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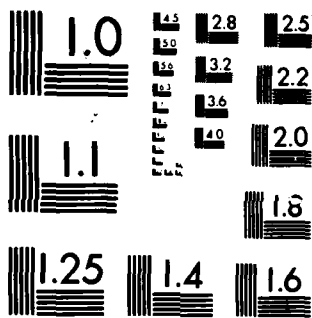   DEPT OF APPLIED MECHANICS A..   A TOLEDANO ET AL. DEC 85
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TECHNICAL REPORT

A COMPOSITE PLATE THEORY FOR ARBITRARY
LAMINATE CONFIGURATIONS

by

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In order to improve the accuracy of in-plane responses of shear deformable composite plate theories, a new laminated plate theory was developed for arbitrary laminate configuration based upon Reissner's (1984) new mixed variational principal. To this end, across each individual layer, piecewise linear continuous displacements and quadratic transverse shear stress distributions were assumed. The accuracy of the present theory was examined by applying it to the cylindrical bending problem of laminated plates which had been solved exactly by Pagano(1969). A comparison with the exact solutions obtained for symmetric, antisymmetric and arbitrary laminates that the present theory accurately estimates in-plane responses, even for small span-to-thickness ratios.

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ABSTRACT

In order to improve the accuracy of in-plane responses of shear deformable composite plate theories, a new laminated plate theory was developed for arbitrary laminate configuration based upon Reissner's (1984) new mixed variational principle. To this end, across each individual layer, piecewise linear continuous displacements and quadratic transverse shear stress distributions were assumed. The accuracy of the present theory was examined by applying it to the cylindrical bending problem of laminated plates which had been solved exactly by Pagano (1969). A comparison with the exact solutions obtained for symmetric, antisymmetric and arbitrary laminates indicates that the present theory accurately estimates in-plane responses, even for small span-to-thickness ratios.

1. INTRODUCTION

Thick laminated plates and shells find extensive applications as structural elements. As a result, various approximate theories have been developed in an effort to properly assess their mechanical behavior under static and dynamic loads.

In a series of papers Pagano (1969, 1970, 1971, 1972) gave the exact solution for the problem of cylindrical bending and simply supported plates. He pointed out the importance of considering shear deformation effects in order to accurately predict the plate lateral deflection and the necessity of improving the thickness variation of in-plane displacements, which are assumed to be C^1-linear functions in both classical plate theory (CPT) and Reissner-Mindlin plate theory (FSD).

One of the earliest attempts in deriving an approximate theory for laminates is credited to Yu (1959). He investigated the plane strain problem of isotropic sandwich plates by assuming piecewise linear displacement distributions. Yang, Norris and Stavsky (1966) extended Reissner-Mindlin plate theory to the case of an arbitrary number of bonded anisotropic layers. Whitney and Pagano (1970) using this latter approach, and later Whitney (1972), concluded that the introduction of shear deformations cannot improve the in-plane stress distributions as determined from classical plate theory.

As a remedy to these difficulties, higher order theories have been proposed in which the displacement assumptions are expressed in terms of power series in the thickness variable. The number of plate equations of such theories does not increase with the number of layers. Theories including quadratic variations (Whitney and Sun, 1973, and Nelson and Lorch, 1977) and cubic variations (Hildebrand, Reissner and Thomas, 1949, and Lo, Christensen and Wu, 1977) of in-plane displacements through the plate thickness belong to this category. Reddy's (1984) high-order theory is obtained by imposing the condition of vanishing transverse shear strains on the top and bottom surfaces of the plate. Whitney (1972) derived in-plane displacements by integrating the transverse shear strains deduced by Whitney and Pagano (1970). This resulted in a higher order approximation which accurately predicted in-plane strains, but the resulting modified stresses did not necessarily satisfy the original plate equilibrium equations.

Another class of higher order theories employs the displacements to be continuous and piecewise smooth functions, i.e., smooth within each layer (Yu, 1959). Durocher and Solecki have followed this approach to analyze transversely isotropic sandwich plates. Mau (1973), Srinivas (1973) and Seide (1980) have considered the case of an arbitrary number of layers. No shear correction factors were introduced by the last two authors. In all these theories, the number of field equations and edge boundary conditions depends on the number of layers.

Most of the theories discussed so far, which are classified as displacement-based theories, suffer from a common deficiency: constitutive equations lead to discontinuous interlaminar stresses. This shortcoming has been overcome in the theory proposed by Murakami (1985) which is based upon Reissner's (1984) new mixed variational principle. This theory is obtained by superposing a zig-zag shaped C^O-linear function to the in-plane displacements given by Reissner-Mindlin plate theory. It was later extended by including Legendre polynomials in the displacement variations across the plate thickness (Toledano and Murakami, 1985). In both these theories, the number of equilibrium equations and edge boundary conditions are independent of the number of layers. Thus, the theories have some limitations when applied to composite plates with arbitrary laminate configuration.

In order to improve the in-plane response of composite plates for arbitrary laminate configuration, a new theory which accounts for transverse shear deformations, has been developed by assuming piecewise continuous in-plane displacement distributions. In order to guarantee continuity of interlaminar stresses, Reissner's (1984) new mixed variational principle has been invoked by taking the transverse stresses to be quadratic functions of a local thickness coordinate across each layer. Governing equations and consistent boundary conditions are then deduced. The advantage of using Reissner's new mixed variational principle is that it automatically yields the appropriate shear correction factors for the transverse shear constitutive equations. A comparison with Pagano's (1969) exact elasticity solution for symmetric, antisymmetric and arbitrary laminates in cylindrical bending, shows that the proposed theory can accurately predict in-plane displacements and stresses at low span-to-thickness ratios. To further assess the range of applicability of the present theory, results previously obtained by Murakami (1985) are also presented

2. FORMULATION

Consider an N-layer laminated composite plate of uniform thickness h, as shown in Fig. 1. A cartesian coordinate system is chosen such that the middle surface of the plate occupies a domain D in the x_1, x_2-plane, the x_3-axis being normal to this plane. The following notation: $(\ )^{(k)}$, $k = 1,2, \ldots, N$ will designate quantities associated with the kth-layer. The thickness of each layer is $n^{(k)}h$, in which the volume fractions $n^{(k)}$ satisfy the relation

$$\sum_{k=1}^{N} n^{(k)} = 1 \tag{1}$$

Unless otherwise specified, the usual cartesian indicial notation is employed where latin and greek indices range from 1 to 3 and 1 to 2, respectively. Repeated indices imply the summation convention and $(\ )_{,i}$ is used to denote partial differentiation with respect to x_i.

With the help of the foregoing notation, the governing equations for the displacement vector $u_i^{(k)}$ and stress tensor $\sigma_{ij}^{(k)}$ associated with the kth-layer are:

a) Equilibrium Equations

$$\sigma_{ij,j}^{(k)} + f_i^{(k)} = 0 \quad ; \quad \sigma_{ji}^{(k)} = \sigma_{ij}^{(k)} \tag{2}$$

where f_i are body forces.

b) Constitutive Equations for Monoclinic Layers

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{bmatrix}^{(k)} \left(\frac{\sigma_{33}}{C_{33}} \right)^{(k)} \tag{3a}$$

$$e_{33}^{(k)} = -\left(\frac{C_{13}}{C_{33}} \right)^{(k)} e_{11}^{(k)} - \left(\frac{C_{23}}{C_{33}} \right)^{(k)} e_{22}^{(k)} - \left(\frac{C_{36}}{C_{33}} \right)^{(k)} 2e_{12}^{(k)} + \left(\frac{\sigma_{33}}{C_{33}} \right)^{(k)} \tag{3b}$$

$$\begin{bmatrix} 2e_{23} \\ 2e_{31} \end{bmatrix}^{(k)} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix}^{(k)-1} \begin{bmatrix} \sigma_{23} \\ \sigma_{31} \end{bmatrix}^{(k)} \equiv \begin{bmatrix} \tilde{C}_{55} & -\tilde{C}_{45} \\ -\tilde{C}_{45} & \tilde{C}_{44} \end{bmatrix}^{(k)} \begin{bmatrix} \sigma_{23} \\ \sigma_{31} \end{bmatrix}^{(k)} \tag{3c}$$

where C_{ij} are the elastic constants and $\tilde{C}_{ij}$ $(i,j = 1,2,6)$ represent the reduced stiffnesses introduced by

Whitney and Pagano (1970):

    c) Strain-Displacement Relations

$$e_{ij}^{(k)} = \frac{1}{2}\left[u_{i,j}^{(k)} + u_{j,i}^{(k)}\right] \tag{4}$$

    d) Interface Continuity Conditions

$$u_i^{(k)} = u_i^{(k+1)} \quad \text{and} \quad \sigma_{3i}^{(k)} = \sigma_{3i}^{(k+1)} \quad ; \quad k = 1,2,\cdots,N-1 \tag{5}$$

    e) Upper and Lower Surface Stress Conditions

$$\sigma_{3i}^{(1)} = T_i^+ \quad \text{on} \quad x_3 = \frac{h}{2} \tag{6a}$$

$$\sigma_{3i}^{(N)} = T_i^- \quad \text{on} \quad x_3 = -\frac{h}{2} \tag{6b}$$

The objectives in developing the present composite plate theory are : first, to improve the assumed variations of in-plane displacements through the plate thickness; second, to take into account the effect of transverse shear deformation and derive appropriate constitutive equations; and finally, for <u>arbitrary</u> laminate configurations, to assess and compare its range of applicability with the laminated plate theory proposed by Murakami (1985).

