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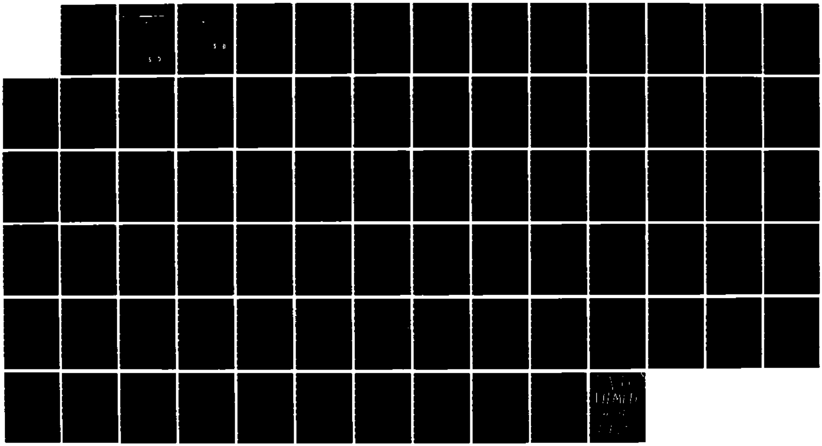

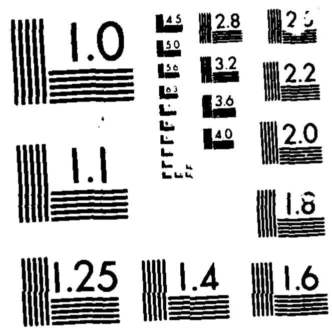

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On the Imaging of Reflectors in the Earth

by

Norman Bleistein

Partially supported by the Consortium Project of the Center for Wave Phenomena and by the Selected Research Opportunities Program of the Office of Naval Research

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ABSTRACT

This paper is based on a paper by Beylkin in which a leading order asymptotic theory for inversion of acoustic data is presented. The method is based on earlier work by Beylkin in the theory of pseudo-differential operators and generalized back projections or Radon transforms. The back projection or inversion is carried out with respect to a general [c(x,y,z)] background sound speed. The asymptotic limit of interest is high frequency. The inversion operator is given as an integral of the observed data over frequency and over the observation surface. Beylkin claims that his result is useful for finding discontinuities in the sound speed, but he does not make clear how this is to be done in practice. I show how to modify Beylkin's inversion operator to obtain an operator whose output is an array of singular functions, one for each reflector (discontinuity surface of the sound speed) in the subsurface. The singular function of a surface is a Dirac delta function whose support lies on that surface. Thus, the array of singular functions produces a reflector map of the subsurface. The validity of modification of Beylkin's inversion operator is verified by applying it to band limited Born-approximate and then Kirchhoff-approximate representations of the upward propagating wave field. Multi-dimensional stationary phase is applied to the spatial integration over the variables of the field representation and the variables of the observation surface. It is confirmed that the output is proportional to the band limited singular functions of the reflectors and further that one can estimate the jump in velocity across each reflector from the peak amplitude of the output on each reflector. This is done for the cases of common (or single fixed) source, common receiver, and common (or fixed) offset between source and receiver, with zero offset or backscatter as a special case of the last of these.

GLOSSARY

$A(\underline{x},\underline{x}_s)$ — Amplitude of ray-theoretic or WKBJ Green's function for background sound speed with source at $\underline{x}_s$ and observation point $\underline{x}$. See Appendix A.

$\alpha(\underline{x})$ — Perturbation in sound speed. Eq. (3).

$c(\underline{x})$ — Reference sound speed. Eq. (3).

$D(\underline{\xi},\omega)$ — Observed data, $u_S(\underline{x}_r(\underline{\xi}),\underline{x}_s(\underline{\xi}),\omega)$. Eq. (5).

$\delta_B(\Phi(\underline{x},\underline{x}',\underline{x}_s\underline{x}_r))$ — Band limited Dirac delta function of Φ (defined below) with $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ evaluated at the stationary point, defined by Eq. (C1), as functions of $\underline{x}$.

$\delta_B(s)$ — Band limited Dirac delta function of s, arc length normal to a surface on which $s = 0$. Band limited singular function of the surface. Eq. (38).

Δ_j — Jump in $\alpha(\underline{x})$ across the surface S_j. Eq. (24).

$F(\omega)$ — Filtered (smoothed and tapered) source function in the Fourier domain.

$\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)$ — Phase of inversion operator applied to Born-approximate or Kirchhoff-approximate field data. Eq. (9).

$[\Phi_{\xi\sigma}]$ — Hessian matrix of the phase of the inversion operator applied to Born-approximate or Kirchhoff-approximate field data. Eq. (33).

g_j — First fundamental form of differential geometry evaluated on the reflector S_j. Eq. (29).

g_s	First fundamental form of differential geometry evaluated on the source/receiver surface evaluated at $\underline{x}_s$. Eq. (B9).
$\gamma_j(\underline{x})$	Singular function of the surface S_j.
$\gamma_{jB}(\underline{x})$	Band limited version of $\gamma_j(\underline{x})$.
$h(\underline{x},\underline{\xi})$	Determinant defining transformation from variables (ω,ξ_1,ξ_2) to Fourier wave vector $\underline{k}$. Eq. (6).
J	Jacobi determinant of ray theory. Eq. (A9).
$\underline{k}$	Approximate Fourier variable of the inversion theory. Eq. (11).
$\underline{n}$	Upward normal vector on reflectors.
$\underline{p}(\underline{x},\underline{x}_r)$	$\nabla\tau(\underline{x},\underline{x}_r)$. Eq. (A5).
$R(\underline{x}',\underline{x}_s)$	Ray-theoretic reflection coefficient. Eq. (47).
S	Reflector in Kirchhoff representation of upward propagationg wave.
S_0	Observation surface.
S_j, $j \geq 1$	Reflectors, surfaces of discontinuity of $\alpha(\underline{x})$ in the subsurface.
S_ξ	Domain of integration in ξ-variables.
σ	Running parameter along rays. Appendix A.
$\underline{\sigma} = (\sigma_1,\sigma_2)$	Parameters used to define a reflecting surface.
$\tau(\underline{x},\underline{x}_s)$	Ray-theoretic travel time between $\underline{x}$ and $\underline{x}_s$. See Appendix A.

θ	Angle between the normal to a surface at the point $\underline{x}'$ and the ray from $\underline{x}_s$ or $\underline{x}_r$ to $\underline{x}'$, under the stationarity conditions (C1). Opening angle between these rays and normal subject to Snell's law of reflection.
$u_S(\underline{x}_r(\underline{\xi}),\underline{x}_s(\underline{\xi}),\omega)$	Observed field data. Eq. (5).
$v(\underline{x})$	Sound speed. Eq. (2).
$\underline{x}$	Point at which the output of inversion operator applied to $D(\underline{\xi},\omega)$ is to be evaluated.
$\underline{x}' = \underline{x}'(\underline{\sigma})$	Point on reflecting surface.
$\underline{x}_s$, $\underline{x}_r$	Source and receiver coordinates, repsectively. Eq. (1).
$\underline{\xi} = (\xi_1,\xi_2)$	Parameters labelling source and/or receiver points; i. e., $\underline{x}_s = \underline{x}_s(\underline{\xi})$ and/or $\underline{x}_r = \underline{x}_r(\underline{\xi})$. Eq. (1).

1. INTRODUCTION

This paper is based on the brilliant paper by Beylkin [1985]. In that paper, the author presented a theory for asymptotic inversion of observations for the acoustic wave equation to estimate discontinuities in the sound speed. The method allows for an assortment of possible source/receiver configurations, broad enough to accomodate most of the cases of interest in seismic exploration and other applications. For example, the method applies to zero-offset data; common (or single) source, multi-receiver array data (or the reverse); or fixed offset data. The inversion of the data is an integral over the source/receiver array. Extension to other wave equations is also quite apparent from the presentation.

Beylkin's results are couched in the language of pseudo-differential operators and generalized back projections or generalized inverse Radon transforms. He also does not make precise the manner in which his method actually produces the discontinuities in the sound speed. The theory only predicts that his integral solution is a leading order high frequency asymptotic inversion operator without relating the output to the information being sought.

The purpose of this paper is twofold. First, I make more precise the manner in which Beylkin's method provides an asymptotic solution of the inverse problem. I start from one of Beylkin's interpretations of his inversion scheme to view the output as a high frequency Fourier inversion of band limited data for the perturbation in sound speed. I show that by modifying the integrand of the operator by a scale factor, it will produce asymptotically an array of scaled **singular functions** of the surfaces of discontinuity of the sound speed. These surfaces are the reflectors in the subsurface. The singular function of a surface is a *Dirac delta function* whose support is on the surface. Thus, knowledge of the singular functions is equivalent to mathematical imaging of the reflecting surfaces. The scaling factor for each reflector is a known function of the jump in sound speed across the reflector. That known function takes a different form depending on whether one has used the Born approximation or the Kirchhoff approximation to represent the input data. In either case, I provide a means of estimating from the output the change in sound speed across the reflector. The estimate is consistent with the form of the input data.

Beylkin derives his inversion operator for upward traveling waves represented by their Born approximation. The "backwards projection" of this approach is with respect to an assumed known reference sound speed. I begin from such a representation, as well. From this starting point, I can only have confidence in the accuracy of our interpretation of the output of our method for small perturbations in sound speed.

The output that is produced depends on a certain open angle between two particular rays in the subsurface. This angle is not known because it varies from point to point in the subsurface. At each point, it depends on the directions of incidence and reflection of a pair of specular rays from an (unknown) source/receiver pair on the datum surface. I introduce an alternative inversion integral which, in the Born limit produces an output whose peak value on a reflector is independent of that opening angle.

I then consider the application of my inversion operator to Kirchhoff approximate data for a single reflector. For such a representation, I need not assume small perturbations, but only high frequency, which is already assumed in this theory. Here, I am able to show that the weighting on the singular function of the surface is, to leading order asymptotically, the full angular dependent reflection coefficient, independent of the size of the jump in velocity across the reflector. Thus, it would seem that what needs be "small" in this formalism is the error between the background sound speed above a given reflector and the "true" sound speed above the reflector.

The angle of the reflection coefficient is as described above. I show how this angle can be determined by exploiting the two inversion integrals already proposed. Thus, in terms of numerical processing one need compute only one additional sum with summand given in terms of previously computed elements. From these two outputs, one obtains a reflector map, an angularly dependent reflection coefficient and an estimate of the (cosine of that) angle.

I then propose a third inversion integral. This one has the property that its peak amplitude on the reflecting surface is equal to the product of the angularly dependent reflection coefficient and the area under the temporal filter of the original time signal. This last result is the most esthetically appealing, but offers little practical advantage over the other two operators. In any case, at least two inversion integrals must be computed to determine the unknown angle at each output point and then the jump in velocity from the angular-dependent reflection coefficient.

This verification also suggests a recursive application of the inversion formalism. That is, starting from the upper surface, each time a "major" reflector is imaged, the background sound speed is updated to account for the new information and data is processed deeper into the section until a new major reflector is imaged. The method is pointwise,hence lending itself to this type of recursive implementation.

In these results, the upper surface is allowed to be curved. Thus, the inversion one produces eliminates two preprocessing steps usually applied to seismic data. The first is a static correction for variable height of the source/receiver array. The second is stacking to produce an "equivalent" zero offset (backscatter) data set.

Central to this derivation of these results is the method of multi-dimensional stationary phase. This brings me to the second point of this paper, namely, to provide a more classical verification of the asymptotic validity of our modifications of Beylkin's inversion operator. Also, the interpretation in terms of imaging of reflectors and estimating reflection coefficients arises in a natural way in this method. Furthermore, the method predicts that such imaging will occur only at those points on the reflector for which there are a specular pair of rays from the source and the receiver to the surface point. This ties the inversion back to the required source/receiver array necessary for imaging the reflectors in the subsurface.

The verification of the **two-and-one-half dimensional** (2.5D) specialization of Beylkin's result has already been presented in Bleistein, Cohen and Hagin [1985b]. In that case, it is assumed that the data is gathered only on a single line on the surface and that all parameters of the subsurface are functions of the transverse variable along that line and depth. Significant simplifications occur in that analysis because fourfold integrals over two surfaces appearing in the analysis here reduce to twofold integrals over two curves in the 2.5D case. Furthermore, a certain 3×3 matrix central to Beylkin's approach, appearing in the present analysis, reduces to a 2×2 matrix in the 2.5D case and can be analyzed much more readily than here.

The modification of Beylkin which I use is a generalization of the method previously employed in Bleistein and Cohen [1979, 1982], Bleistein and Gray [1985], Cohen, Bleistein and Hagin [1985], and Bleistein, Cohen and Hagin [1985a] and [1985b]. The essential feature of this modification is a fundamental result in Fourier analysis, namely, given the Fourier transform of a function with surfaces (3D) or curves (2D) of discontinuity, multiplication of the Fourier data by $\pm ik$ before inversion, produces the array of singular functions of the discontinuity curves or surfaces. Here, k is the magnitude of the spatial transform vector variable and $\pm 1 = sgn\omega$ (time transform variable) in the present application. In the application to Beylkin's result, I need only identify his inversion representation as an inverse Fourier transform and then identify k for this representation. It then remains to carry out the verification of the validity of the application of this theory to upward scattered data in this general context.

This paper contains an extensive appendix on the relation between Beylkin's transformation determinant, $h(\underline{x},\underline{\xi})$, and the Jacobi determinant which naturally arises in ray theory. This analysis is important if the method is to be implemented numerically. Furthermore, the length of this appendix reflects my personal interest in this interplay between ray theory and asymptotic inversion.

