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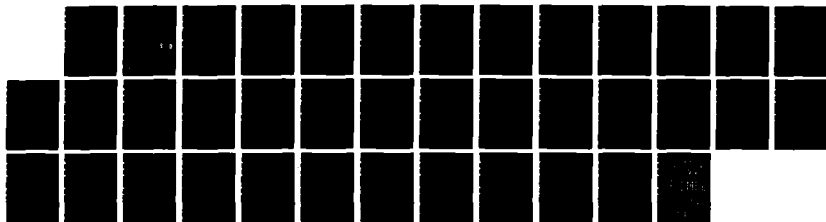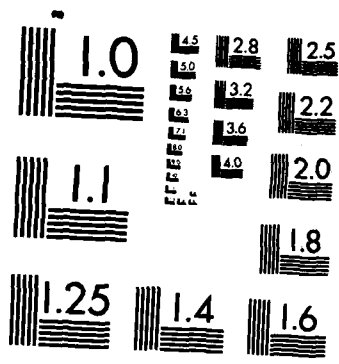   OF APPLIED MECHANICS A.   H MURAKAMI ET AL. DEC 85
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TECHNICAL REPORT

A MIXTURE MODEL

FOR UNIDIRECTIONALLY FIBER-REINFORCED COMPOSITES

by

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A binary mixture theory with microstructure is constructed for unidirectionally fiber-reinforced elastic composites. Model construction is based on a asymptotic scheme with multiple scales and the application of Reissner's new mixed variational principle (1984). In order to assess the accuracy of the model, comparison of the mixture model predictions with available experimental data on dispersion of harmonic waves is made for boron/epoxy and tungusten/aluminum composites. Formulas for the effective moduli are also presented, and the results are compared with the test data and other available predictions.

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TABLE OF CONTENTS

Page

ABSTRACT

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1. Introduction

With the advent of high strength and stiffness fibers such as boron and carbon, and the development of techniques for binding such materials to plastic or metal, fibrous composites have become important elements of modern structures. Such composites, due to their microstructural heterogeneity, may exhibit response phenomena for some environments that are not observed for homogeneous materials. An example of these phenomena for dynamic environments is wave dispersion, and understanding of which is important both from the standpoints of direct response prediction and indirect analyses associated with such topics as nondestructive testing. For fibrous composites, wave dispersion has been amply demonstrated via ultrasonic techniques by such investigators as Tauchert and Guzelse (1972), and Sutherland and Lingle (1972).

Simulation of response phenomena associated with the material microstructure, such as wave dispersion, requires a higher-order continuum description. Several such models have been proposed, some phenomenological, some nonphenomenological.

A higher-order continuum model which simulates wave dispersion was first proposed by Achenback and Herrmann (1968) for unidirectionally fiber-reinforced composites. This theory, called the "effective stiffness theory", has been further studied and applied to fibrous composites by Bartholomew and Torvick (1972), Hlavacek (1975), Achenback (1976), and Aboudi (1981). The aforementioned work concerned linear materials. By modifying the original methodology, Aboudi (1982, 1983) extended the linear model to account for inelastic responses of the composite constituents.

In addition to the effective stiffness modeling concept, a mixture approach has been followed by a number of investigators. A phenomenological version of this model type was adopted by Martin, Bedford and Stern (1971). Deterministic, nonphenomenological mixture theories were introduced by Hegemier, Gurtman and Nayfeh (1973), Hegemier and Gurtman (1974), and Murakami, Maewal and Hegemier (1979). Although capable of simulating nonlinear component responses and interfacial slip, this work was limited to waveguide-type problems. This limitation was removed in the mixture theory developed for laminated composites by Hegemier, Murakami and Maewal (1979), and Murakami, Maewal and Hegemier (1982). In their papers, it was demonstrated that the mixture-type model was

capable of simulating harmonic wave dispersion in laminated composites more accurately than the effective stiffness theories. Further, the mixture-type model requires fewer governing equations. The accuracy and efficiency of the mixture theory is due to the use of appropriate displacement and stress microstructural fields, and a judicious smoothing technique. These are obtained by an asymptotic procedure with multiple scales. This procedure yields a series of microboundary value problems (MBVP's) defined over a unit cell, which in turn represents the (periodic) microstructure of a composite. The lowest order version of the MBVP method is equivalent to the "O(1) homogenization theory" summarized by Bensoussan, Lions, and Papanicolaou (1978), and Sanchez-Palencia (1980). The latter, while it generates appropriate static moduli, is nondispersive. Simulation of wave dispersion requires at least a theory which is classified as an $O(\epsilon)$ homogenization theory in which ϵ denotes the representative ratio of micro-to-macrodimensions of a composite.

To date an $O(\epsilon)$ mixture theory has not been constructed for fibrous composites subject to arbi-trary wave motion. Construction and validation of such a 3D model for unidirectional binary composites with periodic microstructure are the objective of this paper. To facilitate this task, the asymptotic procedure with multiple scales noted previously is combined with a variational technique (Murakami, 1985). Following development of the basic equations, the dispersion of time-harmonic waves is studied and the results are compared with experimental data for boron/epoxy (Tauchert and Guzelse, 1972) and tungusten/aluminum (Sutherland and Lingle, 1972) composites. The good correlation obtained with experimental data indicates that the proposed mixture model furnishes a basic tool by which dynamic responses of elastic composites can be investigated. While the model construction procedure is applicable to inelastic component response and interface slip, extension and investigation of the non-linear problem is deferred to later publications.

