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THERMAL EXPANSION OF ELASTIC-PLASTIC

COMPOSITE MATERIALS

by

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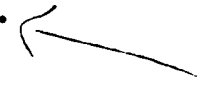
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## Abstract

Exact relationships are derived between instantaneous overall
thermal stress or strain vectors and instantaneous overall mechanical
stiffness or compliance, for two binary composite systems in which one
of the phases may deform plastically. Also, the local instantaneous
thermal strain and stress concentration factors are related in an exact
way to the corresponding mechanical concentration factors. The results
depend on instantaneous thermoelastic constants and volume fractions of
the phases. They are found for fibrous composites with two distinct
elastically isotropic or transversely isotropic phases, and for any
binary composite with elastically isotropic phases. The results indicate
that in the plastic range the thermal and mechanical loading effects are
coupled even if the phase properties do not depend on changes in temper-
ature. The derivation is based on a novel decomposition procedure which
indicates that spatially uniform elastic strain fields can be created in
certain heterogeneous media by superposition of uniform phase eigen-
strains with local strains caused by piecewise uniform stress fields
which are in equilibrium with prescribed surface tractions. The method
is extended to discretized microstructures, and also to analysis of
moisture absorption and phase transformation effects on overall response
and on local fields in the two composite materials.

## 1. INTRODUCTION

The response of elastic composite materials to spatially uniform changes in temperature is well understood. An essential contribution to the solution of this problem was made by LEVIN (1967), who found that macroscopic thermal expansion coefficients of a composite medium, consisting of two distinct isotropic phases of arbitrary shape, depend in a unique way on overall elastic moduli of the aggregate and on thermoelastic constants of the phases. Thus, if the elastic moduli are known, the thermal expansion coefficients can be calculated. This line of inquiry was extended by SHAPERY (1968), who derived bounds on thermal expansion coefficients of multi-phase composites with isotropic phases, while ROSEN and HASHIN (1970) applied LEVIN'S approach to binary composites consisting of anisotropic phases, and they also found bounds on overall thermal expansion coefficients of multiphase materials. BUDIANSKY (1970) gave self-consistent estimates of several thermal and thermoelastic properties of multiphase isotropic mixtures. Among the more recent contributions to the subject are the papers by LAWS (1973) and CRAFT CHRISTENSEN (1980).

The response of elastic-plastic composite materials to uniform thermal changes has been explored only to a limited extent. This is a more difficult problem because at least one of the phases is inelastic, and the deformation of the phases and of the composite is affected both by the overall thermal change and by the current macroscopic mechanical loading. In the plastic state the thermal and mechanical loading effects are coupled, even if the mechanical properties of the phases do not depend on temperature. The problem is, of course, nonlinear and

must be solved in an incremental way. Therefore, the connection between
the two effects must be examined within a combined load increment. A
representative volume element of the composite is first subjected to a
certain uniform initial macroscopic stress or strain and to a uniform
thermal change; then, both the stress or strain and the temperature
experience a small simultaneous change to another uniform state. Over-
all instantaneous stiffness and compliance, and thermal stress and
strain vectors are sought.

Earlier solutions of problems of this kind have been limited to
simple loading situations in fibrous composites, such as pure thermal
change (DE SILVA and CHADWICK 1969) or thermal change combined with
axisymmetric mechanical loads (DVORAK and RAO 1976). More recently
(DVORAK 1983) it was shown that the total overall strain increment
caused in a prestressed fibrous composite by a small uniform thermal
change can be related in an exact way to thermoelastic constants of the
phases and to instantaneous overall compliance. No restrictions need to
be imposed on the type of prestress or on the matrix constitutive law
except for plastic incompressibility, but the fiber must be isotropic or
transversely isotropic and remain elastic. This result has been applied
in analysis of a composite cylinder element (DVORAK and WUNG 1984)
subjected to axisymmetric mechanical loading, uniform thermal changes,
and variations in matrix yield stress.

The present paper develops the connections between overall instan-
taneous mechanical and thermal properties in a more general way. First,
it is shown that the overall thermal stress and strain vectors for an
elastic fibrous composite with transversely isotropic phases can be
obtained through superposition of certain uniform fields in the phases,

and local fields caused by a uniform overall stress or strain. A similar result is derived for any composite consisting of two isotropic phases. These results are then utilized to find instantaneous thermo-plastic properties and local fields of these composite systems for simultaneous mechanical and thermal load increments. Extensions of the results to discretized microstructures, and to additional load effects, such as phase transformations and moisture absorption are discussed as well.

## 2. ELASTIC FIBROUS COMPOSITE

A binary composite material consists of a matrix reinforced by aligned and bonded cylindrical fibers. Both phases are assumed to be homogeneous and transversely isotropic about the fiber direction $x_3$. In the transverse $x_1x_2$-plane, the cross sections and distribution of the phases can be arbitrary providing that the composite is statistically homogeneous, transversely isotropic, and free of voids.

A representative volume element V of the composite is selected and subjected to a certain loading or deformation history which is imposed through application of uniform overall stresses $\bar{\sigma}^0$ or strains $\bar{\epsilon}^0$ to the surface S of volume V. Also, a certain uniform thermal change has been applied such that the current temperature in V is constant and equal to $\theta_0$. At this particular point of the loading sequence simultaneous increments of $d\bar{\sigma}$ and $d\theta$, or $d\bar{\epsilon}$ and $d\theta$, are applied to V.

The response of the composite to these load increments is described by constitutive equations:

$$d\bar{\epsilon} = M d\bar{\sigma} + m d\theta \quad , \quad d\bar{\sigma} = L d\bar{\epsilon} - \ell d\theta , \tag{1}$$

where M,L are (6x6) overall stiffness and compliance matrices, and $m, \ell$ are (6x1) overall thermal strain and stress vectors*.

While M and L are known, we wish to determine the vectors $m$ and $\ell$. To this end it is necessary to specify the constitutive equations for field averages of the phases:

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*We use the customary notation (HILL 1963, LAWS 1973) where, except as noted, (6x6) matrices are denoted by lightface uppercase Latin letters, and (6x1) vectors by boldface lowercase Latin or Greek letters. Top bars denote overall volume averages.

$$d\epsilon_r = M_r \, d\sigma_r + m_r \, d\theta, \qquad d\sigma_r = L_r \, d\epsilon_r - \ell_r d\theta \qquad (r = f,m) \qquad (2)$$

which are analogous to (1); f,m indicate the "fiber" and "matrix" phases. In elastic composites, these phases are interchangeable and f,m are used merely for convenience of notation.