2. MODIFYING BEYLKIN'S RESULT TO OBTAIN THE SINGULAR FUNCTION

I present here Beylkin's solution to the acoustic inverse problem in a constant density medium. Let us consider a seismic experiment carried out on the surface of the earth in which the source/receiver pairs, $\underline{x}_s$ and $\underline{x}_r$, respectively, are identified by a parameter $\underline{\xi} = (\xi_1, \xi_2)$ as follows:

$$\underline{x}_s = \underline{x}_s(\underline{\xi}), \quad \underline{x}_r = \underline{x}_r(\underline{\xi}). \tag{1}$$

For example, for the case of common source, $\underline{x}_s$ would be a constant vector denoting that fixed position and the function $x_r(\underline{\xi})$ would be a parametric representation of the receiver surface; for the common receiver case, the roles of $\underline{x}_s$ and $\underline{x}_r$ would be reversed; for the common offset case or common midpoint case, both $\underline{x}_s$ and $\underline{x}_r$ would vary with $\underline{\xi}$.

It is assumed that $u(\underline{x}, \underline{x}_s, \omega)$ is the response to an impulsive point source at $\underline{x}_s$ satisfying the wave equation

$$\nabla^2 u - \frac{\omega^2}{v^2} u = \delta(\underline{x} - \underline{x}_s) \; . \tag{2}$$

Born-approximate inversion is based on the assumption that the propagation speed can be written in terms of some reference speed, $c(\underline{x})$, and a small perturbation $\alpha(\underline{x})$ as follows:

$$\frac{1}{v^2} = \frac{1}{c^2} \left[1 + \alpha \right] . \tag{3}$$

The total field, u in (2), is then written in terms of an incident field which satisfies (2) with v replaced by c, and a scattered field, $u_S(\underline{x},\underline{x}_s,\omega)$, which is everything else.

It is further assumed that observations of this latter field are made at the points $\underline{x} = \underline{x}_r(\xi)$ for sources at the points $\underline{x} = \underline{x}_s(\xi)$. Beylkin's inversion formula for $\alpha(\underline{x})$ is

$$\alpha(\underline{x}) \sim \frac{c^2(\underline{x})}{8\pi^3} \iint_{S_\xi} d^2\xi \; \frac{|h(\underline{x},\xi)|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)} \cdot \int d\omega \; F(\omega) \exp\{-i\omega[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)]\} \, D(\xi,\omega) \; . \tag{4}$$

In this equation I have used the following notation. The domain of integration S_ξ is the set of ξ-values which are required to cover the source/receiver array. The notation S_0 is reserved for the surface on which $\underline{x}_s$ and $\underline{x}_r$ are located. The domain of integration in ω is limited by the "filter" $F(\omega)$. I take this function to be symmetric and smoothly tapering to zero at the ends of its support. I think of $F(\omega)$ as a smoothly tapered version of the source wavelet. In (2), there should have been some frequency domain source function on the right side but I omitted it there in order not to introduce two functions, one for the source and another for the smoothing filter. Both are contained in this one function in the inversion formula. The functions $\tau(\underline{x},\underline{x}_s)$ and $A(\underline{x},\underline{x}_s)$ [$\tau(\underline{x},\underline{x}_r)$ and $A(\underline{x},\underline{x}_r)$] are the WKBJ or ray-theoretic phase and amplitude of the Green's function with source at $\underline{x}_s$ [$\underline{x}_r$] and observation point $\underline{x}$, discussed in Appendix A. The function, $D(\underline{\xi},\omega)$ is a shortened notation for the observed data at the upper surface:

$$D(\underline{\xi},\omega) = u_S(\underline{x}_r(\underline{\xi}),\underline{x}_s(\underline{\xi}),\omega) \quad . \tag{5}$$

The function, $h(\underline{x},\underline{\xi})$, is the essential element of Beylkin's result. It is the determinant,

$$h(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \dfrac{\partial}{\partial \xi_1} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \dfrac{\partial}{\partial \xi_2} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \end{vmatrix} \tag{6}$$

It is assumed throughout that $h \neq 0$ and is finite. For four configurations of $\underline{x}_s(\underline{\xi})$ and $\underline{x}_r(\underline{\xi})$ of interest in geophysical experiments, we show in Appendix B that this is equivalent to the assumption that there are no caustics in the ray families between the output point $\underline{x}$ and the upper surface, S_0.

I will digress here to provide an interpretation of the result (4). This interpretation is a synthesis of Beylkin's own presentation and the discussion to be found in Cohen, Bleistein and Hagin [1985]. Let us consider the high frequency Born approximation of the data, $D(\underline{\xi},\omega)$, in terms of the perturbation, $\alpha(\underline{x})$. That data is given by

$$D(\underline{\xi},\omega) = -\omega^2 \int c^{-2}(\underline{x}') \, \alpha(\underline{x}') \, A(\underline{x}',\underline{x}_s) \, A(\underline{x}',\underline{x}_r)$$

$$\cdot \exp\{i\omega[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r)]\} \, d^3x' \ . \tag{7}$$

This result is inserted into (4) to produce the following.

$$\alpha(\underline{x}) \sim -\frac{c^2(\underline{x})}{8\pi^3} \iint_{S_\xi} d^2\xi \, \frac{|h(\underline{x},\underline{\xi})|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)} \int \omega^2 \, d\omega \, F(\omega)$$

$$\cdot \int c^{-2}(\underline{x}') \, \alpha(\underline{x}') \, A(\underline{x}',\underline{x}_s) \, A(\underline{x}',\underline{x}_r) \, d^3x' \tag{8}$$

$$\cdot \exp\{i\omega \, \Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\} \ .$$

In this equation,

$$\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r) = \tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r) - [\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \tag{9}$$

is the difference of travel times, source to input point to receiver minus source to output point to receiver.

Beylkin's theoretical approach to the asymptotic inversion predicts that the dominant critical point of the integral (8) is the point where Φ vanishes, namely, where $\underline{x}' = \underline{x}$, for any choice of $\underline{\xi}$. Furthermore, in a neighborhood of that point, linear approximation (or first term of the

Taylor series) for Φ yields

$$\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r) \approx \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \cdot (\underline{x}' - \underline{x}) \ . \tag{10}$$

With this approximation, the phase in (8) has the form of a Fourier phase, $i\underline{k}\cdot(\underline{x}' - \underline{x})$, with the wave vector $\underline{k}$ given by

$$\underline{k} = \omega \ \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \ . \tag{11}$$

I view (11) as defining a change of variables of integration from $(\omega,\underline{\xi})$ -- three variables -- to $\underline{k}$ -- also three variables. The function, $\omega^2 h(\underline{x},\underline{\xi})$ is just the Jacobian of that transformation, so that

$$\omega^2 \left|h(\underline{x},\underline{\xi})\right| \, d\omega d^2\xi = dk^3 . \tag{12}$$

Now the integral in (8) is seen to be of the form of a forward and inverse Fourier transform, producing the integrand evaluated at $\underline{x}' = \underline{x}$. That evaluation, indeed, yields $\alpha(\underline{x})$.

This ends the digression.

It is this interpretation as a Fourier integral which will allow me to deduce a representation for the reflectivity function from the solution (8) for $\alpha(\underline{x})$. I base that result on a theory for identifying surfaces of discontinuity of a function from large $|\underline{k}|$ band limited data for the function [Cohen and Bleistein, 1979, Bleistein, 1984]. The result is that if $\alpha(\underline{x})$ has Fourier transform $\tilde{\alpha}(\underline{k})$, then

$$\frac{\partial\alpha(\underline{x})}{\partial n} \longleftrightarrow i(\mathrm{sign}\,\omega)\,|\underline{k}|\,\tilde{\alpha}(\underline{k}) \ . \tag{13}$$

In this equation, $\partial\alpha/\partial n$ denotes the upward normal derivative of α at the surfaces of discontinuity and the correspondence means that if we have the Fourier data $\tilde{\alpha}(\underline{k})$, we obtain the Fourier data for $\partial\alpha/\partial n$ by the indicated multiplication. In fact, I have now interpreted (8) as a Fourier integral with $\underline{k}$ defined by (11). Thus, to obtain $\partial\alpha/\partial n$, I need only multiply the integrand in (8) by the factor

$$i(\mathrm{sign}\,\omega)\,|\underline{k}| = i\omega\,\left|\nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)]\right| \ . \tag{14}$$

Let us suppose that $\alpha(\underline{x})$ has a discontinuity surface, S. Then the upward normal derivative at any point on S is proportional to a Dirac delta function of arclength along a curve normal to S with peak of that delta function on S itself. This delta function is the <u>singular function</u> of the surface S. It is through this depiction of the singular functions or the discontinuity surfaces of α that processing for $\partial\alpha/\partial n$ provides a reflector map. Furthermore, for each surface the multiplier of the singular function is equal to the upward jump in $\alpha(\underline{x})$ across the surface, namely,

$$\alpha(\underline{x}-) - \alpha(\underline{x}+) \approx c^2(\underline{x})\left[\frac{1}{v^2(\underline{x}-)} - \frac{1}{v^2(\underline{x}+)}\right] \approx \frac{v(\underline{x}+) - v(\underline{x}-)}{2c(\underline{x})} \ . \tag{15}$$

Here, $\alpha(\underline{x}-)$ is the limiting value of α from points above the surface and $a(\underline{x}+)$ is the limiting value of α from points below the surface. I have written approximate equalities in these equations and retained only the linear approximation from (3) because this is, after all, only a linear inversion theory at this point. That will be remedied in Section 5 below; we must content ourselves with this result for the present. One can see, however, that the singular function theory combined with the Born, high frequency inversion theory provides a reflector map and a basis for parameter estimation.

The modification (13,14) is now used in the expression (4) for α to write down the following result for $\partial\alpha/\partial n$:

$$\frac{\partial\alpha}{\partial n} \sim \frac{c^2(\underline{x})}{8\pi^3} \iint_{S_\xi} d^2\xi \; \frac{|h(\underline{x},\xi)| \; |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)}$$

$$\cdot \int i\omega\, d\omega \; F(\omega) \; \exp\{-i\omega[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)]\} \; D(\xi,\omega) \; . \tag{16}$$

This is our first inversion formula. I will analyze this formula in the next three sections but also offer alternative inversion formulas with various computational advantages. The key difference among them will be powers of $|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|$ in the first line.

3. ASYMPTOTIC ANALYSIS - PRELIMINARY RESULTS

For the inversion operator (16), I use for $D(\xi,\omega)$ the Born-approximate data (7) to obtain an expression analogous to (8), expressing the output, $\partial\alpha/\partial n$, in terms of the perturbation, α, itself. The result is

$$\frac{\partial\alpha}{\partial n} \sim -\frac{c^2(\underline{x})}{8\pi^3} \iint_{S_0} d^2\xi \; \frac{|h(\underline{x},\xi)| \; |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)} \int i\omega^3 \, d\omega \, F(\omega) \tag{17}$$

$$\cdot \int c^{-2}(\underline{x}') \; \alpha(\underline{x}') \; A(\underline{x}',\underline{x}_s) \; A(\underline{x}',\underline{x}_r) \; \exp\{i\omega \, \Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\} d^3x'.$$

The phase, Φ, is given by (9); $A(\underline{x},\underline{x}_s) \exp\{i\omega\tau(\underline{x},\underline{x}_s)\}$ is the ray-theoretic or WKBJ Green's function for source point at $\underline{x}s$ and observation point, $\underline{x}$, with a similar description for the other amplitudes and traveltimes; the Jacobian, $h(\underline{x},\xi)$ is defined by (6).

My objective is to show that the multi-fold integral in (17) asymptotically produces an array of scaled singular functions of the surfaces of discontinuity of $\alpha(\underline{x})$. To do this, I will apply the method of stationary phase [Bleistein, 1984] to the five-fold integral in $\underline{x}'$ and ξ under the assumption of "high frequency", and then analyze the remaining ω-integral. I remind the reader that by high frequency I mean that

$$\Lambda = \frac{2\omega L}{c_0} = \frac{4\pi f L}{c_0} \gg 1. \tag{18}$$

In this equation, L is a "typical length scale", c_0 is a local estimate of the background sound speed and f is the frequency in Hz. The dimensionless parameter, Λ, then, measures typical lengths in units of the wave length. In practice, a value of Λ of 3 is sufficient for high frequency asymptotic analysis to adequately approximate a Fourier type integral, such as the one we have here.

The integral (17) does not even converge unless we make some assumption about the nature of $\alpha(\underline{x})$. The reason is that the amplitudes $A(\underline{x}',\underline{x}_s)$ and $A(\underline{x}',\underline{x}_r)$ each decay only as $1/|\underline{x}'|$ at infinity, while d^3x' is $O(|\underline{x}'|^2 d|\underline{x}'|)$ at infinity. Thus, to avoid convergence problems at infinity, I will assume that $\alpha(\underline{x}')$ has finite support; that is, this function is zero outside of some finite domain. This is not a serious constraint. Real data is always of finite extent both spatially and temporally. In a model experiment over all space and time, one simply models this finiteness of data as being a consequence of the finite support of the function $\alpha(\underline{x}')$. Of course, our output may contain "artifacts" which arise from the boundaries of the input data set. We must be alert to these and reject them.

Alternatively, one could taper the data set both spatially and temporally to minimize the effects of the abrupt boundaries. Equivalently, I assume that $\alpha(\underline{x}')$ vanishes smoothly at the boundaries of its support domain, while I will still allow $\alpha(\underline{x}')$ to have discontinuities inside that domain. In this manner, the mathematical asusmption that $\alpha(\underline{x}')$ has finite support and vanishes smoothly at the boundary of its support is seen as a physically reasonable assumption, while providing a useful mathematical constraint for our further analysis.