2. Formulation

Consider a domain $\overline{V}$ which contains a uniaxial periodic array of fibers embedded in the matrix, as shown in Fig. 1. Let a rectangular reference system $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ be selected with $\overline{x}_1$ in the axial direction of the fibers. In the $\overline{x}_2$, $\overline{x}_3$-plane, a typical cell that represents the geometrical microstructure of the

composite is shown in Fig. 2 for a hexogonal array.

For notational convenience forms $(\ )^{(\alpha)}$, $\alpha = 1,2$ denote quantities associated with material α with $\alpha = 1$ representing fiber and $\alpha = 2$ matrix. Cartesian indicial notation will be employed in which Latin indices range from 1 to 3 and repeated indices imply the summation convention unless otherwise stated. In addition, the notations $(\bar{\ })_{,i} \equiv \partial(\bar{\ })/\partial \bar{x}_i$ and $(\bar{\ })_{,t} \equiv \partial(\bar{\ })/\partial \bar{t}$ will be employed in which $\bar{t}$ represents time. Quantities of the form $(\bar{\ })$ and $(\ )$ denote dimensional and nondimensional variables, respectively.

The governing relations for the displacement vector $\bar{u}_i^{(\alpha)}$ and the stress tensor $\bar{\sigma}_{ij}^{(\alpha)}$ in the two constituents are:

(a) Equations of motion

$$\bar{\sigma}_{ji,j}^{(\alpha)} = \bar{\rho}^{(\alpha)}\bar{u}_{i,tt}^{(\alpha)} \quad , \quad \bar{\sigma}_{ji}^{(\alpha)} = \bar{\sigma}_{ij}^{(\alpha)} \tag{1}$$

where $\bar{\rho}^{(\alpha)}$ is the mass density;

(b) Constitutive relations

$$\bar{\sigma}_{ij}^{(\alpha)} = \bar{\lambda}^{(\alpha)}\delta_{ij}\ e_{kk}^{(\alpha)} + 2\bar{\mu}^{(\alpha)}\ e_{ij}^{(\alpha)} \tag{2}$$

where $\bar{\lambda}^{(\alpha)}$, $\bar{\mu}^{(\alpha)}$ are Lame's constants, $e_{ij}^{(\alpha)}$ is the infinitesimal Cauchy strain, and δ_{ij} is the Kronecker delta;

(c) Strain-displacement relations

$$e_{ij}^{(\alpha)} = \frac{1}{2}\ (\bar{u}_{i,j}^{(\alpha)} + \bar{u}_{j,i}^{(\alpha)}) \quad ; \tag{3}$$

(d) Interface continuity relations

$$\bar{u}_i^{(1)} = \bar{u}_i^{(2)} \quad , \quad \bar{\sigma}_{ji}^{(1)}\nu_j^{(1)} = \bar{\sigma}_{ji}^{(2)}\nu_j^{(1)} \quad \text{on} \quad \bar{\mathscr{I}} \tag{4}$$

where $\nu_i^{(1)} \equiv 0$ on the fiber-matrix interface $\bar{\mathscr{I}}$;

(e) Initial conditions at $\bar{t} = 0$ and appropriate boundary data along the boundary $\partial\bar{V}$.

Conditions (a) - (e) define a well posed initial boundary value problem. However, due to the large number of fiber-matrix interfaces the direct solution to this problem is extremely difficult. The

objective of the subsequent analysis is to alleviate such difficulties by deriving a set of partial differential equations with constant coefficients whose solution can be utilized to approximate the solution of the problem. To this end, it will be convenient to nondimensionalize the basic equations by using the following quantities:

$\bar{\Lambda}$ typical macrosignal wavelength

$\bar{\Delta}$ typical fiber spacing or cell dimension

$\bar{C}_{(m)}, \bar{\rho}_{(m)}$ reference wave velocity and macrodensity

$\bar{E}_{(m)} \equiv \bar{\rho}_{(m)} \bar{C}_{(m)}^2$ reference modulus

$\bar{t}_{(m)} \equiv \bar{\Lambda}/\bar{C}_{(m)}$ typical macrosignal travel time

$\epsilon \equiv \bar{\Delta}/\bar{\Lambda}$ ratio of micro-to-macrodimensions.

