AD-A164 389   A HIGH-ORDER MIXTURE MODEL FOR PERIODIC PARTICULATE   1/1
               COMPOSITES(U) CALIFORNIA UNIV SAN DIEGO LA JOLLA DEPT
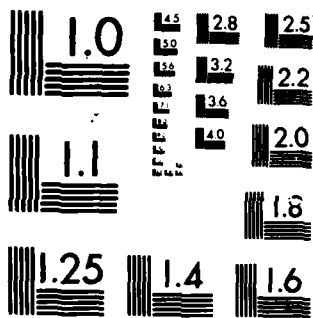   OF APPLIED MECHANICS A..   A TOLEDANO ET AL. DEC 85
UNCLASSIFIED   N00014-84-K-0468   F/G 11/4   NL

END

FILMED

DTIC

MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

TECHNICAL REPORT

A HIGH-ORDER MIXTURE MODEL FOR
PERIODIC PARTICULATE COMPOSITES

by

A. Toledano and H. Murakami

Department of Applied Mechanics and Engineering Sciences
University of California. San Diego
La Jolla. California 92093

DTIC
SELECTED
FEB 1 3 1986

D

DECEMBER 1985

Prepared for

86

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A164389	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A High-Order Mixture Model for Periodic Particulate Composites		5. TYPE OF REPORT & PERIOD COVERED Technical Report July 1984 to December 1985
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. Toledano and H. Murakami		8. CONTRACT OR GRANT NUMBER(s) N00014-84-K-0468
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept of Applied Mechanics & Engineering Sciences University of California, San Diego La Jolla, California 92093		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Mechanics Division, Solid Mechanics Program Office of Naval Research 800 N. Quincy St., Arington, VA 22217		12. REPORT DATE December 1985
		13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS (of this report)
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distributionunlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Particulate composite, a mixture theory, effective elastic moduli, wave dispersion.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
A deterministic mixture theory is presented for periodic particulate composites. The model is constructed by introducing convenient microdisplacement and micro-stress variables, and by using a regular asymptotic technique with multiple scales. Governing equations and appropriate boundary conditions are then deduced from Ressiner's mixed variational principle (1950). In order to test the accuracy of the present model, harmonic wave propagation is examined and compared with available experimental data for Glass/Epoxy and Steel/PMMA composites reported by Kinra and Ker (1983). Also, the effective elastic moduli are computed, and the results compared with other analytical methods. We will

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE
1 JAN 73

S/N 0102-LF-014-6601

TABLE OF CONTENTS

QUALITY
INSPECTED
3

ABSTRACT

A deterministic mixture theory is presented for periodic particulate composites. The model is constructed by introducing convenient microdisplacement and microstress variables. and by using a regular asymptotic technique with multiple scales. Governing equations and appropriate boundary conditions are then deduced from Reissner's mixed variational principle [25]. In order to test the accuracy of the present model. harmonic wave propagation is examined and compared with available experimental data for Glass/Epoxy and Steel/PMMA composites reported by Kinra and Ker [16]. Also. the effective elastic moduli are computed. and the results compared with other analytical methods.

1. INTRODUCTION

Composite materials exhibit very different behavior under static and dynamic loads, as compared to homogeneous isotropic solids. Therefore, proper assessment of the composite's physical properties, such as elastic moduli and wave dispersion spectra, is important for most engineering applications.

Different analytical methods for the determination of the effective elastic moduli of particulate composites have been compiled by Christensen [1]. These are: (i) the variational approach based on extremum principles [2,3,4,5], (ii) the probabilistic approach [6,7,8,9]. and (iii) the self-consistent scheme approach [10,11,12]. At this moment it is safe to conclude that formulas are available which predict accurately the effective elastic moduli of the composites.

In order to clarify dynamic behaviors of particulate composites ultrasonic measurements of longitudinal and shear wave velocities have been recorded by Kinra, Petraitis and Datta [14] at low frequency, and by Kinra and Anand [15] at long and short wavelengths for random particulate composites. Later. Kinra and Ker [16], again using ultrasonic devices, measured longitudinal wave speeds at low and high frequencies for periodic particulate composites. Their experimental investigation revealed the existence of stop and pass bands in the periodic composites [16], which were totally absent in the random composites [14,15]. To date there is no continuum model which accounts for such wave phenomena in particulate composites with periodic microstructure.

The dynamic behavior of random particulate composites has been studied theoretically via harmonic wave propagation. Mal and Bose [17], using a probabilistic approach, gave values for the effective wave speeds for imperfectly bonded spheres randomly dispersed in the matrix. Their analysis is confined to the long wavelength range. Datta [18] obtained longitudinal wave velocities through the scattering of a plane P-wave by a distribution of rigid spheroids. Later. Datta [19], using a probabilistic approach similar to that in [18], computed longitudinal and shear wave speeds for the case of identical elastic ellipsoids. His results [19] are however

nondispersive, since the calculations stem from a long wavelength approximation. Dispersive models for random particulate composites have been proposed by Beltzer *et al.* [20] for a viscoelastic matrix, with dilute particulate concentration, and by Ben-Amoz [21] using asymptotic expansions to derive governing equations for the macro-motion. The latter model fails to predict the experimental data reported later by Kinra and Ker [16]. Gaunard and Uberall [22] have also presented a dispersive model for random particulate distribution at low concentrations using scattering theory.

A deterministic micromechanical theory is proposed here for the simulation of harmonic wave propagation through periodic particulate composites. This method has been successfully applied to angle-ply laminates [23] and fiber-reinforced composites [24]. It consists in using a regular asymptotic technique with multiple scales to derive appropriate trial displacement and stress fields. This procedure yields a sequence of microboundary value problems (MBVP's) defined over a unit cell, which characterizes the periodicity of the composite microstructure. Governing equations and boundary conditions are then deduced from Reissner's mixed variational principle [25]. The lowest order model of this MBVP method is equivalent to the "O(1) homogenization theory." by Bensoussan, Lions and Papanicolaou [26] and Sanchez-Palencia [27]. This gives the effective elastic moduli. However, for dynamic problems where the wave length is of the same order of magnitude as the inclusion's dimensions, a higher order model is necessary to simulate wave dispersion spectra. Such a theory is named as $O(\epsilon)$ (at least) homogenization theory, where ϵ denotes a typical ratio of micro-to-macrodimensions of the composite. To date such a high order continuum model has not been constructed for periodic particulate composites.

Propagation of harmonic waves through periodic particulate composites is examined using the approach outlined above, and the results are compared with the experimental data recorded by Kinra and Ker [16] for Glass/Epoxy and Steel/PMMA composites. Also, in the limit when $\epsilon \to 0$, the effective elastic moduli are computed and compared with Hashin's results [2]. In all

cases, good agreement is observed between the available experimental data and analytical methods.

2. FORMULATION

Consider a volume $\bar{V}$ which contains particulates periodically distributed within the matrix, as shown in Fig. 1. This volume is referred to a cartesian coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3$.

The following notation: $(\ )^{(\alpha)}$, $\alpha = 1.2$ will designate quantities associated with material α, such that $\alpha = 1$ represents the particulate and $\alpha = 2$ the matrix. Barred and unbarred quantities will be associated with dimensional and nondimensional variables, respectively. Unless otherwise specified, the usual cartesian indicial notation is employed where latin indices range from 1 to 3 and repeated indices imply the summation convention. Also, $(\ )_{,i}$ and $(\ )_{,t}$ are used to denote partial differentiation with respect to $\bar{x}_i$ and time $\bar{t}$, respectively.

