deleting all such values. For $\sigma^2 \varepsilon \Omega'$, $pr[\sigma^2 \varepsilon S'(y)] = pr[\sigma^2 \varepsilon S(y)] = 1-\alpha$. Thus, when the parameter space for $\sigma^2$ is $\Omega'$, S'(y), like S(y), is a $100(1-\alpha)\%$ confidence region for $\sigma^2$.

In the special case $\Omega' = \Omega_0$, the $100(1-\alpha)\%$ confidence region S'(y) is obtained from S(y) by deleting all values of $\sigma^2$ for which one or more of the first k variance components $\sigma_1^2,\ldots,\sigma_k^2$ is negative. Expanding the set S'(y) slightly, we obtain the set $S^+(y) =$

$$\{\sigma^2: \sigma_{k+1}^2 > 0,\ \ell_{k+1}<\sigma_{k+1}^2<u_{k+1},\ \sigma_{k+1}^2 \max(\ell_i,0)<\sigma_i^2<\sigma_{k+1}^2 \max(u_i,0)\ (i=1,\ldots,k)\}.$$

We have that $pr[\sigma^2 \varepsilon S^+(y)]$ equals $1-\alpha$, if $\sigma_i^2 > 0$ for $i=1,\ldots,k$, and is greater than or equal to $1-\alpha$, otherwise, as is easily verified. Thus, when the parameter space for $\sigma^2$ is $\Omega_o$, $S^+(y)$ is a conservative $100(1-\alpha)\%$ confidence region. Unlike $S'(y)$, the set $S^+(y)$ is nonempty, with probability one.

5. There will generally be more than one way to express a particular mixed or random linear model as a special case of the general mixed linear model (1.1). Consider, for example, the customary additive model for a two-way crossed classification with both factors regarded as random. We can take the elements of $b_1$ to be the effects of the first factor or, alternatively, the effects of the second factor.

It should be noted that, except for highly structured situations, like those considered in Section 4.2, the confidence region $S(y)$ will vary with the order in which we *assign the various sets of random effects* to the vectors $b_1,\ldots,b_k$.

6. Note that, by exploiting general relationships between confidence regions and tests of hypothesis or significance, the proposed procedure for forming a confidence region can be used to test, against appropriate alternatives, a null hypothesis of the general form $H_o$: $\sigma_1^2 = c_1,\ldots,\sigma_{k+1}^2 = c_{k+1}$ or $H_o$: $\gamma_1 = c_1,\ldots,\gamma_k = c_k$, where $c_1,\ldots,c_{k+1}$ represent specified constants.

Also, following Harville and Fenech (1984, Section 7), we can obtain point estimators of the variance components or variance ratios by equating the pivotal quantities $Q_1/\sigma_{k+1}^2,\ldots,Q_{k+1}/\sigma_{k+1}^2$ or $(r_{k+1}/r_1)Q_1/Q_{k+1}$, $\ldots,(r_{k+1}/r_k)Q_k/Q_{k+1}$ to appropriately chosen constants. In particular, if we equate $Q_1/\sigma^2_{k+1},\ldots,Q_{k+1}/\sigma^2_{k+1}$ to $r_1,\ldots,r_{k+1}$, respectively, we obtain estimators which, in the special case where there exists an analysis of variance for the variance components and where $s = k+1$, simplify to the usual analysis-of-variance estimators. Alternatively, if we equate these quantities to the medians of their respective chi-square distributions, we obtain estimators that can be interpreted as the coordinates of a degenerate, single-point confidence region.

7. Let $Q_* = Q_1 + \ldots + Q_k$ and $r_* = r_1 + \ldots + r_k$, and define $F^{(\ell)}$ and $F^{(u)}$ to be any constants such that $\mathrm{pr}[F^{(\ell)} < F < F^{(u)}] = 1-\alpha$, where F represents an F random variable with degrees of freedom $r_*$ and $r_{k+1}$, respectively. Clearly, the distribution of $Q_*/\sigma^2_{k+1}$ is $\chi^2(r_*)$, and $Q_*$ and $Q_{k+1}$ are distributed independently. It follows that an exact $100(1-\alpha)\%$ confidence region for $\gamma_1,\ldots,\gamma_k$, alternative to the confidence region $S^*(y)$, is the set $S^\#(y)$ consisting of those values of $\gamma$ that satisfy the inequality

$$F^{(\ell)} < (r_{k+1}/r_k)Q_*/Q_{k+1} < F^{(u)}.$$

It can be shown that, in the special case where $F^{(\ell)} = 0$ (and hence where $F^{(u)}$ is the upper-$\alpha$ point of the distribution of F), the confidence region $S^\#(y)$ is the same as the exact confidence region devised by Hartley and Rao (1967, Sec. 9). We do not recommend the confidence region $S^\#(y)$

since, as previously indicated in the special case of the Hartley-Rao region, it has a seemingly unappealing form and can, with high probability, produce confidence regions of infinite volume.