With the aid of the above notation, nondimensional variables are now introduced according to

$$(x_1, x_2, x_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)/\bar{\Lambda} \quad , \quad t = \bar{t}/\bar{t}_{(m)} \quad ,$$

$$(\lambda, \mu)^{(\alpha)} = (\bar{\lambda}, \bar{\mu})^{(\alpha)}/\bar{E}_{(m)} \quad , \quad \rho^{(\alpha)} = \bar{\rho}^{(\alpha)}/\bar{\rho}_{(m)} \quad . \tag{5}$$

With the variables defined according to (5), the material properties are seen to be periodic in the x_2, x_3-plane in which the periodicity of the fiber lattice structure may be defined by the cell. It is expected that stress and deformation fields will vary significantly with respect to two basic length scales: (1) a "global" or "macro" length typical of the body size or loading condition, and (2) a "micro" length typical of "cell" planar dimensions. Further, it is expected that these scales will differ by at least one order of magnitude in most cases. This suggests the use of multivariable asymptotic techniques (Bensoussan, Lion and Papanicolaou, 1978, Hegemier, Murakami and Maewal, 1979, Sanchez-Palencia, 1980). This approach commences by introducing new independent microvariables according to

$$x_i^* \equiv x_i/\epsilon \quad . \tag{6}$$

Therefore, all field variables are considered to be functions of the microvariables x_2^* and x_3^*, as well as the macrovariables x_i, $i = 1-3$:

$$f(x_1, x_2, x_3, t) = f^*(x_1, x_2, x_3, x_2^*, x_3^*, t ; \epsilon) \tag{7a}$$

Spacial derivatives of a function f then takes the form

$$\frac{\partial}{\partial x_i} f (x_k, t) = \frac{\partial}{\partial x_i} f^* (x_k, x_j^*, t ; \epsilon)$$

$$+ \frac{1}{\epsilon} \frac{\partial}{\partial x_i^*} f^* (x_k, x_j^*, t ; \epsilon) \tag{7b}$$

where $\partial( )/\partial x_1^* \equiv 0$. By introducing the notation $( )_{,i^*} \equiv \partial( )/\partial x_i^*$ equation (7b) can be rewritten as:

$$f_{,i} = f_{,i}^* + \frac{1}{\epsilon} f_{,i^*}^* . \tag{7c}$$

In the sequel f^* will be written as f for notational simplicity.

The operations (7), when applied to all field variables, lead to the following "synthesized" governing field relations:

(a) Equations of motion

$$\sigma_{ji,j}^{(\alpha)} + \frac{1}{\epsilon} \sigma_{ji,j^*}^{(\alpha)} = \rho^{(\alpha)} u_{i,tt}^{(\alpha)} \quad , \quad \sigma_{ji}^{(\alpha)} = \sigma_{ij}^{(\alpha)} \quad ; \tag{8}$$

(b) Constitutive relations

$$\sigma_{ij}^{(\alpha)} = \lambda^{(\alpha)} \delta_{ij} e_{kk}^{(\alpha)} + 2\mu^{(\alpha)} e_{ij}^{(\alpha)} \quad ; \tag{9}$$

(c) Strain-displacement relations

$$e_{ij}^{(\alpha)} = \frac{1}{2} \left\{ u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)} + \frac{1}{\epsilon} (u_{i,j^*}^{(\alpha)} + u_{j,i^*}^{(\alpha)}) \right\} \quad ; \tag{10}$$

(d) Interface continuity conditions

$$u_i^{(1)} = u_i^{(2)} \quad , \quad \sigma_{ji}^{(1)} \nu_j^{(1)} = \sigma_{ji}^{(2)} \nu_j^{(1)} \quad \text{on} \quad \mathscr{I} . \tag{11}$$

At this point, the variation of field variables which satisfy the periodicity with respect to x_i^* is assumed. According to this condition field variables take equal values on opposite sides of the cell boundary. The premise allows one to analyze a single cell in an effort to determine the distribution of any field variable with respect to the microcoordinates x_i^*. The x^*-periodicity condition is motivated by the Floquet and

Block theorems (Brillouin, 1946) for harmonic wave in periodic structures. Certainly, it eliminates boundary layer effects. However, it is expected to provide a good model for the global wave phenomena in fibrous composites with periodic microstructure.

For the construction of a mixture model it is convenient to cast the field equations in a variational form by using the Reissner new mixed variational principle (Reissner, 1984). In the Reissner variational principle the variations of displacement, strain with (10) as definition and transverse stresses, i.e., all stress-components except $\sigma_{11}^{(\alpha)}$, are considered. Thus, it is convenient to rewrite the constitutive relation (9) in terms of the axial strain $e_{11}^{(\alpha)}$ and the transverse stresses:

$$\sigma_{11}^{(\alpha)} \equiv (\lambda + 2\mu)^{(\alpha)} e_{11}^{(\alpha)} + \lambda^{(\alpha)} \left\{ e_{22}^{(\alpha)}(\cdots) + e_{33}^{(\alpha)}(\cdots) \right\} \quad,$$

$$\begin{bmatrix} e_{22}^{(\alpha)}(\cdots) \\ e_{33}^{(\alpha)}(\cdots) \end{bmatrix} \equiv \begin{bmatrix} (\lambda + 2\mu)^{(\alpha)} & \lambda^{(\alpha)} \\ \lambda^{(\alpha)} & (\lambda + 2\mu)^{(\alpha)} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \sigma_{22} \\ \sigma_{33} \end{bmatrix}^{(\alpha)} - \lambda^{(\alpha)} e_{11}^{(\alpha)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad,$$

$$[2e_{23}^{(\alpha)}(\cdots) \ , \ 2e_{31}^{(\alpha)}(\cdots) \ , \ 2e_{12}^{(\alpha)}(\cdots)]$$

$$\equiv \frac{1}{\mu^{(\alpha)}} [\sigma_{23}^{(\alpha)} \ , \ \sigma_{31}^{(\alpha)} \ , \ \sigma_{12}^{(\alpha)}] \tag{12}$$