In order to carry out this task, Reissner's (1984) new mixed variational principle for displacements and transverse stresses was applied to the N-layer composite plate:

$$\iint_D \left[\sum_{k=1}^N \int_{A^{(k)}} \left\{\delta e_{ij}^{(k)}\sigma_{ij}^{(k)} + [u_{3,\alpha}^{(k)} + u_{\alpha,3}^{(k)} - 2e_{3\alpha}^{(k)}(\cdots)]\delta\tau_{3\alpha}^{(k)} + [u_{3,3}^{(k)} - e_{33}^{(k)}(\cdots)]\delta\tau_{33}^{(k)}\right\}dx_3\right]dx_1 dx_2$$

$$= -\iint_D \left[\sum_{k=1}^N \int_{A^{(k)}} \delta u_i^{(k)} f_i^{(k)} dx_3\right]dx_1 dx_2 + \iint_D \left[\delta u_i^{(1)}(x_1,x_2,\frac{h}{2})T_i^+ - \delta u_i^{(N)}(x_1,x_2,-\frac{h}{2})T_i^-\right]dx_1 dx_2$$

$$\tag{7}$$

$$+ \int_{\partial D_T}\left[\sum_{k=1}^N \int_{A^{(k)}} \delta u_i^{(k)} \bar{T}_i^{(k)} dx_3\right]ds$$

where ∂D_T denotes the boundary of D with outward normal ν_α on which tractions $\bar{T}_i$ are prescribed and $A^{(k)}$ is the x_3-domain occupied by the kth-layer. Also $\tau_{3i}^{(k)}$ denote the approximate transverse

stresses and $e_{3i}^{(k)}(\cdots)$ are given by Eqs. (3b,c). Due to the nature of Reissner's mixed variational principle, Eqs. (3a) are taken to be the definitions of $\sigma_{\alpha\beta}^{(k)}$ used in connection with (7).

3. TRIAL DISPLACEMENT AND TRANSVERSE STRESS FIELDS

The present laminated plate theory which accounts for transverse shear effects is obtained by assuming a linear variation of the in-plane displacements across each individual layer, as shown in Fig. 2. Transverse displacements are taken to be constant throughout the entire thickness of the plate. Therefore shear strains are constant within each layer, but differ from layer to layer. This approach has been previously adopted by Seide (1980) for the N-layer case. However no mention of shear correction factors was made in his paper.

The appropriate trial functions used in connection with Reissner's mixed variational principle Eq. (7) are chosen to be:

a) Trial Displacement Field

$$u_\alpha^{(k)}(x_i) = U_\alpha^{(k-1)}(x_\beta)g_1^{(k)}(x_3^{(k)}) + U_\alpha^{(k)}(x_\beta)g_2^{(k)}(x_3^{(k)})$$ (8a)

$$u_3^{(k)}(x_i) = U_3(x_\beta)$$ (8b)

where $$g_\alpha^{(k)}(x_3^{(k)}) \equiv \frac{1}{2} + (-1)^{\alpha+1}\frac{x_3^{(k)}}{n^{(k)}h}$$ (9)

$x_3^{(k)}$ is a local x_3-coordinate system with its origin at the center $x_{30}^{(k)}$ of the kth-layer, i.e.

$$x_3^{(k)} \equiv x_3 - x_{30}^{(k)}$$ (10)

From Eqs. (8a) and (9) it is seen that $U_\alpha^{(k)}$ ($k = 1,2,\cdots,N-1$), $U_\alpha^{(0)}$ and $U_\alpha^{(N)}$ represent the values of $u_\alpha^{(k)}$ at the interface, top and bottom surfaces of the plate, respectively. Also, Eq. (8) satisfy interface displacement continuity conditions Eq. (5a).

b) Trial Transverse Stress Field

$$\tau_{3\alpha}^{(k)}(x_i) = Q_\alpha^{(k)}(x_\beta)F_1(z) + T_\alpha^{(k-1)}(x_\beta)F_2(z) + T_\alpha^{(k)}(x_\beta)F_3(z)$$ (11a)

$$\tau_{33}^{(k)}(x_i) \equiv 0$$ (11b)

where $$F_1(z) = \frac{3}{2n^{(k)}h}(1-z^2)$$ (12a)

$$F_i(z) = \frac{3}{4}z^2 + \frac{1}{2}(-1)^i z - \frac{1}{4} \quad , \quad i = 2,3$$ (12b)

and
$$z \equiv \frac{2x_3^{(k)}}{n^{(k)}h} \; ; \; -1 \leqslant z \leqslant 1 \qquad (13)$$

Also
$$Q_\alpha^{(k)} \equiv \int_{A^{(k)}} \tau_{3\alpha}^{(k)} \, dx_3 \qquad (14)$$

In Eq. (11a), $T_\alpha^{(k-1)}$ and $T_\alpha^{(k)}$ are the values of $\tau_{3\alpha}^{(k)}$ at the top and bottom surfaces of the kth layer, respectively. From Eq. (6) one has

$$T_i^{(0)} = T_i^+ \quad \text{and} \quad T_i^{(N)} = T_i^- \qquad (15)$$

Eq. (11) satisfy the interface stress continuity conditions Eq. (5b). Due to the approximation for $u_3^{(k)}$ which yields $e_{33}^{(k)} = 0$, $\sigma_{33}^{(k)}$ becomes a reactive stress. Consequently, $\sigma_{33}^{(k)}$ can be determined by integrating the third equilibrium equation.

4. LAMINATED PLATE EQUATIONS

Substituting Eqs. (8) and (11) into Eq. (7) and using Gauss' theorem, one obtains:

a) Equilibrium Equations

$$\frac{1}{2} N_{\beta\alpha,\beta}^{(1)} + \frac{1}{n^{(1)}h} \left[M_{\beta\alpha,\beta}^{(1)} - N_{3\alpha}^{(1)} \right] + \frac{1}{2} \overset{\backprime}{F}_{\alpha}^{(1)} + \frac{1}{n^{(1)}h} \overset{M}{F}_{\alpha}^{(1)} + T_{\alpha}^{-} = 0 \tag{16a}$$

$$\frac{1}{2} \left[N_{\beta\alpha,\beta}^{(k)} + N_{\beta\alpha,\beta}^{(k-1)} \right] - \frac{1}{h} \left[\frac{M_{\beta\alpha,\beta}^{(k)}}{n^{(k)}} - \frac{M_{\beta\alpha,\beta}^{(k-1)}}{n^{(k-1)}} \right] + \frac{1}{h} \left[\frac{N_{3\alpha}^{(k)}}{n^{(k)}} - \frac{N_{3\alpha}^{(k-1)}}{n^{(k+1)}} \right]$$

$$+ \frac{1}{2} \left[\overset{\backprime}{F}_{\alpha}^{(k)} + \overset{\backprime}{F}_{\alpha}^{(k-1)} \right] - \frac{1}{h} \left[\frac{\overset{M}{F}_{\alpha}^{(k)}}{n^{(k)}} - \frac{\overset{M}{F}_{\alpha}^{(k-1)}}{n^{(k-1)}} \right] = 0 \quad ; \quad k = 1,2,\cdots,N-1 \tag{16b}$$

$$\frac{1}{2} N_{\beta\alpha,\beta}^{(N)} - \frac{1}{n^{(N)}h} \left[M_{\beta\alpha,\beta}^{(N)} - N_{3\alpha}^{(N)} \right] + \frac{1}{2} \overset{\backprime}{F}_{\alpha}^{(N)} - \frac{1}{n^{(N)}h} \overset{M}{F}_{\alpha}^{(N)} - T_{\alpha}^{-} = 0 \tag{16c}$$

