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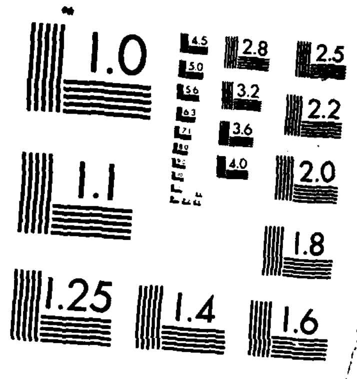

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PRODUCT OF TWO RALEIGH QUOTIENTS

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square root of

ABSTRACT

A computational algorithm is developed for finding the stationary values of the function $x'Cx/(x'Ax)^{\frac{1}{2}}(x'Bx)^{\frac{1}{2}}$ where A and B are positive definite and C is a symmetric matrix. The square of the function under consideration is the product of two Raleigh coefficients $x'Cx/x'Ax$ and $x'Cx/x'Bx$. The general problem occurs in multivariate analysis in the computation of homologous canonical variates in studying relationships between two sets of homologous measurements. The special case with $C = I$ occurs in designing control systems with minimum norm feedback matrices.

1. INTRODUCTION

If A is a positive definite matrix and C is a symmetric matrix of order p, then it is well known that the stationary values of the Raleigh coefficient $x'Cx/x'Ax$ are the eigen values of C with respect to A (Rao, 1973, p. 74). In particular, if $\lambda_1 \geq \ldots \geq \lambda_p$ are the ordered eigen values, then

$$\lambda_1 = \max_x \frac{x'Cx}{x'Ax}, \quad \lambda_p = \min_x \frac{x'Cx}{x'Ax}. \tag{1.1}$$

In this paper, we consider the problem of obtaining the stationary values of

$$\frac{x'Cx}{(x'Ax)^{\frac{1}{2}}(x'Bx)^{\frac{1}{2}}} \tag{1.2}$$

where A and B are positive definite matrices and C is a symmetric matrix of order p. The square of (1.2) is the product of the two Raleigh coefficients $(x'Cx/x'Ax)$ and $(x'Cx/x'Bx)$.

The special case of (1.2) with $C = I$ originally arose in attempts to design control systems with minimum norm feedback matrices (Kouvaritakis and Cameron, 1980; Cameron and Kouvaritakis, 1980) and also in the study of the stability of multivariable non-linear feedback systems (Cameron, 1983). The general case of (1.2) occurs in the analysis of familial data when multiple homologous measurements are available on say father and son and the object is to determine a linear combination of the measurements which has the maximum parent-offspring correlation. In this case, the dispersion matrix of (Y,Z), the vectors of p homologous measurements on father andson, can be written as

$$\begin{pmatrix} A & C_1 \\ C_1' & B \end{pmatrix}. \tag{1.3}$$

The correlation between two homologous linear functions $x'Y$ and $x'Z$ is then

$$\frac{x'Cx}{(x'Ax)^{\frac{1}{2}}(x'Bx)^{\frac{1}{2}}}\ , \quad C = (C_1+C_1')/2, \tag{1.4}$$

and the problem is one of maximizing or minimizing (1.4) over $x \in R^p$. We call such optimizing linear functions homologous canonical variates (HCV's).

To obtain the stationary values of (1.2), we equate the derivative of (1.2) with respect to x to the zero vector (Rao, 1973, p. 72). This yields the equation

$$\frac{x'Cx}{x'Ax} Ax + \frac{x'Cx}{x'Bx} Bx = 2Cx \tag{1.5}$$

which can be written in the equivalent form

$$\left.\begin{aligned} \lambda Ax + \mu Bx &= 2Cx \\ \lambda x'Ax &= x'Cx \end{aligned}\right\} \tag{1.6}$$

introducing two additional variables λ and μ, or in the form

$$\left.\begin{aligned} \lambda(A+\nu B)x &= 2Cx \\ x'Ax &= \nu x'Bx \end{aligned}\right\} \tag{1.7}$$

introducing two additional variables λ and ν.