Let us now consider applying the method of stationary phase to (17) in the variables $\underline{x}'$ and ξ. Then the first derivatives of Φ with respect to all of these variables would have to be equal to zero. I claim that, except, in unusual (pathological) cases this cannot happen. More specifically, the derivatives with respect to all of the $\underline{x}'$ variables cannot all simultaneously be zero. To see why this is so, I write down those derivatives, using (9):

$$\nabla'\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r) = \nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r). \tag{19}$$

In this equation, ∇' denotes the gradient with respect to $\underline{x}'$. The reader is reminded that $\nabla'\tau(\underline{x}',\underline{x}_s)$ $[\nabla'\tau(\underline{x}',\underline{x}_r)]$ is the tangent vector to the geometrical optics ray from $\underline{x}_s$ $[\underline{x}_r]$ to $\underline{x}'$ with background sound speed $c(\underline{x}')$. In order for $\nabla'\Phi$ to be zero, then, the ray directions would have to be anti-colinear. Equivalently, both rays would have to be segments of a single ray which emanates from one of the points $\underline{x}_s$, $\underline{x}_r$, travels into the subsurface and turns back to the surface to emerge at the other of these two points. We will assume that $\underline{x}_s$, $\underline{x}_r$ and $\underline{x}'$ are not a triple of points for which this can occur. ON the other hand, however, for vertical seismic profiling or other tomographic-like inversion problems, this would be exactly the stationary point of interest.

The method of stationary phase dictates that when there are no interior stationary points, one applies integration by parts (the divergence theorem in more than one dimension) to replace the integral over the interior by an integral over certain boundary surfaces. There are two such types of boundaries: the actual boundary of the domain of integration (where we have assumed that $\alpha(\underline{x}') = 0$) and the <u>discontinuity surfaces</u> of the integrand.

For the present, I assume that the only discontinuities are in $\alpha(\underline{x}')$, itself. At the end of Section 4, I will discuss the effects of discontinuities in the background speed c, as well.

Let us consider, for a moment, only the $\underline{x}'$-integral in (17):

$$I = \int c^{-2}(\underline{x}')\ \alpha(\underline{x}')\ A(\underline{x}',\underline{x}_s)\ A(\underline{x}',\underline{x}_r)\ \exp\{i\omega\,\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\}\ d^3x'. \tag{20}$$

I denote by S_1, S_2..., the discontinuity surfaces of $\alpha(\underline{x}')$. For simplicity, we sahould think of them as extending across the support domain of $\alpha(\underline{x}')$ and not intersecting. (This will simplify our analysis somewhat. It will be clear below that the case of intersecting surfaces -- lenses, for example -- can easily be included in the discussion.) I assume, therefore, that $\alpha(\underline{x}')$ is given by different functions between the surfaces:

$$\alpha(\underline{x}') = \begin{cases} \alpha_0(\underline{x}'), & \underline{x}' \text{ between } S_0 \text{ and } S_1\ , \\ \alpha_1(\underline{x}'), & \underline{x}' \text{ between } S_1 \text{ and } S_2\ , \\ \quad\cdots\cdot & \\ \alpha_n(\underline{x}'), & \underline{x}' \text{ between } S_n \text{ and } S_{n+1}, \end{cases} \tag{21}$$

with S_0 the observation surface on which $\underline{x}_s$ and $\underline{x}_r$ reside.

Let us consider the integral in $\underline{x}'$ over the domain between S_j and S_{j+1} where $\alpha(\underline{x}') = \alpha_j(\underline{x}')$. In order to apply the divergence theorem, we must first write the integrand in the $\underline{x}'$ integral as a divergence. To do so, consider, first, the identity

$$W\exp\{i\omega\Phi\} = \nabla'\cdot\left[\frac{\nabla'\Phi}{i\omega|\nabla'\Phi|^2} W\exp\{i\omega\Phi\}\right] - \nabla'\cdot\left[\frac{\nabla'\Phi}{i\omega|\nabla'\Phi|^2} W\right]\exp\{i\omega\Phi\}. \quad (22)$$

In this expression, I have used W to denote the amplitude of the $\underline{x}'$-integral in (20). Note that the second term on the right here is of the same form as the left side of this equation with a different amplitude function and a factor $(i\omega)^{-1}$. Thus, asymptotically, one should expect that the integral of this term will be of lower order in ω $[O(\omega^{-1})]$ than the integral of the left side. Therefore, to leading order asymptotically, I need only integrate the first term on the right. It is to this integral that the divergence theoremcan be applied, to obtain

$$I = \int_{S_j - S_{j+1}} c^{-2}(\underline{x}')\,\alpha_j(\underline{x}')\,A(\underline{x}',\underline{x}_s)\,A(\underline{x}',\underline{x}_r)\exp\{i\omega\,\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\} \quad (23)$$

$$\cdot\frac{\hat{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]}{\left|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)\right|^2}\,dS' .$$

In this equation, $\hat{n}$ denotes the upward normal on the surface, S_j or S_{j+1}. This is the outward normal tc the domain of integration at the upper surface, but it is the inward normal at the lower surface. Thus, one obtains surface integrals of opposite sign when the result of the divergence theorem is expressed in terms of this upward normal. In particular, I assume that $\alpha = 0$ on S_0 so that the integral over S_0 is equal to zero. I have also explicitly written $\nabla'\Phi$ in terms of the gradients of the separate travel times, from (9).

It is now necessary to sum integrals of the form (23) over all of the separate domains of definition of $\alpha(\underline{x}')$. For each domain integral we obtain a pair of surface integrals. When this sum is re-ordered by surfaces instead of domains, we obtain a pair of integrals over each surface in which the only difference in the integrand arises from the discontinuities of $\alpha(\underline{x}')$. Let us introduce the notation

$$\Delta_j = [\alpha(\underline{x}'-) - \alpha(\underline{x}'+)], \ \underline{x}' \text{ on } S_j . \tag{24}$$

As in the previous section, (-) denotes the limit through lower values of z (from above) and (+) denotes the limit from below. Now,

$$\frac{\partial \alpha}{\partial n} = \sum \frac{\partial \alpha_j}{\partial n} , \tag{25}$$

where

$$\frac{\partial \alpha_j}{\partial n} \sim -\frac{c^2(\underline{x})}{8\pi^3} \int\int_{S_\xi} d^2\xi \ \frac{|h(\underline{x},\xi)| \ |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)} \int \omega^2 \, d\omega \, F(\omega)$$
$$\cdot \int_{S_j} c^{-2}(\underline{x}') \ \Delta_j \ A(\underline{x}',\underline{x}_s) \ A(\underline{x}',\underline{x}_r) \exp\{i\omega \, \Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\}$$
$$\cdot \frac{\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]}{|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)|^2} \, dS' . \tag{26}$$

In this integral, the surface S_j is parametrized by two parameters, $\underline{\sigma} = (\sigma_1,\sigma_2)$:

$$\underline{x}' = \underline{x}'(\underline{\sigma}), \ \underline{x}' \text{ on } S_j. \tag{27}$$

It might be more proper to index $\underline{\sigma}$ by j, as well. However, I will continue the discussion below focused on only one surface and hence dispense with indexing on this variable. In terms of these parameters,

$$dS' = \sqrt{g_j}\, d\sigma_1 d\sigma_2, \tag{28}$$

with g_j the first fundamental form of differential geometry for S_j,

$$g_j = \left| \frac{d\underline{x}'}{d\sigma_1} \times \frac{d\underline{x}'}{d\sigma_2} \right|^2 = \left| \det \left[\frac{d\underline{x}'}{d\sigma_k} \cdot \frac{d\underline{x}'}{d\sigma_m} \right] \right|, \ k,m = 1,2 \ . \tag{29}$$

Here $\times$ denotes the vector cross product and $\cdot$ denotes the vector dot product.

4. ASYMPTOTIC ANALYSIS OF OUTPUT FOR BORN-APPROXIMATE DATA

I will now apply the method of stationary phase to (26) in the four variables $(\underline{\xi},\underline{\sigma})$. The phase Φ is a function of these variables through the dependence of $\underline{x}'$ on $\underline{\sigma}$ and the dependence of $\underline{x}_s$ and $\underline{x}_r$ on ξ. Equation (9) is used to write the four first derivatives of Φ in terms of the derivatives of the travel times:

$$\frac{d\Phi}{d\xi_m} = \nabla_s\left[\tau(\underline{x}',\underline{x}_s) - \tau(\underline{x},\underline{x}_s)\right]\cdot\frac{d\underline{x}_s}{d\xi_m} + \nabla_r\left[\tau(\underline{x}',\underline{x}_r) - \tau(\underline{x},\underline{x}_r)\right]\cdot\frac{d\underline{x}_r}{d\xi_m}$$

$$\frac{d\Phi}{d\sigma_m} = \nabla'\left[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r)\right]\cdot\frac{d\underline{x}'}{d\sigma_m}\ , \ m = 1,2. \tag{30}$$

The stationary points in $(\underline{\xi},\underline{\sigma})$ are determined by requiring that these first derivatives all be equal to zero.

In Appendix C, I discuss the conditions under which Φ is stationary. The stationary phase conditions are stated as equation (C1). Also, in that appendix I show that, for $\underline{x}$ on the surface S_j for some fixed j, there is a unique stationary triple, $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$, with $\underline{x}' = \underline{x}$. This is shown for the following source/receiver configurations of practical interest: common source, common receiver and common offset. Although I have only considered here the fully three dimensional problem, this analysis specializes to the cases of 2.5D inversion.

I will proceed below by focusing attention on this stationary point on S_j when $\underline{x}$ is in the neighborhood of S_j. That is, this is the stationary point which has limit $\underline{x}' = \underline{x}$ as $\underline{x}$ approaches S_j. If there were no source/receiver pair in the seismic survey under consideration which included the particular $\underline{x}_s$ and $\underline{x}_r$ needed to complete the stationary triple, then the asymptotic contribution for that point $\underline{x}$ would be of lower order in ω and almost always of smaller value numerically after the ω integration than the result we will obtain below. Thus, I proceed under the assumption that such a stationary triple has been determined and that the corresponding values of $\underline{\sigma}$ and $\underline{\xi}$ are interior points of the respective domains of integration.

Stationary Phase Evaluation

The integral in (26) is evaluated by the method of multi-dimensional stationary phase in the four variables $\underline{\sigma}$ and $\underline{\xi}$. The result is

$$\frac{\partial \alpha_j}{\partial n} \sim -\Delta_j \frac{c^2(\underline{x})}{c^2(\underline{x}')} \frac{A(\underline{x}',\underline{x}_s)\,A(\underline{x}',\underline{x}_r)}{A(\underline{x},\underline{x}_s)\,A(\underline{x},\underline{x}_r)} \frac{|h(\underline{x},\underline{\xi})|\;|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|}{\left|\det\left[\Phi_{\xi\sigma}\right]\right|^{1/2}}$$

$$\cdot \frac{\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]}{|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)|^2} \sqrt{g_j}\; I(\underline{x}). \tag{31}$$

In this equation, g_j is defined by (28) and $I(\underline{x})$ denotes the integral

$$I(\underline{x}) = \frac{1}{2\pi} \int F(\omega) \exp\{i\omega\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r) + i(\operatorname{sign}\omega)(\pi/4)\operatorname{sig}(\Phi_{\xi\sigma})\}\, d\omega\,, \tag{32}$$

This integral, as well as the entire right side of (31), is a function of $\underline{x}$, alone, because $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ are determined as functions of $\underline{x}$ from the stationarity conditions, (C1). The matrix, $[\Phi_{\xi\sigma}]$, is a 4×4 matrix,

$$\left[\Phi_{\xi\sigma}\right] = \begin{bmatrix} \dfrac{\partial^2\Phi}{\partial\xi_i\partial\xi_m} & \dfrac{\partial^2\Phi}{\partial\xi_i\partial\sigma_m} \\[2ex] \dfrac{\partial^2\Phi}{\partial\xi_i\partial\sigma_m} & \dfrac{\partial^2\Phi}{\partial\sigma_i\partial\sigma_m} \end{bmatrix}, \quad i,m = 1,2; \tag{33}$$

$\det[\Phi_{\xi\sigma}]$ denotes the determinant of this matrix and $\mathrm{sig}(\Phi_{\xi\sigma})$ denotes the signature of the matrix, which is the number of positive eigenvalues minus the number of negative eigenvalues of the matrix.

Determinant and Signature for $\underline{x}$ near S_j

Since it is expected that $\partial\alpha_j/\partial n$ peaks for $\underline{x}$ on S_j, we are interested in evaluating the result (31, 32, 33) for $\underline{x}$ near S_j. Let us first consider the behavior of the matrix $[\Phi_{\xi\sigma}]$ in (33) when $\underline{x}$ is on S_j. In this case, σ can be fixed before evaluating the second derivatives with respect to ξ_i, ξ_m. In that limit, $\Phi = 0$; the entire 2X2 matrix in the upper left hand corner of $[\Phi_{\xi\sigma}]$ is a matrix of zeroes; the determinant of $[\Phi_{\xi\sigma}]$ is just the square of the determinant of the 2X2 matrix in the upper left hand corner:

$$\det\left[\Phi_{\xi\sigma}\right] = \left[\det\left[\frac{\partial^2\Phi}{\partial\xi_k\partial\sigma_m}\right]\right]^2 = \left[\det\left[\frac{\partial^2\left[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r)\right]}{\partial\xi_k\partial\sigma_m}\right]\right]^2, \tag{34}$$

with $\underline{x}'$ evaluated at the stationary point $\underline{x}$ on S_j and $\underline{x}_s$ and $\underline{x}_r$ evaluated so as to make the phase stationary.

From this result, we see that the determinant is positive, so that the eigenvalues of each sign must occur in pairs. Thus, the only choices for $\mathrm{sig}(\Phi_{\xi\sigma})$ are ± 4 and 0 and the only effect that the signature factor can have on the final result in (31,32) is a multiplication by -1 or +1, respectively. In Appendix D, I show that, in fact, the signature is zero and the multiplier is +1. In this case, the integral $I(\underline{x})$ defined by (32) becomes

$$I(\underline{x}) = \frac{1}{2\pi} \int F(\omega) \exp \{i\omega\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\}\, d\omega \ . \tag{35}$$

I will assume that the original source was impulsive. Thus, from the assumptions about $F(\omega)$ in Section 2, it can be seen that $I(\underline{x})$ is a band limited Dirac delta function of the argument, $\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)$. Therefore I set

$$I(\underline{x}) = \delta_B(\Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)), \tag{36}$$

where I have used the subscript B to remind us that this is a band limited delta function.