With the help of the foregoing notation, the governing equations for the displacement vector $\bar{u}_i^{(\alpha)}$ and stress tensor $\bar{\sigma}_{ij}^{(\alpha)}$ associated with the αth-constituent are:

a) Equations of motion

$$\bar{\sigma}_{ji,j}^{(\alpha)} = \bar{\rho}^{(\alpha)}\bar{u}_{i,tt}^{(\alpha)} \ ; \ \bar{\sigma}_{ji}^{(\alpha)} = \bar{\sigma}_{ij}^{(\alpha)} \tag{1}$$

where $\bar{\rho}^{(\alpha)}$ represents the mass density:

b) Constitutive equations for isotropic constituents

$$\bar{\sigma}_{ij}^{(\alpha)} = \bar{\lambda}^{(\alpha)}e_{kk}^{(\alpha)}\delta_{ij} + 2\bar{\mu}^{(\alpha)}e_{ij}^{(\alpha)} \tag{2}$$

where $\bar{\lambda}^{(\alpha)}$, $\bar{\mu}^{(\alpha)}$ are Lamé's constants, $e_{ij}^{(\alpha)}$ the infinitesimal strain tensor and δ_{ij} the Kronecker delta.

c) Strain-displacement relations

$$e_{ij}^{(\alpha)} = \frac{1}{2}\left(\bar{u}_{i,j}^{(\alpha)} + \bar{u}_{j,i}^{(\alpha)}\right) \tag{3}$$

d) Interface continuity conditions

$$\bar{u}_i^{(1)} = \bar{u}_i^{(2)} \ , \ \bar{\sigma}_{ji}^{(1)}\nu_j^{(1)} = \bar{\sigma}_{ji}^{(2)}\nu_j^{(1)} \quad \text{on} \quad \bar{A}_I \tag{4}$$

where $\bar{A}_I$ denotes the particulate-matrix interface surface.

e) Initial conditions at $t = 0$ and appropriate boundary conditions along the boundary $\partial\Gamma$.

Conditions (a) through (e) constitute an initial boundary value problem. However, due to the large number of heterogeneities in the medium, its direct solution represents a formidable task. The purpose of the following analysis is to circumvent this difficulty by deriving a set of partial differential equations with constant coefficients, whose solution will thus represent an approximation to the original problem. To this end, the basic equations are nondimensionalized by introducing the following parameters:

$\bar{\lambda}$ typical macrosignal wavelength

$\bar{\Delta}$ typical particulate spacing or cell dimension

$\bar{C}_{(m)}$, $\bar{\rho}_{(m)}$ reference wave velocity and macro mass density

$\bar{E}_{(m)} \equiv \bar{\rho}_{(m)}\bar{C}^2_{(m)}$ reference modulus

$\bar{t}_{(m)} \equiv \bar{\lambda}/\bar{C}_{(m)}$ typical macrosignal travel time

$\epsilon \equiv \bar{\Delta}/\bar{\lambda}$ ratio of micro-to- macrodimension

With the help of these parameters, nondimensional variables are now defined, such that

$$z_i = \frac{\bar{z}_i}{\bar{\lambda}} \quad , \quad t = \frac{\bar{t}}{\bar{t}_{(m)}}$$

$$(\lambda,\mu)^{(a)} = \frac{1}{\bar{E}_{(m)}}(\bar{\lambda},\bar{\mu})^{(a)} \quad , \quad \rho^{(a)} = \frac{\bar{\rho}^{(a)}}{\bar{\rho}_{(m)}}$$

(5)

Equation (5) shows that the material properties are periodic in the z_i-space in which the periodicity can be characterized by the cell. This representation implies that stress and displacement fields will vary according to two basic length scales: (1) a macro length characteristic of the body size, and (2) a micro length characteristic of cell spatial dimensions. Further, in most applications of interest, the microscale is much smaller than the macroscale. Therefore, it appears convenient to use an asymptotic technique based on the two scales involved in the problem. To this end, new independent micro-coordinates are introduced, such that:

$$z_i^* = \frac{z_i}{\epsilon}$$

(6)

As a consequence, all field variables now become functions of both the macrovariables z_i and microvariables z_i^*:

$$F(z_i, t) = F^*(z_i, z_j^*, t; \epsilon) \tag{7a}$$

Spatial derivatives of such a function F then take the form:

$$F_{,i}(z_{j,i}) = F^*_{,i}(z_j, z_k^*, t; \epsilon) + \frac{1}{\epsilon} F^*_{,i^*}(z_j, z_k^*, t; \epsilon) \tag{7b}$$

where $( )_{,i^*}$ is used to denote partial differentiation with respect to z_i^*. For notational simplicity F^* will be written as F in what follows.

Applying Eqs. (5) and (7b) to Eqs. (1) - (4) yields a new set of "synthesized" governing equations, given by:

a) Equations of motion

$$\sigma_{ji,j}^{(\alpha)} + \frac{1}{\epsilon} \sigma_{ji,j^*}^{(\alpha)} = \rho^{(\alpha)} u_{i,tt}^{(\alpha)}, \quad \sigma_{ji}^{(\alpha)} = \sigma_{ij}^{(\alpha)} \tag{8}$$

b) Constitutive equations for isotropic constituents

$$\sigma_{ji}^{(\alpha)} = \lambda^{(\alpha)} e_{kk}^{(\alpha)} \delta_{ij} + 2\mu^{(\alpha)} e_{ij}^{(\alpha)} \tag{9}$$

c) Strain-displacement relations

$$e_{ij}^{(\alpha)} = \frac{1}{2} \left[u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)} + \frac{1}{\epsilon} \left(u_{i,j^*}^{(\alpha)} + u_{j,i^*}^{(\alpha)} \right) \right] \tag{10}$$

d) Interface continuity conditions

$$u_i^{(1)} = u_i^{(2)}, \quad \sigma_{ji}^{(1)} \nu_j^{(1)} = \sigma_{ji}^{(2)} \nu_j^{(1)} \quad \text{on} \quad A_i \tag{11}$$

The periodicity condition implies that all field variables will assume equal values on opposite sides of the cell boundary. Therefore it is only necessary to consider a single cell in order to determine the dependence of these field variables on the microcoordinates z_i^*. As a consequence edge layer effects are not included in the model.

The construction of the present mixture model is accomplished by resorting to Reissner's mixed variational principle [25] for displacements and stresses. This variational principle. applied to the synthesized fields using the multiscale representation, takes the form:

$$
\int \left\{ \left[\sum_{\alpha=1}^{2} \int_{V^{*(\alpha)}} \left\{ \delta e_{ij}^{(\alpha)} \hat{\sigma}_{ij}^{(\alpha)} - \delta \hat{\sigma}_{ij}^{(\alpha)} \left[\frac{1}{2} \left(u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)} - \frac{1}{\epsilon} \left(u_{i,j^*}^{(\alpha)} + u_{j,i^*}^{(\alpha)} \right) \right) - e_{ij}^{(\alpha)}(\cdots) \right] \right\} dx_1^* dx_2^* dx_3^* \right. \right.
$$

$$
\left. + \frac{1}{\epsilon} \int_{A_I} \left[\overset{\nu}{T_i^I}(\delta u_i^{(2)} - \delta u_i^{(1)}) - \delta \overset{\nu}{T_i^I}(u_i^{(2)} - u_i^{(1)}) \right] dA \right] dx_1 dx_2 dx_3
$$

(12)

$$
= \int_V \left[\sum_{\alpha=1}^{2} \int_{V^{*(\alpha)}} \delta u_i^{(\alpha)}(-\rho^{(\alpha)} u_{i,tt}^{(\alpha)}) dx_1^* dx_2^* dx_3^* \right] dx_1 dx_2 dx_3
$$

$$
- \int_{\partial V_T} \left[\sum_{\alpha=1}^{2} \int_{V^{*(\alpha)}} \overset{\nu}{T_i^{(\alpha)}} \delta u_i^{(\alpha)} dx_1^* dx_2^* dx_3^* \right] dA
$$

where $V^{*(\alpha)}$ denotes the x_i^*-domain occupied by constituent α (see Fig. 2), $\hat{\sigma}_{ij}^{(\alpha)}$ is used for the approximate stress field. $\overset{\nu}{T_i^{(\alpha)}}$ represents the traction vector on the surface ∂V_T where tractions are specified and $\overset{\nu}{T_i^I}$ is the interface traction vector defined by

$$
\overset{\nu}{T_i^I} \equiv \hat{\sigma}_{ji}^{(\alpha)} \nu_j^{(1)} \quad \text{on} \quad A_I \quad .
$$