Since A and B are positive definite matrices there exists a nonsingular transformation S such that $A = S\Delta S'$ and $B = SS'$ where Δ is a diagonal matrix (Rao, 1973, p. 41). Then writing x for $S'x$ and C for $S^{-1}C(S^{-1})'$, the equation (1.7) assumes the simpler form

$$\left.\begin{aligned} \lambda(\Delta+\nu I)x &= 2Cx \\ x'\Delta x &= \nu x'x. \end{aligned}\right\} \tag{1.8}$$

If $\delta_1,\ldots,\delta_p$ are the diagonal elements of Δ and $x_1,\ldots,x_p$ are the components of x, then eliminating λ and ν from (1.8), we have the equations for $x_1,\ldots,x_p$

$$2x'x\,[(e_i'Cx)x_1-(e_1'Cx)x_i\,] = x_1x_i(\delta_i-\delta_1)x'Cx, \qquad (1.9)$$

$$i = 1,\ldots,p,$$

where e_i is the elementary vector with unity as the i-th component and zeroes elsewhere. In (1.9) we have (p-1) quartic equations in (p-1) ratios (x_2/x_1), $\ldots,(x_{p-1}/x_1)$. *The solution of these equations is in general not easy except* in the case of p = 2 when there is only one quartic equation as observed by Kouvaritakis and Cameron (1980).

In this paper, we provide a computational algorithm for solving the equations (1.7) in the general case. The computer output gives all the solutions of (1.7) and the corresponding stationary values of (1.2).

2. THE CASE WHERE ALL THE MATRICES ARE DIAGONABLE

When all the matrices A, B and C are diagonable by a common transformation, the equation (1.8) reduces to

$$\left.\begin{aligned} 2Fx &= \lambda\Delta x + \mu x \\ x'Fx &= \lambda x'\Delta x \end{aligned}\right\} \qquad (2.1)$$

where F is a diagonal matrix with say $f_1,\ldots,f_p$ as its diagonal elements. In terms of the components of x, the first equation in (2.1) can be written as

$$2f_ix_i = (\lambda\delta_i + \mu)x_i\ ,\ i = 1,\ldots,p. \qquad (2.2)$$

There can be several types of solutions to (2.2).

(1) $x = e_i$ satisfies (2.1) with $\lambda = f_i/\delta_i$ and $\mu = f_i$ giving the stationary value $f_i/\sqrt{\delta_i}$.

(2) There can be solutions of the form $x = ae_i + be_j$. In such a case

$$2f_i = \lambda\delta_i + \mu\ ,\ 2f_j = \lambda\delta_j + \mu\ ,\ \delta_i \neq \delta_j \text{ and } f_i \neq f_j \qquad (2.3)$$

giving

$$\lambda = 2(f_i - f_j)/(\delta_i - \delta_j),\ \mu = 2(f_i\delta_j - f_j\delta_i)/(\delta_j - \delta_i). \tag{2.4}$$

A solution of the form $ae_i + be_j$ exists only if $\nu = [(f_i\delta_j - f_j\delta_i)/(f_j - f_i)] \in (\delta_i, \delta_j)$. If this happens, then $x = ae_i \pm be_j$ are solutions to (2.1), where $(a/b) = [(\nu - \delta_j)/(\nu - \delta_i)]^{\frac{1}{2}}$, yielding the same stationary value $\lambda\sqrt{\nu}$.

(3) There can be solutions of the form $ae_i + be_j + ce_k$ but they lead to the same stationary values as in (2).

An interesting case is that of Kantarovich (1948), where $\delta_i = \lambda_i^2$, $f_i = \lambda_i$ giving the stationary values 1 corresponding to solutions of type (1), and $2\sqrt{\lambda_i\lambda_j}/(\lambda_i + \lambda_j)$ corresponding to solutions of type (2). In this case the largest value is 1 and the smallest is $2\sqrt{\lambda_1\lambda_p}/(\lambda_1 + \lambda_p)$ where $\lambda_1 = \max\{\lambda, \ldots, \lambda_p\}$ and $\lambda_p = \min\{\lambda_1, \ldots, \lambda_p\}$, which gives the celebrated inequality of Kantarovich.

Thus, when all the matrices A, B and C are simultaneously diagonable, we have a closed form solution to the optimization problem. *Otherwise the solutions* to (1.7) have to be obtained through a suitable algorithm which we develop in the next section.