The function, Φ, is equal to zero on the surface S_j. Thus, the support of this delta function includes S_j. In fact, this is the only zero and it is isolated. To see why this is so, let us take the gradient of Φ with respect to $\underline{x}$, with $\underline{x}',\underline{x}_s$ and $\underline{x}_r$ defined by the stationarity conditions (C1):

$$\frac{d\Phi}{dx_j} = \frac{\partial\Phi}{\partial x_j} + \sum_{k,m} \frac{\partial\Phi}{\partial x'_k}\frac{\partial x'_k}{\partial\sigma_m}\frac{\partial\sigma_m}{\partial x_j} + \frac{\partial\Phi}{\partial x_{rk}}\frac{\partial x_{rk}}{\partial\xi_m}\frac{\partial\xi_m}{\partial x_j} + \frac{\partial\Phi}{\partial x_{sk}}\frac{\partial x_{sk}}{\partial\xi_m}\frac{\partial\xi_m}{\partial x_j}\ , \quad j = 1,2,3. \tag{37}$$

In this equation, each sum on k is zero by the stationarity conditions (C1).

Thus, the total derivative with respect to x_j is just the partial derivative with respect to the explicit x_j in Φ. We can now conclude that $\nabla\Phi$ is not zero by the same assumptions as were made to conclude that $\nabla'\Phi$ (19) was not zero. Consequently, the only zero of Φ, subject to the stationarity conditions, (C1), is the surface S_j, itself. By standard rules about delta functions we can now write $I(\underline{x})$ in terms of a delta function of a single arclength variable having the property that it measures arclength along a curve normal to S_j. If we denote that arclength by s, then

$$I(\underline{x}) = \frac{\delta_B(s)}{|\nabla\Phi|} = \frac{\delta_B(s)}{|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|} . \tag{38}$$

This delta function, with support on S_j is the singular function of the surface, S_j, introduced below equation (14). Below, we will denote this function by $\gamma_j(\underline{x})$. Determination of the singular function of a surface constitutes mathematical imaging of the surface. A plot of the band limited delta function $\gamma_{jB}(\underline{x})$ will, indeed, depict the surface. In fact, standard seismic output depicts the reflectors by plotting their singular functions within a scale factor. By using the result (38) in (30) with $\delta_B(s)$ replaced by $\gamma_{jB}(\underline{x})$ one obtains

$$\begin{aligned} \frac{\partial\alpha_j}{\partial n} \sim -\Delta_j \frac{c^2(\underline{x})}{c^2(\underline{x}')} \frac{A(\underline{x}',\underline{x}_s)\, A(\underline{x}',\underline{x}_r)}{A(\underline{x},\underline{x}_s)\, A(\underline{x},\underline{x}_r)} \frac{|h(\underline{x},\xi)|}{\left|\det\left[\Phi_{\xi\sigma}\right]\right|^{1/2}} \\ \cdot \frac{\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]}{\left|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)\right|^2} \sqrt{g_j}\, \gamma_{jB}(\underline{x}) . \end{aligned} \tag{39}$$

Again the reader is reminded that $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ are determined here as functions of $\underline{x}$ by the stationarity conditions (C1), so that the entire result is a function of $\underline{x}$. At this point we have a result which images the jth reflector through the dependence of $\partial\alpha_j/\partial n$ on the function $\gamma_{jB}(\underline{x})$. It only remains to determine the peak amplitude of this result when $\underline{x}$ is on the reflector.

Peak Amplitude

I will now analyze the multiplier of $\gamma_{Bj}(\underline{x})$ at the peak of this function, that is, on the reflector, itself. To do so, let us first introduce the acute angle θ between the upward normal to the surface and the incident and reflected rays on the surface. Note that the downward gradients $\nabla'\tau(\underline{x}',\underline{x}_s)$ and $\nabla'\tau(\underline{x}',\underline{x}_r)$ make angles of $\pi-\theta$ with this normal and make an angle of 2θ with one another. Therefore,

$$\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)] = -\frac{2\cos\theta}{c(\underline{x}')} ,$$

$$\left|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)\right|^2 = \frac{2}{c^2(\underline{x}')}[1+\cos 2\theta] = \left[\frac{2\cos\theta}{c(\underline{x}')}\right]^2 . \tag{40}$$

Finally, in Appendix E, I show that

$$\frac{|h(\underline{x},\underline{\xi})|}{\left|\det\left[\Phi_{\xi\sigma}\right]\right|^{1/2}}\sqrt{g_j} = |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)| = \frac{2\cos\theta}{c(\underline{x})} , \quad \underline{x} \text{ on } S_j. \tag{41}$$

By inserting the results (40, 41) into (38), one obtains the following result for $\partial\alpha_j/\partial n$ at its peak; that is for $\underline{x}$ on S_j:

$$\frac{\partial \alpha_j}{\partial n} \sim \Delta_j \, \gamma_{jB}(\underline{x}) \ , \ \underline{x} \text{ on } S_j . \tag{42}$$

With Δ_j defined by (24), this result confirms the discussion at the end of Section 2. Namely, the modification (16) of Beylkin's original inversion equation (4) asymptotically produces a singular function at each reflector multiplied by the upward jump in $\alpha(\underline{x})$ across the reflector.

I will now proceed to evaluate $\gamma_{jB}(\underline{x})$, itself, on S_j. To do so, I use (35), (38) and (40) with $\underline{x}' = \underline{x}$ to conclude that

$$\frac{\partial \alpha_j}{\partial n} \sim \frac{2\cos\,\theta}{c(\underline{x})} \Delta_j \, \frac{1}{2\pi} \int F(\omega) \, d\omega \ . \tag{43}$$

We see here that the actual numerical value at the peak depends on the opening angle θ between the normal and each of the rays from $\underline{x}_s$ to $\underline{x}$ on S_j and $\underline{x}_r$ to $\underline{x}$ and S_j. Unfortunately, we do not know this angle. From (41) and (11), we recognize the first fraction in (43) as just the factor relating $|\omega|$ and $|\underline{k}|$ in our heuristic transformation (eq. 11) from $(\omega,\underline{\xi})$ to $\underline{k}$. The reason for the appearance of this factor now becomes more apparent. In (43), I have expressed our answer in terms of a band limited frequency domain delta function. I have done this, because that is the way the data is provided in the seismic experiment. On the other hand, $\gamma_B(\underline{x})$ should be expressed as a band limited wave number domain delta function. This first factor is the scale between these two transform variables.

With (40) as a guide, the modification of our processing formula to obtain a result which is θ-independent at the peak is apparent. We need only modify the integral formula in (16) by eliminating the factor of $|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|$! This effectively changes the scale of the normal derivative so that we are no longer differentiating with respect to n, but with respect to $\Phi_j(\underline{x}) = \Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)$ evaluated at the stationary point related to S_j. Thus, symbolically, I set

$$\frac{\partial\alpha}{\partial\Phi} \sim \frac{c^2(\underline{x})}{8\pi^3} \int\int_{S_\xi} d^2\xi \; \frac{|h(\underline{x},\underline{\xi})|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)} \cdot \int i\omega \, d\omega \, F(\omega) \exp\{-i\omega\,[\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)]\}\, D(\underline{\xi},\omega) \;. \tag{44}$$

The symbolic differentiation here is with respect to Φ_j in the neighborhood of S_j. (No such differentiation is actually performed since I only mean to give a suggestive name to the indicated operations on the right side of the equation.) All of the asymptotic analysis carried out above for $\partial\alpha/\partial n$ is identical for $\partial\alpha/\partial\Phi$, except that in the evaluation at the stationary point one must eliminate one factor of $|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)| = 2\cos\theta/c(\underline{x})$ (eq. (42)). Consequently, the peak value of the reflectivity function $\partial\alpha_j/\partial\Phi_j$ in response to data from the jth surface is

$$\frac{\partial\alpha_j}{\partial\Phi_j} = \Delta_j \frac{1}{2\pi} \int F(\omega)\, d\omega \;, \quad x \text{ on } S_j \;, \tag{45}$$

and this replaces (43). Since we know the filter, we know the integral on the right. Correcting the peak output by this scale factor provides an estimate for Δ_j. In (15), the relationship between Δ_j and the change in sound speed across the reflector is stated. Thus, if we know the sound speed above the reflector we obtain an estimate of the sound speed below the reflector consistent with the Born approximation on which the analysis was based.

In summary, then, I have proposed a *modification* of Beylkin's fundamental inversion formula forto Born-approximate data and then proceeded to analyze the output as applied to such data by asymptotic methods. My conclusions are as follows.

(i) The output $\partial\alpha/\partial n$ is proportional to the sum of bandlimited scaled singular functions of the reflectors plus possible lower order (smaller) asymptotic contributions.

(ii) At the peak value of the output on a given reflector, the scale factor of each singular function is the peak value of the bandlimited singular function multiplied by the jump in α across that reflector. The peak value of the singular function is proportional to the cosine of an unknown angle. By a slight modification of the inversion operator, the data can be processed for a function I defined as $\partial\alpha/\partial\Phi$, whose peak value is just the jump in α multiplied by the area under the frequency domain filter of the data.

I repeat that $F(\omega)$ is typically a tapered version of the original source. Thus, this integral, within smoothing is just the source in the time domain, $F(t)$, evaluated at $t = 0$.

I return now to the question of a discontinuous background sound speed. Let us consider the effect on the output of a $c(\underline{x})$ which might be discontinuous above S_j but remains continuous in the neighborhood of S_j. As long as the ray-theoretic phases and amplitudes used in our inversion formulas include the effects of those disontinuities -- e.g., refraction of rays and transmission coefficients -- then the results obtained here remain valid. After the surface S_j has been identified, the effects (f that surface are incorporated into the integral operator to invert for points below S_j.

More generally, given a set of surfaces of discontinuity for the background velocity, the upper velocity is used for a small region below each input surface and then effects of that surface are included to process deeper. This was the method successfully used by Docherty [1985].

5. ASYMPTOTIC ANALYSIS FOR KIRCHHOFF-APPROXIMATE DATA

Let us consider, now, the consequences of applying the inversion formulas (16) to Kirchhoff-approximate data $D(\omega,\underline{\xi})$ for a single reflector, rather than to Born-approximate data. On the one hand, such an application is suspect, since the derivation of the results (16) was based on a Born approximation of the solution to the forward scattering problem for the upward scattered data from the subsurface. On the other hand, Kirchhoff data is not constrained to small perturbations, but only to high frequency, which we have used throughout, in any case. Furthermore, Kirchhoff data more accurately represents the upward scattered field from a single reflector (especially for separated source and receiver) than does Born data. Thus, a useful output of this analysis will allow us to dispense with the small perturbation constraint in the forward scattering problem, a constraint which was imposed on the original derivation of (16) while providing us a better estimate of the effect of applying our inversion operator to field data.

In order that the reflector in question be properly located, it will still be necessary that the background sound speed not vary too much from the "true" sound speed above the reflector in question. However, this is a somewhat less severe constraint when one contemplates a theory that will produce something better than a linear approximation to the velocity variations and we could then contemplate applying a three dimensional "layer stripping" method to proceed from one reflector to another, progressively deeper in the subsurface.

Kirchhoff-approximate data for a seperated source and receiver and single reflector can be found in many sources, including Bleistein [1986], eq. (49). In our present notation the result is

$$D(\xi,\omega) \sim i\omega \int_S R(\underline{x}',\underline{x}_s)\, A(\underline{x}',\underline{x}_s)\, A(\underline{x}',\underline{x}_r)\, \exp\{i\omega[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r)]\} \tag{46}$$

$$\cdot\, \tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]\, dS' \;.$$

In this equation, $R(\underline{x}',\underline{x}_s)$ is the geometrical optics reflection coefficient,

$$R(\underline{x}',\underline{x}_s) = \frac{\left|\partial\tau(\underline{x}',\underline{x}_s)/\partial n\right| - \sqrt{v^{-2}(\underline{x}'+) - v^{-2}(\underline{x}'-) + \left[\partial\tau(\underline{x}',\underline{x}_s)/\partial n\right]^2}}{\left|\partial\tau(\underline{x}',\underline{x}_s)/\partial n\right| + \sqrt{v^{-2}(\underline{x}'+) - v^{-2}(\underline{x}'-) + \left[\partial\tau(\underline{x}',\underline{x}_s)/\partial n\right]^2}} \;. \tag{47}$$

This result into (16) to obtain the following multifold integral representation of the output $\partial\alpha/\partial n$:

$$\frac{\partial\alpha}{\partial n} \sim -\frac{c^2(\underline{x})}{8\pi^3} \int\!\!\int_{S_\xi} d^2\xi\, \frac{|h(\underline{x},\xi)|\; |\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)} \int \omega^2\, d\omega\, F(\omega) \tag{48}$$

$$\cdot \int_S R(\underline{x}',\underline{x}_s)\, A(\underline{x}',\underline{x}\ )\, A(\underline{x}'.\underline{x}\ )\, \exp\{i\omega\, \Phi(\underline{x},\underline{x}',\underline{x}_s,\underline{x}_r)\}$$

$$\tilde{n}\cdot[\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)]\, dS' \quad .$$

This result is to be compared to the integral (26) for the analysis of Born-approximate data from a single reflector S_j. We can see that the integrals are identical except for the following factors:

$$\frac{c^{-2}(\underline{x}')\,\Delta_j}{\left|\nabla'\tau(\underline{x}',\underline{x}_s) + \nabla'\tau(\underline{x}',\underline{x}_r)\right|^2} \longleftrightarrow R(\underline{x}',\underline{x}_s) \ . \tag{49}$$

The left side of the arrow is the factor appearing in the Born-approximate output (26) while the right side is its replacement in the Kirchhoff-approximate output (48). Thus, no further asymptotics need be done. The result of stationary phase applied to (48) can be determined by making the same replacement in the results of applying stationary phase to (26); that is, to (39) and to (43).