Since both the composite and each of the phases are transversely isotropic about $x_3$, it is possible to write a subset of (1) and (2) which relates the first two stress and strain invariants. With top bars and subscripts r,f,m omitted in (1) and (2) one obtains (DVORAK and BAHEI-EL-DIN 1979):

$$\begin{Bmatrix} d\epsilon_1 \\ d\epsilon_2 \end{Bmatrix} = \frac{1}{kE} \begin{bmatrix} n & -\ell \\ -\ell & k \end{bmatrix} \begin{Bmatrix} d\sigma_1 \\ d\sigma_2 \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} d\theta \qquad (3)$$

$$\begin{Bmatrix} d\sigma_1 \\ d\sigma_2 \end{Bmatrix} = \begin{bmatrix} k & \ell \\ \ell & n \end{bmatrix} \begin{Bmatrix} d\epsilon_1 \\ d\epsilon_2 \end{Bmatrix} - \begin{bmatrix} k\alpha + \ell\beta \\ \ell\alpha + n\beta \end{bmatrix} d\theta \qquad (4)$$

where $k,\ell,n$ are Hill's (1964) elastic moduli, $E = n - \ell^2/k$, $\alpha = 2\alpha_T$, $\beta = \alpha_L$, and $\alpha_T$ and $\alpha_L$ are linear coefficients of thermal expansion in the transverse plane and longitudinal direction, respectively. The strain and stress invariants are defined as:

$$d\epsilon_1 = d\epsilon_{11} + d\epsilon_{22} \qquad\qquad d\epsilon_2 = d\epsilon_{33} \qquad (5)$$

$$d\sigma_1 = \frac{1}{2}(d\sigma_{11} + d\sigma_{22}) \qquad\qquad d\sigma_2 = d\sigma_{33} \qquad (6)$$

With appropriate values of elastic moduli and coefficients $\alpha,\beta$, equations (3) to (6) can be applied either to the composite medium or to

each of the two phases.

As long as M,L do not depend on $d\theta$, $\underset{\sim}{m}$ on $d\bar{\sigma}$, and $\underset{\sim}{\ell}$ on $d\bar{\epsilon}$, the thermal and mechanical contributions to $d\bar{\epsilon}$ and $d\bar{\sigma}$ in (1) can be found separately and superimposed. By assumption M and L are known, hence the first terms on right-hand side in (1) are evaluated without difficulty for any given $d\bar{\sigma}$ or $d\bar{\epsilon}$. To find $\underset{\sim}{m}$ and $\underset{\sim}{\ell}$, and the second terms in (1), we utilize the decomposition procedure of DVORAK (1983).

In the first step of the procedure the fiber and matrix phases are separated and surface tractions which preserve the current local stresses $\overset{o}{\underset{\sim}{\sigma_r}}$ and strains $\overset{o}{\underset{\sim}{\epsilon_r}}$ are applied to each phase $r = f,m$. Alternatively, surface displacements corresponding to $\overset{o}{\underset{\sim}{\epsilon_r}}$ may be prescribed to preserve $\overset{o}{\underset{\sim}{\sigma_r}}$. In addition, a uniform thermal change $d\theta$ is applied to both phases. The local strains caused by $d\theta$ would make the phases incompatible if the composite was to be reassembled. Therefore, uniform tractions $d\hat{\sigma}^r_1$, $d\hat{\sigma}^r_2$ of as yet unknown magnitude are applied to the phases simultaneously with $d\theta$. (The top hats indicate auxilliary uniform fields used in the decomposition and reassembly of the composite.) This leads to the following uniform strain increments in the separated phases:

$$d\hat{\epsilon}^f_1 = (n_f \, d\hat{\sigma}^f_1 - \ell_f \, d\hat{\sigma}^f_2)/k_f E_f + \alpha_f d\theta$$

$$d\hat{\epsilon}^f_2 = (-\ell_f \, d\hat{\sigma}^f_1 - k_f \, d\hat{\sigma}^f_2)/k_f E_f + \beta_f d\theta$$

$$(7)$$

$$d\hat{\epsilon}^m_1 = (n_m \, d\hat{\sigma}^m_1 - \ell_m \, d\hat{\sigma}^m_2)/k_m E_m + \alpha_m d\theta$$

$$d\hat{\epsilon}^m_2 = (-\ell_m \, d\hat{\sigma}^m_1 - k_m \, d\hat{\sigma}^m_2)/k_m E_m + \beta_m d\theta$$

In the second step of the procedure, the tractions $d\hat{\sigma}_1^r$ and $d\hat{\sigma}_2^r$ must be adjusted to assure compatibility of the phases and equilibrium of these tractions at phase interfaces and on the surface S of the representative volume V. Compatibility and equilibrium require that

$$d\hat{\varepsilon}_1^m = d\hat{\varepsilon}_1^f \quad , \qquad d\hat{\varepsilon}_2^m = d\hat{\varepsilon}_2^f \tag{8}$$

$$d\hat{\sigma}_1^f = d\hat{\sigma}_1^m = dS_T \tag{9}$$

$$c_f \, d\hat{\sigma}_2^f + c_m \, d\hat{\sigma}_2^m = dS_A \tag{10}$$

where $dS_T$ and $dS_A$ are surface tractions which need to be added at S to preserve overall equilibrium of V while $d\hat{\sigma}_1^r$ and $d\hat{\sigma}_2^r$ are applied to the phases. The magnitudes of phase volume fractions $c_f + c_m = 1$ need also be known at this point.

All strain and stress increments in (7) are uniform, hence equations (8) to (10) are exact for any transverse plane geometry. These relations suggest that spatially uniform strain fields can be created in certain heterogeneous media by superposition of uniform eigenstrains $a_r d\theta$, $\beta_r d\theta$ in the phases, with local strains caused by piecewise uniform stress fields which are in equilibrium with surface tractions $dS_A$, $dS_T$.

Internal equilibrium and compatibility of the phases in V depend only on the eight unknown strains and stresses $d\hat{\varepsilon}_1^r$, $d\hat{\varepsilon}_2^r$, $d\hat{\sigma}_1^r$, $d\hat{\sigma}_2^r$, and not on $dS_A$, $dS_T$. Therefore, (7), (8) and (9$_1$) represent a system of seven equations for the eight unknwons. To find a solution, it is necessary to introduce an additional constraint.