Using the equations of motion (8), Gauss' theorem, and the $x^\bullet$-periodicity condition, it can be demonstrated that the Reissner mixed variational principle, applied to the synthesized fields by the multivariable representation, takes the form:

$$\iiint_V \left[\sum_{\alpha=1}^{2} \iint_{A^{(\alpha)}} \left\{ \delta e_{11}^{(\alpha)} \ \sigma_{11}^{(\alpha)} + \delta e_{22}^{(\alpha)} \ \hat{\sigma}_{22}^{(\alpha)} + \delta e_{33}^{(\alpha)} \ \hat{\sigma}_{33}^{(\alpha)} + 2\delta e_{23}^{(\alpha)} \ \hat{\sigma}_{23}^{(\alpha)} \right. \right.$$

$$+ 2\delta e_{31}^{(\alpha)} \ \hat{\sigma}_{31}^{(\alpha)} + 2\delta e_{12}^{(\alpha)} \ \hat{\sigma}_{12}^{(\alpha)}$$

$$+ \delta\hat{\sigma}_{22}^{(\alpha)}(u_{2,2}^{(\alpha)} + \frac{1}{\epsilon} u_{2,2^\bullet}^{(\alpha)} - e_{22}^{(\alpha)}(\cdots)) + \delta\hat{\sigma}_{33}^{(\alpha)}(u_{3,3}^{(\alpha)} + \frac{1}{\epsilon} u_{3,3^\bullet}^{(\alpha)} - e_{33}^{(\alpha)}(\cdots))$$

$$+ \delta\hat{\sigma}_{23}^{(\alpha)}(u_{2,3}^{(\alpha)} + u_{3,2}^{(\alpha)} + \frac{1}{\epsilon} u_{2,3^\bullet}^{(\alpha)} + \frac{1}{\epsilon} u_{3,2^\bullet}^{(\alpha)} - 2e_{23}^{(\alpha)}(\cdots))$$

$$+ \delta\hat{\sigma}_{31}^{(\alpha)}(u_{1,3}^{(\alpha)} + u_{3,1}^{(\alpha)} + \frac{1}{\epsilon} u_{1,3^\bullet}^{(\alpha)} - 2e_{31}^{(\alpha)}(\cdots))$$

$$+ \delta \hat{\sigma}_{12}^{(\alpha)} (u_{1,2}^{(\alpha)} + u_{2,1}^{(\alpha)} + \frac{1}{\epsilon} u_{1,2}^{(\alpha)} - 2e_{12}^{(\alpha)}(\cdots))) \Big\} dx_2^* \, dx_3^*$$

$$\cdots + \int_{\mathscr{I}} \Big\{ (\delta u_i^{(2)} - \delta u_i^{(1)}) \overset{\nu}{T_i^*} + \delta \overset{\nu}{T_i^*} (u_i^{(2)} - u_i^{(1)}) \Big\} ds^* \Big] dx_1 \, dx_2 \, dx_3$$

$$= \iiint_V \Big\{ \sum_{\alpha=1}^{2} \iint_{A^{(\alpha)}} \delta u_i^{(\alpha)} (-\rho^{(\alpha)} u_{i,tt}^{(\alpha)}) dx_2^* \, dx_3^* \Big\} dx_1 \, dx_2 \, dx_3$$

$$+ \iint_{\partial V_T} (\sum_{\alpha=1}^{2} \iint_{A^{(\alpha)}} \delta u_i^{(\alpha)} \overset{\nu}{T_i^{(\alpha)}} dx_2^* \, dx_3^*) dA \quad , \tag{13}$$

where $A^{(\alpha)}$ denotes the x_2^*, x_3^*-domain of the cell occupied by material α (Fig. 2), $\hat{\sigma}_{ij}^{(\alpha)}$ is used for the approximate transverse stresses, $\overset{\nu}{T_i^{(\alpha)}}$ denotes the traction vector on the surface ∂V_T where the traction is specified, ds^* is an infinitesimal line element on $\mathscr{I}$, and dA is an infinitesimal surface element on the boundary of $V : \partial V$. In (13) basic variables are the displacement $u_i^{(\alpha)}$, the transverse stresses $\hat{\sigma}_{ij}^{(\alpha)}$ and the interface traction vector $\overset{\nu}{T_i^*}$. The Euler-Lagrange equations of (13) include (8a), (11a), (12), and

$$\overset{\nu}{T_i^*} \equiv \hat{\sigma}_{ji}^{(\alpha)} \nu_j^{(1)} \quad \text{on} \quad \mathscr{I} \, . \tag{14}$$

The above variational equation (13) furnishes a tool with which a mixture model can be obtained with appropriate trial displacement and transverse stress fields. The basic requirement for the variables is the x^*-periodicity condition on the cell boundary ∂A. The microstructural variation of the trial functions can be obtained by the asymptotic procedure (Murakami, Maewal and Hegemier, 1981).