$$\sum_{k=1}^{N} N_{3\alpha,\alpha}^{(k)} + \overset{\backprime}{F}_3 + T_3^{-} - T_3^{-} = 0 \tag{16d}$$

where $\quad \left[N_{\alpha\beta}^{(k)}, M_{\alpha\beta}^{(k)} \right] \equiv \int_{A^{(k)}} (1, x_3^{(k)}) \sigma_{\alpha\beta}^{(k)} dx_3 \quad ; \quad N_{3\alpha}^{(k)} \equiv \int_{A^{(k)}} \tau_{3\alpha}^{(k)} dx_3 \tag{17a,b}$

$$\left[\overset{\backprime}{F}_{\alpha}^{(k)}, \overset{M}{F}_{\alpha}^{(k)} \right] \equiv \int_{A^{(k)}} (1, x_3^{(k)}) f_{\alpha}^{(k)} dx_3 \quad ; \quad \overset{\backprime}{F}_3 \equiv \sum_{k=1}^{N} \int_{A^{(k)}} f_3^{(k)} dx_3 \tag{17c,d}$$

b) Boundary Conditions

Specify $U_{\alpha}^{(k)}$ or $\left[\frac{1}{2} (N_{\beta\alpha}^{(k)} + N_{\beta\alpha}^{(k-1)}) - \frac{1}{h} \left[\frac{M_{\beta\alpha}^{(k)}}{n^{(k)}} - \frac{M_{\beta\alpha}^{(k-1)}}{n^{(k-1)}} \right] \right] \nu_\beta \tag{18a}$

Specify U_3 or $\sum_{k=1}^{N} N_{3\alpha}^{(k)} \nu_\alpha \tag{18b}$

Eq. (18) constitute $(2N + 3)$ conditions. It can be seen that the natural edge traction boundary conditions are coupled. i.e. force and moment resultants acting on two adjacent layers are involved.

c) Constitutive Equations for Monoclinic Layers

$$Q_1^{(k)} - \left[\frac{\tilde{C}_{45}}{\tilde{C}_{44}} \right]^{(k)} Q_2^{(k)} - \frac{n^{(k)}h}{12} \left[T_1^{(k-1)} + T_1^{(k)} \right] + \frac{n^{(k)}h}{12} \left[\frac{\tilde{C}_{45}}{\tilde{C}_{44}} \right]^{(k)} \left[T_2^{(k-1)} + T_2^{(k)} \right]$$

$$= \frac{5}{6\,\bar{C}_{44}^{(k)}} \left[U_1^{(k-1)} - U_1^{(k)} + n^{(k)}h\, U_{3.1} \right] \tag{19a}$$

$$- \left[\frac{\bar{C}_{45}}{\bar{C}_{55}} \right]^{(k)} Q_1^{(k)} + Q_2^{(k)} + \frac{n^{(k)}h}{12} \left[\frac{\bar{C}_{45}}{\bar{C}_{55}} \right]^{(k)} \left[T_1^{(k-1)} + T_1^{(k)} \right] - \frac{n^{(k)}h}{12} \left[T_2^{(k-1)} + T_2^{(k)} \right]$$

$$= \frac{5}{6\,\bar{C}_{55}^{(k)}} \left[U_2^{(k-1)} - U_2^{(k)} + n^{(k)}h\, U_{3.2} \right] \tag{19b}$$

$$\frac{1}{10} \left[\bar{C}_{44}^{(k)} Q_1^{(k)} + \bar{C}_{44}^{(k-1)} Q_1^{(k-1)} \right] - \frac{1}{10} \left[\bar{C}_{45}^{(k)} Q_2^{(k)} + \bar{C}_{45}^{(k-1)} Q_2^{(k-1)} \right]$$

$$+ \frac{h}{30} \left[n^{(k)} \bar{C}_{44}^{(k)} T_1^{(k-1)} - 4 \left(n^{(k)} \bar{C}_{44}^{(k)} + n^{(k+1)} \bar{C}_{44}^{(k-1)} \right) T_1^{(k)} + n^{(k+1)} \bar{C}_{44}^{(k-1)} T_1^{(k-1)} \right] \tag{19c}$$

$$- \frac{h}{30} \left[n^{(k)} \bar{C}_{45}^{(k)} T_2^{(k-1)} - 4 \left(n^{(k)} \bar{C}_{45}^{(k)} + n^{(k+1)} \bar{C}_{45}^{(k+1)} \right) T_2^{(k)} + n^{(k+1)} \bar{C}_{45}^{(k-1)} T_2^{(k-1)} \right] = 0$$

$$- \frac{1}{10} \left[\bar{C}_{45}^{(k)} Q_1^{(k)} + \bar{C}_{45}^{(k-1)} Q_1^{(k-1)} \right] + \frac{1}{10} \left[\bar{C}_{55}^{(k)} Q_2^{(k)} + \bar{C}_{55}^{(k-1)} Q_2^{(k-1)} \right]$$

$$- \frac{h}{30} \left[n^{(k)} \bar{C}_{45}^{(k)} T_1^{(k-1)} - 4 \left(n^{(k)} \bar{C}_{45}^{(k)} + n^{(k+1)} \bar{C}_{45}^{(k-1)} \right) T_1^{(k)} + n^{(k+1)} \bar{C}_{45}^{(k-1)} T_1^{(k-1)} \right] \tag{19d}$$

$$+ \frac{h}{30} \left[n^{(k)} \bar{C}_{55}^{(k)} T_2^{(k-1)} - 4 \left(n^{(k)} \bar{C}_{55}^{(k)} + n^{(k+1)} \bar{C}_{55}^{(k-1)} \right) T_2^{(k)} + n^{(k+1)} \bar{C}_{55}^{(k-1)} T_2^{(k-1)} \right] = 0$$

In Eqs. (19a,b) k ranges from 1 to N, while in Eqs. (19c,d) k ranges from 1 to $(N-1)$. Eqs. (19) can be solved for $Q_\alpha^{(k)}$ and $T_\alpha^{(k)}$ in terms of $U_\alpha^{(k)}$ and $U_{3,\alpha}$. As a result, the quantities $N_{3\alpha}^{(k)}$ can be determined as functions of these displacement variables. Such expressions will automatically include appropriate shear correction factors by virtue of Reissner's mixed variational principle.

The remaining constitutive equations for $N_{\alpha\beta}^{(k)}$ and $M_{\alpha\beta}^{(k)}$ are obtained by substituting (3a), (4) and (8) into (17a) to yield

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix}^{(k)} = \frac{n^{(k)}h}{2} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} U_{1.1}^{(k-1)} + U_{1.1}^{(k)} \\ U_{2.2}^{(k-1)} + U_{2.2}^{(k)} \\ U_{1.2}^{(k-1)} + U_{2.1}^{(k-1)} + U_{1.2}^{(k)} + U_{2.1}^{(k)} \end{bmatrix} \tag{20a}$$

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix}^{(k)} = \frac{(n^{(k)}h)^2}{12} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} U_{1.1}^{(k-1)} - U_{1.1}^{(k)} \\ U_{2.2}^{(k-1)} - U_{2.2}^{(k)} \\ U_{1.2}^{(k-1)} + U_{2.1}^{(k-1)} - U_{1.2}^{(k)} - U_{2.1}^{(k)} \end{bmatrix} \tag{20b}$$

5. CYLINDRICAL BENDING OF LAMINATED PLATES

In order to test the accuracy and assess the range of applicability of the present theory, cylindrical bending of composite plates under sinusoidal loading is considered. The plate is simply supported at the ends $x_1 = 0$ and l, and is infinitely long in the x_2-direction. The prescribed boundary conditions on the top and bottom surfaces of the plate are

$$T^- = 0 \ , \ \ T_3^- = q \, \sin \frac{\pi x_1}{l} \qquad \text{on } x_3 = \frac{h}{2} \qquad (21a)$$

$$T_1^- = T_3^- = 0 \qquad \text{on } x_3 = -\frac{h}{2} \qquad (21b)$$

The boundary conditions for the simply supported ends are, from (18)

$$U_3' = 0 \quad \text{at} \ \ x_1 = 0, l \qquad (22a)$$

$$\frac{1}{2}\left(N_{11}^{(k)} + N_{11}^{(k-1)}\right) - \frac{1}{h}\left(\frac{M_{11}^{(k)}}{n^{(k)}} - \frac{M_{11}^{(k-1)}}{n^{(k-1)}}\right) = 0 \quad \text{at} \ \ x_1 = 0, l \qquad (22b)$$