A particular choice which will be useful in the sequel is:

$$d\hat{\sigma}_2^m = \rho \; d\hat{\sigma}_1^m \; , \tag{11}$$

where $\rho \neq 0$ is a constant.

Now, $dS_T$ and $dS_A$ follow from (7), after a substitution of (9) and (10), and from (8):

$$a_1 \; dS_T + a_2 \; dS_A + a_3 \; d\theta = 0 \tag{12}$$

$$b_1 \; dS_T + b_2 \; dS_A + b_3 \; d\theta = 0 \tag{13}$$

where:

$$a_1 = \frac{n_f}{k_f E_f} - \frac{n_m}{k_m E_m} + \rho \left[ \frac{c_m \, \ell_f}{c_f k_f E_f} + \frac{\ell_m}{k_m E_m} \right]$$

$$a_2 = -\frac{1}{c_f} \frac{\ell_f}{k_f E_f} \quad , \quad a_3 = \alpha_f - \alpha_m$$

$$b_1 = \frac{\ell_f}{k_f E_f} - \frac{\ell_m}{k_m E_m} + \rho \left[ \frac{c_m}{c_f E_f} + \frac{1}{E_m} \right]$$

$$b_2 = -\frac{1}{c_f E_f} \quad , \quad b_3 = -\beta_f + \beta_m$$

and:

$$dS_T = s_T \; d\theta \quad , \quad dS_A = s_A \; d\theta \tag{14}$$

$$s_T = (a_2 b_3 - a_3 b_2)/(a_1 b_2 - a_2 b_1) \tag{15}$$

$$s_A = (a_3 b_1 - a_1 b_3)/(a_1 b_2 - a_2 b_1) \tag{16}$$

At this point we change from the invariants (5), (6) to the (6x1) stress and strain vectors and write

$$d\hat{\epsilon}_{11}^{f} = d\hat{\epsilon}_{22}^{f} = \frac{1}{2} d\hat{\epsilon}_{1}^{f} = g_1 \, d\theta , \qquad d\hat{\epsilon}_{33}^{f} = d\hat{\epsilon}_{2}^{f} = g_2 \, d\theta \qquad (17)$$

$$d\hat{\epsilon}_{11}^{m} = d\hat{\epsilon}_{22}^{m} = \frac{1}{2} d\hat{\epsilon}_{1}^{m} = h_1 \, d\theta , \qquad d\hat{\epsilon}_{33}^{m} = d\hat{\epsilon}_{2}^{m} = h_2 \, d\theta \qquad (18)$$

From (7), with (8) to (10), (15), and (16):

$$g_1 = \left[ \frac{1}{2} (n_f - \gamma \, \ell_f)/(k_f \, E_f) \right] s_T + \frac{1}{2} \alpha_f$$

$$g_2 = \left[ (-\ell_f + \gamma \, k_f)/(k_f \, E_f) \right] s_T + \beta_f$$

$$(19)$$

$$h_1 = \left[ \frac{1}{2} (n_m - \rho \, \ell_m)/(k_m \, E_m) \right] s_T + \frac{1}{2} \alpha_m$$

$$h_2 = \left[ (-\ell_m + \rho \, k_m)/(k_m \, E_m) \right] s_T + \beta_m$$

where

$$\gamma = (s_A - \rho \, c_m \, s_T)/(c_f \, s_T) , \qquad (20)$$

and, according to (8):

$$g_1 = h_1 \quad , \quad g_2 = h_2 \quad . \qquad (21)$$

Analogous results for stresses are:

$$d\hat{\sigma}_{11}^{f} = d\hat{\sigma}_{22}^{f} = d\hat{\sigma}_{1}^{f} = s_T \, d\theta$$

$$(22)$$

$$d\hat{\sigma}_{33}^{f} = d\hat{\sigma}_{2}^{f} = \gamma \, s_T \, d\theta$$

$$d\hat{\sigma}_{11}^{m} = d\hat{\sigma}_{22}^{m} = d\hat{\sigma}_{1}^{m} = s_T \, d\theta$$

$$(23)$$

$$d\hat{\sigma}_{33}^{m} = d\hat{\sigma}_{2}^{m} = \rho \, s_T \, d\theta$$

In the final step of the decomposition procedure, the composite is reassembled and the surface tractions $dS_T$, $dS_A$ removed. Of course, the local strains and stresses (17) to (23) already assure that the phases are compatible and in internal equilibrium, in fact they are equal to local fields caused in the composite by simultaneous application of $d\theta$, $dS_A$ and $dS_T$. They must be now added to local fields caused in the composite by surface tractions $-dS_T$, $-dS_A$.

The final results assume a concise form with the definitions

$$\underset{\sim}{h} = [h_1 \; h_1 \; h_2 \; 0 \; 0 \; 0]^T$$

$$\underset{\sim}{s_a} = [s_T \; s_T \; s_A \; 0 \; 0 \; 0]^T$$

$$\underset{\sim}{\gamma} = [1 \; 1 \; \gamma \; 0 \; 0 \; 0]^T \tag{24}$$

$$\underset{\sim}{\rho} = [1 \; 1 \; \rho \; 0 \; 0 \; 0]^T$$

where $[\;]^T$ denotes a transpose and the coefficients appear in (11), (15), (16), and (19) to (21).

Therefore, for $d\overline{\sigma} = 0$, $d\theta \neq 0$ in $(1_1)$:

$$d\overline{\epsilon} = \underset{\sim}{m} \, d\theta \tag{25}$$

$$\underset{\sim}{m} = \underset{\sim}{h} - M s_a \tag{26}$$

where $m$ is the overall thermal strain vector and M is the known overall compliance.

Also, suppose that the local stresses in the phases are written in terms of concentration factors:

$$d\sigma_r = B_r \, d\overline{\sigma} + b_r \, d\theta \quad , \quad (r = f, m) \tag{27}$$

and that $B_m$, $B_f$ are known.

With regard to (22) and (23) one obtains:

$$b_f = s_T \underset{\sim}{\gamma} - B_f s_a$$

$$b_m = s_T \underset{\sim}{\rho} - B_m s_a \tag{28}$$

Similar results can be found for a fully constrained composite subjected to a uniform thermal change. Recall that the strains (17) and (18) are actually equal to overall strains under $d\theta$, $dS_A$, $dS_T$. This follows immediately from (8) and (21). These overall strains must be removed, and the local fields adjusted accordingly.