The conclusion of this comparison is that for the Kirchhoff data, the formalism (16) produces a singular function of the reflecting surface scaled by some function of $\underline{x}$. That is, the output provides a reflector map. It remains only to determine what the peak value of the output is when $\underline{x}$ is on the reflector. To do so, we must make the replacement (49) when $\underline{x} = \underline{x}'$ and the stationarity conditions (C1) are satisfied. In order to make this substitution, note first that

$$c^2(\underline{x}) \left|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)\right|^2 = 4\cos^2\theta \ . \tag{50}$$

When this result is combined with (49), we see that we need only make the replacement

$$\Delta_j \longleftrightarrow 4\cos^2\theta \; R(\underline{x},\underline{x}_s) \tag{51}$$

at the peak. By using this observation in (43) one concludes that

$$\frac{\partial\alpha}{\partial n} = \frac{8R(\underline{x},\underline{x}_s)\cos^3\theta}{c(\underline{x})} \; \frac{1}{2\pi}\int F(\omega)\, d\omega \;, \; \underline{x} \text{ on } S \quad . \tag{52}$$

I now show how this peak value is related to the previous result, (43). To do so, I use the fact that $\nabla\tau$ makes an angle of $\pi - \theta$ with the normal at the stationary point to rewrite (47) as

$$R(\underline{x},\underline{x}_s) = \frac{c^{-1}(\underline{x})\cos\theta - \sqrt{v^{-2}(\underline{x}+) - v^{-2}(\underline{x}-) + c^{-2}(\underline{x})\cos^2\theta}}{c^{-1}(\underline{x})\cos\theta + \sqrt{v^{-2}(\underline{x}+) - v^{-2}(\underline{x}-) + c^{-2}(\underline{x})\cos^2\theta}} \quad . \tag{53}$$

Now let us assume for a moment that the jump in $v(\underline{x})$ across S is small. One then obtains the leading order approximation of $R(\underline{x},\underline{x}_s)$ for small values of the jump by expanding the square root both in the numerator and the denominator, retaining two terms in the numerator and one in the denominator. The result is that

$$4R(\underline{x},\underline{x}_s)\cos^2\theta \approx c^2(\underline{x}) \left[\frac{1}{v^2(\underline{x}-)} - \frac{1}{v^2(\underline{x}+)} \right] \approx \frac{v(\underline{x}+) - v(\underline{x}-)}{2c(\underline{x})} = \Delta_j \tag{54}$$

where I have used (15) and (24) with $\underline{x} = \underline{x}'$ to obtain the last result. That is, to leading order for small jumps, the present result agrees with the previous result. However, the result (52) predicts that the output will, in fact, produce the geometrical optics reflection coefficient at the peak, even though this coefficient is a nonlinear function of the jump in sound speed.

On the other hand, an estimate of the angularly dependent reflection coefficient is not useful unless we have a means of estimating the angle. Fortunately, we do have such a means. What one must do is compute $\partial\alpha_j/\partial\Phi$ [Eq. (44)] as well as $\partial\alpha/\partial n$, since we know that the peak output of the former by a factor of $2\cos\theta/c(\underline{x})$. That is, if the surface S is one of the surfaces S_j then the peak value (45) is replaced by the result,

$$\frac{\partial\alpha_j}{\partial\Phi_j} = 4R(\underline{x},\underline{x}_s)\cos^2\theta\ \frac{1}{2\pi}\int F(\omega)\,d\omega\ ,\ \underline{x} \text{ on } S\ . \tag{55}$$

Now, by taking the ratio of the results in (52) and (55) one obtains

$$\frac{\dfrac{\partial\alpha_j}{\partial n}}{\dfrac{\partial\alpha_j}{\partial\Phi_j}} \sim \frac{2\cos\theta}{c(\underline{x})}\ ,\ \underline{x} \text{ on } S = S_j\ . \tag{56}$$

From this ratio, $\cos\theta$ is computed. Given $\cos\theta$, (53) is used to compute the sound speed below the reflector in terms of the sound speed above the reflector. This is a fully nonlinear estimate of sound speed consistent with the high frequency asymptotic analysis that has been carried out.

For esthetic reasons it would also be desirable to obtain a result in which the peak value of the output was equal to the reflection coefficient multiplied by the area under the filter. By examining (52), we see that we need only eliminate a factor of $8\cos^3\theta/c(\underline{x})$ at the peak. From (50), we see that the way to do this is to introduce into the inversion operator (16) a divisor of $c^2(\underline{x})\left|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)\right|^2$. Thus, I propose one other operator, which I call the <u>reflectivity function</u> and denote by $\beta(\underline{x})$:

$$\beta(\underline{x}) \sim \frac{1}{8\pi^3} \int\!\!\int_{S_\xi} d^2\xi \, \frac{|h(\underline{x},\xi|}{A(\underline{x},\underline{x}_s)A(\underline{x},\underline{x}_r)\left|\nabla\tau(\underline{x},\underline{x}_s) + \nabla\tau(\underline{x},\underline{x}_r)\right|^2} \cdot \int i\omega \, d\omega \, F(\omega) \exp\{-i\omega[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)]\} \, D(\xi,\omega) \, . \tag{57}$$

With no further analysis, we can be assured that the peak value of the reflectivity function is given by

$$\beta(\underline{x}) \sim R(\underline{x},\underline{x}_s) \frac{1}{2\pi} \int F(\omega) \, d\omega, \ \underline{x} \text{ on } S = S_j \, . \tag{58}$$

I note, further, from comparing this result with (55), that we could as easily determine $\cos\theta$ through the ratio

$$\frac{\dfrac{\partial\alpha_j}{\partial\phi_j}}{\beta(\underline{x})} \sim \cos^2\theta, \ \underline{x} \text{ on } S = S_j \, . \tag{59}$$

In summary, then, the operator derived on the basis of the Born approximation produces a reflector map when applied to Kirchhoff approximate data with a nonlinear estimate of the jump in sound speed across reflectors, consistent with the geometrical optics reflection coefficient of each reflector. The angular dependence of the reflection coefficient is resolved by simultaneously computing two inversion outputs and taking their ratio.

6. CONCLUSIONS

Starting from an inversion operator proposed by Beylkin, I have proposed three modifications of that operator. Each of those operators are shown by asymptotic analysis to produce a reflector map when applied either to Born-approximate input data or Kirchhoff-approximate input data. The peak value of the output of these operators is proportional to the Born approximate reflection coefficient (or jump in $\alpha(\underline{x})$) in the former case and to the geometrical optics reflection coefficient in the latter case. The output also depends upon the opening angle between specular rays from the source to the reflecting surface and from the receiver to the reflecting surface. This opening angle is unknown, but can be eliminated for either type of data. These results are valid for three source/receiver configurations of interest: common (or fixed) source, common receiver or common offset, with the last of these including the zero offset or backscatter case. The analysis also assumes that there are no caustics in the ray trajectories from subsurface points to sources or receivers. The analysis allows for a curved datum surface.

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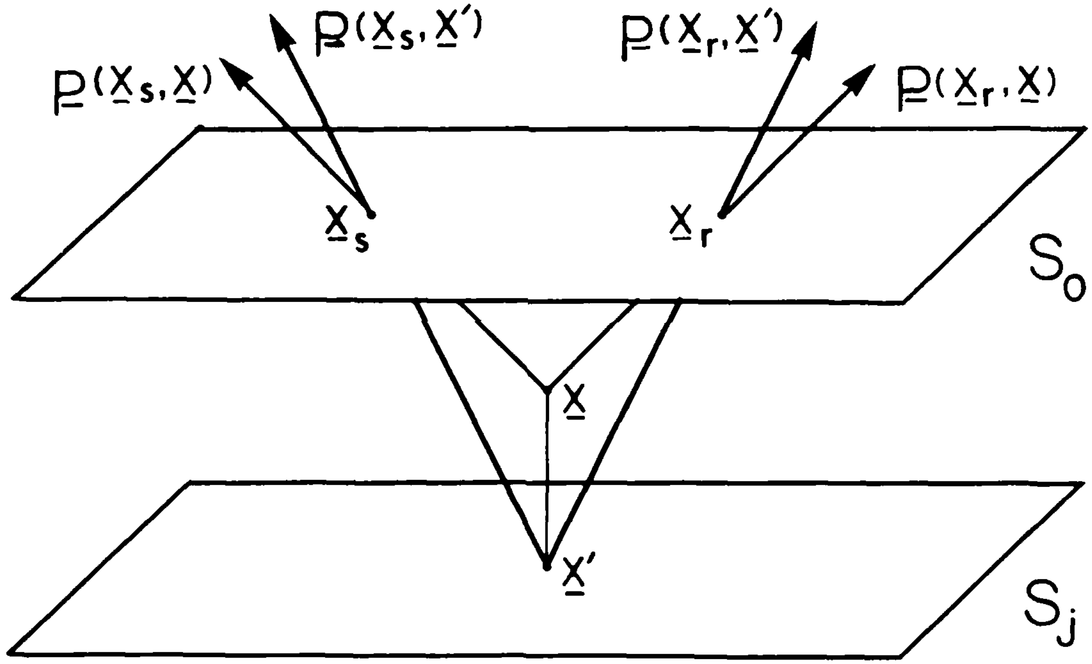
FIGURE CAPTION

Figure C1: Triple $\underline{x}'$, $\underline{x}_s$, $\underline{x}_r$ satisfying stationary phase conditions, Eq. (C1) for a horizontal observation surface, horizontal reflector and constant background sound speed.



$\underline{P}(\underline{X}_s, \underline{X})$
$\underline{P}(\underline{X}_s, \underline{X}')$
$\underline{P}(\underline{X}_r, \underline{X}')$
$\underline{P}(\underline{X}_r, \underline{X})$
$\underline{X}_s$
$\underline{X}_r$
S_o
$\underline{X}$
$\underline{X}'$
S_j

FIGURE C1

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APPENDIX A: RAY-THEORETIC GREEN'S FUNCTION

In this appendix I present some results about the WKBJ or ray-theoretic Green's function,

$$G(\underline{x},\underline{x}',\omega) = A(\underline{x},\underline{x}')\exp\{i\omega\tau(\underline{x},\underline{x}')\} \quad , \tag{A1}$$

satisfying the inhomgeneous Helmholtz equation,

$$\nabla^2 G + \frac{\omega^2}{c^2} G = \delta(\underline{x} - \underline{x}'). \tag{A2}$$

In this equation, c denotes the background sound speed, $c(\underline{x})$, with its argument omitted when it is clear in context. The travel time τ and the amplitude A are chosen so that this equation is satisfied to order ω^2 and ω. The equations they satisfy are called, respectively, the eikonal and transport equations:

$$[\nabla\tau]^2 = 1/c^2, \quad 2\nabla\tau\cdot\nabla A + A\nabla^2\tau = 0 \quad . \tag{A3}$$

A solution of these equations is required to describe a wave emanating from the source point, $\underline{x}'$. Thus, the initial conditions for τ and A are

$$\tau(\underline{x}',\underline{x}') = 0, \quad |\underline{x} - \underline{x}'|A(\underline{x},\underline{x}') \to 1/4\pi, \text{ as } \underline{x} \to \underline{x}' \ . \tag{A4}$$

The problem for τ is solved by the method of characteristics, [Bleistein, 1984]. The characteristics satisfy the system of ordinary differential equations,

$$\frac{d\underline{x}}{d\sigma} = \underline{p}, \ \underline{x} = \underline{x}', \text{ for } \sigma = 0; \ \frac{d\underline{p}}{d\sigma} = \nabla[1/c^2], \quad \underline{p} = \nabla\tau \ . \tag{A5}$$

Let us denote the initial value of $\underline{p}$ by $\underline{p}_0$; that is,

$$\underline{p}_0 = \underline{p}(\underline{x}',\underline{x}'). \tag{A6}$$

This initial value of $\underline{p}$ is not determined, except for its magnitude which must equal $1/c(\underline{x}')$ from the eikonal equation. Each choice of the direction of $\underline{p}$ or, equivalently, each choice of p_{10} and p_{20} within the constraint on $|\underline{p}|$, determines or "labels" an initial direction of a ray, which is the solution curve $\underline{x}(\sigma)$, or, more completely $\underline{x}(\sigma,p_{10},p_{20})$. The rays "sweep out" a volume of space as we vary the three ray parameters. If rays for different choices of (p_{10},p_{20}) do not cross one another -- i.e., <u>if there are no caustics</u> -- then away from $\underline{x} = \underline{x}'$, there is a one-to-one correspondence between values of $\underline{x}$ and values of the triple (σ,p_{10},p_{20}). It is assumed throughout the analysis that there are no caustics in the region of interest.

The travel time also satsifies an ordinary differential equation with respect to σ -- i.e., an ordinary differential equation along the ray -- namely,

$$\frac{d\tau}{d\sigma} = \frac{1}{c^2} \quad , \tag{A7}$$

and thus is determined by integration along the ray. The transport equation, the second equation in (A3), can also be written as an ordinary differential equation along the ray, with solution,

$$A(\underline{x},\underline{x}') = \frac{1}{4\pi\left|p_{30}J\right|^{1/2}} \quad . \tag{A8}$$

The Jacobi determinant, J, is given by

$$J = \frac{\partial(x_1,x_2,x_3)}{\partial(\sigma,p_{10},p_{20})} = \det \begin{vmatrix} \underline{p} \\ \dfrac{\partial \underline{x}}{\partial p_{10}} \\ \dfrac{\partial \underline{x}}{\partial p_{20}} \end{vmatrix} \quad . \tag{A9}$$

The assumption that there are no caustics assures us that J does not vanish, except at $\sigma = 0$, that is, at $\underline{x} = \underline{x}'$.