For simplicity, only cylindrical bending of laminated plates consisting of _orthotropic_ layers will be considered in the subsequent analysis. However, no additional restrictions are imposed on layer thicknesses, elastic moduli and stacking sequence. In this case, there holds

$$Q_2^{(k)} = T_2^{(k)} \equiv 0 \qquad (23)$$

$$\tilde{C}_{45}^{(k)} = 0 \quad \text{and} \quad \tilde{C}_{44}^{(k)} = 1 \ C_{55}^{(k)} \qquad (24)$$

As a result, terms involving $Q_2^{(k)}$ and $T_2^{(k)}$ will drop out from Eqs. (19a,c), while Eqs. (19b,d) will not appear altogether. The remaining equations for $Q_1^{(k)}$ and $T_1^{(k)}$ can be written in matrix form as

$$Q_1 - h\,[B_1]T_1 = b \qquad (25a)$$

$$[D_1]Q_1 + h\,[F_1]T_1 = 0 \qquad (25b)$$

where

$$Q_1 \equiv [Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(N)}]^T \tag{26a}$$

$$T_1 \equiv [T_1^{(1)}, T_1^{(2)}, \ldots, T_1^{(N-1)}]^T \tag{26b}$$

and $[B_1]$, $[D_1]$ and $[F_1]$ are matrices of dimensions $N \times (N-1)$, $(N-1) \times N$ and $(N-1) \times (N-1)$, respectively. These matrices are only functions of $n^{(k)}$ and $\tilde{C}_{55}^{(k)}$. The right-hand side of Eq. (25a) contains the displacement variables $U_1^{(k)}$ and $U_{3,1}$. Eqs. (25) can be solved for $Q_1^{(k)}$ and $T_1^{(k)}$ to yield

$$h \, T_1 = - [F_1^{-1}][D_1]Q_1 \tag{27a}$$

$$Q_1 = ([I] + [B_1][F_1^{-1}][D_1])^{-1} \, b \tag{27b}$$

where $[I]$ is the $N \times N$ identity matrix.

The remaining constitutive equations are simply, from (20)

$$N_{11}^{(k)} = \frac{(n^{(k)}h)}{2} \tilde{C}_{11}^{(k)}\left[U_{1,1}^{(k-1)} + U_{1,1}^{(k)} \right] \tag{28a}$$

$$M_{11}^{(k)} = \frac{(n^{(k)}h)^2}{12} \tilde{C}_{11}^{(k)} \left[U_{1,1}^{(k-1)} - U_{1,1}^{(k)} \right] \tag{28b}$$

$$N_{12}^{(k)} = N_{22}^{(k)} = M_{12}^{(k)} = M_{22}^{(k)} \equiv 0 \tag{28c}$$

Using surface boundary conditions (21), the equilibrium equations (16) for cylindrical bending in the absence of body forces reduce to

$$\frac{1}{2} N_{11,1}^{(1)} + \frac{1}{n^{(1)}h} \left[M_{11,1}^{(1)} - N_{31}^{(1)} \right] = 0 \tag{29a}$$

$$\frac{1}{2} \left[N_{11,1}^{(k)} + N_{11,1}^{(k+1)} \right] - \frac{1}{h} \left[\frac{M_{11,1}^{(k)}}{n^{(k)}} - \frac{M_{11,1}^{(k+1)}}{n^{(k+1)}} \right] + \frac{1}{h} \left[\frac{N_{31}^{(k)}}{n^{(k)}} - \frac{N_{31}^{(k+1)}}{n^{(k+1)}} \right] = 0 \; ; \; k = 1,2,\ldots,N-1 \tag{29b}$$

$$\frac{1}{2} N_{11,1}^{(N)} - \frac{1}{n^{(N)}h} \left[M_{11,1}^{(N)} - N_{31}^{(N)} \right] = 0 \tag{29c}$$

$$\sum_{k=1}^{N} N_{31,1}^{(k)} + q \sin \frac{\pi x_1}{l} = 0 \tag{29d}$$

The form of the dependence on the displacement variables $U_1^{(k)}$ and U_3 of the constitutive equations (27b) and (28a,b), and the nature of the applied load suggests the following expressions for the

displacements

$$U_1^{(k)} = h \; \hat{U}_1^{(k)} \cos \frac{\pi x_1}{l} \quad \text{and} \quad U_3 = h \; \hat{U}_3 \sin \frac{\pi x_1}{l} \tag{30}$$

where $\hat{U}_1^{(k)}$ and $\hat{U}_3$ are nondimensional quantities by definition. It is easily proven that the boundary conditions (22) are satisfied when (30) are substituted therein

Finally, inserting (30) into the constitutive equations (27b) and (28a,b), and these in turn into the equilibrium equations (29), yields a system of $(N+2)$ algebraic equations with the $(N+2)$ nondimensional amplitudes $\hat{U}_1^{(k)}$ and $\hat{U}_3$ as unknowns. This system is conveniently written in matrix form as

$$[X] \; \hat{U} = F \tag{31}$$

$$\text{where} \quad \hat{U} = [\hat{U}_1^{(o)}, \hat{U}_1^{(1)}, \cdots, \hat{U}_1^{(N)}, \hat{U}_3]^T \tag{32a}$$

$$F = [0, 0, \ldots, 0, q]^T \tag{32b}$$

and $[X]$ is a $(N+2) \times (N+2)$ matrix.

6. NUMERICAL RESULTS AND DISCUSSION

In order to test the accuracy of the present theory, the problem of cylindrical bending of an infinitely long strip under sinusoidal loading is re-examined. The exact elasticity solution has been given by Pagano (1969), where a symmetric three layer cross-ply laminate was considered, the $0°$ layers being at the outer surfaces of the plate. The elastic properties are

$$\text{for the } 0° \text{ layers} \quad \frac{\tilde{C}_{11}}{E_T} = 25.062657 \quad , \quad \frac{C_{55}}{E_T} = 0.5 \tag{33a}$$

$$\text{and for the } 90° \text{ layers} \quad \frac{\tilde{C}_{11}}{E_T} = 1.002506 \quad , \quad \frac{C_{55}}{E_T} = 0.2 \tag{33b}$$

where E_T is a reference modulus.

Following Pagano's (1969) nondimensionalization, the displacements and stresses are written in the form

$$\bar{u}_1^{(k)} = \left[\frac{E_T}{q}\right] \frac{u_1^{(k)}(o,x_3)}{h} \quad , \quad \bar{u}_3^{(k)} = \left[\frac{E_T}{q}\right] \frac{100\, h^3}{l^4} u_3^{(k)} \left(\frac{l}{2}, 0\right)$$

$$\bar{\sigma}_{11}^{(k)} = \frac{1}{q} \sigma_{11}^{(k)} \left(\frac{l}{2}, x_3\right) \tag{34}$$

$$\text{Also} \quad \bar{x}_3 = \frac{x_3}{h} \quad \text{and} \quad S = \frac{l}{h} \tag{35}$$

In the various curves the solid line represents the exact solution while the results of the present theory are shown by a broken line. Also shown, for comparison purposes, are the results given by the laminated plate theory proposed by Murakami (1985) which are represented by a dashed-dotted line. It is a shear deformable theory obtained by superposing to the linear variations of the Reissner-Mindlin theory a zig-zag in-plane displacement variation across the plate thickness. For brevity, this theory will be called here "The First-Order Zig-Zag Theory" and abbreviated as ZZ

For a symmetric 3-layer cross-ply laminate (0/90/0) with layers of equal thickness, Table 1 shows the values of the central deflection $\bar{u}_3$ obtained from the different theories for a span-to-thickness ratio $S = 4$. As observed, the present theory and the First-Order Zig-Zag theory yield exactly the same

numerical result. This is also true for the thickness variations of the in-plane displacement $\bar{u}_1^{(k)}$ and normal stress $\bar{\sigma}_{11}^{(k)}$. As shown in Figs. 3a,b, very close agreement is found between both theories and Pagano's exact solution. It should be pointed out that the numerical results obtained by Seide (1980) differ slightly from those given here since no shear correction factors were introduced by him.

The present theory was next tested for a symmetric 5-layer cross-ply laminate (0/90/0/90/0) with layers of equal thickness. The central deflection $\bar{u}_3$ for $S = 4$ is shown in Table 1, where closer agreement between the present theory and the exact solution is observed as compared to the first-order zig-zag theory. The distributions across the plate thickness of in-plane variables $\bar{u}_1^{(k)}$ and $\bar{\sigma}_{11}^{(k)}$ are compared in Figs. 4a,b. It is seen that the present theory has improved upon ZZ, especially in the interior layers of the plate.