3. Asymptotic Analysis

The premise that the composite macrodimension is much larger than the microdimension, $\epsilon \ll 1$, and the form of scaled equations (8) and (10), suggest the expansion of the dependent variables in the asymptotic series:

$$\{ u_i, \sigma_{ij} \}^{(\alpha)} (x_k, x_i^*, t; \epsilon) = \sum_{n=o}^{\infty} \epsilon^n \{ u_{i(n)}, \sigma_{ij(n)} \}^{(\alpha)} (x_k, x_i^*, t) \quad . \tag{15}$$

If (15) is substituted into (8)-(11) and the coefficients of different powers of ϵ are equated to zero, a sequence of problems defined on the cell is obtained. The first of the equations in this sequence furnishes

$$u_{i(o),j^*}^{(\alpha)} = 0 \quad , \quad \sigma_{ji(o),j^*}^{(\alpha)} = 0 \quad . \tag{16}$$

Equation (16a) implies that $u_{i(o)}^{(\alpha)}$ is independent of x_j^* and yields with the zero-th order expansion of (11a):

$$u_{i(o)}^{(\alpha)} = U_{i(o)}(x_k , t) \quad . \tag{17}$$

The remaining systems of equations obtained from (8)-(10) are, for $n \geqslant 0$:

$$\sigma_{ji(n+1),j^*}^{(\alpha)} = \rho^{(\alpha)} u_{i(n),tt}^{(\alpha)} - \sigma_{ji(n),j}^{(\alpha)} \ , \ \sigma_{ji(n)}^{(\alpha)} = \sigma_{ij(n)}^{(\alpha)} \quad , \tag{18}$$

$$\sigma_{ij(n)}^{(\alpha)} = \lambda^{(\alpha)} \delta_{ij} \, e_{kk(n)}^{(\alpha)} + 2\mu^{(\alpha)} e_{ij(n)}^{(\alpha)} \quad , \tag{19}$$

$$e_{ij(n)}^{(\alpha)} = \frac{1}{2} \left(u_{i(n),j}^{(\alpha)} + u_{j(n),i}^{(\alpha)} + u_{i(n+1),j^*}^{(\alpha)} + u_{j(n+1),i^*}^{(\alpha)} \right) \quad . \tag{20}$$

To be added to the foregoing are the interface conditions and the x^*-periodicity conditions for $n \geqslant 0$:

$$u_{i(n)}^{(1)} = u_{i(n)}^{(2)} \quad , \quad \sigma_{ji(n)}^{(1)} \nu_j^{(1)} = \sigma_{ji(n)}^{(2)} \nu_j^{(1)} \quad \text{on } \mathscr{I} \ , \tag{21}$$

$$u_{i(n)}^{(2)} \quad \text{and} \quad \sigma_{ji(n)}^{(2)} \nu_j^{(2)} \quad \text{are } x^*-\text{periodic on } \partial A \quad . \tag{22}$$

The first set of microboundary value problems (MBVP's) for $\sigma_{ji(o)}^{(\alpha)}$ and $u_{i(1)}^{(\alpha)}$, called the O(1) MBVP's, is defined by (16b), (18b), (19)-(20), (21b), (22b) with $n = 0$, and (21a), (22a) with $n = 1$. The O(1) MBVP's are excited by $U_{i(o),j}$. Similarly, a sequence of MBVP's is defined for each n from (18)-(22). With appropriate integrability and normalization conditions, higher order terms may be computed by solving the MBVP's. In particular, the O(1) MBVP's are the ones solved for the O(1) homogenization theory proposed by Bensoussan, Lion and Papanicolaou (1978) and Sanchez-Palencia (1980), and, also, form the basis of the mixture theory which may be classified as an O(ϵ) homogenization theory. The asymptotic approach yields the microstructures of displacement and stress fields after solving a multitude of MBVP's which are complicated.

In order to use the approximate solutions of the MBVP's in the course of developing a mixture model, and to ease the burden of solving the MBVP's exactly, a variational procedure was adopted by Murakami (1985) for laminated composites with the help of the Reissner new mixed variational principle (Reissner, 1984). A similar approach is adopted here for fibrous composites. To obtain the lowest order mixture theory by using (13), it is necessary to obtain trial displacement and transverse stresses to $O(\epsilon)$. In the sequel, the trial functions are obtained for a hexagonal cell with a concentric cylinders approximation as shown in Fig. 2. In Fig. 2, (r,θ) are micropolar coordinates:

$$r = \sqrt{x_2^{*2} + x_3^{*2}} \quad , \quad \tan\theta = x_3^{*}/x_2^{*} \quad , \tag{23}$$

by which $r = 1$ constitutes the cell boundary and $r = \sqrt{n^{(1)}}$, denotes the interface $\mathscr{I}$. The quantities $n^{(\alpha)}$ indicate the volume fraction of material α and satisfy

$$n^{(1)} + n^{(2)} = 1 \quad . \tag{24}$$

In terms of the polar coordinates the x^{*}-periodicity conditions for a hexagonal cell with the concentric cylinders approximation reduce to the form:

$$f(x_r,\theta,t) = f(x_k,r,\pi + \theta,t) \quad \text{at} \quad r = 1 \quad . \tag{25}$$