Therefore, for $d\overline{\underset{\sim}{\varepsilon}} = \underset{\sim}{0}$, $d\theta \neq 0$ in $(1_2)$:

$$d\overline{\underset{\sim}{\sigma}} = - \underset{\sim}{\ell} \, d\theta$$

$$\underset{\sim}{\ell} = -s_a + L \underset{\sim}{h} \tag{29}$$

where $\underset{\sim}{\ell}$ is the overall thermal stress vector, and $L$ is the known overall stiffness.

Also, if the local strains are written in terms of concentration factors:

$$d\varepsilon_r = A_r \, d\overline{\underset{\sim}{\varepsilon}} - \underset{\sim}{a}_r \, d\theta , \qquad (r = f,m) \tag{30}$$

and if $A_m$, $A_f$ are known, one obtains with the help of (17) to (21):

$$a_f = (A_f - I) \underset{\sim}{h} , \qquad a_m = (A_m - I) \underset{\sim}{h} \tag{31}$$

To facilitate applications we note that

$$m = [\alpha_T \ \alpha_T \ \alpha_L \ 0 \ 0 \ 0]^T \tag{32}$$

where $\alpha_T$, $\alpha_L$ are linear coefficients of thermal expansion in the transverse plane, and in longitudinal direction.

For any binary fibrous composite with known phase preperties and phase volume fractions, the effect of thermal change is reduced to equivalent mechanical loads and to certain uniform fields in the phases. Thus $m$ and $\ell$ are found in terms of M and L, and $a_r$, $b_r$ in terms of $A_r$, $B_r$. All these relations are exact. While the constant $\rho$ is a free parameter, none of the results actually depend on $\rho$. For each $\rho$ one obtains by superposition a solution to the same boundary value problem. According to the uniqueness theorem in the theory of elasticity, all such solutions must coincide. This can be verified by numerical calculations.

## 3.   TWO ISOTROPIC ELASTIC PHASES

Suppose that a composite aggregate consists of two perfectly bonded elastic phases, which are distinct but isotropic. The microstructural geometry can be arbitrary, providing that the composite is statistically homogeneous and free of voids. The composite itself need not be isotropic, it can be reinforced by aligned, braided, or otherwise distributed continuous fibers, short fibers, particles of any shape, and by combinations of such reinforcements.

Assume that the overall constitutive relations are again given by (1) and that the overall compliance M and stiffness L are known. Also, let local fields be described by (27) and (30), and assume that $A_r$, $B_r$ are known. As in the previous section we utilize the subscripts f,m to identify the two phases, even though we no longer require that either phase be of cylindrical shape. Local phase properties need be known only in terms of bulk moduli $K_f$, $K_m$, and linear thermal expansion co-efficients $\alpha_f$, $\alpha_m$.

We again pose the problem described in the previous section: The composite has been loaded by a certain uniform overall stress $\bar{\sigma}^0$, or strain $\bar{\epsilon}^0$, and uniform temperature $\theta_0$. Simultaneous increments $d\bar{\sigma}$ and $d\theta$, or $d\bar{\epsilon}$ and $d\theta$ are applied. The response of the aggregate is sought in terms of the overall thermal strain and stress vectors $m$ and $\ell$, and phase concentration factors $a_r$, $b_r$, $(r = f,m)$.

As in Section 2, the phases are first separated, and loaded by $d\theta$ and by certain unknown tractions $d\sigma_r$. The nonvanishing stress and strain increments are

$$\hat{d\sigma}^f_{11} = \hat{d\sigma}^f_{22} = \hat{d\sigma}^f_{33} = dS_f$$

$$\hat{d\sigma}^m_{11} = \hat{d\sigma}^m_{22} = \hat{d\sigma}^f_{33} = dS_m$$

$$\hat{d\epsilon}^f_{11} = \hat{d\epsilon}^f_{22} = \hat{d\epsilon}^f_{33} = dS_f/(3K_f) + \alpha_f \, d\theta$$

$$\hat{d\epsilon}^m_{11} = \hat{d\epsilon}^m_{22} = \hat{d\epsilon}^m_{33} = dS_m/(3K_m) + \alpha_m \, d\theta$$

(33)

To assure equilibrium and compatibility:

$$dS_f = dS_m = dS, \qquad d\epsilon^f_{ij} = d\epsilon^m_{ij}, \qquad (34)$$

and

$$dS = s \, d\theta \quad , \quad s = -3(\alpha_f - \alpha_m)/(1/K_f - 1/K_m) \qquad (35)$$

The composite is now loaded by three equal overall normal stresses dS, and by dθ. Local strain and stress fields follow from (33) and (34).

Finally, the composite is reassembled and surface tractions dS are removed.

Let

$$q = s/(3K_f) + \alpha_f = s/(3K_m) + \alpha_m$$

$$\underset{\sim}{q} = q \, [1\ 1\ 1\ 0\ 0\ 0]^T \qquad (36)$$

$$\underset{\sim}{s} = s \, [1\ 1\ 1\ 0\ 0\ 0]^T$$

In analogy with (25) to (30); one obtains

For $d\overline{\sigma} = 0$, $d\theta \neq 0$, in (1)

$$d\overline{\epsilon} = m \, d\theta \tag{37}$$

$$m = q - Ms$$

$$d\sigma_r = b_r \, d\theta \qquad (r = f,m)$$

$$b_f = (I - B_f) \, s \quad , \qquad b_m = (I - B_m) \, s \tag{38}$$

For $d\overline{\epsilon} = 0$, $d\theta \neq 0$ in (1):

$$d\overline{\sigma} = - \ell \quad d\theta \tag{39}$$

$$\ell = - s + Lq$$

$$d\epsilon_r = a_r \, d\theta \qquad (r = f,m)$$

$$a_f = (A_f - I) \, q \quad , \qquad a_m = (A_m - I) \, q \tag{40}$$

If the composite is macroscopically anisotropic, then M and L can depend on up to 21 elastic constants  and

$$m = [\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4 \; \alpha_5 \; \alpha_6]^T \tag{41}$$

where $\alpha_1$ to $\alpha_6$ are overall linear thermal expansion coefficients, de-fined by $(1_1)$ at $d\overline{\sigma} = 0$. For a fibrous composite which is transversely isotropic one recovers (32). For an isotropic composite all $\alpha$ in (32) become equal to

$$\alpha = \alpha_m + \frac{(\alpha_f - \alpha_m)}{\left(\frac{1}{K_f} - \frac{1}{K_m}\right)} \left(\frac{1}{K} - \frac{1}{K_m}\right) \tag{42}$$

where K is the overall bulk modulus.