An antisymmetric 4-layer cross-ply laminate (0/90/0/90) with layers of equal thickness, was also examined. In this case, the error in the central deflection between the exact solution and ZZ is quite large ($\sim$ 21%), while the present theory still gives a satisfactory value (see Table 1). The variations across the plate thickness of the in-plane displacement $\bar{u}_1^{(k)}$ and normal stress $\bar{\sigma}_{11}^{(k)}$ are shown in Figs. 5a,b for $S = 4$, respectively. From the curves for $\bar{u}_1^{(k)}$, it is seen that the first-order zig-zag theory deviates significantly from the exact solution at the bottom layer of the plate. On the other hand, the present theory is in very good agreement with the exact solution.

To further assess the range of applicability of both the present theory and the first-order zig-zag model, arbitrary laminate configurations consisting of 3, 4 and 5 layers were tested. Three different materials were used, with the following elastic properties

$$\text{Material 1} : \quad \frac{\bar{C}_{11}}{E_T} = 1.002506 \quad , \quad \frac{C_{55}}{E_T} = 0.2 \tag{36a}$$

$$\text{Material 2} : \quad \frac{\bar{C}_{11}}{E_T} = 32.631 \quad , \quad \frac{C_{55}}{E_T} = 8.21 \tag{36b}$$

$$\text{Material 3} : \quad \frac{\bar{C}_{11}}{E_T} = 25.062657 \quad , \quad \frac{C_{55}}{E_T} = 0.5 \tag{36c}$$

The laminate configurations corresponding to the three cases examined are shown in Table 2. In all cases, $S = 4$.

The values of the central deflection for the three laminate configurations are given in Table 3. It can be observed that the discrepancies with the exact solution are larger in the case of the first-order zig-zag theory than in the case of the present theory. In particular, the error in $\bar{u}_3$ for $N = 5$ is as high as 49%. The variations across the plate thickness of the in-plane displacement $\bar{u}_1^{(k)}$ and normal stress $\bar{\sigma}_{11}^{(k)}$ are shown in Figs. 6, 7 and 8 for $N = 3, 4$ and 5, respectively. As expected, the symmetric distributions obtained by the present theory and ZZ for $N = 3$ and 5 no longer hold for arbitrary laminate configurations. In all the cases considered, the present theory is still in good agreement with the exact solution, except possibly at the top layer of the 4-ply laminate (see Fig. 7). On the other hand, it is seen that the first-order zig-zag theory deviates considerably from the exact solution.

Some important points are now discussed. The accuracy of the present theory can be improved by dividing each ply into a finite number of sub-layers, but at the expense of increasing computer storage. Also, the proposed theory possesses two main drawbacks, which are : first, the number of equilibrium equations and edge boundary conditions increases with the number of layers; and second, due to the coupling of the natural edge boundary conditions (see Eq. (18)), no clear physical meaning seems to be associated with them.

Next, as a possible explanation for the increased discrepancies in the case of arbitrary laminates between the exact solution and the first-order zig-zag theory, the following argument is proposed. The inclusion of the zig-zag shaped C^0-function was motivated by the displacement microstructue of periodic laminated composites (Murakami, Maewal and Hegemier, 1981). Obviously, for arbitrary laminate configurations, this periodicity is destroyed. Therefore, the first-order zig-zag theory should be expected to break down in these particular cases.

Finally, it is worth mentioning that an advantage of the zig-zag model is that the number of equations to be solved for the displacement-type variables are independent of the number of layers. This is certainly not true for the present theory. In design problems involving laminates with a large number of layers, these considerations should be taken into account.

7. CONCLUSION

A composite plate theory, which accurately predicts in-plane responses of arbitrary laminates, was developed by assuming a linear variation of in-plane displacements and a quadratic variation of transverse stresses across each individual lamina. Transverse displacements were kept constant throughout the entire plate thickness. Governing equations and appropriate boundary conditions were then deduced from Reissner's (1984) new mixed variational principle. The accuracy of the theory was examined for the case of cylindrical bending of an infinitely long strip and compared with the exact solution given by Pagano (1969). The results obtained by the laminated plate theory proposed by Murakami (1985), and called "The First-Order Zig-Zag Theory" were also shown. Values for the central deflection and in-plane displacement and normal stress for symmetric, antisymmetric and arbitrary laminates were presented. It was observed that the first-order zig-zag theory gave satisfactory results only for symmetric cross-ply laminates. The discrepancies between this theory and the exact solution were more pronounced for laminates of arbitrary configuations than for antisymmetric cross-ply laminates. On the other hand, in all the cases considered, the present theory was in close agreement with the exact solution. Despite certain shortcomings, the good correlation with the exact elasticity solution indicates that the proposed theory may prove useful in the investigation of the mechanical response of arbitrary laminates.

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Table 1. Central Deflection $\bar{u}_3$ for Symmetric 3 and 5-Layer and Antisymmetric 4-Layer Cross-Ply Laminates $(S = 4)$

	$N = 3$ (0/90/0)	$N = 5$ (0/90/0/90/0)	$N = 4$ (0/90/0/90)
Exact Solution	2.887	3.044	4.181
Present Theory	2.907	3.059	4.202
1st-Order Zig-Zag	2.907	3.018	3.316

Table 2. Geometric and Physical Properties of Arbitrary 3, 4 and 5-Layer Laminates ($S = 4$)

Number of Layers N	Layer	Volume Fraction $n^{(k)}$	Material
3	1	0.25	3
	2	0.40	1
	3	0.35	2
4	1	0.25	1
	2	0.35	3
	3	0.30	2
	4	0.10	1
5	1	0.10	1
	2	0.25	2
	3	0.15	3
	4	0.20	1
	5	0.30	3

Table 3. Central Deflection $\bar{u}_3$ for Arbitrary 3, 4 and 5-Layer Laminates ($S = 4$)

	$N = 3$	$N = 4$	$N = 5$
Exact Solution	2.341	1.665	2.456
Present Theory	2.364	1.644	2.467
1st-Order Zig-Zag	1.992	1.303	1.261

LIST OF FIGURE CAPTIONS

Figure 8a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of an Arbitrary 5-Layer Laminate for $S = 4$.

Figure 8b. Thickness Variation of In-Plane Normal Stress $\bar{\sigma}_{11}^{(k)}$ of an Arbitrary 5-Layer Laminate for $S = 4$.