(13)

Also.

$$
\delta e_{ij}^{(\alpha)} \equiv \frac{1}{2} \left[\delta u_{i,j}^{(\alpha)} + \delta u_{j,i}^{(\alpha)} - \frac{1}{\epsilon} \left(\delta u_{i,j^*}^{(\alpha)} + \delta u_{j,i^*}^{(\alpha)} \right) \right] \quad .
$$

(14)

and

$$
e_{ij}^{(\alpha)}(\cdots) = - \left[\frac{\lambda}{2\mu(3\lambda - 2\mu)} \right]^{(\alpha)} \hat{\sigma}_{kk}^{(\alpha)} \delta_{ij} - \frac{1}{2\mu^{(\alpha)}} \hat{\sigma}_{ij}^{(\alpha)}
$$

(15)

3. ASYMPTOTIC ANALYSIS

The asymptotic technique starts by assuming the following expansions for the field variables:

$$\{u_i, \sigma_{ij}\}^{(\alpha)}(x_k, x_i^*, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \{u_{i(n)}, \sigma_{ij(n)}\}^{(\alpha)}(x_k, x_i^*, t) \quad ; \quad \epsilon \ll 1 \tag{16}$$

Substituting (16) into the governing equations (8)-(11) and grouping terms in equal powers of ϵ, a sequence of problems is obtained. The first of these yields

$$u_{i(0),j}^{(\alpha)} = 0 \quad , \quad \sigma_{ji(0),j}^{(\alpha)} = 0 \tag{17}$$

Equation (17a) implies that $u_{i(0)}^{(\alpha)}$ are independent of the microcoordinates x_j^*. Together with the zeroth order expansion of (11a), one has then

$$u_{i(0)}^{(\alpha)} = U_{i(0)}(x_k, t) \tag{18}$$

The remaining sets of equations obtained from (8)-(11) are, for $n \geq 0$:

$$\sigma_{ji(n+1),j}^{(\alpha)} = \rho^{(\alpha)} u_{i(n),tt}^{(\alpha)} - \sigma_{ji(n),j}^{(\alpha)} \quad , \quad \sigma_{ji(n)}^{(\alpha)} = \sigma_{ij(n)}^{(\alpha)} \tag{19}$$

$$\sigma_{ij(n)}^{(\alpha)} = \lambda^{(\alpha)} e_{kk(n)}^{(\alpha)} \delta_{ij} - 2\mu^{(\alpha)} e_{ij(n)}^{(\alpha)} \tag{20}$$

$$e_{ij(n)}^{(\alpha)} = \frac{1}{2}\left(u_{i(n),j}^{(\alpha)} - u_{j(n),i}^{(\alpha)} - u_{i(n+1),j}^{(\alpha)*} + u_{j(n+1),i}^{(\alpha)*}\right) \tag{21}$$

Equations (19)-(21) are supplemented with interface continuity and x^*-periodicity conditions. These are, for $n \geq 0$:

$$u_{i(n)}^{(1)} = u_{i(n)}^{(2)} \quad , \quad \sigma_{ji(n)}^{(1)}\nu_j^{(1)} = \sigma_{ji(n)}^{(2)}\nu_j^{(1)} \quad \text{on} \quad A_i \tag{22}$$

$$u_{i(n)}^{(2)} \text{ and } \sigma_{ji(n)}^{(2)}\nu_j^{(2)} \text{ are } x^* \text{-periodic on cell boundary} \tag{23}$$

Equations (19) through (23) generate a series of microboundary value problems (MBVP's) for $\sigma_{ij(n)}^{(\alpha)}$ and $u_{i(n)}^{(\alpha)}$, which are to be solved sequentially. The first of the MBVP's, corresponding to $\sigma_{ij(0)}^{(\alpha)}$ and $u_{i(1)}^{(\alpha)}$, is called the O(1) MBVP's and is defined by (17b), (19b), (20), (21), (22b), (23b) with $n = 0$ and (22a), (23a) with $n = 1$.

Higher order terms in the asymptotic expansions for the displacement and stress fields' microstructures, can be computed with suitable integrability and normalization conditions. However. due to the complexity of these MBVP's. a variational approach, based on Reissner's mixed principle [25] is preferred here in order to construct the mixture model. This construction can be carried out by supplying Eq. (12) with appropriate trial displacement and stress fields which must satisfy the interface continuity condition (22) and the x^*-periodicity condition (23) on the cell boundary. As a result only approximate solutions of the MBVP's are necessary.

These trial functions are obtained by modeling the cell as two concentric spheres. as shown in Fig. 2. where (r,θ,ϕ) are spherical coordinates. such that

$$x_1^* = r\ \sin\phi\ \cos\theta$$

$$x_2^* = r\ \sin\phi\ \sin\theta \tag{24}$$

$$x_3^* = r\ \cos\phi$$

The volume of the cell is denoted by V^*, its boundary is defined by $r = 1$ and its interface surface by $r = (n^{(1)})^{1/3}$. The quantities $n^{(\alpha)}$ denote the volume fraction of material α and satisfy the relation:

$$n^{(1)} + n^{(2)} = 1 \tag{25}$$

In terms of the spherical coordinates (r,θ,ϕ). the x^*-periodicity condition for the concentric-spheres approximation can be expressed as

$$F(r,\theta,\phi) = F(r,\pi + \theta,\pi + \phi) \tag{26}$$

4. TRIAL DISPLACEMENT AND STRESS FIELD

The $O(1)$ stress and $O(\epsilon)$ displacement fields can be determined by solving the $O(1)$ MBVP's which are defined by (17b), (19b), (20)-(23). As shown in [26], these MBVP's are excited by $U_{i(0),j}$. Therefore, the mixture formulation becomes more tractable by introducing microdisplacement variables which represent $U_{i(0),j} + U_{j(0),i}$, such that

$$\overset{j}{S}_i(z_k,t) = \frac{1}{\epsilon V^*} \int_{A_I} u_i^{(o)} \nu_j^{(1)} dA^* = \frac{1}{V^*} \int_{A_I} u_i^{(o)}_{\{1\}} \nu_j^{(1)} dA^* \tag{27}$$

where V^* is the volume of the cell. Since $u_{i\{1\}}^{(o)}$ is excited by $U_{i(0),j} + U_{j(0),i}$, there holds

$$\overset{j}{S}_i = \overset{i}{S}_j \tag{28}$$

As an $O(\epsilon)$ trial displacement field, the following form may be used

$$u_{i\{1\}}^{(o)} = \overset{j}{S}_i (z_k,t) g_j^{(o)}(z_l^*) \tag{29a}$$

where

$$g_i^{(o)}(z_j^*) = \frac{1}{n^{(o)}} \left[(-1)^{\alpha+1} z_i^* + \delta_{\alpha 2} \frac{z_i^*}{r^3} \right] \tag{29b}$$

Considering the $O(\epsilon^2)$ difference of the average of $u_i^{(o)}$ over $V^{*(o)}$, Eqs. (18) and (29) yield the following trial displacement field

$$u_i^{(o)}(z_k, z_l^*, t, \epsilon) = U_i^{(o)}(z_k,t) + \epsilon\, u_{i\{1\}}^{(o)}(z_k, z_l^*, t) \tag{30}$$

where $u_{i\{1\}}^{(o)}$ is given by (29). Eqs. (29) and (30) show that the mixture displacement variables are $U_i^{(1)}$, $U_i^{(2)}$, $\overset{j}{S}_i$ subject to the constraint (28). Inserting (30) into (20) and (21) with $n = 0$ and considering the $O(\epsilon^2)$ differences of the average stresses, the $O(1)$ trial stress field may be written as

$$\sigma_{ij(0)}^{(o)} = \tau_{ij}^{(o)}(z_k,t) + \tau_{ijk}^{(o)}(z_p,t) g_{k,l}^{(o)} \tag{31}$$