There are two practical ways to calculate J. First, J can be related to the spreading of ray tubes. That is, J is proportional to the normal cross-sectional area of the differential ray tube. Thus, if one is "shooting" rays, a reasonably accurate approximation of J can be determined by measuring the area of the tube of neighboring rays at the arrival point. The constant of proportionality depends upon the ray parameters that are used in the shooting method.

Alternatively, the components of the Jacobi matrix from which J is computed also satisfy ordinary differential equations with respect to σ; that is ordinary differential equations on the rays. See, Červeny, etal. [1977] or Bleistein, [1984].

One can see from this brief discussion that ray theory lends itself naturally to initial value problems, that is, to the "shooting of rays" from a prescribed point. On the other hand, in the application to inverse problems, one is constantly thinking in terms of a travel time and amplitude between two fixed points, say, $\underline{x}$ and $\underline{x}_s$, for example. Indeed, it will, at times, be convenient to think of the rays as emanating at $\underline{x}$ and arriving at $\underline{x}_s$, and sometimes, the reverse. There are both theoretical and computational difficulties associated with the transition between initial value problems from each point and the boundary value problem between the two points. We will see some of this in the next appendix.

APPENDIX B: ANALYSIS OF $h(\underline{x},\underline{\xi})$

The purpose of this appendix is to introduce simplifications of the determinant $h(\underline{x},\underline{\xi})$, defined by (6), for four cases of interest in seismic exploration. In particular, It will be shown how this matrix is related to the consitutents of the ray theoretic Green's function developed in the previous appendix.

Case 1: Zero-Offset

Let us suppose that we are considering the idealized case of coincident source and receiver. In this case,

$$\underline{x}_s(\underline{\xi}) = \underline{x}_r(\underline{\xi}), \tag{B1}$$

the sum of travel times becomes just twice the travel time to the source/receiver point and the determinant in (6) becomes the simpler result,

$$h(\underline{x},\underline{\xi}) = 8 \det \begin{vmatrix} \nabla\tau(\underline{x},\underline{x}_s) \\ \dfrac{\partial}{\partial\xi_1}\nabla\tau(\underline{x},\underline{x}_s) \\ \dfrac{\partial}{\partial\xi_2}\nabla\tau(\underline{x},\underline{x}_s) \end{vmatrix} . \tag{B2}$$

The gradient $\nabla\tau(\underline{x},\underline{x}_s)$ is the value of the p-vector (introduced in Appendix A) on the ray initiated at $\underline{x}_s$ and arriving at depth at the point, $\underline{x}$. Since this vector satisfies the eikonal equation with right side $1/c^2(\underline{x})$, independent of ξ, we have the result,

$$0 = \frac{\partial}{\partial \xi_j} \frac{1}{c^2} = \frac{\partial}{\partial \xi_j} \underline{p} \cdot \underline{p} = \sum_{k=1}^{3} p_k \frac{\partial}{\partial \xi_j} p_k = 0, \; j = 1,2. \tag{B3}$$

To exploit this identity I multiply the matrix in (B2) by another matrix,

$$P = \begin{vmatrix} 1 & 0 & p_1 \\ 0 & 1 & p_2 \\ 0 & 0 & p_3 \end{vmatrix} , \tag{B4}$$

before computing the determinant. Since $\det P = p_3$, the result of this calculation is

$$h(\underline{x},\underline{\xi}) = \frac{8}{p_3 c^2(\underline{x})} \det \begin{vmatrix} p_1 & p_2 & 1 \\ \dfrac{\partial p_1}{\partial \xi_1} & \dfrac{\partial p_2}{\partial \xi_1} & 0 \\ \dfrac{\partial p_1}{\partial \xi_2} & \dfrac{\partial p_2}{\partial \xi_2} & 0 \end{vmatrix} = \frac{8}{p_3 c^2(\underline{x})} \frac{\partial(p_1, p_2)}{\partial(\xi_1, \xi_2)} . \tag{B5}$$

In this equation, the factor $1/c^2(\underline{x})$ arises as the coefficient in the first row third column, which makes it a factor of the entire third column.

Let us now view the ray connecting $\underline{x}$ and $\underline{x}_S$ as starting from the point at depth, $\underline{x}$, and arriving at the surface at the point, $\underline{x}_S$. The vector denoted by $\underline{p}$, here, is just the negative of the initial value of the $\underline{p}$-vector on that ray, say, $\underline{p}_0^S$. The superscript, s, to denote a ray which emanates from $\underline{x}$ and arrives at $\underline{x}_S$. (Later, I will have need of the notation

p_0^r, as well.) In terms of this new vector, the result (5) can be rewritten as

$$h(\underline{x},\xi) = -\frac{8}{p_{30}^{s}c^{2}(\underline{x})}\frac{\partial(p_{10}^{s},p_{20}^{s})}{\partial(\xi_1,\xi_2)} \tag{B6}$$

The determinant $h(\underline{x},\xi)$ has been written in terms of derivatives of the two ray parameters at depth which label the ray propagating from $\underline{x}$ to $\underline{x}_s$. In fact, under the assumption that there are no caustics, the variables ξ serve equally well as parameters to label the ray from $\underline{x}$ which emerges at the upper surface at the point $\underline{x}_s(\xi)$. Thus, $h(\underline{x},\xi)$ can be interpreted as being proportional to the Jacobian of transformation between two "ray-labeling" sets of parameters. Each parameter pair can be supplemented with a third running parameter along a fixed ray, namely, σ, of Appendix A. Therefore, without changing the value of $h(\underline{x},\xi)$, two-by-two determinant in (6) can be expanded into the three-by-three determinant,

$$h(\underline{x},\xi) = -\frac{8}{p_{30}^{s}c^{2}(\underline{x})}\frac{\partial(p_{10}^{s},p_{20}^{s},\sigma)}{\partial(\xi_1,\xi_2,\sigma)}, \tag{B7}$$

since the third row and column of this augmented determinant have only zeroes except for a one on the main diagonal.

Having used $\underline{x}$ to denote the initial point of this ray, let us think of the running coordinate along the ray as being some other variable, say $\underline{\tilde{x}}$. The derivatives in (B7) are to be evaluated when $\underline{\tilde{x}} = \underline{x}_s$. Thus, I write

$$\frac{\partial(p^s_{10},p^s_{20},\sigma)}{\partial(\xi_1,\xi_2,\sigma)} = \left[\frac{\partial(\tilde{\underline{x}})}{\partial(p_{10},p_{20},\sigma)}\right]^{-1}\left[\frac{\partial(\tilde{\underline{x}})}{\partial(\xi_1,\xi_2,\sigma)}\right]\Bigg|_{\tilde{\underline{x}}=\underline{x}_s} . \tag{B8}$$

The first bracket on the right can be recognized from Appendix A as just the Jacobian of the ray family initiated from $\underline{x}$ and evaluated at the surface point, $\underline{x}_s$. This Jacobian will be denoted by $J(\underline{x}_s,\underline{x})$. The second determinant can be re-interpreted as a triple scalar product of two tangents in the surface S_0 with the ray tangent $d\underline{x}_s/d\sigma = \underline{p}(\underline{x}_s,\underline{x})$. The cross product of the tangents is in the direction of the normal vector at $\underline{x}_s$ and has magnitude $\sqrt{g(\underline{x}_s)} = \sqrt{g_s}$, in complete analogy with the result (29) for the surface S_j. Thus,

$$\frac{\partial(p^s_{10},p^s_{20},\sigma)}{\partial(\xi_1,\xi_2,\sigma)} = \frac{\underline{p}(\underline{x}_s,\underline{x})\cdot\tilde{n}_s\sqrt{g_s}}{J(\underline{x}_s,\underline{x})} \quad , \quad \tilde{n}_s\sqrt{g_s} = \frac{\partial\underline{x}_s}{\partial\xi_1}\times\frac{\partial\underline{x}_s}{\partial\xi_2} \quad . \tag{B9}$$

With this result, $h(\underline{x},\underline{\xi})$ can be rewritten as

$$h(\underline{x},\underline{\xi}) = -\frac{\underline{p}(\underline{x}_s,\underline{x})\cdot\tilde{n}_s\sqrt{g_s}}{p^s_{30}c^2(\underline{x})\ J(\underline{x}_s,\underline{x})} \quad . \tag{B10}$$

What can be seen, here, is that the matrix $h(\underline{x},\underline{\xi})$ is expressible in terms of variables which are computed as part of the ray theoretic Green's function and in terms of parameters defining the structure of the observation surface. For h to be nonzero, J must remain finite and the rays should not emerge tangent to the upper surface. For h to remain finite, J must be nonzero -- no caustics. The singularity at $p^s_{30} = 0$ is only apparent; the product $p^s_{30}J(\underline{x}_s,\underline{x})$ must remain finite in the limit as p^s_{30} approaches zero. This product is exactly what appears in the denominator of the ray-theoretic amplitude, (A8), and must therefore remain nonzero in this limit because one can find alternative representations of the amplitude which confirm that the amplitude remains finite along a ray that is initially horizontal. In the special case where the upper surface is flat, the dot product in (B10) is simply $p_3(\underline{x}_s,\underline{x})$ and the numerator being nonzero requires that the ray direction have some vertical component at the upper surface. Furthermore, in this case, it would be natural to take $\xi_1 = x_{1s}$ and $\xi_2 = x_{2s}$ and then $g_s = 1$. Consequently,

$$h(\underline{x},\underline{\xi}) = -\frac{p_3(\underline{x}_s,\underline{x})}{p^s_{30}c^2(\underline{x})\ J(\underline{x}_s,\underline{x})}\ , \text{ flat upper surface.} \qquad \text{(B11)}$$

As a further simplification, if $c(\underline{x})$ is a constant, p_3 is a constant on each ray. In this case, the result (B11) simplifies still further to

$$h(\underline{x},\underline{\xi}) = -\left[c^2 J(\underline{x}_s,\underline{x})\right]^{-1}, \text{ flat upper surface, constant background.} \qquad \text{(B12)}$$

In this simplest form, the reciprocal dependence between h and J provides the total structure of the relationship between them.

Case 2: Common Source

Let us consider, next, the case in which the source point is fixed. This case was discussed in Cohen, Bleistein and Hagin [1985]. It is included here for completeness.

For this case, the parametrization of the source and receiver points becomes

$$\underline{x}_s(\underline{\xi}) = \text{const.}, \quad \underline{x}_r = \underline{x}_r(\underline{\xi}), \tag{B13}$$

with the function $\underline{x}_r(\underline{\xi})$ ranging over the observation surface, S_0, as $\underline{\xi}$ ranges over its set of values on S_ξ. For this case, the determinant $h(\underline{x},\underline{\xi})$ in (6) becomes

$$h(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \dfrac{\partial}{\partial \xi_1} \nabla\tau(\underline{x},\underline{x}_r) \\ \dfrac{\partial}{\partial \xi_2} \nabla\tau(\underline{x},\underline{x}_r) \end{vmatrix} . \tag{B14}$$

Thus, while the first row remains a sum of two vectors, the second and third row are identified as derivatives of single vectors exactly of the form discussed above. Just as in the previous case, the matrix on the right is to be multiplied by the matrix P defined in (B4) before the determinant is computed. However, I now have to be more precise because the source and receiver are separated. The $\underline{p}$-vector I use in P is the vector $\underline{p}(\underline{x}_r,\underline{x})$. By carrying out the matrix multiplication and then taking determinants, the following result is obtained:

$$h(\underline{x},\underline{\xi}) = \frac{1}{c^2(\underline{x})p_{3r}} \det \begin{vmatrix} p_{1s}+p_{1r} & p_{2s}+p_{2r} & 1+c^2(\underline{x})\underline{p}_s\cdot\underline{p}_r \\ \dfrac{\partial p_{1r}}{\partial\xi_1} & \dfrac{\partial p_{2r}}{\partial\xi_1} & 0 \\ \dfrac{\partial p_{1r}}{\partial\xi_2} & \dfrac{\partial p_{2r}}{\partial\xi_2} & 0 \end{vmatrix} \tag{B15}$$

In this equation,

$$\underline{p}_s = \underline{p}(\underline{x},\underline{x}_s), \qquad \underline{p}_r = \underline{p}(\underline{x},\underline{x}_r) \ . \tag{B16}$$

Except for the subscripts, the two-by-two determinant in the lower left hand corner in (B15) is exactly of the form of *the two-by-two treated in* Case 1. Thus, use of the same method as in Case 1 yields

$$h(\underline{x},\underline{\xi}) = -\frac{\left[1+c^2(\underline{x})\underline{p}_0^s\cdot\underline{p}_0^r\right]\underline{p}(\underline{x}_r,\underline{x})\cdot\hat{\underline{n}}_r\sqrt{g_r}}{p_{30}^r c^2(\underline{x})\ J(\underline{x}_r,\underline{x})} \ . \tag{B17}$$

The subscript zero is again used to designate vectors at $\underline{x}$ pointing along the rays to $\underline{x}_s$ and $\underline{x}_r$ and superscripts on the p-vectors to denote that these are the vectors on the rays oriented *from* $\underline{x}$ to $\underline{x}_s$ and $\underline{x}_r$, reverse of the orientation on the subscripted vectors above. The vector $\underline{p}(\underline{x}_r,\underline{x})$, is the final value of the vector $\underline{p}_0^r$ at the surface point, $\underline{x}_r$.

As above, the determinant $h(\underline{x},\underline{\xi})$ is expressed in terms of elementary parameters of the ray-theoretic Green's function and parameters which characterize the upper surface. Note that the same simplifications as were made above can be made here, too.

When this result is compared to the previous result, (B10), we see that there is one additional way in which h can be zero in (B17). The first factor of the numerator might vanish. This can only occur if the vectors $\underline{p}_0^s$ and $\underline{p}_0^r$ are anti-colinear. That possibility was eliminated in Section 3.