Figure 1.   Plate Geometry and Coordinate System.
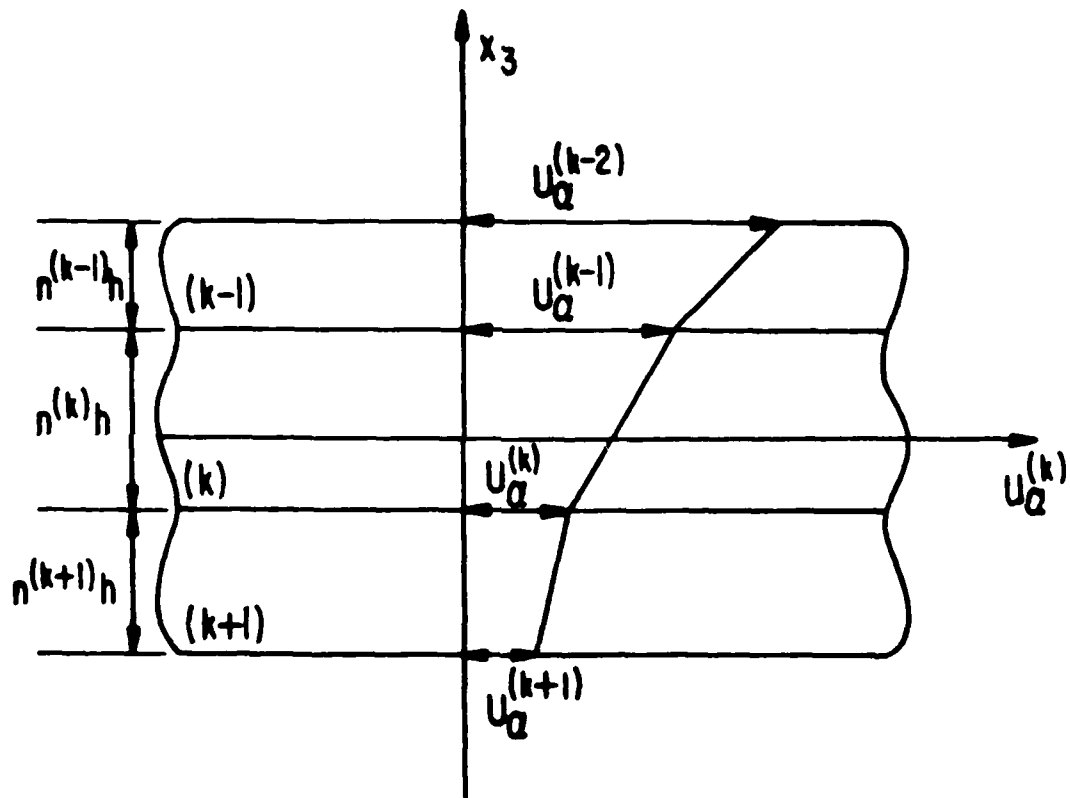
Figure 2. Approximation of In-Plane Displacements.
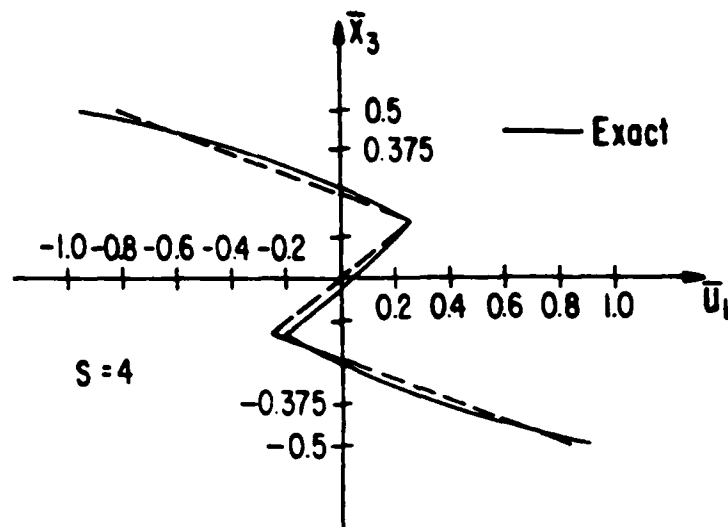
Figure 3a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of a Symmetric 3-Layer Cross-Ply Laminate for $S = 4$.
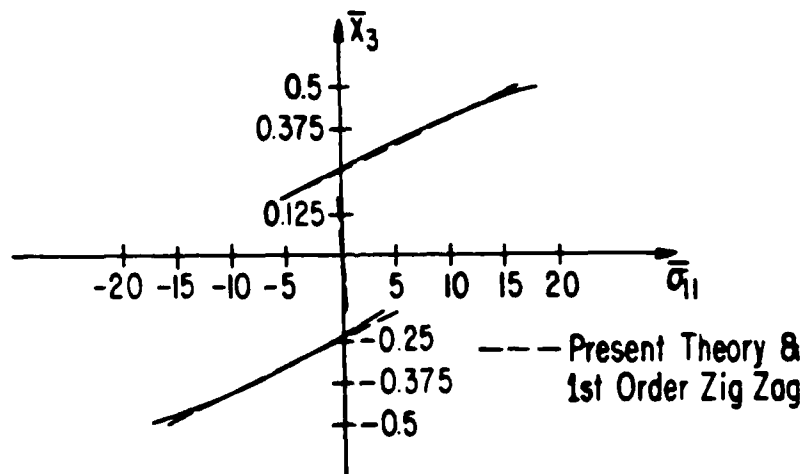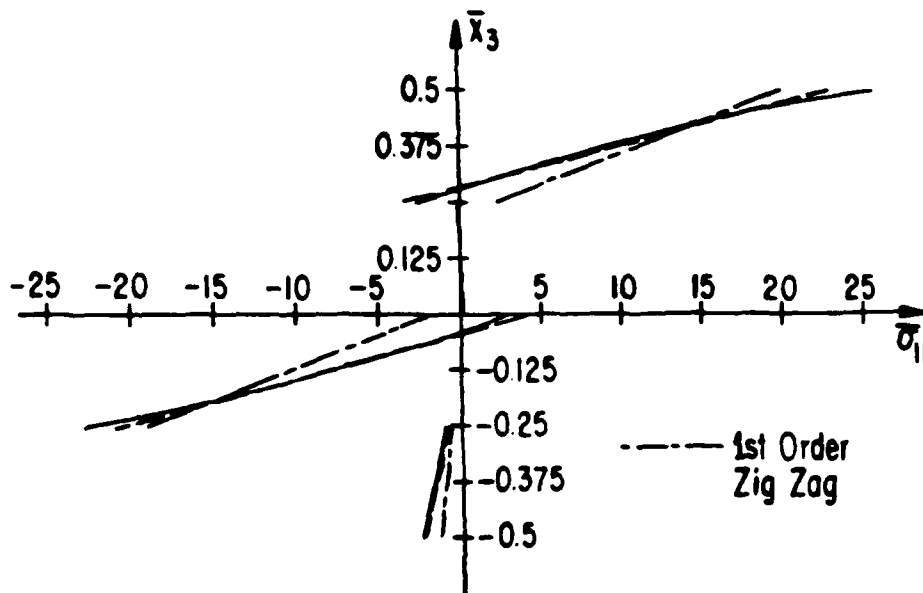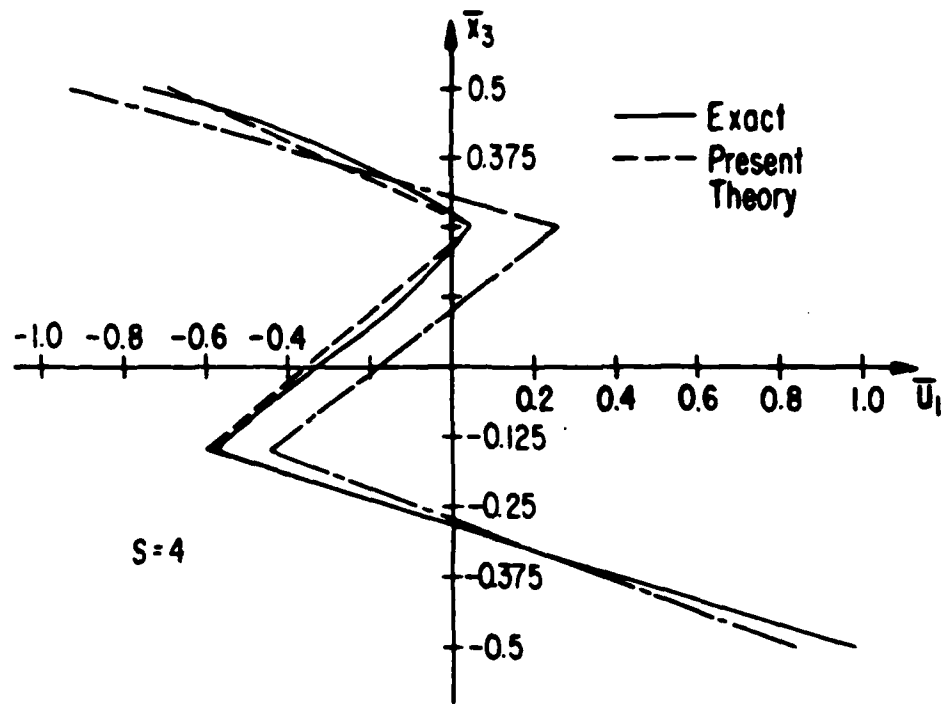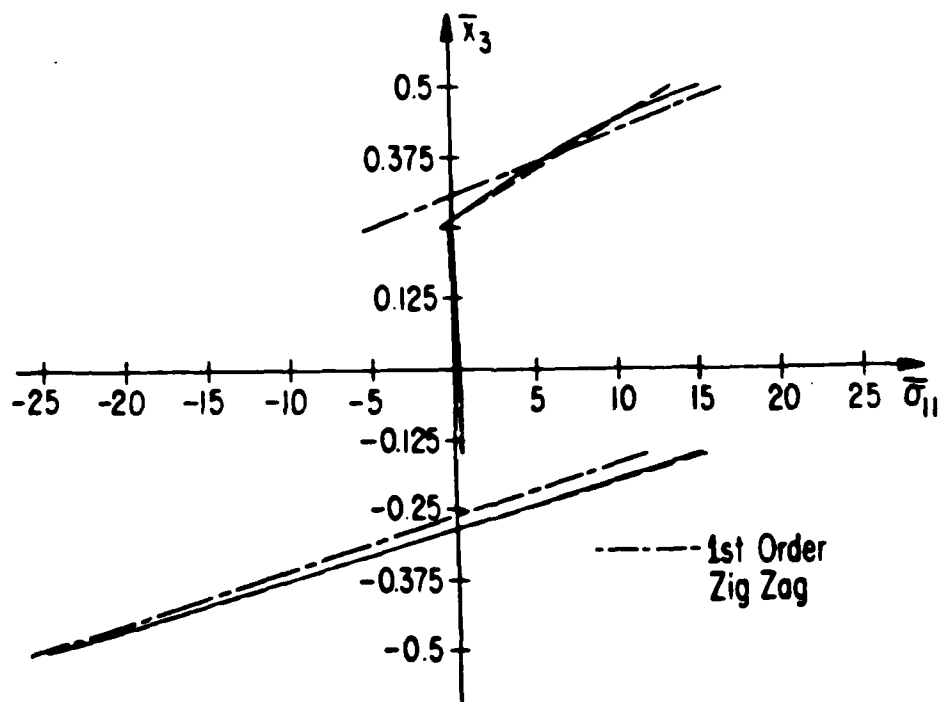