3. COMPUTATIONAL ALGORITHM IN THE GENERAL CASE

Let us consider the basic equation (1.5) in the form (1.7)

$$\left.\begin{aligned} 2Cx &= \lambda(A + \nu B)x \\ x'Ax &= \nu x'Bx \end{aligned}\right\} \tag{3.1}$$

where we recall that A and B are positive definite and C is a symmetric matrix all of order p. From the second equation in (3.1), we find that $\nu \in [\nu_p, \nu_1]$, where ν_1 and ν_p are the largest and smallest eigen values of A with respect to B. For any given $\nu \in [\nu_p, \nu_1]$, the first equation in (3.1) provides p eigen values

$$\lambda_1(\nu) \geq \ldots \geq \lambda_p(\nu) \tag{3.2}$$

of 2C with respect to $A + \nu B$, and p associated eigen vectors

$$x_1(\nu), \ldots, x_p(\nu). \tag{3.3}$$

The pair $(\nu, x_i(\nu))$ will be a solution of (3.1) if and only if

$$\nu = \frac{x_i'(\nu)Ax_i(\nu)}{x_i'(\nu)Bx_i(\nu)}. \tag{3.4}$$

Our computational algorithm is basically a search for ν and a suitable eigen vector $x_i(\nu)$ such that (3.4) holds. The complexity of the algorithm depends on the nature of the p eigen value functions

$$\lambda_i(\nu),\ \nu \in [\nu_p, \nu_1],\ i = 1, \ldots, p \tag{3.5}$$

each of which is a continuous function of ν (see Kato, 1980, Chapter 2 for various results used in this section).

If the rank of C is $s < p$, then $(p-s)$ functions in (3.5) identically vanish. All the solutions of $Cx = 0$ with $\lambda = 0$ (i.e., eigen vectors of C corresponding to its zero eigen value) satisfy (3.1), and the stationary value of (1.2) corresponding to each such solution is zero. Then there is a fixed number s_1 of the functions (3.5) such that

$$\lambda_1(\nu) \geq \ldots \geq \lambda_{s_1}(\nu) > 0 \tag{3.6}$$

and a fixed number s_2 such that

$$\lambda_p(\nu) \leq \ldots \leq \lambda_{p-s_2+1}(\nu) < 0. \tag{3.7}$$

Let us start with $\lambda_1(\nu)$. If $\lambda_1(\nu) \neq \lambda_2(\nu)$ for all ν, then there is a unique continuous eigen vector function $x_1(\nu)$ associated with $\lambda_1(\nu)$. We can then construct the continuous function

$$f_1(\nu) = [x_1'(\nu)\ A\ x_1(\nu)/x_1'(\nu)\ B\ x_1(\nu)] - \nu \tag{3.8}$$

which is ≥ 0 when $\nu = \nu_{min}$ and ≤ 0 when $\nu = \nu_{max}$, so that there is at least one value of ν, say ν_1, which makes (3.8) vanish and provides the solution $[\nu_1, x_1(\nu_1)]$ to (3.1). There may be more than one solution to the equation $f_1(\nu) = 0$, each of which leads to a solution of (3.1). Since the value of $f_1(\nu)$ for any given ν is uniquely computable, the solutions of $f_1(\nu) = 0$ can be easily found through a suitable computer program. We then consider $\lambda_2(\nu)$ and if $\lambda_2(\nu) \neq \lambda_3(\nu)$ for any ν, then the above procedure can be implemented leading to additional solutions. Now we go to $\lambda_3(\nu)$ to find additional solutions and so on. Thus in the case when the eigen value functions $\lambda_i(\nu)$ are distinct (no two meet anywhere) all the solutions can be obtained by considering the individual ordered eigen value functions. This is probably the case which often arises in practice leading to at least p solutions of (3.1). Otherwise we proceed as follows.

The above procedure can be implemented starting with $\lambda_1(\nu)$ so long as two successive eigen functions do not meet. Let us suppose that at the i-th stage we first encounter the case

$$\lambda_i(\nu) = \ldots = \lambda_{i+h-1}(\nu) \tag{3.9}$$

for some value of ν. Associated with this repeated root, there are h eigen vectors which may be written as columns of a matrix

$$X_i = (\ _i(\nu) \ :\ldots:\ x_{i+h-1}(\nu)). \tag{3.10}$$

Note that the choice of the individual vectors in (3.10) is not unique, but any choice would generate the same eigen space. We then form the matrices

$$E_i = X_i'\ A\ X_i,\ F_i = X_i'\ B\ X_i \tag{3.11}$$

and find the largest and smallest eigen values α_1 and α_h of E_i with respect to F_i and the associated eigen vectors y_1 and y_h. Then ν will be a solution iff $(\alpha_1-\nu)(\alpha_2-\nu) \leq 0$. If this happens,

$$\nu,\ x = X_i(c_1y_1 \pm c_2y_2) \tag{3.12}$$

are solutions to (3.1), where $c_1^2(\alpha_1-\nu) = c_2^2(\nu-\alpha_h)$, leading to the same stationary value $\sqrt{\nu}\ \lambda_i(\nu)$. If $(\alpha_1-\nu)(\alpha_2-\nu) > 0$, then ν is not a solution.