This last equation was derived in a different way by LEVIN (1967).

## 4.   ELASTIC-PLASTIC COMPOSITES

### (i)   Fibrous composites

Consider again the fibrous composite system of Section 2.   Suppose that the matrix phase is elastic within a certain stress region, but becomes elastic-plastic when a given yield condition has been satisfied. The fiber remains elastic until failure.   This suggests a metal matrix, which is usually elastically isotropic.   Thus the matrix elastic moduli in (3) and (4) become related as follows:

$$\ell_m = k_m - m_m, \qquad n_m = k_m + m_m, \qquad E_m = n_m - \ell_m^2/k_m$$

$$(43)$$

$$\frac{n_m}{k_m E_m} = \frac{2(1-\nu_m)}{E_m}, \qquad \frac{\ell_m}{k_m E_m} = \frac{2\nu_m}{E_m}, \qquad \alpha_m = 2\beta_m$$

where $E_m$, $\nu_m$ are the isotropic constants, and $\beta_m$ is the linear thermal expansion coefficient of the matrix.

In the plastic range the matrix response is assumed to be the piecewise linear and given by (2), but $M_m$ and $L_m$, $m_m$ and $\ell_m$ are now instantaneous compliance and stiffness matrices, and thermal vectors, at a particular point of a loading path.   We assume that $M_m$ and $L_m$ are symmetric, satisfy the requirement of plastic incompressibility of the matrix, and do not depend on $d\theta$; $m_m$ and $\ell_m$ do not depend on $d\theta$, but may be functions of $\theta_0$.

Furthermore, we assume that the response of the composite to any purely mechanical loading by uniform $d\bar{\sigma}$ or by $d\bar{\varepsilon}$ is also piecewise linear and described by (1).   Suppose that instantaneous overall properties M

and L in (1), as well as instantaneous phase concentration factors $A_r$, $B_r$ in (27) can be evaluated for any given mechanical loading step. The instantaneous thermal properties m, $\ell$, and concentration factors $a_r$ and $b_r$ are to be determined.

Therefore, we again pose the problem stated in Section 2: A representative volume V of the composite has been subjected to a certain loading or deformation history such that the current overall stresses and strains in V are uniform and have magnitudes $\bar{\sigma}^o$ and $\bar{\varepsilon}^o$. Also, the current temperature in V is constant and equal to $\theta_0$. At this particular point of the loading sequence we apply simultaneous increments of $d\bar{\sigma}$ and $d\theta$, or $d\bar{\varepsilon}$ and $d\theta$ in V, and wish to evaluate instantaneous values of M, L, m, $\ell$, and of the concentration factors $A_r$, $B_r$, $a_r$, $b_r$ during the loading step.

This problem is solved by the decomposition procedure of Section 2. Initially, the composite is subjected to the prescribed thermal change $d\theta$ and to simultaneously applied surface tractions $dS_T$ and $dS_A$ given by (14) to (16). These thermal and mechanical loads create local strain increments $d\hat{\varepsilon}_m$ in (18) and stress increments $d\hat{\sigma}_m$ in (23). In general, these strain increments may be inelastic. However, since the matrix is plastically incompressible, it is possible to assure that these increments correspond to purely elastic deformation in both phases under $d\theta$, $dS_A$ and $dS_T$. This is obviously the case when one chooses $\rho = 1$ in (11), so that the stress and strain increments in the matrix are isotropic.

The tractions $dS_A$ and $dS_T$ must now be removed. This may lead to plastic straining in the matrix, which corresponds to or is caused by $d\theta$. Also, if an overall stress increment $d\bar{\sigma}$ is applied simultaneously

with $d\theta$, then $dS_A$, $dS_T$, and $d\bar{\sigma}$ must be added and applied together.

The final results for a plastically deforming composite can now be written on the basis of (24) to (31). With $\rho = 1$ in (11) to (23), we retain the definitions (24) of vectors $h$, $s_a$ and $\gamma$ and of their components but replace that of $\rho$ with

$$1 = [1\ 1\ 1\ 0\ 0\ 0]^T. \tag{44}$$

For loading by $d\bar{\sigma}$ and $d\theta$, the overall strain increment $d\bar{\varepsilon}$ is:

$$d\bar{\varepsilon} = h\ d\theta + M(d\bar{\sigma} - s_a\ d\theta). \tag{45}$$

A comparison with $(1_1)$ again yields the form (26)

$$m = h - Ms_a \tag{46}$$

We note that for $\rho = 1$ and an isotropic matrix one obtains from (19) and (43) the following expression for $h$ in (24):

$$h = h\ 1 \tag{47}$$

where

$$h = h_1 = h_2 = s_T/(3K_m) + \beta_m,$$

and $s_T$ is given by (15).

Therefore, the first term in (45) is an overall isotropic strain increment, and, according to (18), it is equal to the matrix strain increment. The loading vector $d\bar{\sigma} - s_a\ d\theta$ represents total mechanical load that must be applied to the composite to reflect the effect of simultaneous application of $d\bar{\sigma}$ and $d\theta$. M is the instantaneous overall compliance corresponding to this loading vector.

The stresses caused in the phases by simultaneous application of $d\bar{\sigma}$

and $d\theta$ are:

$$d\sigma_f = s_T \underset{\sim}{\gamma} d\theta + B_f(d\overline{\sigma} - \underset{\sim}{s_a} d\theta)$$

$$\tag{48}$$

$$d\sigma_m = s_T \underset{\sim}{1} d\theta + B_m(d\overline{\sigma} - \underset{\sim}{s_a} d\theta)$$

where $B_f$, $B_m$ are the instantaneous stress concentration factors for the overall mechanical load increment $d\overline{\sigma} - \underset{\sim}{s_a} d\theta$. If these increments are described by (27), then the instantaneous thermal stress concentration factors become:

$$\underset{\sim}{b_f} = s_T \underset{\sim}{\gamma} - B_f \underset{\sim}{s_a}$$

$$\tag{49}$$

$$\underset{\sim}{b_m} = s_T \underset{\sim}{1} - B_m \underset{\sim}{s_a}$$

Equations (45) to (49) convert in an exact way the thermomechanical problem into a mechanical loading problem along the incremental path $d\overline{\sigma} - \underset{\sim}{s_a} d\theta$.