Figure 3b. Thickness Variation of In-Plane Normal Stress $\bar{\sigma}_{11}^{(k)}$ of a Symmetric 3-Layer Cross-Ply Laminate for $S = 4$.
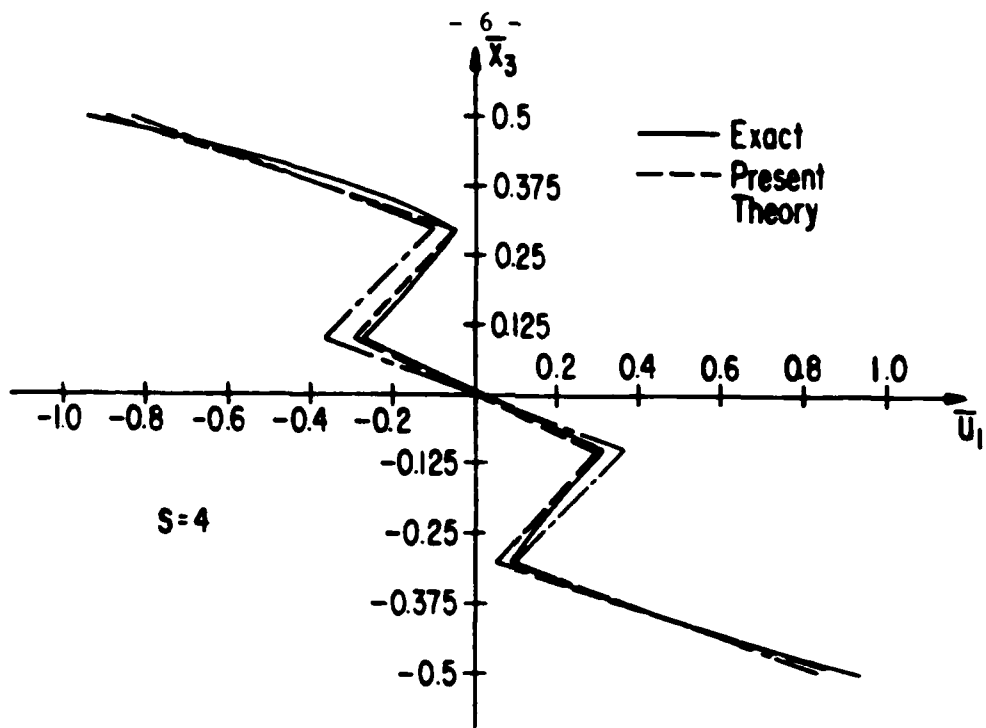
Figure 4a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of a Symmetric 5-Layer Cross-Ply Laminate for $S = 4$.
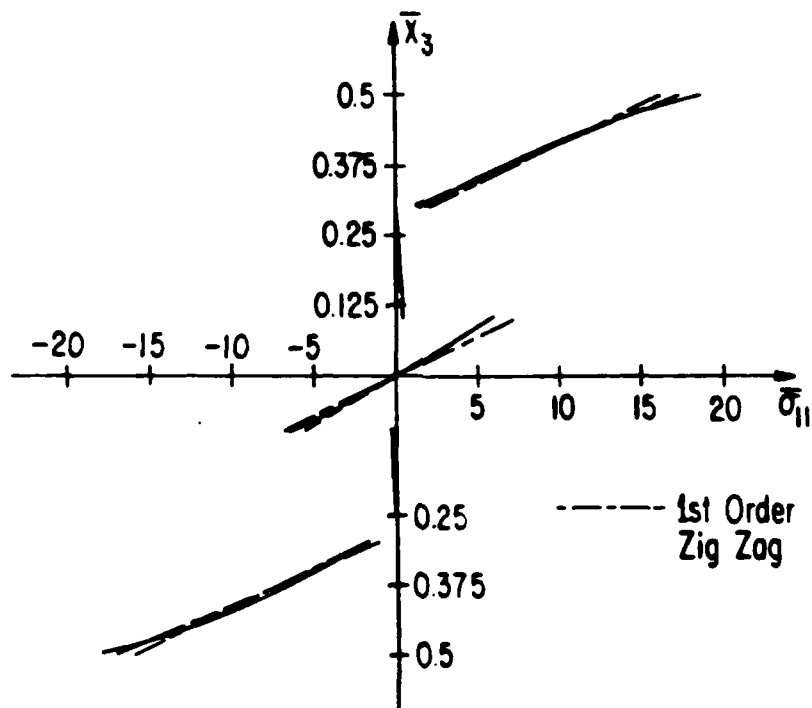
Figure 4b. Thickness Variation of In-Plane Normal Stress $\bar{\sigma}_{11}^{(k)}$ of a Symmetric 5-Layer Cross-Ply Laminate for $S = 4$.
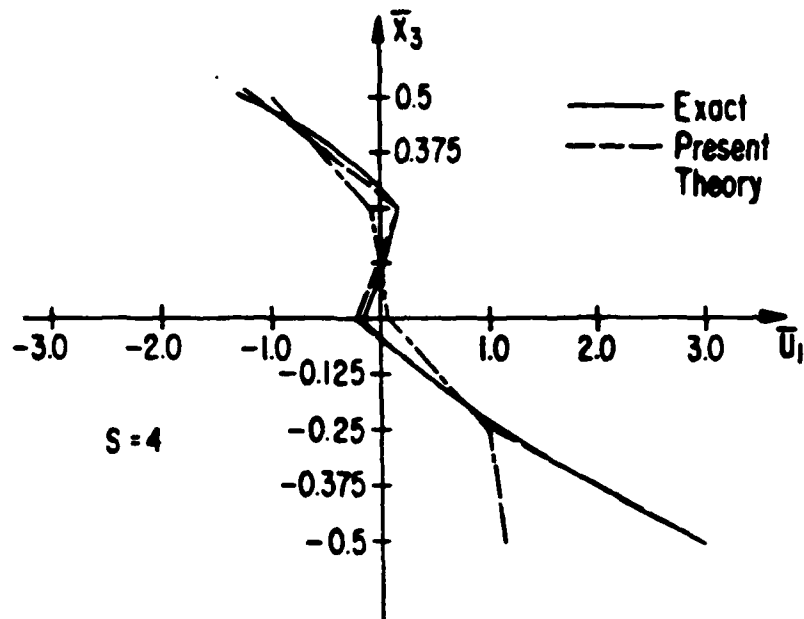
Figure 5a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of an Antisymmetric 4-Layer Cross-Ply Laminate for $S = 4$.