As with displacement variables, an $O(\epsilon)$ stress variables are conveniently introduced according

to

$$P_i(x_k,t) \equiv \frac{1}{\epsilon V^*} \int_{X_I} \sigma_{ji}^{(o)}\nu_j^{(1)}dA^* = \frac{1}{V^*} \int_{X_I} \sigma_{ji(1)}^{(o)}\nu_j^{(1)}dA^* \tag{32}$$

Integrating Eq. (8a) over $V^{*(o)}$ and making use of the x^*-periodicity condition. the following mixture equation is obtained

$$n^{(o)}\sigma_{ji,j}^{(oa)} - (-1)^{a+1} P_i = n^{(o)} \rho^{(o)} u_{i,tt}^{(oa)} \tag{33}$$

where the average operation is defined by

$$F^{(oa)}(x_k,t) \equiv \frac{1}{n^{(o)}V^*} \int_{V^{*(o)}} F^{(a)}(x_k,x_j^*,t)dx_1^* dx_2^* dx_3^* \tag{34}$$

From (33) it is seen that P_i plays the role of an interaction body force between the two constituents through the interface.

As an $O(\epsilon)$ trial stress field. the following form may be used

$$\hat{\sigma}_{ij(1)}^{(o)}(x_k,x_l^*,t) = P_n(x_k,t) g_n^{(o)}(x_l^*)\delta_{ij} \tag{35}$$

As a result, the trial stress field may now be written in the form

$$\hat{\sigma}_{ij}^{(o)}(x_k,x_l^*,t,\epsilon) = \hat{\sigma}_{ij(0)}^{(o)}(x_k,x_l^*,t) + \epsilon\, \hat{\sigma}_{ij(1)}^{(o)}(x_k,x_l^*,t) \tag{36}$$

where $\hat{\sigma}_{ij(0)}^{(o)}$ and $\hat{\sigma}_{ij(1)}^{(o)}$ are defined by (31) and (35) respectively.

5. MIXTURE EQUATIONS

Substituting the trial displacement and stress fields defined by (30) and (36), respectively, into Reissner's mixed variational principle Eq. (12) and using Eqs. (14) and (15), the following governing equations are obtained

a) Equations of motion

$$n^{(\alpha)} \tau_{ji,j}^{(\alpha)} + (-1)^{\alpha+1} \tau_{jikk,j}^{(\alpha)} - (-1)^{\alpha+1} P_i = n^{(\alpha)} \rho^{(\alpha)} U_{i,tt}^{(\alpha)} \tag{37a}$$

$$\frac{1}{\epsilon^2} \left[\tau_{ij}^{(2)} - \tau_{ij}^{(1)} - \frac{1}{n^{(1)}} \tau_{ij,kk}^{(1)} - \frac{1}{n^{(2)}} \tau_{ij,kk}^{(2)} - \kappa_{mlkj} \tau_{skml}^{(2)} \right] + A_1 P_{j,i} = A_2 \ddot{S}_{i,tt} \tag{37b}$$

where

$$(A_1, A_2) \equiv \sum_{\alpha=1}^{2} A^{(\alpha)}(1, \rho^{(\alpha)}) \tag{38a}$$

and

$$A^{(\alpha)} = \frac{1}{n^{(\alpha)2}} \left[\frac{1}{5} (-1)^{\alpha+1} n^{(1)5/2} + \delta_{\alpha 2} \left(\frac{n^{(1)} + 1}{n^{(1)1/3}} - \frac{9}{5} \right) \right] \tag{38b}$$

Also κ_{mlkj} are constants that depend on $n^{(1)}$ and $n^{(2)}$ only. The non-zero values of these constants are given in the Appendix. It should be noted that in Eq. (37), the superscript "a" used to denote "average" has been dropped from $\tau_{ij}^{(\alpha)}$ and $\tau_{ijkl}^{(\alpha)}$ since these are independent of the microcoordinates x_i^*.

b) Constitutive equations

$$\tau_{ij}^{(\alpha)} = \lambda^{(\alpha)} U_{k,k}^{(\alpha)} \delta_{ij} + \mu^{(\alpha)} (U_{i,j} + U_{j,i}^{(\alpha)}) \tag{39a}$$

$$\tau_{ijkl}^{(\alpha)} = \lambda^{(\alpha)} \overset{l}{S}_k \delta_{ij} + \mu^{(\alpha)} (\delta_{ki} \overset{l}{S}_i + \delta_{ki} \overset{l}{S}_j) \tag{39b}$$

$$P_i = A_3 \left(\frac{U_i^{(2)} - U_i^{(1)}}{\epsilon^2} + A_1 \overset{i}{S}_{k,k} \right) \tag{39c}$$

- 15 -

where

$$A_3 \equiv 1 / \left[3 \sum_{\alpha=1}^{2} \left(\frac{A^{(\alpha)}}{3\lambda^{(\alpha)} + 2\mu^{(\alpha)}} \right) \right] \qquad (40)$$

c) Boundary conditions along ∂V

$$\delta U_i^{(\alpha)} = 0 \quad \text{or} \quad n^{(\alpha)} \tau_{ij}^{(\alpha)} \nu_j - (-1)^{\alpha-1} \tau_{ij,kk}^{(\alpha)} \nu_j = \overset{\nu}{T_i}^{(\alpha p)} \qquad (41a)$$

$$\delta \overset{J}{S_i} = 0 \quad \text{or} \quad A_1 P_{,i} \nu_i = \overset{\nu J}{T_i} \qquad (41b)$$

where

$$\overset{\nu}{T_i}^{(\alpha F)} \equiv \frac{1}{V^*} \int_{V^*(\alpha)} \overset{\nu}{T_i}^{(\alpha)} dx_1^* dx_2^* dx_3^* \qquad (42a)$$

$$\overset{\nu J}{T_i} \equiv \frac{1}{V^*} \sum_{\alpha=1}^{2} \int_{V^*(\alpha)} \overset{\nu}{T_i}^{(\alpha)} g_j^{(\alpha)} dx_1^* dx_2^* dx_3^* \qquad (42b)$$

e) Initial conditions

$$\text{specify} \quad U_i^{(\alpha)} , \quad U_{i,t}^{(\alpha)} , \quad \overset{J}{S_i} , \quad \overset{J}{S_{i,t}} \quad \text{at} \quad t = 0 \qquad (43)$$

Equations (37), (39), (41) and (43) constitute a well-posed boundary value problem with respect to time t and macrocoordinates x_k.

6. EFFECTIVE ELASTIC MODULI

The effective elastic moduli, which relate stress averages to strain averages, are defined by the $O(1)$ homogenization theory. They can thus be obtained by letting $\epsilon \to 0$ in the mixture Eq. (37) and setting

$$U_i^{(1)} = U_i^{(2)} = U_i \tag{44}$$

Equation (37a) yields

$$\sigma_{ij}^{(m)} = \rho^{(m)} U_{i,tt} \tag{45}$$

where

$$\sigma_{ij}^{(m)} = \sum_{\alpha=1}^{2} \left[n^{(\alpha)} \tau_{ij}^{(\alpha)} + (-1)^{\alpha+1} \tau_{ij,kk}^{(\alpha)} \right] \tag{46a}$$

$$\rho^{(m)} \equiv \sum_{\alpha=1}^{2} n^{(\alpha)} \rho^{(\alpha)} \tag{46b}$$

while Eq. (37b) yields

$$\tau_{ij}^{(2)} - \tau_{ij}^{(1)} - \frac{1}{n^{(1)}} \tau_{ij,kk}^{(1)} - \frac{1}{n^{(2)}} \tau_{ij,kk}^{(2)} - \kappa_{mlkj} \tau_{ikm,l}^{(o)} = 0 \tag{47}$$

This last equation can be solved for the $\overset{j}{S}_i$ by inserting therein Eqs. (39a,b). Substituting in turn these expressions for $\overset{j}{S}_i$ into (46a), one obtains

$$\underset{\sim}{\sigma}^{(m)} = [C^{(m)}] \underset{\sim}{e}^{(m)} \tag{48}$$

where

$$\underset{\sim}{\sigma}^{(m)} = \left[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12} \right]^T \tag{49a}$$

$$\underset{\sim}{e}^{(m)} = \left[U_{1,1}, U_{2,2}, U_{3,3}, U_{2,3} + U_{3,2}, U_{3,1} + U_{1,3}, U_{1,2} + U_{2,1} \right]^T \tag{49b}$$

and $C^{(m)}$ is the effective elastic modulus matrix. Although the medium is heterogeneous, it is isotropic. Consequently only two independent constants should suffice, as is indeed the case.