Next, consider loading by $d\overline{\epsilon}$ and $d\theta$. The composite is first subjected to loading by $d\theta$, $dS_A$, and $dS_T$, which causes isotropic strains $h d\theta$ in both the composite and matrix. Since the overall strain increment is now prescribed, the $h d\theta$ and any additional overall strains must be equal to $d\overline{\epsilon}$. Hence, the overall stress increment is:

$$d\overline{\sigma} = \underset{\sim}{s_a} d\theta + L(d\overline{\epsilon} - \underset{\sim}{h} d\theta) \tag{50}$$

A comparison with ($1_2$) again yields (29):

$$\underset{\sim}{\ell} = -\underset{\sim}{s_a} + L \underset{\sim}{h} . \tag{51}$$

While $s_a$ is not isotropic, together with $d\theta$ it causes an isotropic stress increment $s_T \underset{\sim}{1} d\theta$ in the matrix. This is found from (23) at $\rho = 1$. Accordingly, plastic loading of the composite is caused only by the second term in (50). The overall mechanical strain is equal to $\overline{d\epsilon} - h d\theta$. L is the instantaneous overall stiffness corresponding to this strain increment.

The strain increments in the phases are:

$$d\epsilon_f = \underset{\sim}{h} d\theta + A_f(\overline{d\epsilon} - \underset{\sim}{h}d\theta)$$

$$\tag{52}$$

$$d\epsilon_m = \underset{\sim}{h} d\theta + A_m(\overline{d\epsilon} - \underset{\sim}{h}d\theta)$$

where $A_f$, $A_m$ are instantaneous strain concentration factors for overall strain $\overline{d\epsilon} - hd\theta$. If (30) is used, then the instantaneous thermal strain concentration factors are:

$$a_f = (A_f - I) \underset{\sim}{h}$$

$$\tag{53}$$

$$a_m = (A_m - I) \underset{\sim}{h}$$

Inasmuch as the instantaneous M and L may have as many as 21 independent coefficients, the vectors $\underset{\sim}{m}$ in (46) and $\underset{\sim}{\ell}$ in (51) may have 6 independent coefficients. For example, $\underset{\sim}{m}$ assumes the form (41), with $\alpha_i$ ($i = 1$ to 6) representing instantaneous thermal expansion coefficients of the composite.

As in the previous case, (50) to (52) convert in an exact way the thermomechanical problem into a mechanical deformation problem along the path $\overline{d\epsilon} - hd\theta$. It is seen that in both cases the thermal and mechanical effects are coupled, even though phase mechanical properties do not

depend directly on temperature.

## (ii) Two-phase composites

Finally, we consider the two-phase composite with isotropic phases of arbitrary geometry, Section 3. The reinforcement phase (f) is assumed to remain elastic, while the matrix phase (m) may become elastic-plastic when a given yield condition has been satisfied. In the plastic region, the matrix constitutive relation is described by (2), with $M_m$ and $L_m$ replaced by instantaneous compliance and stiffness. Again, $M_m$ and $L_m$ are assumed to be piecewise linear, symmetric, and satisfy the requirement of plastic incompressibility of the matrix. Also, we assume that overall instantaneous properties M and L of the composite, as well as the instantaneous concentration factors $A_r$, $B_r$ can be evaluated for any purely mechanical overall stress or strain increment in the elastic and plastic range.

To find instantaneous thermal properties m, $\ell$, and the concentration factors $a_r$, $b_r$, we again consider a representative volume V of the composite which has been loaded to current uniform overall stress $\bar{\sigma}^0$, strain $\bar{\varepsilon}^0$, and temperature $\theta_0$. The volume V is now subjected to additional increments of $d\bar{\sigma}$ and $d\theta$, or $d\bar{\varepsilon}$ and $d\theta$. As in Section 3, we apply overall increments of temperature $d\theta$ and of isotropic stress dS, with dS given by (35). Resulting phase stresses and strains follow from (33), they are isotropic and by assumption cannot cause plastic deformation in the matrix. The surface tractions or strains must now be adjusted to satisfy the prescribed $d\bar{\sigma}$ or $d\bar{\varepsilon}$ at the boundary S of V.

For the case of $d\theta$ and $d\bar{\sigma}$ applied simultaneously one obtains the overall composite strain increment

$$d\underline{\bar{\varepsilon}} = \underline{q}\ d\theta + M(d\underline{\bar{\sigma}} - \underline{s}\ d\theta) \tag{54}$$

where $\underline{q}$ and $\underline{s}$ are given by (35) and (36), and M is the overall instantaneous compliance for the mechanical stress increment $d\underline{\bar{\sigma}} - \underline{s}\ d\theta$.

A comparison with $(1_1)$ again yields the form (37)

$$\underline{m} = \underline{q} - M\underline{s} \tag{55}$$

The stresses caused in the phases are:

$$d\underline{\sigma}_f = \underline{s}\ d\theta + B_f(d\underline{\bar{\sigma}} - \underline{s}\ d\theta)$$

$$\tag{56}$$

$$d\underline{\sigma}_m = \underline{s}\ d\theta + B_m(d\underline{\bar{\sigma}} - \underline{s}\ d\theta)$$

where $B_f$, $B_m$ are instantaneous stress concentration factors for the overall mechanical load increment $d\underline{\bar{\sigma}} - \underline{s}\ d\theta$. From (27) and (56), the instantaneous thermal stress concentration factors are:

$$\underline{b}_f = (I - B_f)\underline{s} \qquad , \qquad \underline{b}_m = (I - B_m)\underline{s} \tag{57}$$

For the case of $d\theta$ and $d\underline{\bar{\varepsilon}}$ applied together, one obtains the overall stress increment

$$d\underline{\bar{\sigma}} = \underline{s}\ d\theta + L(d\underline{\bar{\varepsilon}} - \underline{q}\ d\theta) \tag{58}$$

and

$$\underline{\ell} = -\underline{s} + L\underline{q} \tag{59}$$

where L is the instantaneous composite stiffness for the overall strain increment $d\underline{\bar{\varepsilon}} - \underline{q}\ d\theta$, and $\underline{q}$ is given by (36). The phase strain increments in this case are

$$d\varepsilon_f = q \ d\theta + A_f(d\bar{\varepsilon} - q \ d\theta)$$

(60)

$$d\varepsilon_m = q \ d\theta + A_m(d\bar{\varepsilon} - q \ d\theta)$$

and the instantaneous concentration factors:

$$a_f = (A_f - I)q \quad , \quad a_m = (A_m - I)q$$

(61)

where $A_f$, $A_m$ are instantaneous strain concentration factors for an over-all mechanical strain increment equal to $d\bar{\varepsilon} - q \ d\theta$.