Case 3: Common Receiver

For the common receiver case, we need only interchange the source and receiver point in the previous case. When this is done in (16), the following result is obtained:

$$h(\underline{x},\underline{\xi}) = - \frac{\left[1 + c^2(\underline{x})\underline{p}_0^r \cdot \underline{p}_0^s\right] \underline{p}(\underline{x}_s,\underline{x})\cdot\tilde{n}_s\sqrt{g_s}}{p_{30}^s c^2(\underline{x})\, J(\underline{x}_s,\underline{x})} \quad . \tag{B18}$$

Case 4: The General Case

When both $\underline{x}_s$ and $\underline{x}_r$ are more general functions of $\underline{\xi}$, there is not a great deal of simplification that can be achieved in the representation of $h(\underline{x},\underline{\xi})$. The determinant h in (6) can be rewritten as a sum of four determinants:

$$h(\underline{x},\underline{\xi}) = h_1(\underline{x},\underline{\xi}) + h_2(\underline{x},\underline{\xi}) + h_3(\underline{x},\underline{\xi}) + h_4(\underline{x},\underline{\xi}) \quad , \tag{B19}$$

$$h_1(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \frac{\partial}{\partial \xi_1} \nabla\tau(\underline{x},\underline{x}_s) \\ \frac{\partial}{\partial \xi_2} \nabla\tau(\underline{x},\underline{x}_s) \end{vmatrix} , \qquad (B20)$$

$$h_2(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \frac{\partial}{\partial \xi_1} \nabla\tau(\underline{x},\underline{x}_r) \\ \frac{\partial}{\partial \xi_2} \nabla\tau(\underline{x},\underline{x}_r) \end{vmatrix} , \qquad (B21)$$

$$h_3(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \frac{\partial}{\partial \xi_1} \nabla\tau(\underline{x},\underline{x}_s) \\ \frac{\partial}{\partial \xi_2} \nabla\tau(\underline{x},\underline{x}_r) \end{vmatrix} , \qquad (B22)$$

$$h_4(\underline{x},\underline{\xi}) = \det \begin{vmatrix} \nabla[\tau(\underline{x},\underline{x}_s) + \tau(\underline{x},\underline{x}_r)] \\ \frac{\partial}{\partial \xi_1} \nabla\tau(\underline{x},\underline{x}_r) \\ \frac{\partial}{\partial \xi_2} \nabla\tau(\underline{x},\underline{x}_s) \end{vmatrix} . \qquad (B23)$$

The determinant, $h_2(\underline{x},\xi)$, is exactly like the result (B14) for the entire determinant $h(\underline{x},\xi)$ in the case of a common source; $h_1(\underline{x},\xi)$ is the same determinant as for the common receiver case. Thus, (B18) and (B17) can be used to write these determinants in terms of the parameters of ray propagation:

$$h_1(\underline{x},\xi) = -\frac{\left[1 + c^2(\underline{x})\underline{p}_0^r \cdot \underline{p}_0^s\right] \underline{p}(\underline{x}_s,\underline{x})\cdot\hat{\underline{n}}_s\sqrt{g_s}}{p_{30}^s c^2(\underline{x})\; J(\underline{x}_s,\underline{x})} , \tag{B24}$$

$$h_2(\underline{x},\xi) = -\frac{\left[1 + c^2(\underline{x})\underline{p}_0^s \cdot \underline{p}_0^r\right] \underline{p}(\underline{x}_r,\underline{x})\cdot\hat{\underline{n}}_r\sqrt{g_r}}{p_{30}^r c^2(\underline{x})\; J(\underline{x}_r,\underline{x})} . \tag{B25}$$

The remaining two determinants, $h_3(\underline{x},\xi)$ and $h_4(\underline{x},\xi)$, are not so easily dealt with. The reason is that the second and third rows do not involve derivatives of the same travel time. That is, one row has $\tau(\underline{x},\underline{x}_s)$ while the other has $\tau(\underline{x},\underline{x}_r)$. Thus, a simplification to merely a ray determinant, J, is not possible since these rows are related to different ray families. On the other hand, each of the matrix entries is separately expressible in terms of ray parameters and surface parameters. That derivation now follows.

Let us consider a typical term, $\partial p_i(\underline{x},\tilde{\underline{x}})/\partial\xi_j$, for $i,j = 1,2$, and $\tilde{\underline{x}}$ to be evaluated along the ray to (and ultimately at) $\underline{x}_s$ or $\underline{x}_r$. It is not necessary consider derivatives of p_3, because the eikonal equation (A3) can be used to express them in terms of p_1 and p_2:

$$\frac{\partial p_3}{\partial \xi_j} = -\frac{p_1}{p_3}\frac{\partial p_1}{\partial \xi_j} - \frac{p_2}{p_3}\frac{\partial p_2}{\partial \xi_j} \quad . \tag{B26}$$

For the derivatives of interest,

$$\frac{\partial p_i}{\partial \xi_j} = \sum_{k=1}^{3} \frac{\partial p_i}{\partial \tilde{x}_k} \frac{\partial \tilde{x}_k}{\partial \xi_j} \quad , \; i,j = 1,2. \tag{B27}$$

When $\underline{\tilde{x}}$ is evaluated at $\underline{x}_s$ or $\underline{x}r$, the derivatives $\partial\tilde{x}_k/\partial\xi_j$ simply become derivatives of the expressions $\underline{x}_s(\xi)$ and $\underline{x}_r(\xi)$ with respect to the components of $\underline{\xi}$. Thus, these derivatives are known from the surface geometry. For the other factor on the right in (B27), I again take the point of view that $\underline{p}(\underline{x},\underline{\tilde{x}})$ is the negative of the initial value of the $\underline{p}$-vector along the ray starting from $\underline{x}$ and propagating to $\underline{\tilde{x}}$, denoted above by $\underline{p}_0^s$ or $\underline{p}_0^r$. As noted in Appendix A, the first two components of these vectors may be used as ray-labeling parameters: see the discussion below (A6). Thus, as a first step,

$$\frac{\partial p_i}{\partial \tilde{x}_k} = -\frac{\partial p_{i0}}{\partial \tilde{x}_k} \quad . \tag{B28}$$

Now, as in Appendix A, the family of rays emanating from $\underline{x}$ and covering a volume around that point are viewed as defining a transformation or change of variables from $\underline{\tilde{x}}$ to (p_{10},p_{20},σ), with the inverse of that transformation mapping (p_{10},p_{20},σ) to $\underline{\tilde{x}}$. The expression in (B28) is an element of the Jacobi matrix of the latter transformation. On the other hand, the matrix of the former transformation is more familiar. It is the matrix whose determinant $J(\underline{x},\tilde{\underline{x}})$ plays a crucial role in the propagation of amplitude along rays. Thus, let us introduce the matrix

$$\left[J_{ij}\right] = \begin{bmatrix} \dfrac{\partial \tilde{\underline{x}}}{\partial \sigma} \\ \\ \dfrac{\partial \tilde{\underline{x}}}{\partial p_{10}} \\ \\ \dfrac{\partial \tilde{\underline{x}}}{\partial p_{20}} \end{bmatrix}, \ \det \left[J_{ij}\right] = J(\tilde{\underline{x}},\underline{x}). \tag{B29}$$

Since the two transformations in question here are inverses of one another, the matrices of transformation must also be inverses of one another. (In making this claim, I am assuming that there are no caustics in the ray family over the domain of interest.) Thus, the derivative in (B28) can be rewritten in terms of the inverse of the matrix in (B29). The result is

$$\frac{\partial p_i}{\partial \tilde{x}_k} = -\frac{\partial p_{io}}{\partial \tilde{x}_k} = J^{-1}(\tilde{\underline{x}},\underline{x}) \ \mathrm{cof} \left[\frac{\partial \tilde{x}_k}{\partial p_{io}}\right] \tag{B30}$$

In this equation, cof denotes <u>cofactor</u> in the matrix $[J_{ij}]$ in (B29).

This result is now used in (B26) to write

$$\frac{\partial p_i}{\partial \xi_j} = J^{-1}(\tilde{\underline{x}},\underline{x}) \sum_{k=1}^{3} \frac{\partial \tilde{x}_k}{\partial \xi_j} \ \mathrm{cof} \left[\frac{\partial \tilde{x}_k}{\partial p_{io}}\right], \quad i,j = 1,2. \tag{B31}$$

4. Two-and-one-half dimensions

There is a whole class of special cases where $h(\underline{x},\underline{\xi})$ can be written more simply. These are the so-called two and one half dimensional (2.5D) cases. Here, it is assumed that the earth parameters, c and α are functions of (x,z) only (essentially two dimensional) while the wave propagation is three dimensional. Furthermore, only one line of data is gathered, say, for $y_s = y_r = 0$. However, for such a medium, all lines parallel to the line $y = 0$ would, of necessity, produce identical data and only the output of the algorithm in-plane, that is, for $y = 0$, is required.

In the inversion formula, (16), then, the data, $D(\underline{\xi},\omega)$, is independent of ξ_2 and the integration in ξ_2 can be carried out by the method of stationary phase. The stationary point turns out to be $\xi_2 = 0$ and the remainder of the integrand need only be evaluated in-plane. In this case, $h(\underline{x},\underline{\xi})$ can be evaluated in terms of the in-plane ray Jacobians associated with two dimensional wave propagation. Out-of-plane effects are accounted for through a scaling by $\sqrt{\sigma}$. In particular, $h(\underline{x},\underline{\xi})$ is expressible in terms of 2×2 determinants with the first row always being related to a $\underline{p}$-vector. Thus, a case like $h_3(\underline{x},\underline{\xi})$ or $h_4(\underline{x},\underline{\xi})$ cannot occur and h is expressible in terms of the two Jacobians associated with rays between $\underline{x}$ and $\underline{x}_s$ or between $\underline{x}$ and $\underline{x}_r$.

In Bleistein, Cohen and Hagin, [1985], the details of this computation were carried out for the cases of common source, common receiver and common offset. It should be noted that the 2.5D common source or common receiver cases do not arise directly from the 3D common source or common receiver case. In the 3D common source case, for example, there is only one source in the plane with $y = 0$. Thus, the data gathered on lines parellel to $y = 0$ is not identical. To make lines of identical data as the 2.5D model requires, the source must be moved to each new line, $y =$ const. when the receivers are moved. Thus, 2.5D common source corresponds to the 3D case in which there is a line of sources, say with $x = 0$, and the data from each source is gathered along an orthogonal line, $y =$ const. Now each line of experiments will be identical.

The results for $h(x,\xi)$ in 2.5D will not be restated here because the entire processing formula in 2.5D changes. That result is an integral over the source/receiver line (or curve) with an adjustment of the inversion integrands provided in this paper to account for the out-of-plane stationary phase computation.

APPENDIX C: ANALYSIS OF THE STATIONARY PHASE CONDITIONS

The purpose of this Appendix is to discuss the conditions of stationary phase, that is, the conditions that the four first partial derivatives of Φ in (30) are all equal to zero. The reader is reminded that the traveltime τ is symmetric in its initial and final coordinates. Thus, each of the gradients appearing in (30) is a $\underline{p}$-vector directed tangent to the ray. For example, $\nabla_s\tau(\underline{x}',\underline{x}_s) = \underline{p}(\underline{x}_s,\underline{x}')$, is a $\underline{p}$-vector tangent to the ray from $\underline{x}'$ to $\underline{x}_s$ (from second argument to first argument) evaluated at $\underline{x}_s$ (evaluated at first argument). It has magnitude $1/c^2(\underline{x}_s)$ and is directed away from the initial point, $\underline{x}'$. Similarly, $\nabla'\tau(\underline{x}',\underline{x}_s) = \underline{p}(\underline{x}',\underline{x}_s)$ is evaluated at $\underline{x}'$, has magnitude $1/c^2(\underline{x}')$ and is directed away from $\underline{x}_s$.

The result (30) and the notation for gradients introduced here are used to write the conditions that the phase be stationary as follows:

$$\underline{p}(\underline{x}_s,\underline{x}') \cdot \frac{d\underline{x}_s}{d\xi_m} + \underline{p}(\underline{x}_r,\underline{x}') \cdot \frac{d\underline{x}_r}{d\xi_m} = \underline{p}(\underline{x}_s,\underline{x}) \cdot \frac{d\underline{x}_s}{d\xi_m} + \underline{p}(\underline{x}_r,\underline{x}) \cdot \frac{d\underline{x}_s}{d\xi_m} ,$$

$$\left[\underline{p}(\underline{x}',\underline{x}_s) + \underline{p}(\underline{x}',\underline{x}_r) \right] \cdot \frac{d\underline{x}'}{d\sigma_m} = 0 \quad , \quad m = 1,2. \tag{C1}$$

It is assumed that a proper parameterization has been used for which the two vectors in each case ($m = 1,2$) are linearly independent.

The second condition is easier to interpret. It states that the tangents to the rays from $\underline{x}_s$ and $\underline{x}_r$ to the surface point $\underline{x}'$ have equal projections on two linearly independent tangents in the reflecting surface. Consequently, the projections of these two vectors onto S_j must be equal.

This is just Snell's law for reflection. The magnitudes of the $\underline{p}$-vectors must be equal (to $1/c^2(\underline{x}'))$ and hence the out-of-plane components must be equal in magnitude, as well. Indeed, the normal components of these vectors are of the same sign and must, in fact, be equal.

The first condition in (C1) ties the points on the two surfaces to the output point, $\underline{x}$. Let us consider the rays from $\underline{x}$ to the upper surface points, $\underline{x}_s$ and $\underline{x}_r$. Similarly, we consider the rays from $\underline{x}'$ to the upper surface points, $\underline{x}_s$ and $\underline{x}_r$. For each pair of rays we take projections on tangents at their respective emergence points. The sum of these projections for each pair of rays must be equal to one another. This must be true for two linearly independent tnagents at each point.