8. In practice, we may want to display graphically the confidence region $S(y)$, or when $k$ exceeds one or two, to display, for each of various fixed values of the vector $(\sigma^2_{i+1},\ldots,\sigma^2_k, \sigma^2_{k+1})$ and the integer $i$, the interval of $\sigma^2_i$ values, or perhaps the two- or three-dimensional set of $\sigma^2_{i-1}$, $\sigma^2_i$ or $\sigma^2_{i-2}$, $\sigma^2_{i-1}$, $\sigma^2_i$ values, that are represented in $S(y)$. We are then faced with the computational problem of determining the numerical values of $\ell_i$ and $u_i$ corresponding to each value of $(\sigma^2_{i+1},\ldots,\sigma^2_k,\sigma^2_{k+1})$ and each value of $i$.

In the special case where there exists an analysis of variance for the variance components and where $s = k+1$, this problem reduces, in effect, to that of computing the entries in the analysis-of-variance table. More generally, we can use the approach discussed, in the special case $i = k$, by Harville and Fenech (1984, Section 4), to compute the values of $\Delta_{i1},\ldots,\Delta_{ir_i}$ and of $\tau_{i1},\ldots,\tau_{ir_i}$ and to then determine the values of $\ell_i$ and $u_i$ graphically or iteratively. How a series of such determinations, involving various values of the vector $(\sigma^2_{i+1},\ldots,\sigma^2_k,\sigma^2_{k+1})$ and/or the integer $i$, might be accomplished most efficently is a question for possible investigation.

## APPENDIX

### Simplification of the quadratic forms

We now derive, for the special case where there exists an analysis of variance for the variance components, the simplified representations given in Section 4.2 for the quadratic forms $Q_1,\ldots,Q_k$.

Let $U_0^{(0)} = C(X_0)$, $U_i^{(0)} = C(X_{i-1}^*)^\perp \cap C(X_i^*)$ $(i=1,\ldots,k)$, and $U_{k+1}^{(0)} = C(X_k^*)^\perp$, so that, for $i=1,\ldots,k+1$, $U_i^{(0)}$ represents $U_i$ when $\gamma_{i+1} = \cdots = \gamma_k = 0$. Define $M_i$ to be an $n \times r_i$ matrix whose columns form an orthonormal basis for $U_i^{(0)}$ with respect to the ordinary inner product.

Note that the columns of the matrix $M_i^* = (M_0,\ldots,M_i)$ form an orthonormal basis for $C(X_i^*)$ $(i=0,\ldots,k)$ and that the columns of the matrix $M = (M_1,\ldots,M_{k+1})$ form an orthonormal basis for $C(X_0)^\perp$. Note also that, for $i=0,\ldots,k$,

$$M_i^* M_i^{*T} = M_i^*(M_i^{*T}M_i^*)^{-1}M_i^{*T} = X_i^*(X_i^{*T}X_i^*)^- X_i^{*T} = P_i^{(0)}$$

and that, for $i=1,\ldots,k$,

$$M_i M_i^T = M_i^* M_i^{*T} - M_{i-1}^* M_{i-1}^{*T} = P_i^{(0)} - P_{i-1}^{(0)},$$

$$X_i^T M_i M_i^T X_i = X_i^T(P_i^{(0)} - P_{i-1}^{(0)})X_i = X_i^T(I - P_{i-1}^{(0)})X_i,$$

$$\text{(A.1)} \quad \operatorname{rank}(M_i^T X_i X_i^T M_i) = \operatorname{rank}(X_i^T M_i M_i^T X_i) = \operatorname{rank}[X_i^T(I-P_{i-1}^{(0)})X_i] = r_i.$$

Suppose now that there exists an analysis of variance for the variance components. Then, the k matrices $M^TX_iX_i^TM =$ $\mathrm{diag}[(M_1,\ldots,M_i)^T X_i^TX_i(M_1,\ldots,M_i), 0]$ $(i=1,\ldots,k)$ commute in pairs (Brown, 1984, Corollary 2), which implies that, for $i=1,\ldots,k$,

$$M_i^TX_iX_i^TM = \mathrm{diag}(M_i^TX_1X_1^TM_1,\ldots,M_i^TX_iX_i^TM_i, 0,\ldots,0), \tag{A.2}$$

as we now show.