Equations (54), or (58), again convert in an exact way the thermo-mechanical loading problem into a mechanical one along a loading path $d\bar{\sigma} - s \ d\theta$, or a strain path $d\bar{\varepsilon} - q \ d\theta$, respectively. As in the case of a fibrous composite, the thermal and mechanical loading effects are coupled in the instantaneous M, L, and $A_r$, $B_r$.

## 5.  DISCRETIZATION OF THE PHASES

Results of the previous sections depend on the availability of
overall instantaneous mechanical stiffnesses, compliances, and phase
concentration factors in each loading step. These quantities need to be
evaluated for a certain model geometry of the composite material. An
important consideration in the choice of a material model is the fact
that the thermal loading paths $d\bar{\sigma} - s_a\ d\theta$ in (45) and $d\bar{\sigma} - s\ d\theta$ in (54),
as well as the thermal strain paths $d\bar{\varepsilon} - h\ d\theta$ in (50) and $d\bar{\varepsilon} - q\ d\theta$ in
(58), may have a significant isotropic component. That is easily seen
from the definitions (24), (36) and (47) of $s_a$, $s$, $h$ and $q$. It follows
that the material model chosen for analysis of the mechanical response
must give reasonably accurate predictions when the composite is loaded
by isotropic overall stresses or strains. This restriction may exclude
certain models which are primarily useful in predicting the behavior of
a fibrous lamina under in-plane loads, such as the VFD model (DVORAK and
BAHEI-EL-DIN 1982).

Another important consideration in the choice of a material model
is the fact that when the matrix phase becomes plastic, the local pro-
perties (2) are stress-dependent, and therefore, $L_m$ and $M_m$ are no longer
spatially uniform. Even if (2) are regarded as relations for averages
in the phases, the phase properties need to be determined for the actual
local fields or their approximations. This excludes application of
certain averaging techniques, such as the self-consistent method, which
assume that phase fields are uniform.

These considerations suggest that the chosen composite model should
be based on a specific representative geometry of the microstructure,

which allows for discretization of each phase into a number of sub-elements with locally uniform fields. An example of such an approach was outlined by DVORAK and TEPLY (1985). In general, if the representative volume of the composite and the phase geometry in this volume are specified, then each phase can be subdivided into a certain number of finite elements, and the overall properties L, M, local properties in the plastically deforming subelements, as well as the concentration factors for each subelement can be calculated for any load or strain increment.

Suppose that the subelement stresses, strains, as well as the stiffnesses and compliances of plastically deformed subelements have been found for a certain increment $d\bar{\sigma}$ or $d\bar{\epsilon}$ applied to the representative volume at $d\theta = 0$. Let subscripts i, and j, denote subelements in the matrix, and fiber, respectively. If the partial contributions of ea.h row of $d\bar{\sigma}$ or $d\bar{\epsilon}$ are identified, one can write the uniform subelement fields in the form

$$d\sigma_{im} = B_{im} \, d\bar{\sigma} \qquad\qquad d\sigma_{jf} = B_{jf} \, d\bar{\sigma}$$

$$\tag{62}$$

$$d\epsilon_{im} = A_{im} \, d\bar{\sigma} \qquad\qquad d\epsilon_{jf} = A_{jf} \, d\bar{\epsilon}$$

where the A, B are instantaneous subelement concentration factors.

One can also write the following relations between the overall averages and the uniform local fields in the subelements of the repesentative volume:

$$d\bar{\sigma} = \Sigma \, c_i \, d\sigma_{im} + \Sigma \, c_j \, d\sigma_{jf}$$

$$\tag{63}$$

$$d\bar{\epsilon} = \Sigma \, c_i \, d\epsilon_{im} + \Sigma \, c_j \, d\epsilon_{jf}$$

where $c_i$, $c_j$ are subelement volume fractions such that

$$\Sigma \ c_i = c_m \qquad , \qquad \Sigma \ c_j = c_f \qquad , \qquad c_f + c_m = 1 \qquad (64)$$

Using (62) and (63) one can obtain the average phase concentration factors

$$A_m = \frac{1}{c_m} \Sigma \ c_i \ A_{im} \qquad\qquad B_m = \frac{1}{c_m} \Sigma \ c_i \ B_{im}$$

$$\qquad\qquad (65)$$

$$A_f = \frac{1}{c_f} \Sigma \ c_i \ A_{jf} \qquad\qquad B_f = \frac{1}{c_f} \Sigma \ c_j \ B_{jf}$$

From the local instantaneous $L_{im}$, $M_{im}$ in the plastically deforming subelements, known $L_m$, $M_m$, $L_f$, $M_f$ in the elastic subelements, and (62) to (65), one can find the overall instantaneous properties as:

$$L = \Sigma \ c_i \ L_{im} \ A_{im} + \Sigma \ c_j \ L_f \ A_{jf}$$

$$\qquad\qquad (66)$$

$$M = \Sigma \ c_i \ M_{im} \ B_{im} + \Sigma \ c_j \ M_f \ B_{jf}$$

The local thermal strain vectors $\underset{\sim}{m}_m$ and $\underset{\sim}{m}_f$ remain constant in each sub-element, at least for a given $d\theta$, and equal to those of the elastic phase. These thermal stress vectors are:

$$\underset{\sim}{\ell}_{im} = L_{im} \ \underset{\sim}{m}_m \qquad , \qquad \underset{\sim}{\ell}_f = L_f \ \underset{\sim}{m}_f \qquad (67)$$

The decomposition procedure can now be applied to the discretized representative volume. The results follow from those presented in Section 4.