$|C^{(m)}|$ is conveniently written in the standard form:

$$|C^{(m)}| = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}^{(m)} \tag{50}$$

The composite Lamé's constants $\lambda^{(m)}$ and $\mu^{(m)}$ are given in the Appendix.

In order to assess the accuracy of the present O(1) mixture theory, the results for the elastic constants of a tungsten carbide cobalt particulate composite are presented and compared with those obtained by Hashin [2]. Young's modulus E and Poisson's ratio ν are.

$$\text{for tungsten carbide} \quad E^{(1)} = 102 \times 10^6 \text{ psi} , \; \nu^{(1)} = 0.22 \tag{51a}$$

$$\text{and for cobalt} \quad E^{(2)} = 30 \times 10^6 \text{ psi} , \; \nu^{(2)} = 0.30 \tag{51b}$$

The variations in terms of the particulate volume fraction $n^{(1)}$ of the composite bulk modulus $K^{(m)}$ and shear modulus $\mu^{(m)}$ are shown in Fig. 3, where very good agreement is observed with Hashin's results [2]. For the bulk modulus $K^{(m)}$, the two curves are identical. The expression for $K^{(m)}$ derived by Hashin [2] represents an exact result when the medium is completely filled out with concentric spheres.

7. HARMONIC WAVE PROPAGATION

In order to test the accuracy of the present $O(\epsilon)$ mixture model, phase velocity spectra have been compared with experimental data for harmonic wave propagation. Due to the composite isotropy, it is sufficient to consider harmonic waves propagating along, say, the x_1-direction. Therefore let

$$\{U_i^{(\alpha)}(x_k,t) , \overset{j}{S_i}(x_k,t)\} = \{\hat{U}_i^{(\alpha)} , (ik)\overset{j}{\hat{S}_i}\} e^{(ikx_1 - i\omega t)} \tag{52}$$

where $\hat{U}_i^{(\alpha)}$ and $\overset{j}{\hat{S}_i}$ represent constant amplitudes. Also, k and ω denote the wave number and the angular frequency, respectively.

Substituting (52) into (37) and using the constitutive equations (39), yields an eigenvalue problem for $(\epsilon\omega)$ of the form:

$$[K]\underset{\sim}{U} = (\epsilon\omega)^2[M]\underset{\sim}{U} \tag{53}$$

where $\underset{\sim}{U} = [\hat{U}_1^{(1)} , \hat{U}_1^{(2)} , \overset{1}{\hat{S}_1}{}^* , \overset{2}{\hat{S}_2}{}^* , \overset{3}{\hat{S}_3}{}^* , \hat{U}_2^{(1)} , \hat{U}_2^{(2)} , \overset{2}{\hat{S}_1}{}^* , \hat{U}_3^{(1)} , \hat{U}_3^{(2)} , \overset{3}{\hat{S}_1}{}^* , \overset{3}{\hat{S}_2}{}^*]^T \tag{54}$

$[K]$ and $[M]$ are (12×12) real symmetric matrices, whose elements are functions of the volume fractions, constituents elastic properties and wave number. Furthermore, $[M]$ is a diagonal matrix. For a given (ϵk), a corresponding eigenvalue $(\epsilon\omega)$ can be computed. The phase velocity C_p is then determined from

$$C_p = \frac{(\epsilon\omega)}{(\epsilon k)} \tag{55}$$

In the present analysis the typical cell dimension $\bar{\Delta}$ was chosen to be the cell radius by using the concentric-spheres model. The reference elastic modulus and density used for the nondimensionalization are:

$$\bar{E}_{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\bar{E}^{(\alpha)} \quad , \quad \bar{\rho}_{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\bar{\rho}^{(\alpha)} \tag{56}$$

The dimensional frequency $f(Hz)$ is then given by

$$f = \frac{(\epsilon\,\omega)}{2\pi\,\bar{\Delta}} \sqrt{\frac{L_{(m)}}{\bar{\rho}_{(m)}}} \tag{57}$$

Numerical results were obtained for two composites: Glass/epoxy and Steel/PMMA, for which experimental data have been recorded by Kinra and Ker [16] for longitudinal waves. The constituents' physical properties are given in Table 1, in which the values for Poisson's ratio are estimates. In the simulation $\bar{\Delta}$ was computed from the particulate's radius which was 1 mm for glass and 0.55 mm for steel.

Phase velocity profiles for each composite are shown in Figs. 4 and 5, where two distinct modes are displayed: a longitudinal (P- mode) wave and a shear (S- mode) wave.

The first two branches in the spectrum for the P- mode, which are believed to be sufficiently accurate within the present calculation, are plotted as functions of frequency. Figures 4 and 5 clearly bring out the dispersion characteristic of the composite. The experimental data correspond to the longitudinal mode, and it can be seen that the correlation with the present results is quite satisfactory. In their paper, Kinra and Ker [16] have predicted, at higher frequencies where the wavelength approaches the particulate's radius, additional second (f=1.4 MHz) and third (f=2.0 MHz) stop bands for Glass/Epoxy, and additional third (f=1.5 MHz) and fourth (f=1.9 MHz) stop bands for Steel/PMMA. Their experiments however did not reveal these higher stop bands. They have conjectured "that this is probably due to the excitation of resonances of the spherical inclusions which, at higher frequencies, tend to dominate the resonances of the unit cell." Figure 4 (Glass/Epoxy) shows an additional stop band at f=1.53 MHz, which is in fairly close agreement with the value predicted by Kinra and Ker [16].

Finally, for completeness, the first branch of the shear wave velocity spectrum is also shown in Figs. 4 and 5, for each composite. Due to the composite isotropy, the vertically (SV) and horizontally (SH) polarized wave speeds are identical.

8. CONCLUSION

A micromechanical mixture model for periodic particulate composites was presented based on a two-scale asymptotic expansions. Governing equations and appropriate boundary conditions were deduced from Reissner's mixed variational principle [25]. Expressions for the effective elastic moduli were derived and compared with Hashin's results [2]. Also. harmonic wave dispersion spectra were obtained. and a good correlation with the experimental data of Kinra and Ker [16] for Glass Epoxy and Steel PMMA composites was observed.

ACKNOWLEDGEMENT

REFERENCES

1. R. M. Christensen. Mechanics of Composite Materials. a Wiley-Interscience Publication. New York (1979).

2. Z. Hashin. "The Elastic Moduli of Heterogeneous Materials." ASME J. Appl. Mech.. 29. 143-150 (1962).

3. Z. Hashin and S. Shtrikman. "A Variational Approach to the Theory of the Elastic Behavior of Multiphase Materials." J. Mech. Phys. Solids. 11. 127-140 (1963).

4. L. J. Walpole. "On Bounds for the Overall Elastic Moduli of Inhomogeneous Systems." J. Mech. Phys. Solids. 14. 151-162 (1966).

5. M. H. Miller, "Bounds for Effective Bulk Modulus of Heterogeneous Materials." J. Math. Phys.. 10. 2005-2013 (1969).

6. G. K. Batchelor and J. T. Green. "The Determination of the Bulk Stress in a Suspension of Spherical Particles to Order C^2". J. Fluid Mech.. 56. 401-407 (1972).