At first glance, it may not seem apparent that such a condition can ever be satisfied. However, consider the case in which $\underline{x}$ is on the reflecting surface, S_j. Then, for $\underline{x}' = \underline{x}$ (and $\underline{\sigma}$ chosen accordingly) the two pairs of rays overlay one another and these stationarity conditions are automatically satisfied for <u>any</u> pair of surface points, $\underline{x}_s$ and $\underline{x}_r$. Thus, we would only have to find such a pair for which Snell's law is satisfied, as well. Indeed, if there were no such pair in the seismic experiment being modeled, then that subsurface point would not be one for which the stationarity conditions are satisfied <u>and that point would not be imaged</u>.

On the other hand, there are many <u>candidates</u> for source/receiver pairs on the upper surface when $\underline{x}' = \underline{x}$. To find them, proceed as follows. At $x' = \underline{x}$, pass a plane through the normal to S_j. In the plane, choose two directions making equal angles with the normal. Use these as initial directions for rays from the point. Snell's law is satisfied for this pair of rays. The pair of emergence points at the upper surface are candidates for a source/receiver pair. Vary the opening angle of the ray pair in the normal plane and rotate the plane. Thereby, obtain a two-dimensional continuum of candidate source/receiver pairs in the upper surface.

Let us suppose now that such a pair is available in a given seismic survey when $\underline{x}$ is on S_j. Given that pair, it is argued by continuity that for $\underline{x}$ nearby S_j there must by points $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ satisfying (C1) and nearby the solution obtained in the limit when $\underline{x}$ is on S_j.

Constant Background Soundspeed

Further insight into the stationarity conditions is gained by considering the case of constant background speed and flat layers, as in Figure C1. Given a point, $\underline{x}$, a perpendicular is dropped to the surface S_j. This determines a point, $\underline{x}'$. Pass a plane through the normal and draw the rays at equal angles to the upper surface. This determines a pair of points, as candidates for $\underline{x}_s$ and $\underline{x}_r$. For this pair of points. the sum of projections on either side of the first line of (C1) is equal to zero. Thus, this triple of points satisfies both conditions of stationarity.

The three points, $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ must be in the same plane in order that Snell's law be satisfied. If $\underline{x}$ were not in the same plane, then the projections of its p-vectors would no longer be colinear and could not sum to zero. On the other hand, the sum of projections of the $\underline{p}$-vectors from $\underline{x}'$ would remain zero. Thus, the first condition in (C1) could not be satisfied. Similarly, if $\underline{x}$ is in the normal plane but not on the normal line the first condition could not be satisfied. That is, the conditions of stationarity are satisfied by three points $\underline{x}'$, $\underline{x}_s$ and $\underline{x}_r$ which, along with $\underline{x}$ lie in a plane normal to the reflector with $\underline{x}'$ at the foot of the normal to S_j drawn from $\underline{x}$. The only freedom left in these conditions, then, are the opening angle of the rays at $\underline{x}'$ and the orientation of the normal plane. Below, I discuss how these are further constrained for particular source/receiver configurations and this flat reflector constant background model.

Case 1: Zero-Offset

When the source/receiver pair are coincident, the opening angle of the rays at $\underline{x}'$ must both be zero; both rays from $\underline{x}'$ to $\underline{x}_s$ and $\underline{x}_r$ must be the normal ray to the surface, passing through $\underline{x}$. The stationary point on the upper surface and the point $\underline{x}'$ must have the same transverse coordinates as $\underline{x}$, itself. The stationarity conditions are completely satisfied by these three points. Because of the degeneracy of this case, a specific normal plane is not determined. However, that is secondary to determining the actual triple of points, itself.

The generalization of this result to curved surfaces and variable background is fairly straightforward. Given $\underline{x}$, find a normal ray from S_j which passes through $\underline{x}$. The initial point of that ray on S_j is the point $\underline{x}'$. The point where the ray emerges on the upper surface S_0 is the source/receiver point which completes the triple of points satisfying (C1). For $\underline{x}$ on S_j, there is clearly only one stationary triple. On the other hand for $\underline{x}$ on the evolute of S_j (the envelope of normals to S_j) there will be more than one triple. In order for the asymptotic methods used here to be valid, it is necessary to assume that this evolute is a few wavelengths (at least three) away from S_j. Thus, it is assumed that the reflector is not severely curved; that is, the principal radii of curvature of the reflector must be a few wavelengths long.

Case 2: Common Source

Let us suppose now that the source point is fixed. Given $\underline{x}$, drop the normal to S_j and thereby determine $\underline{x}'$. Pass a plane through $\underline{x}_s$, $\underline{x}$ and $\underline{x}'$. This plane is normal to S_j. Draw the ray from $\underline{x}_s$ to $\underline{x}'$. Draw the reflected ray in the given normal plane. The emergence point on S_0 is the point $\underline{x}_r$. If $\underline{x}$ is on S_j, set $\underline{x}' = \underline{x}$ and use the normal at that point and the fixed point $\underline{x}_s$ to determine the normal plane. Then proceed to determine $\underline{x}_r$ as before, with $\underline{x}$ not on S_j.

In a theoretical model, receivers are spread over the entire upper surface. In practice, the spread is finite. Thus, the spread need not extend to the determined $\underline{x}_r$. In that case, the determined point, $\underline{x}'$ will not be part of a triple satisfying (C1) and will not be imaged. In the text I have proceeded as if such candidate points are indeed stationary points.

As above, I argue by continuity that for curved surfaces and variable $c(\underline{x})$, differing "not greatly" from the constant background case, the essential features of this analysis still apply.

Case 3: Common Receiver

As in the previous appendix, one need only interchange the subscripts s and r in the discussion of Case 2 to obtain a completely analogous conclusion here.

Case 4: Common Offset

It is assumed that all of the offset pairs lie on lines that are parallel. We rotate the normal plane containing $\underline{x}$ and $\underline{x}'$ until it is parallel to this set of lines. Indeed, the intersection of the normal plane and the upper surface contains one of those lines. Choose the opening angle of the rays from $\underline{x}'$ so that the rays emerge at the upper surface at a separation distance equal to the common offset distance. The emergence points are the pair $\underline{x}_s$ and $\underline{x}_r$.

Case 5: Common Midpoint

There will only be a solution to (C1) in this case if the common midpoint and $\underline{x}$ lie along a common normal to S_j. Furthermore, in that case, all source/receiver pairs are stationary points. The method of stationary phase breaks down since the stationary points are no longer isolated. This is a case which requires further investigation.

APPENDIX D: MATRIX SIGNATURE

The purpose of this appendix is to show that the signature of the matrix $[\Phi_{\xi\sigma}]$ defined by (33) is equal to zero. To do so, I consider first the special case in which the background sound speed c in the region between the upper surface and the reflecting surface is constant, the layers are flat and there is zero offset between sources and receivers. In this case, the upper surface and the reflecting surface are defined, respectively, by

$$
\begin{aligned}
&x_{1s} = x_{1r} = \xi_1, \quad x_{2s} = x_{12} = \xi_{2}, \quad x_{3s} = x_{3r} = 0, \\
&x_1' = \sigma_1 \;, \quad x_2' = \sigma_2 \;, \; x_3' = H \;.
\end{aligned}
\tag{D1}
$$

Furthermore, the travel times are just the distances between initial and final point, divided by c:

$$
\begin{aligned}
&\tau(\underline{x},\underline{x}_s) = |\underline{x} - \underline{x}_s|/c, \quad \tau(\underline{x},\underline{x}_r) = |\underline{x} - \underline{x}_r|/c, \\
&\tau(\underline{x}',\underline{x}_s) = |\underline{x}' - \underline{x}_s|/c, \quad \tau(\underline{x}',\underline{x}_r) = |\underline{x}' - \underline{x}_r|/c.
\end{aligned}
\tag{D2}
$$

These results are used to simplify Φ, as defined by (9) and then to computed the determinant in (33). The analysis is further simplified by setting $\underline{x}' = \underline{x}$. The result is

$$
\left[\Phi_{\xi\sigma}\right] = \begin{bmatrix} 0 & 0 & -1/Hc & 0 \\ 0 & 0 & 0 & -1/Hc \\ -1/Hc & 0 & 1/Hc & 0 \\ 0 & -1/Hc & 0 & 1/Hc \end{bmatrix} . \tag{D3}
$$

For this matrix it is fairly straightforward to calculate the characteristic equation. The result is

$$\det\left[\Phi_{\xi\sigma} - \frac{\lambda}{Hc} I\right] = [\lambda(1 - \lambda) + 1]^2 \Big/ [Hc]^4 = 0 \ . \tag{D4}$$

This equation has two double roots, $\lambda = [1 \pm \sqrt{5}]/2$. Since two of the roots are positive and two are negative, $\mathrm{sig}[\Phi_{\xi\sigma}] = 0$.

Let us now consider deforming this constant background, zero offset, flat layer model into the true model. If the signature is to change as the model is deformed, then at some point in the deformation, at least one eigenvalue must be zero. In fact, exactly two eigenvalues would have to be zero at this point, since $\det[\Phi_{\xi\sigma}]$ is nonnegative and by assumption, the signature changes.

In the next section, it is shown that $\det[\Phi_{\xi\sigma}]$ is proportional to $h(\underline{x},\underline{\xi})$. It has been assumed that h is nonzero for the true model. I now add to that the assumption that our true model is not so severely different from the flat earth case for h to have passed through a zero on the way from one model to the other. Thus, $\mathrm{sig}[\Phi_{\xi\sigma}] = 0$ for the true model, as well.

APPENDIX E: RELATION BETWEEN $h(\underline{x},\xi)$ AND $\det[\Phi_{\xi\sigma}]$ AT THE STATIONARY POINT

In this appendix (41) will be verified. To do so, isnecessary to evaluate $|h(\underline{x},\xi)|$ as defined by (6) subject to the stationarity conditions, (C1) and the additional condition that $\underline{x} = \underline{x}'$. As a first step, $\underline{x}$ is replaced by $\underline{x}'$ in (6) and that equation is rewritten in terms of p-vectors. The result is

$$h(\underline{x}',\xi) = \det \begin{vmatrix} \underline{p}(\underline{x}',\underline{x}_s) + \underline{p}(\underline{x}',\underline{x}_r) \\ \frac{\partial}{\partial\xi_1}\left[\underline{p}(\underline{x}',\underline{x}_s) + \underline{p}(\underline{x}',\underline{x}_r)\right] \\ \frac{\partial}{\partial\xi_2}\left[\underline{p}(\underline{x}',\underline{x}_s) + \underline{p}(\underline{x}',\underline{x}_r)\right] \end{vmatrix} . \tag{E1}$$

To calculate this determinant, the matrix is multiplied by a matrix whose determinant is known. That matrix is

$$K = \left[\frac{d\underline{x}'}{d\sigma_1}, \frac{d\underline{x}'}{d\sigma_2}, \frac{d\underline{x}}{dn} \right] , \tag{E2}$$

where each vector here represents a column of K. We remark that

$$\det K = \tilde{n} \cdot \frac{d\underline{x}'}{d\sigma_1} \times \frac{d\underline{x}'}{d\sigma_2} = \sqrt{g_j} , \tag{E3}$$

with the second equality being equivalent to (29).

Now, in multiplying K by the matrix in (E1), we see that the first two elements of the first row are both zero by (C1), while the third element is given by

$$\left[\underline{p}(\underline{x}',\underline{x}_s) + \underline{p}(\underline{x}',\underline{x}_r)\right] \cdot \tilde{\underline{n}} = \frac{2\cos\theta}{c(\underline{x})} \quad , \tag{E4}$$

which follows from (40). Thus, in expanding the determinant of the product by the first row, it is only necessary to consider the lower left 2X2 matrix after multiplication. Thus, let us now consider a typical term,

$$\begin{aligned} \frac{\partial}{\partial\xi_k}\left[\underline{p}(\underline{x}',\underline{x}_s) + \underline{p}(\underline{x}',\underline{x}_r)\right]\cdot\frac{d\underline{x}'}{d\sigma_m} &= \frac{\partial}{\partial\xi_k}\frac{\partial}{\partial\sigma_m}\left[\tau(\underline{x}',\underline{x}_s) + \tau(\underline{x}',\underline{x}_r)\right] \\ &= \frac{\partial^2\Phi}{\partial\xi_k\partial\sigma_m}\ ,\quad k,m = 1,2. \end{aligned} \tag{E5}$$

It now follows that if the matrix in (E1) is multiplied by the matrix K before calculating the determinant, the following result is obtained:

$$\det h(\underline{x}',\xi)\sqrt{g_j} = \frac{2\cos\theta}{c(\underline{x}')}\det\left[\Phi_{\xi\sigma}\right] , \tag{E6}$$

for $\underline{x}' = \underline{x}$ on S_j. The outer equality in (41) follows from this result. The right equality in (41) follows from (40).

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ABSTRACT

This paper is based on a paper by Beylkin in which a leading order asymptotic theory for inversion of acoustic data is presented. The method is based on earlier work by Beylkin in the theory of pseudo-differential operators and generalized back projections or Radon transforms. The back projection or inversion is carried out with respect to a general [c(x,y,z)] background sound speed. The asymptotic limit of interest is high frequency. The inversion operator is given as an integral of the observed data over frequency and over the observation surface. Beylkin claims that his result is useful for finding discontinuities in the sound speed, but he does not make clear how this is to be done in practice. I show how to modify Beylkin's inversion operator to obtain an operator whose output is an array of singular functions, one for each reflector (discontinuity surface of the sound speed) in the subsurface. The singular function of a surface is a Dirac delta function whose support lies on that surface. Thus, the array of singular functions produces a reflector map of the subsurface. The validity of modification of Beylkin's inversion operator is verified by applying it to band limited Born-approximate and then Kirchhoff-approximate representations of the upward propagating wave field. Multi-dimensional stationary phase is applied to the spatial integration over the variables of the field representation and the variables of the observation surface. It is confirmed that the output is proportional to the band limited singular functions of the reflectors and further that one can estimate the jump in velocity across each reflector from the peak amplitude of the output on each reflector. This is done for the cases of common (or single fixed) source, common receiver, and common (or fixed) offset between source and receiver, with zero offset or backscatter as a special case of the last of these.

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