4. Trial Displacements and Transverse Stresses

The $O(1)$ stress and $O(\epsilon)$ displacement fields are obtained by solving the $O(1)$ MBVP's which are defined by (16b), (18b), (19)-(22) and (24). These MBVP's are excited by $U_{i(o),j}$. The exact solution of $u_i\{_1^1\}$ is furnished in the Appendix. For the mixture formulation it is convenient to introduce an $O(\epsilon)$ displacement variable which represents $U_{i(o),j} + U_{j(o),i}$ according to:

$$\overset{j}{S_i}(x_k,t) \equiv \frac{1}{\epsilon A}\int_{\mathscr{I}} u_i^{(\alpha)}\nu_j^{(1)}ds^{*} = \frac{1}{A}\int_{\mathscr{I}} u_i\{_1^1\}\nu_j^{(1)}ds^{*} \tag{26}$$

where $A(=\pi)$ is the area of the cell. Due to the fact that $u_i\{_1^1\}$ is excited by $U_{i(o),j} + U_{j(o),i}$ one obtains

$$\overset{3}{S_2} = \overset{2}{S_3} \quad . \tag{27}$$

Equation (27) can also be obtained if one substitutes the exact $u_i\{\}$ in the Appendix into (26) and eliminates $U_{i(o),j}$. To render the analysis tractable, it is preferable to utilize an approximate form of the exact solution for $u_i\{\}$. The exact solution indicates that the following form of the $O(\epsilon)$ displacement yields a good approximation:

$$u_i\{\} (x_k, x_{j^*}, t) = \overset{2}{S}_i (x_k, t) g^{(\alpha)}(r) \cos\theta + \overset{3}{S}_i (x_k, t) g^{(\alpha)}(r) \sin\theta \qquad (28a)$$

where

$$g^{(1)}(r) = \frac{r}{n^{(1)}} \quad , \quad g^{(2)}(r) = \frac{1}{n^{(2)}} (-r + \frac{1}{r}) \quad . \qquad (28b)$$

Anticipating the $O(\epsilon^2)$ difference of the average of $u_i^{(\alpha)}$ on $A^{(\alpha)}$, equations (17) and (27) yield the following trial displacement field:

$$u_i^{(\alpha)} (x_k, x_{j^*}, t) = U_i^{(\alpha)} (x_k, t) + \epsilon\, u_i\{\} (x_k, x_{j^*}, t) \qquad (29)$$

where $u_i\{\}$ is defined by (28). Equations (29) and (28) indicate that the mixture displacement variables are $U_i^{(1)}$, $U_i^{(2)}$, $\overset{2}{S}_i$ and $\overset{3}{S}_i$ with the constraint (27).

By using (29) in (19) with $n = 0$ and considering the $O(\epsilon^2)$ differences of the average transverse stresses, the $O(1)$ trial stress field may be expressed as:

$$
\begin{bmatrix} \hat{\sigma}_{22(o)} \\ \hat{\sigma}_{33(o)} \\ \hat{\sigma}_{23(o)} \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \tau_{22}(x_k,t) \\ \tau_{33}(x_k,t) \\ \tau_{23}(x_k,t) \end{bmatrix}^{(\alpha)} + \frac{\delta_{\alpha 2}}{r^2} \left\{ t_{22}^{(2)}(x_k,t) \begin{bmatrix} \cos 2\theta \\ \cos 2\theta \\ 0 \end{bmatrix} + t_{33}^{(2)}(x_k,t) \begin{bmatrix} \cos 2\theta \\ -\cos 2\theta \\ \sin 2\theta \end{bmatrix} + t_{23}^{(2)}(x_k,t) \begin{bmatrix} \sin 2\theta \\ \sin 2\theta \\ 0 \end{bmatrix} \right\}
$$

$$(30a)$$

$$
\begin{bmatrix} \hat{\sigma}_{31(o)} \\ \hat{\sigma}_{12(o)} \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \tau_{31}(x_k,t) \\ \tau_{12}(x_k,t) \end{bmatrix}^{(\alpha)} + \frac{\delta_{\alpha 2}}{r^2} \left\{ t_{12}^{(2)}(x_k,t) \begin{bmatrix} \sin 2\theta \\ \cos 2\theta \end{bmatrix} + t_{31}^{(2)}(x_k,t) \begin{bmatrix} \cos 2\theta \\ -\sin 2\theta \end{bmatrix} \right\} \qquad (30b)
$$

In order to define the $O(\epsilon)$ trial stress field it is convenient to define the $O(\epsilon)$ stress variable according to

$$P_i (x_k, t) \equiv \frac{1}{\epsilon A} \int_{\mathscr{I}} \sigma_{ji}^{(\alpha)} \nu_j^{(1)} ds^* = \frac{1}{A} \int_{\mathscr{I}} \sigma_{ji}\{\} \nu_j^{(1)} ds^* \quad . \qquad (31)$$

If one integrates (8a) over $A^{(\alpha)}$ and utilizes the x^*-periodicity condition, one obtains the mixture momentum equations:

$$n^{(\alpha)} \sigma_{ji,j}^{(\alpha a)} + (-1)^{\alpha+1} P_i = n^{(\alpha)} \rho^{(\alpha)} u_{i,tt}^{(\alpha a)} \tag{32}$$

where the average operation is defined by

$$f^{(\alpha a)}(x_k,t) \equiv \frac{1}{n^{(\alpha)} A} \iint_{A^{(\alpha)}} f^{(\alpha)}(x_k,x_j^*,t) \, dx_2^* dx_3^* \tag{33}$$