Having noted the computational procedures involved in testing whether a given ν is a solution or not depending on the multiplicity of the roots of

$$|2C-\lambda(A+B\nu)| = 0 \tag{3.13}$$

we make a few remarks on the complexity of the problem one may run into. From the results in perturbation theory of symmetric operators it is known that:

(1) The number of distinct roots of (3.12) *are the same for every* ν *except* for a finite number of "exceptional values" in $[\nu_{min}, \nu_{max}]$ where it can be less.

(2) The eigen value functions $\lambda_1(\nu) \geq \ldots \geq \lambda_p(\nu)$ are well behaved (holomorphic) in the intervals between the "exceptional points." In each such interval some consecutive eigen functions may coincide, and the set of identical eigen functions may be different in different intervals.

We consider some examples and make some general remarks in the next section.

4. ILLUSTRATIVE EXAMPLES

To illustrate the computations, we first consider the Kantarovich problem where all the matrices can be chosen to be diagonal:

$$A = \begin{pmatrix} 2 & & \\ & 4 & \\ & & 5 \end{pmatrix},\ B = \begin{pmatrix} .50 & & \\ & .25 & \\ & & .20 \end{pmatrix},\ C = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{4.1}$$

In this case, $\nu \in [4,25]$, where 4 is the smallest and 25 is the largest eigen value of A with respect to B. The graphs of $\lambda_1(\nu)$, $\lambda_2(\nu)$ and $\lambda_3(\nu)$, the eigen value functions of 2C with respect to $A + \nu B$, are shown in Figure 1.

[Here Figure 1]

We note that there are three possible exceptional points at which repeated roots occur where the type of computations described in (3.9) - (3.12) have to be done. Further, between the exceptional points the eigen value functions are distinct and well behaved.

The next step is to locate the exact values of the exceptional points, i.e., where $\lambda_1(\nu) = \lambda_2(\nu)$, and also the values of ν at which $f_1(\nu)$ vanishes between the exceptional points. [Note that $f_1(\nu)$ is uniquely defined between the exceptional points as in (3.8).] This can be done by tabulating λ_2/λ_1 and $f_1(\nu)$ at short intervals of ν, locating the intervals in which $f_1(\nu) = 0$ or $\lambda_2(\nu)/\lambda_1(\nu) = 1$, and find the values of ν where equalities are attained through a suitable program for finding the roots.

If ν is found such that $f_1(\nu) = 0$, then $(\nu, x_1(\nu))$ is a solution giving the stationary value $\sqrt{\nu}\ \lambda\ (\nu)$.

If ν is found such that $\lambda_1(\nu) = \lambda_2(\nu)$, then we have two eigen vectors say x_1, x_2 associated with this repeated root. We compute the matrices $E_1 = (x_1 : x_2)'\ A\ (x_1 : x_2)$ and $F_1 = (x_1 : x_2)'\ B\ (x_1 : x_2)$ which provide two eigen values $\alpha_1 \geq \alpha_2$ and the associated eigen vectors y_1, y_2 of E_1 with respect to F_1. If $(\alpha_1 - \nu)(\alpha_2 - \nu) \leq 0$, then ν is a solution giving the stationary value $\sqrt{\nu}\ \lambda_1(\nu)$. The vectors at which this value is attained are

$$x = (x_1 : x_2)(c_1y_1 \pm c_2y_2)$$

where $c_1^2(\alpha_1-\nu) = c_2^2(\nu-\alpha_2)$. If $(\alpha_1-\nu)(\alpha_2-\nu) > 0$, then ν is not a solution.

We then proceed to the next eigen value function $\lambda_2(\nu)$ and locate the values of ν at which $\lambda_3(\nu)/\lambda_2(\nu) = 1$, and the non-exceptional values of ν at which $f_2(\nu) = 0$ and repeat the above analysis. Finally, we consider $\lambda_3(\nu)$ and locate the non-exceptional values of ν at which $f_3(\nu) = 0$. The final tabulation leading to the stationary values of the function

$$(x'Cx)/(x'Ax)^{\frac{1}{2}}(x'Bx)^{\frac{1}{2}} \qquad (4.2)$$

is as follows.