7. D. J. Jeffrey. "Group Expansions for the Bulk Properties of a Statistically Homogeneous, Random Suspension." Proc. R. Soc. Lond.. A.338. 503-516 (1974).

8. H. S. Chen and A. Acrivos. "The Solution of the Equations of Linear Elasticity for an Infinite Region Containing Two Spherical Inclusions." Int. J. Solids Struct.. 14. 331-348 (1978).

9. H. S. Chen and A. Acrivos. "The Effective Elastic Moduli of Composite Materials Containing Spherical Inclusions at non-Dilute Concentrations." Int. J. Solids Struct.. 14. 349-364 (1978).

10. R. Hill, "A Self-consistent Mechanics of Composite Materials." J. Mech. Phys.. 13. 213-222 (1965).

11. B. Budiansky, "On the Elastic Moduli of Some Heterogeneous Materials," J. Mech. Phys. Solids, 13, 223-227 (1963).

12. T. T. Wu, "The Effect of Inclusion Shape on the Elastic Moduli of a Two-Phase Material," Int. J. Solids Struct., 2, 1-8 (1966).

13. R. M. Christensen and K. H. Lo, "Solution for Effective Shear Properties in Three Phase Sphere and Cylinder Models," J. Mech. Phys. Solids, 27, 315-330 (1979).

14. V. K. Kinra, M. S. Petraitis, and S. K. Datta, "Ultrasonic Wave Propagation in a Random Particulate Composite," Int. J. Solids Struct., 16, 301-312 (1980).

15. V. K. Kinra and A. Anand, "Wave Propagation in a Random Particulate Composite at Long and Short Wavelengths," Int. J. Solids Struct., 18, 367-380 (1982).

16. V. K. Kinra and E. L. Ker, "An Experimental Investigation of Pass Bands and Stop Bands in Two Periodic Particulate Composites," Int. J. Solids Struct., 19, 393-410 (1983).

17. A. K. Mal and S. K. Bose, "Dynamic Elastic Moduli of a Suspension of Imperfectly Bonded Spheres," Proc. Camb. Phil. Soc., 76, 587-600 (1974).

18. S. K. Datta, "Scattering of Elastic Waves by a Distribution of Inclusions," Archives Mech., 28, 317-324 (1978).

19. S. K. Datta, "A Self-Consistent Approach to Multiple Scattering by Elastic Ellipsoidal Inclusions," J. Appl. Mech., 44, 657-662 (1977).

20. A. J. Beltzer, C. W. Bert and A. G. Striz, "On Wave Propagation in Random Particulate Composites," Int. J. Solids Struct., 19, 785-791 (1983).

21. M. Ben-Amoz, "A Dynamic Theory for Composite Materials," J. Appl. Math. Phys. (ZAMP), 27, 83-99 (1976).

22. G. C. Gaunard and H. Uberall, "Resonance Effects and the Ultrasonic Effective Properties of Particulate Composites," J. Acoust. Soc. Am., 74, 305-313 (1983).

23. H. Murakami. "A Mixture Theory for Wave Propagation in Angle-Ply Laminates, Part I: Theory," ASME J. Appl. Mech., 52. 331-337 (1985).

24. H. Murakami and G. A. Hegemier. "A Mixture Model for Unidirectionally Fiber-Reinforced Composites." Submitted for Publication.

25. E. Reissner. "On a Variational Theorem in Elasticity." J. Math. and Phys., 29. 90-95 (1950).

26. A. Bensoussan. J. L. Lions and G. Papanicolaou. "Asymptotic Analysis for Periodic Structures." North-Holland Publishing Co.. Amsterdam (1978).

27. E. Sanchez-Palencia. "Non-Homogeneous Media and Vibration Theory." Lecture Notes in Physics 127. Springer-Verlag. Berlin (1980).

APPENDIX

A. Non-Zero Values of Constants κ_{mlkj} appearing in Eq. (37b):

$$\kappa_{iiii} = \frac{1}{5n^{(1)}n^{(2)}} \quad : \quad \text{no sum on } i \tag{A1}$$

$$\kappa_{iijj} = \kappa_{jjii} = \frac{1}{15n^{(1)}n^{(2)}} \quad : \quad i \neq j \ , \quad \text{no sum on } i \text{ and } j \tag{A2}$$

$$\kappa_{ijij} = \kappa_{ijji} = \kappa_{jiij} = \kappa_{jiji} = \frac{3}{5n^{(1)}n^{(2)}} \quad : \quad i \neq j \ , \quad \text{no sum on } i \text{ and } j \tag{A3}$$

B. Expression for Composite Lamé's Constants $\lambda^{(m)}$ and $\mu^{(m)}$ appearing in Eq. (50):

$$\lambda^{(m)} = \hat{\lambda} + (3a_1 + 2a_2)\rho_1 + 2a_1\rho_2$$

$$\mu^{(m)} = \hat{\mu} + 2a_2\rho_2 \tag{B1}$$

where $\quad (\hat{\lambda} , \hat{\mu}) \equiv \sum_{\alpha=1}^{2} n^{(\alpha)}(\lambda^{(\alpha)} , \mu^{(\alpha)})$

$$a_1 \equiv \lambda^{(1)} - \lambda^{(2)} \ , \quad a_2 \equiv \mu^{(1)} - \mu^{(2)}$$

$$\rho_1 \equiv \frac{-a_1\beta + (a_1 + 2a_2)\gamma}{\beta^2 - 2\gamma^2 + \beta\gamma} \ , \quad \rho_2 \equiv \frac{-a_2}{\beta - \gamma} \tag{B2}$$

and

$$\beta \equiv \sum_{\alpha=1}^{2} \left(\frac{\lambda^{(\alpha)} - 2\mu^{(\alpha)}}{n^{(\alpha)}} \right) + \frac{2(2\lambda^{(2)} - 7\mu^{(2)})}{5n^{(1)}n^{(2)}}$$

$$\tag{B3}$$

$$\gamma \equiv \sum_{\alpha=1}^{2} \left(\frac{\lambda^{(\alpha)}}{n^{(\alpha)}} \right) - \frac{(3\mu^{(2)} - 2\lambda^{(2)})}{5n^{(1)}n^{(2)}}$$