Figure 5b. Thickness Variation of In-Plane Normal Stress $\bar{\sigma}_{11}^{(k)}$ of an Antisymmetric 4-Layer Cross-Ply Laminate for $S = 4$.
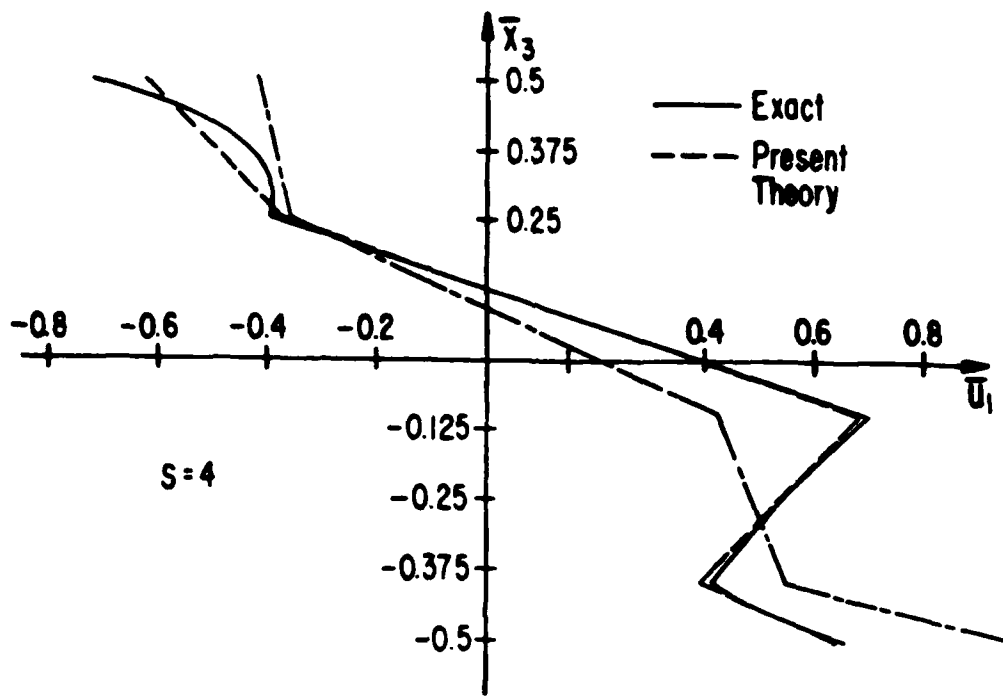
Figure 6a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of an Arbitrary 3-Layer Laminate for

S – 4.

Figure 6b. Thickness Variation of In-Plane Norma' Stress $\bar{\sigma}_{11}^{(k)}$ of ar. Arbitrary 3-Layer Laminate for

S – 4.

Figure 7a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of an Arbitrary 4-Layer Laminate for

$S = 4$
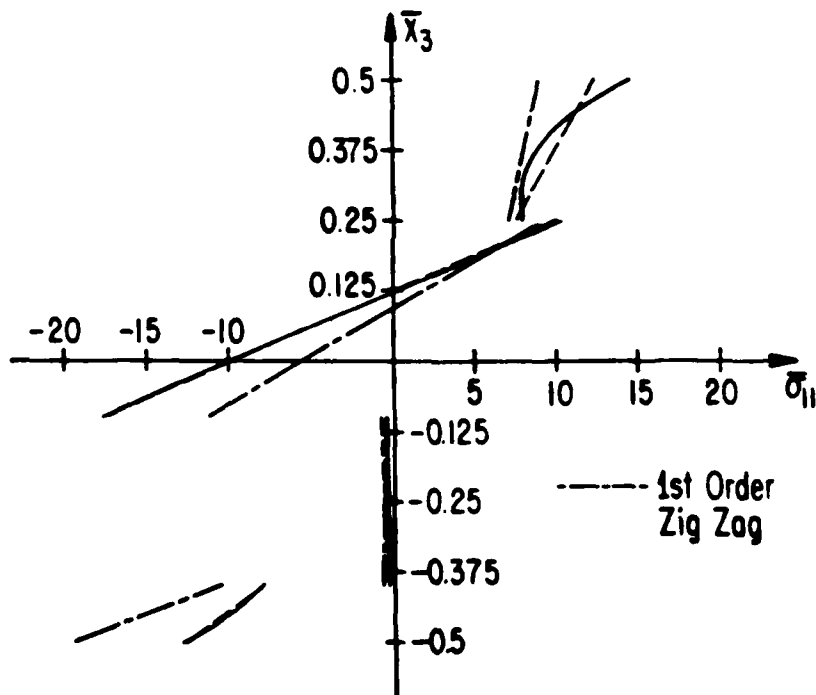

Figure 7b. Thickness Variation of In-Plane Normal Stress $\bar{\sigma}_{11}^{(k)}$ of an Arbitrary 4-Layer Laminate for
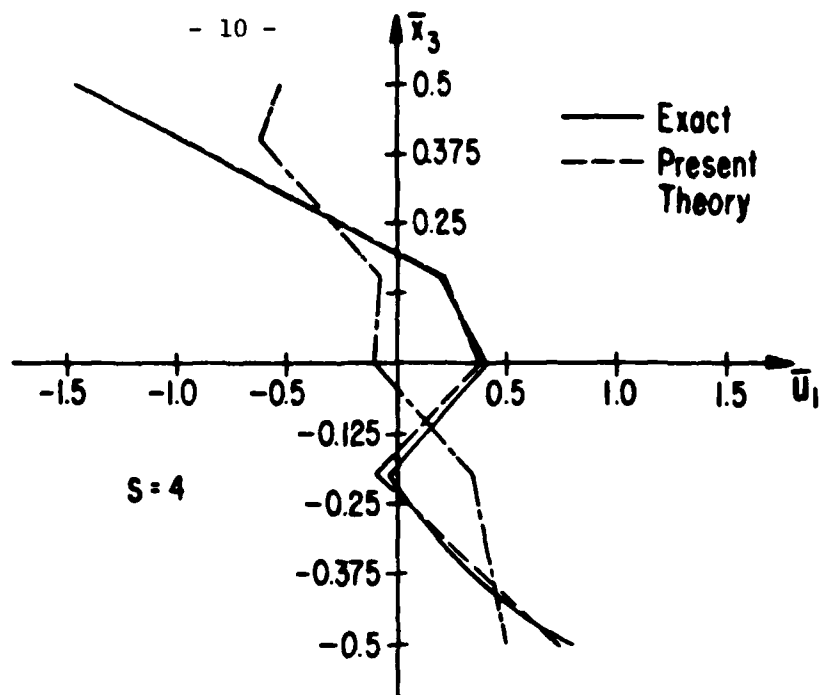
$S = 4$

Figure 8a. Thickness Variation of In-Plane Displacement $\bar{u}_1^{(k)}$ of an Arbitrary 5-Layer Laminate for
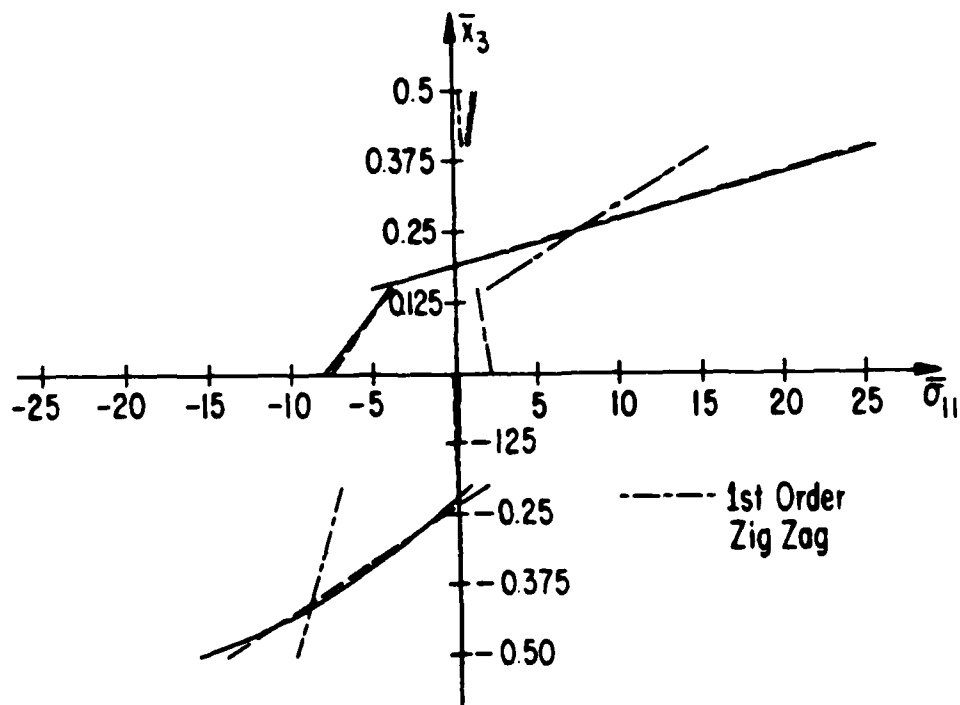
$S = 4$.

Figure 8b. Thickness Variation of In-Plane Normal Stress $\bar{\sigma}_{11}^{(k)}$ of an Arbitrary 5-Layer Laminate for

$S = 4$.

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