Table 1. Stationary values of (4.2)

ν	x - vector			stationary value
4	1	0	0	1
16	0	1	0	1
25	0	0	1	1
8*	$1/\sqrt{2}$	$1/\sqrt{2}$	0	.9428
10*	$1/\sqrt{2}$	0	$1/\sqrt{2}$	.9036
20*	0	$1/\sqrt{2}$	$1/\sqrt{2}$	.9938

*Exceptional points

The x - vectors are standardized to have unit norm. The graphs of $f_1(\nu)$, $f_2(\nu)$ and $f_3(\nu)$ are shown in Figures 2, 3 and 4. Note the discontinuities of each function at the exceptional points.

[Here Figures 2, 3, and 4]

The next example is concerned with the evaluation of what we call homologous canonical variates (HCV). The following table gives the correlation matrix of the measurements on head length (HL), head width (HW), face width (FW) and stature (St) taken on father and son. The problem is to find a linear function of the four measurements which shows the highest correlation between father and son.

Table 2. Correlation Matrix

		Son				Father			
		HL	HW	FW	St	HL	HW	FW	St
Son	HL	1.000							
	HW	0.288	1.000						
	FW	0.410	0.604	1.000					
	St	0.325	0.311	0.219	1.000				
				A					
Father	HL	0.341	0.145	0.243	0.055	1.000			
	HW	0.194	0.045	0.066	0.248	0.137	1.000		
	FW	0.057	-0.033	0.111	0.028	0.027	0.657	1.000	
	St	0.174	0.181	0.187	0.581	0.130	0.325	0.190	1.000
				C_1			B		

The function to be maximized is

$$\rho(x) = (x'Cx)/(x'Ax)^{\frac{1}{2}}(x'Bx)^{\frac{1}{2}}, \; C = \tfrac{1}{2}(C_1+C_1'). \qquad (4.3)$$

In this case, the four eigen value functions are distinct and corresponding to each function there is only one root. The stationary values of the correlation function (4.3) and the standardized vectors at which they are attained are given in Table 3.

Table 3. Homologous Canonical Variates

Stationary Value of $\rho(x)$	HL	HW	FW	St
.5874	-.0076	.1236	-.1209	.9826
.3564	.9549	.1060	.1679	-.2207
.1675	-.1237	-.5772	.8001	.1062
-.0949	-.3531	.8841	.1361	-.2742

5. CONCLUDING REMARKS

In practical problems, the following situations may arise.

(1) The matrices A, B and C are simultaneously diagonable in which case closed form expressions are available as discussed in Section 2.

(2) The eigen value functions $\lambda_1(\nu),\ldots,\nu_p(\nu)$ are distinct (no two have a common point) in $\nu \in [\nu_{min}, \nu_{max}]$ in which case the method described in the paragraph containing the equation (3.8) is applicable leading to at least p solutions. This is probably the simplest and the most frequent case.

(3) The eigen value functions are distinct except at a finite set of "exceptional points." In such a case, the exceptional points are dealt with as in (3.9) - (3.12) and the non-intersecting eigen functions between the exceptional points are treated as in (2) above, except that each eigen value function in a sub-interval may not yield a root.

(4) A complicated situation is when some of the eigen value functions coincide in intervals between exceptional points. Note that the number of distinct eigen functions in each interval will be the same, although the eigen functions $\lambda_i(\nu)$ that coincide may be different in different intervals. When a number, say h, of eigen value functions coincide in an interval, we tabulate $\alpha_1(\nu)$ and $\alpha_h(\nu)$ defined in (3.11) at a number of points within the interval and locate the roots if any by considering the product $(\alpha_1(\nu)-\nu)(\alpha_h(\nu)-\nu)$. For distinct eigen value functions within an interval the procedure indicated in (2) is followed.

The computational algorithm developed in this paper has been implemented through FORTRAN program using the standard routines for eigen value computations and iterative methods for determining the roots of an equation with one variable.

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A computational algorithm is developed for finding the stationary values of the function $x'Cx/(x'Ax)^{\frac{1}{2}}(x'Bx)^{\frac{1}{2}}$ where A and B are positive definite and C is a symmetric matrix. The square of the function under consideration is the product of two Raleigh coefficients $x'Cx/x'Ax$ and $x'Cx/x'Bx$. The general problem occurs in multivariate analysis in the computation of homologous canonical variates in studying relationships between two sets of homologous measurements. The special case with C = I occurs in designing control systems with minimum norm feedback matrices.

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