Table 1. Physical Properties for Glass/Epoxy and Steel/PMMA Composites

	Volume fraction $n^{(\alpha)}$	Young's Modulus $\bar{E}^{(\alpha)}$ (GPa)	Poisson's ratio $\nu^{(\alpha)}$	Mass Density $\bar{\rho}^{(\alpha)}$ (g/cm³)
$^{(1)}$Glass	0.256	62.784	0.20	2.492
$^{(2)}$Epoxy	0.744	4.541	0.43	1.18
$^{(1)}$Steel	0.152	113.534	0.292	7.8
$^{(2)}$PMMA	0.848	3.032	0.30	1.16
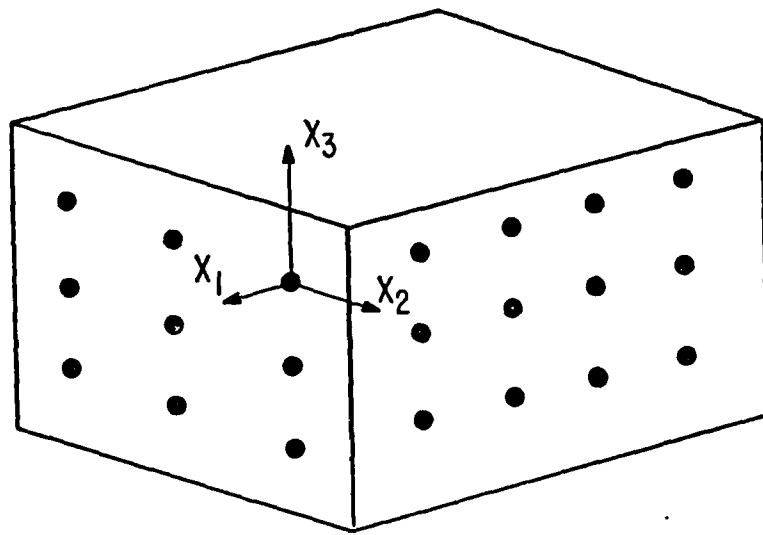
Figure 1. A Periodic Particulate Composite.
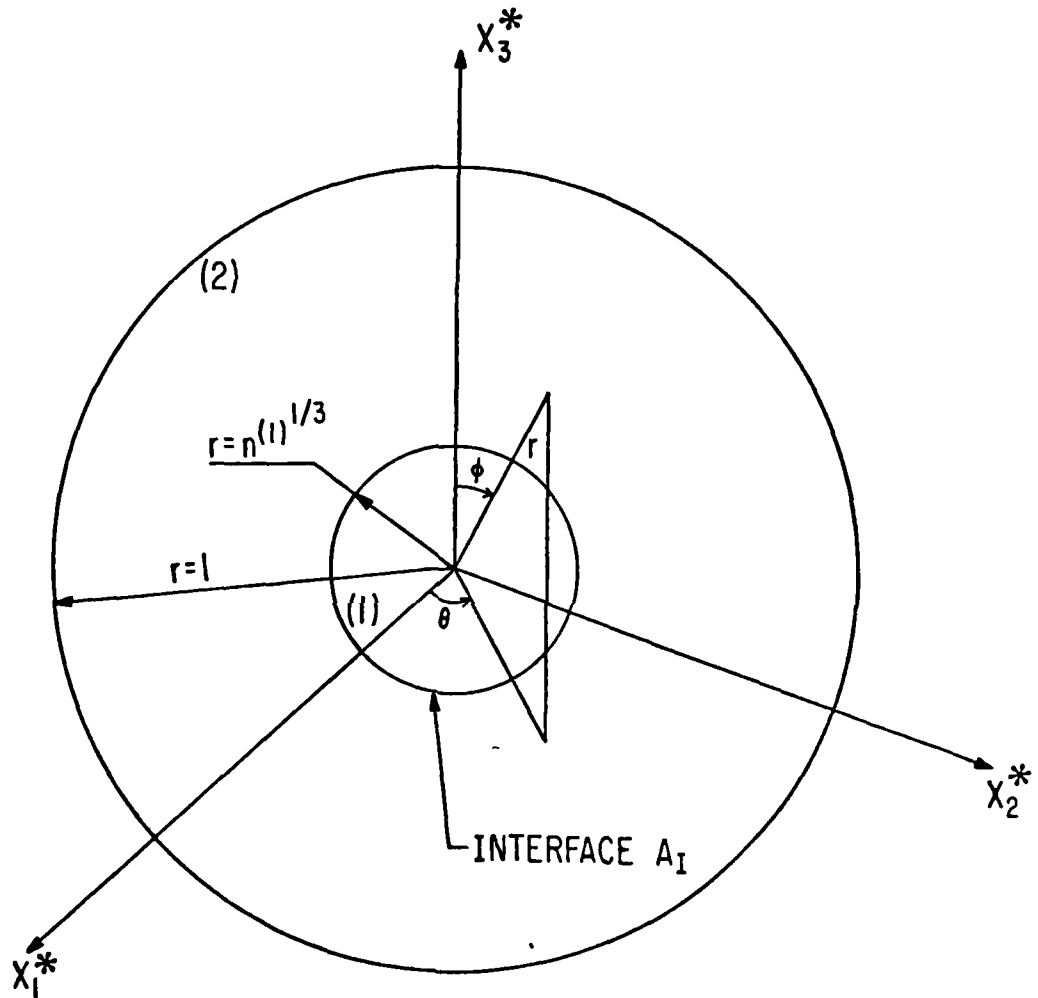
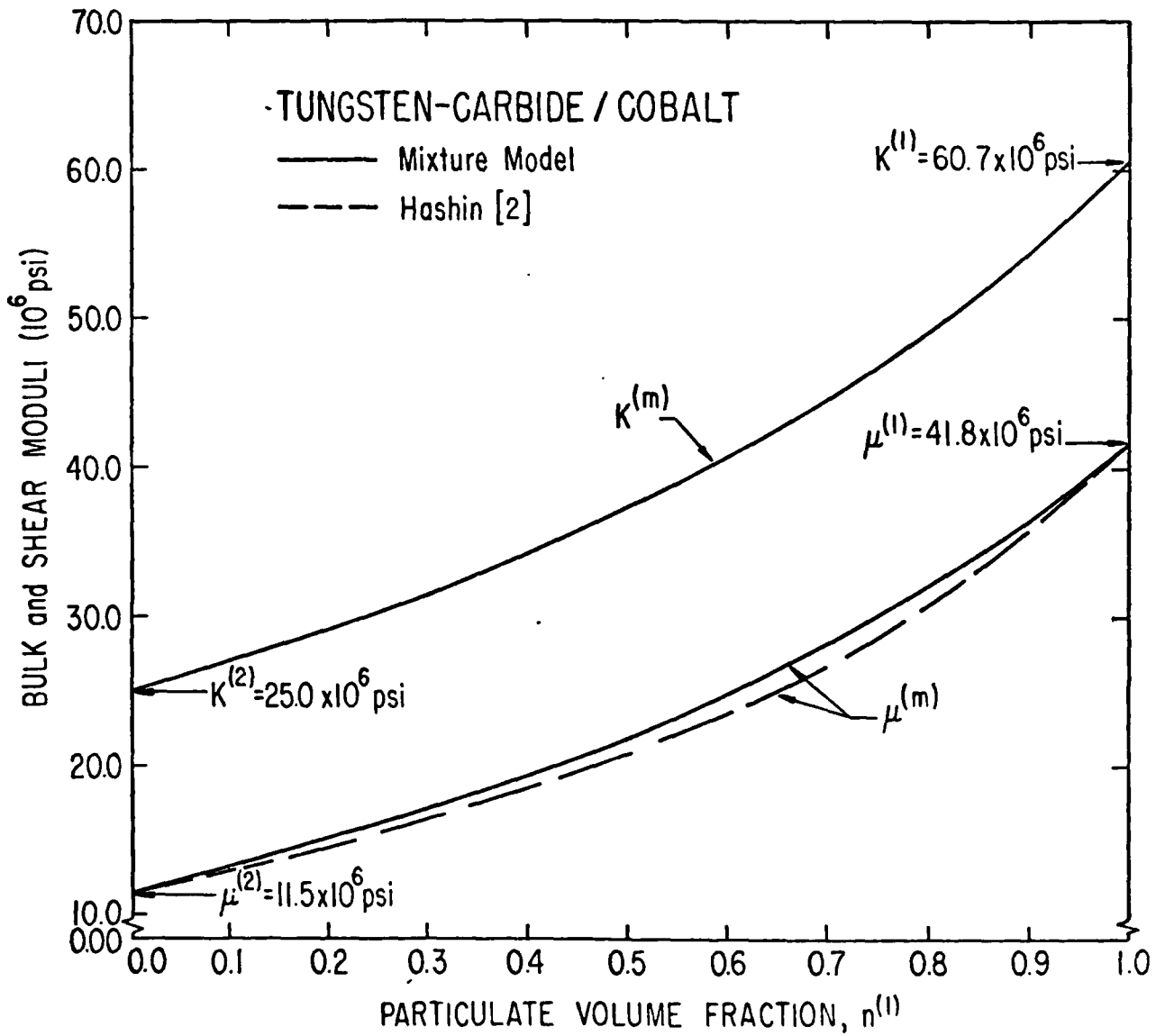Figure 2. Typical Cell ans Concentric-Spheres Model.