From (32) it can be seen that P_i represents an interaction body force between the two constituents across the interface. Also, the form of (32) with P_i defined by (31) satisfies the integrability condition adopted by the O(1) MBVP's for $\sigma_{ij(1)}^{(\alpha)}$ which are defined by (18)-(20) with appropriate n's and (31). As an O(ϵ) trial stress field which satisfies (31b) one may use the following approximate fields:

$$\begin{bmatrix} \hat{\sigma}_{22(1)} \\ \hat{\sigma}_{33(1)} \\ \hat{\sigma}_{23(1)} \end{bmatrix}^{(\alpha)} = \frac{1}{4} \left\{ P_2(x_k,t)g^{(\alpha)}(r) \cos\theta \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + P_3(x_k,t)g^{(\alpha)}(r) \sin\theta \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\} , \tag{34a}$$

$$\begin{bmatrix} \hat{\sigma}_{31(1)} \\ \hat{\sigma}_{12(1)} \end{bmatrix}^{(\alpha)} = \frac{1}{2} P_1(x_k,t)g^{(\alpha)}(r) \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} . \tag{34b}$$

As a result, the trial transverse stresses are expressed as:

$$\hat{\sigma}_{ij}^{(\alpha)} \equiv \hat{\sigma}_{ij(o)}(x_k,x_l^*,t) + \epsilon \, \hat{\sigma}_{ij(1)}^{(\alpha)}(x_k,x_l^*,t) \tag{35}$$

where $\hat{\sigma}_{ij(o)}^{(\alpha)}$ and $\hat{\sigma}_{ij(1)}^{(\alpha)}$ are defined by (30) and (34), respectively.

5. Mixture Equations

By substituting the displacement and transverse stress trial functions defined by (29) and (35), respectively, into the Reissner variational equation (13), one obtains the following relations as the Euler-Lagrange equations:

(a) Equations of motion

$$n^{(\alpha)}\sigma_{ji,j}^{(\alpha a)} + (-1)^{\alpha+1} P_i = n^{(\alpha)}\rho^{(\alpha)} U_{i,tt}^{(\alpha)} \quad , \quad i = 1-3 \quad , \tag{36}$$

$$\overset{2}{M}_{ji,j} + \frac{1}{\epsilon^2} (\sigma_{ji}^{(2\alpha)} - \sigma_{ji}^{(1\alpha)} + R_{ji}^{(2)}) = I \overset{2}{S}_{i,n} \quad , \quad i = 1,2 \quad , \tag{37a,b}$$

$$\overset{3}{M}_{ji,j} + \frac{1}{\epsilon^2} (\sigma_{3i}^{(2a)} - \sigma_{3i}^{(1a)} + R_{3i}^{(2)}) = I \overset{3}{S}_{i,tt} \quad , \quad i = 1,3 \quad , \tag{37c,d}$$

$$\frac{1}{2} (\overset{3}{M}_{j2,j} + \overset{2}{M}_{j3,j}) + \frac{1}{\epsilon^2} (\sigma_{23}^{(2a)} - \sigma_{23}^{(1a)} + R_{23}^{(2)}) = I \overset{3}{S}_{2,tt} \quad . \tag{37e}$$

where

$$\sigma_{ij}^{(\alpha a)} \equiv \frac{1}{n^{(\alpha)}A} \iint_{A^{(\alpha)}} \sigma_{ij}^{(\alpha)} dx_2^* \; dx_3^* \quad ,$$

$$\epsilon (\overset{2}{M}_{ij} , \overset{3}{M}_{ij}) \equiv \frac{1}{A} \sum_{\alpha=1}^{2} \iint_{A^{(\alpha)}} \sigma_{ij}^{(\alpha)} g^{(\alpha)} (\cos\theta \, , \sin\theta) \, dx_2^* \; dx_3^* \quad , \tag{38}$$

and

$$I \equiv \sum_{\alpha=1}^{2} h^{(\alpha)} \rho^{(\alpha)} \; , \; h^{(1)} = \frac{1}{4} \; , \; h^{(2)} = \frac{-1}{4n^{(2)}} \left(2 + n^{(2)} + \frac{2}{n^{(2)}} \ln n^{(1)}\right) \; ; \tag{39}$$