Our proof of result (A.2) is by induction. The result is clearly valid for $i=1$. Suppose that it is valid for $i=1,\ldots,i'-1$. Then, since $M^TX_iX_i^TM$ and $M^TX_{i'}X_{i'}^TM$ commute, implying that the ijth block of the matrix $M^TX_iX_i^TMM^TX_{i'}X_{i'}^TM$ is the same as that of $M^TX_{i'}X_{i'}^TMM^TX_iX_i^TM$, we have, for $i=1,\ldots,i'-1$ and $j=i+1,\ldots,i'$, that

$$M_i^TX_iX_i^TM_iM_i^TX_{i'}X_{i'}^TM_j = 0$$

and hence, in light of result (A.1), that $M_i^TX_{i'}X_{i'}^TM_j = 0$. Observing that $M^TX_iX_i^TM$ is symmetric, we conclude that result (A.2) is valid for $i=i'$, which completes the proof of this result.

Now, since the matrices $M^TX_1X_1^TM,\ldots,M^TX_kX_k^TM$ commute in pairs, we have, in light of result (A.2), that the matrices $M_i^TX_iX_i^TM_i,\ldots,M_i^TX_kX_k^TM_i$ commute in pairs. Consequently, there exists an orthogonal matrix that simultaneously diagonalizes the matrices $M_i^TX_iX_i^TM_i,\ldots,M_i^TX_kX_k^TM_i$; that is, there exists an orthogonal matrix $O_i$ such that

$$O_i^TM_i^TX_jX_j^TM_iO_i = \mathrm{diag}(\lambda_{ji1},\ldots,\lambda_{jir_i})$$

for some nonnegative real numbers $\lambda_{ji1},\ldots,\lambda_{jir_i}$ $(j=i,\ldots,k)$.

As a further consequence of result (A.2), we have that

$$M_i^T V_i(M_{i+1},\ldots,M_{k+1}) = M_i^T(I+\Sigma_{s=i+1}^{k}\ \gamma_s X_s X_s^T)(M_{i+1},\ldots,M_{k+1}) = 0,$$

implying that the columns of $M_i$, and hence those of $M_i O_i$, form a basis for $U_i$ $(i=1,\ldots,k)$.

For $i=1,\ldots,k$, define

$$w_i = (w_{i1},\ldots,w_{ir_i})^T = (M_i O_i)^T y,$$

define $c_{i1},\ldots,c_{ir_i^*}$ to be the distinct linear functions represented among the $r_i$ linear functions $1+\Sigma_{j=i}^{k}\lambda_{ji1}\gamma_j,\ldots,1+\Sigma_{j=i}^{k}\lambda_{jir_i}\gamma_j$, and, for $m=1,\ldots,r_i^*$, take $T_{im}$ to be the collection of values of the index t for which $1+\Sigma_{j=i}^{k}\lambda_{jit}\gamma_j = c_{im}$, take $f_{im}$ to be the dimension of this collection, and let $W_{im} = \Sigma_{t\epsilon T_{im}} w_{it}^2$.

It follows from the results of Section 3.4 that

$$Q_i = Q_i^* = y^T M_i O_i(I+\Sigma_{j=i}^{k}\gamma_j O_i^T M_i^T X_j X_j^T M_i O_i)^{-1} O_i^T M_i^T y$$

$$= \Sigma_{m=1}^{r_i^*} W_{im}/c_{im} \qquad (i=1,\ldots,k),$$

$$Q_{k+1} = Q_{k+1}^* = y^T M_{k+1}(M_{k+1}^T M_{k+1})^{-1} M_{k+1}^T y = y^T M_{k+1} M_{k+1}^T y.$$

Further we find that

$$\Sigma_{i=1}^{k}\Sigma_{m=1}^{r_i^*} W_{im} + Q_{k+1} = y^T M M^T y = y^T(I-M_0 M_0^T)y = y^T(I-P_0^{(0)})y$$

and, proceeding as in Section 3.4, that $W_{im}(i=1,\ldots,k;\ m=1,\ldots,r_i^*)$, $Q_{k+1}$ are distributed independently and $W_{im}/(c_{im}\sigma_{k+1}^2) \sim \chi^2(f_{im})$, implying that $W_{im}(i=1,\ldots,k;\ m=1,\ldots,r_i^*)$, $Q_{k+1}$ are the sums of squares in the analysis of variance for the variance components.