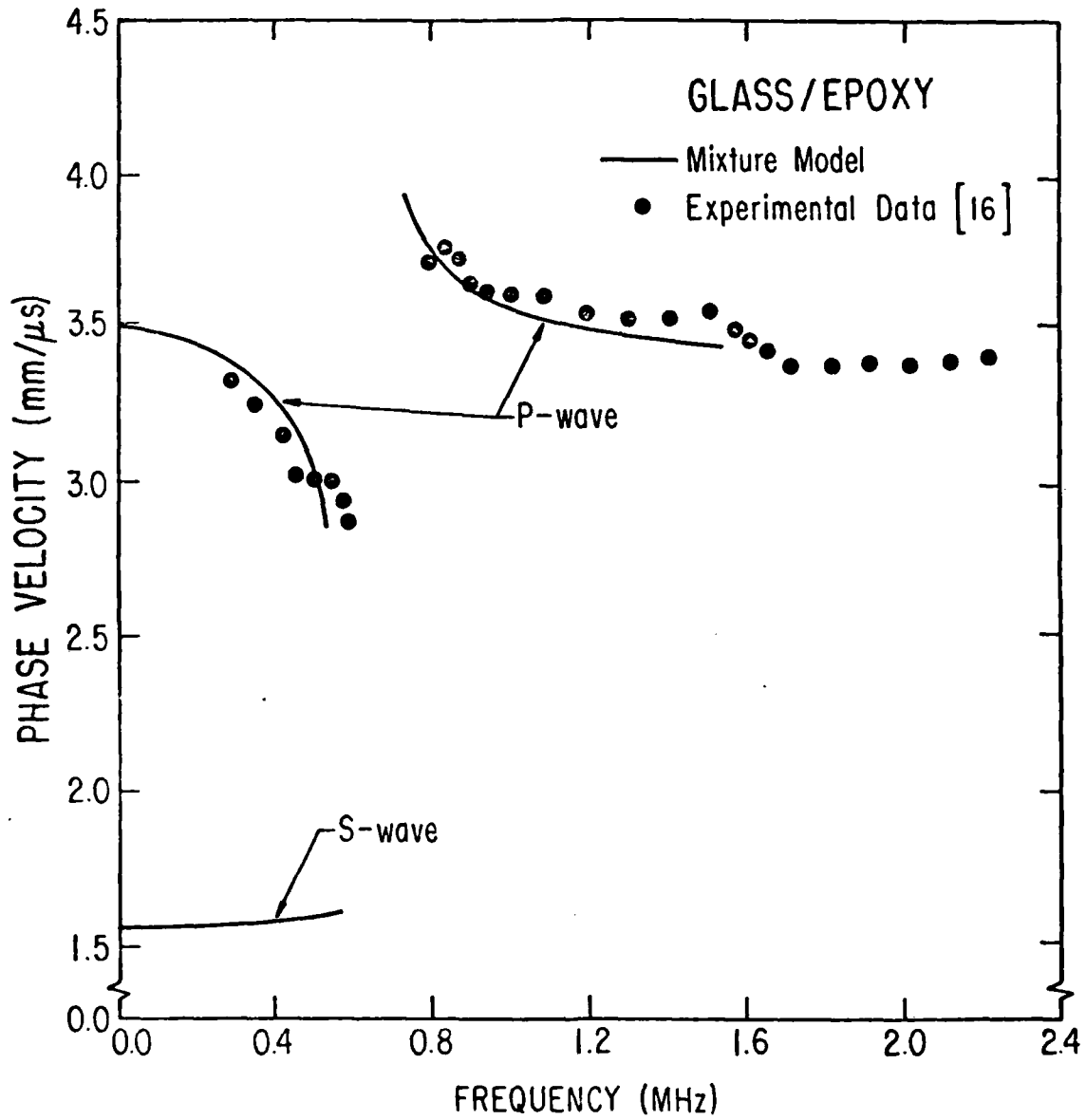Figure 3. Bulk and Shear Moduli as a Function of Particulate Volume Fraction.

Figure 4. Phase Velocity as a Function of Frequency in Glass/Epoxy
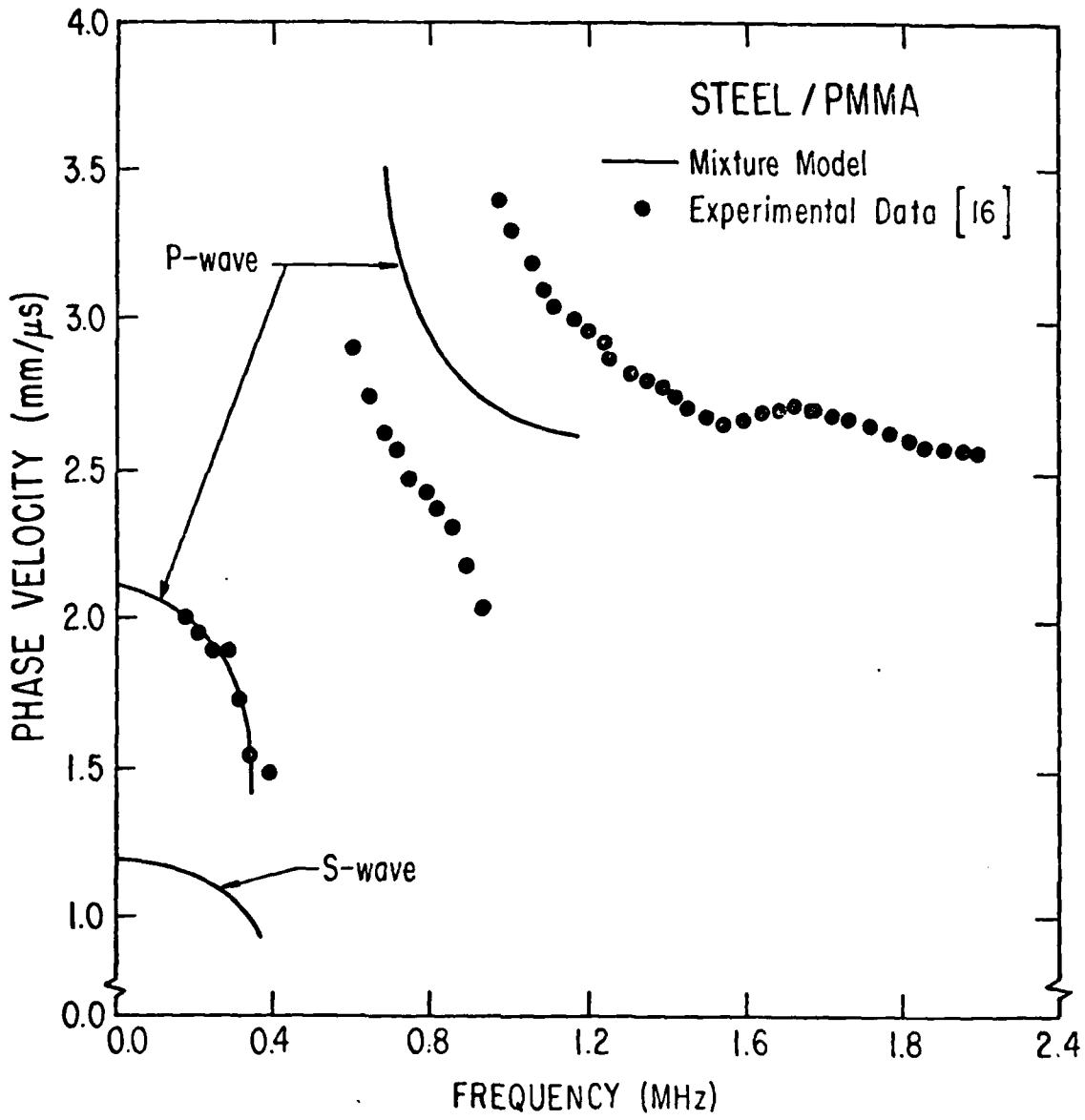Composite.

Figure 5. Phase Velocity as a Function of Frequency in Steel/PMMA
Composite.

END

FILMED

3-86

DTIC