<u>In the fibrous composite</u> one obtains: For $d\overline{\underset{\sim}{\sigma}} \neq 0$, $d\theta \neq 0$, the overall strain increment $d\underset{\sim}{\epsilon}$, and overall $\underset{\sim}{m}$, follow from (45) and (46), with M taken from (66). The local subelement stresses and thermal

stress concentration factors are, in analogy with (48) and (49):

$$d\overline{\sigma}_{jf} = s_T \underset{\sim}{\gamma} \, d\theta + B_{jf}(d\overline{\underset{\sim}{\sigma}} - \underset{\sim}{s}_a \, d\theta)$$

$$d\overline{\sigma}_{im} = s_T \underset{\sim}{1} \, d\theta + B_{im}(d\overline{\underset{\sim}{\sigma}} - \underset{\sim}{s}_a \, d\theta)$$

$$\underset{\sim}{b}_{jf} = s_T \underset{\sim}{\gamma} - B_{jf} \underset{\sim}{s}_a$$

$$\underset{\sim}{b}_{im} = s_T \underset{\sim}{\gamma} - B_{im} \underset{\sim}{s}_a$$

(68)

For $d\overline{\epsilon} \neq 0$ $d\theta \neq 0$, the overall stress increment $d\overline{\sigma}$ and overall $\underset{\sim}{\ell}$, follow from (50), (51) and (66), and the local fields and thermal strain concentration factors are as in (52), (53):

$$d\overline{\epsilon}_{jf} = \underset{\sim}{h} \, d\theta + A_{jf}(d\overline{\underset{\sim}{\epsilon}} - \underset{\sim}{h} \, d\theta)$$

$$d\overline{\epsilon}_{im} = \underset{\sim}{h} \, d\theta + A_{im}(d\overline{\underset{\sim}{\epsilon}} - \underset{\sim}{h} \, d\theta)$$

$$\underset{\sim}{a}_{jf} = (A_{jf} - I) \underset{\sim}{h}$$

$$\underset{\sim}{a}_{im} = (A_{im} - I) \underset{\sim}{h}$$

(69)

These results can be utilized to find average instantaneous thermal strain concentration factors in the phases as:

$$\underset{\sim}{a}_f = \frac{1}{c_f} \Sigma \, c_j \, \underset{\sim}{a}_{jf} \qquad\qquad \underset{\sim}{b}_f = \frac{1}{c_f} \Sigma \, c_j \, \underset{\sim}{b}_{jf}$$

$$\underset{\sim}{a}_m = \frac{1}{c_m} \Sigma \, c_i \, \underset{\sim}{a}_{im} \qquad\qquad \underset{\sim}{b}_m = \frac{1}{c_m} \Sigma \, c_i \, \underset{\sim}{b}_{im}$$

(70)

## 6.  RELATED APPLICATIONS

In addition to mechanical and thermal loading, the composite may also undergo a phase transformation such that one or both phases, if free, experience a volume change over an increment of temperature $d\theta$:

$$dV_r/V = d\varepsilon_{kk}^T = 3 \omega_r \, d\theta \qquad (r = f,m) \tag{71}$$

Furthermore, composites with polymer matrices may absorb moisture. This causes swelling of the unconstrained matrix material. If the moisture concentration is uniform, then

$$d\varepsilon_{kk}^H = 3 \gamma_m \, dc \tag{72}$$

where $\gamma_m$ is the linear swelling coefficient and $c$ is moisture concentration. If the matrix remains elastic in dilatation, and inviscid, then the above theory may be applied with the following adjustments.

Suppose that $d\theta$ and $dc$ are applied simultaneously and that a free phase $r$ undergoes total volume change

$$d\varepsilon_{kk}^r = (d\varepsilon_{kk}^\theta + d\varepsilon_{kk}^T + d\varepsilon_{kk}^H)_r \tag{73}$$

If this superposition holds in a piecewise linear manner, then one can write in phase $r$ for each loading step

$$d\varepsilon_{kk}^r = 3(\alpha_r + \xi_n^r \omega_r + \zeta_n^r \gamma_r) \, d\theta \tag{74}$$

where, in a particular loading step $n$:

$$\xi_n^r = (d\varepsilon_{kk}^\theta/d\varepsilon_{kk}^T)_r \, , \qquad \zeta_n^r = (d\varepsilon_{kk}^\theta/d\varepsilon_{kk}^H)_r$$

are known distribution coefficients. Hence,

$$d\varepsilon_{kk}^r = 3 (1 + \varsigma_n^r + \varsigma_n^r) \alpha_r d\theta. \tag{75}$$

This suggests that for each loading step n one can evaluate a certain multiplier of $3d\theta$ that can be substituted for the instantaneous linear thermal expansion coefficient of phase r in (2). Indeed, even in the case of thermal loading alone it may be appropriate to change $\alpha_r$ with temperature, and that is obviously possible in the present theory.

7.  DISCUSSION

Although the results are valid only for the two binary systems,
they apply to most composite materials of practical interest.  For the
composite systems in question, the elastic values of m and $\ell$ found from
(26) and (29) are identical with those that can be calculated from LEVIN
(1967) formulae, or equation (2.20) in ROSEN and HASHIN (1970) and
equation (33) in LAWS (1973).  However, the methods used in deriving
these respective equations, and their internal structure, are entirely
different.  The decomposition used herein makes it possible to find
overall thermomechanical response of the composite in the plastic range
in terms of instantaneous overall mechanical properties and thermo-
elastic constants of the phases.  Also, average instantaneous phase
stresses and strains are found in terms of mechanical concentration
factors.  All these relations are exact.  They make it possible to con-
vert any available facility for analysis of isothermal elastic-plastic
behavior of the two composite systems to one which can analyze the
effect of both mechanical loads and uniform changes in temperature, as
well as other transformation strains in the phases.

It has not escaped our attention that equations similar to (7) to
(10) can be written for three phase fibrous composites.  The resulting
system has at most one solution, and if it exists it leads to evaluation
of overall thermal vectors and local thermal concentration factors for
the elastic three phase materials which are analogous to those derived
in Section 2.  However, no additional constraints are allowed in this
case hence (11) cannot be introduced, and, therefore, it is not possible
to analyse elastic-plastic deformation of the three phase aggregate by the

method of Section 4(i).

8.    ACKNOWLEDGEMENT

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Exact relationships are derived between instantaneous overall thermal stress or strain vectors and instantaneous overall mechanical stiffness or compliance, for two binary composits systems in which one of the phases may deform plastically. Also, the local instantaneous thermal strain and stress concentration factors are related in an exact way to the corresponding mechanical concentration factors. The results depend on instantaneous thermoelastic constants and volume fractions of the phases. They are found for fibrous composites with two distinct elastically isotropic or transversely isotropic phases, and for any binary composite with elastically isotropic phases. The results indicate that in the plastic range the thermal and mechanical loading effects are coupled even if the phase properties do not depend on changes in temperature. The derivation is based on a novel decomposition procedure which indicates that spatially uniform elastic strain fields can be created in certain heterogeneous media by superposition of uniform phase eigenstrains with local strains caused by piecewide uniform stress fields which are in equilibrium with prescribed surface tractions. The method

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is extended to discretized microstructures, and also to analysis of moisture absorotion and phase transformation effects on overall resoonse and on local fields in the two composite materials.

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