Now, suppose that the analysis of variance for the variance components is such that $s = k+1$. Then, for $i=1,\ldots,k$, $\lambda_{ji1} = \ldots = \lambda_{jir_i} = \lambda_{ji}$ for some real number $\lambda_{ji}$ given by

$$\lambda_{ji} = r_i^{-1} \Sigma_{t=1}^{r_i} \lambda_{jit} = r_i^{-1} tr(O_i^T M_i^T X_j X_j^T M_i O_i)$$

$$= r_i^{-1} tr(X_j^T M_i O_i O_i^T M_i^T X_j)$$

$$= r_i^{-1} tr[X_j^T (P_i^{(0)} - P_{i-1}^{(0)}) X_j] \quad (j=i,\ldots,k)$$

in which case

$$Q_i = w_i^T w_i/(1+\Sigma_{j=i}^{k} \lambda_{ji}\gamma_j) = y^T M_i M_i^T y/(1+\Sigma_{j=i}^{k} \lambda_{ji}\gamma_j)$$

$$= y^T (P_i^{(0)} - P_{i-1}^{(0)}) y/(1+\Sigma_{j=i}^{k} \lambda_{ji}\gamma_j).$$

## REFERENCES


BROEMELING, L. D. (1969). Confidence regions for variance ratios of random models. J. Am. Statist. Assoc. 64, 660-64.

BROEMELING, L. D. & BEE, D. E. (1976). Simultaneous confidence intervals for parameters of a balanced incomplete block design. J. Am. Statist. Assoc. 71, 425-28.

BROWN, K. G. (1984). On analysis of variance in the mixed model. Annals of Statistics, 12, 1488-1499.

HARTLEY, H. O. & RAO, J. N. K. (1967). Maximum-likelihood estimation for the mixed analysis of variance model. Biometrika 54, 93-108.

HARVILLE, D. A. & FENECH, A. P. (1984). Confidence intervals for a variance ratio, or for heritability, in an unbalanced mixed linear model. Biometrics, 41, 137-152.

JESKE, D.R. (1985). Prediction intervals for the realization of a random variable under a general mixed linear model. Ph.D. thesis, Department of Statistics, Iowa State University.

KIMBALL, A. W. (1951). On dependent tests of significance in the analysis of variance. Ann. Math. Statist. 22, 600-2.

KHURI, A. I. (1981). Simultaneous confidence intervals for functions of variance components in random models. J. Am. Statist. Assoc. 76, 878-85.

MILLER, J.J. (1977). Asymptotic properties of maximum likelihood estimates in the mixed model of the analysis of variance. Annals of Statistics, 5, 746-762.

SAHAI, H. & ANDERSON, R. L. (1973). Confidence regions for variance ratios of random models for balanced data. J. Am. Statist. Assoc. 68, 951-2.

SEELY, J. F. & EL-BASSIOUNI, Y. (1983). Applying Wald's variance component test. Ann. Statist. 11, 197-201.

SPJOTVOLL, E. (1968). Confidence intervals and tests for variance ratios in unbalanced variance components models. Rev. Internat. Statist. Inst. 36, 37-42.

SPJOTVOLL, E. (1972). Multiple comparison of regression functions. Ann. Math. Statist. 43, 1076-88.

THOMPSON, W. A., JR. (1955). On the ratio of variances in the mixed incomplete block model. Ann. Math. Statist. 26, 721-33.

WALD, A. (1940). A note on the analysis of variance with unequal class frequencies. Ann. Math. Statist. 11, 96-100.

WALD, A. (1947). A note on regression analysis. Ann. Math. Statist. 18, 586-9.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>86-12 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>CONFIDENCE REGIONS FOR VARIANCE COMPONENTS IN UNBALANCED MIXED LINEAR MODELS | | 5. TYPE OF REPORT & PERIOD COVERED<br>Department of Statistics Preprint Series |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>Alan P. Fenech (University of California-Davis) and David A. Harville | | 8. CONTRACT OR GRANT NUMBER(s)<br>Contract N0014-85-K-0418 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Iowa State University<br>Ames, Iowa 50011 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>Task Number R & T 042-548 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Mathematical Sciences Division<br>Office of Naval Research - Dept. of the Navy<br>Arlington, Virginia 22217-5000 | | 12. REPORT DATE<br>January 1986 |
| | | 13. NUMBER OF PAGES<br>27 |
| 14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Analysis of variance, confidence regions, mixed linear models, unbalanced data, variance components

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We present a general procedure for obtaining exact confidence regions for the variance components in unbalanced mixed linear models. The procedure utilizes, as pivotal quantities, quadratic forms that may depend on the variance components in a complicated way and that are distributed independently as chi-square variates. In the special case of balanced classificatory models, the pivotal quantities simplify to scalar multiples of sums of squares from the usual analysis of variance. The procedure can be easily modified so as to obtain an exact confidence region for ratios

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. (continued)

of variance components and can be regarded as a generalization of Wald's procedure for obtaining a confidence interval for a single variance ratio.

END
FILMED
4-86
DTIC