(b) Constitutive relations

$$\begin{bmatrix} \sigma_{22} \\ \sigma_{33} \end{bmatrix}^{(\alpha a)} = \begin{bmatrix} \tau_{22} \\ \tau_{33} \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \lambda + 2\mu & \lambda \\ \lambda & \lambda + 2\mu \end{bmatrix}^{(\alpha)} \begin{bmatrix} U_{2,2}^{(\alpha)} + (-1)^{\alpha+1} \overset{2}{S}_y / n^{(\alpha)} \\ U_{3,3}^{(\alpha)} + (-1)^{\alpha+1} \overset{3}{S}_y / n^{(\alpha)} \end{bmatrix} + \lambda^{(\alpha)} U_{1,1}^{(\alpha)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ,$$

$$\begin{bmatrix} \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}^{(\alpha a)} = \begin{bmatrix} \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}^{(\alpha)} = \mu^{(\alpha)} \begin{bmatrix} U_{2,3} + U_{3,2} \\ U_{3,1} + U_{1,3} \\ U_{1,2} + U_{2,1} \end{bmatrix}^{(\alpha)} + \frac{(-1)^{\alpha+1}}{n^{(\alpha)}} \begin{bmatrix} 2\overset{3}{S}_2 \\ \overset{3}{S}_1 \\ 2\overset{2}{S}_1 \end{bmatrix} \quad ; \tag{40}$$

$$P_1 = \beta_1 [(U_1^{(2)} - U_1^{(1)})/\epsilon^2 + (h/2)\{\overset{2}{S}_{1,2} + \overset{2}{S}_{2,1}) + (\overset{3}{S}_{3,1} + \overset{3}{S}_{1,3})\}]$$

$$P_2 = \beta_2 [(U_2^{(2)} - U_2^{(1)})/\epsilon^2 + \gamma \overset{2}{S}_{1,1} + h (\overset{2}{S}_{2,2} + \overset{3}{S}_{2,3})]$$

$$P_3 = \beta_3 [(U_3^{(2)} - U_3^{(1)})/\epsilon^2 + \gamma \overset{3}{S}_{1,1} + h (\overset{3}{S}_{2,2} + \overset{3}{S}_{3,3})] \tag{41}$$

where

$$\beta_1 = 1/\{\sum_{\alpha=1}^{} h^{(\alpha)}/(2\mu^{(\alpha)})\} \; , \; \beta_2 = \beta_3 = 1/\{\sum_{\alpha=1}^{} h^{(\alpha)}/(\lambda + \mu)^{(\alpha)}\} \quad ,$$

$$\gamma = \sum_{\alpha=1}^{2} h^{(\alpha)} \lambda^{(\alpha)}/(\lambda + \mu)^{(\alpha)} \; , \; h = \sum_{\alpha=1}^{2} h^{(\alpha)} \; ; \tag{42}$$

$$\overset{2}{M}_{22} = hP_2 \ , \ \overset{2}{M}_{33} = \frac{h}{2} P_2 \ , \ \overset{2}{M}_{12} = \frac{h}{2} P_1 \ , \ \overset{3}{M}_{22} = \frac{h}{2} P_3 \ ,$$

$$\overset{3}{M}_{33} = hP_3 \ , \ \overset{3}{M}_{31} = \frac{h}{2} P_1 \ , \ \overset{2}{M}_{23} = \overset{2}{M}_{23} = \overset{3}{M}_{31} = \overset{3}{M}_{23} = \overset{3}{M}_{12} = 0 \tag{43}$$

where it is understood that

$$\overset{2}{M}_{ij} = \overset{2}{M}_{ji} \ , \ \overset{3}{M}_{ij} = \overset{3}{M}_{ji} \ ; \tag{44}$$

$$R_{21}^{(2)} \equiv t_{12}^{(2)}/n^{(2)} \ ; \ R_{22}^{(2)} \equiv (t_{22}^{(2)}/2 + t_{33}^{(2)})/n^{(2)} \ , \ R_{31}^{(2)} \equiv - t_{31}^{(2)}/n^{(2)} \ ,$$

$$R_{33}^{(2)} \equiv (- t_{22}^{(2)}/2 + t_{33}^{(2)})/n^{(2)} \tag{45}$$

and

$$t_{12}^{(2)} = - \mu^{(2)} \overset{2}{S}/n^{(2)} \ , \ t_{31}^{(2)} = \mu^{(2)} \overset{3}{S}/n^{(2)} \ ,$$

$$t_{22}^{(2)} = - (\lambda + \mu)^{(2)}(\overset{2}{S}_2 - \overset{3}{S}_3))/n^{(2)} \ , \ t_{33}^{(2)} = - \mu^{(2)}(\overset{2}{S}_2 + \overset{3}{S}_3)/n^{(2)} \ ,$$

$$t_{23}^{(2)} = - (\lambda + \mu)^{(2)}(2\overset{3}{S}_2)/n^{(2)} \ . \tag{46}$$

The remaining constitutive relations associated with $\sigma_{11}^{(\alpha)}$ are obtained from (12a); the results are

$$\sigma_{11}^{(\alpha a)} = (\lambda + 2\mu)^{(\alpha)}U_{1,1}^{(\alpha)} + \lambda^{(\alpha)}\left\{ U_{2,2}^{(\alpha)} + U_{3,3}^{(\alpha)} + (-1)^{\alpha+1}(\overset{2}{S}_2 + \overset{3}{S}_3)/n^{(\alpha)}\right\} \ , \tag{47}$$

$$\begin{bmatrix} \overset{2}{M}_{11} \\ \overset{3}{M}_{11} \end{bmatrix} = \sum_{\alpha=1}^{2} h^{(\alpha)}(\lambda + 2\mu)^{(\alpha)} - \frac{\lambda^{(\alpha)2}}{(\lambda + \mu)^{(\alpha)}} \begin{bmatrix} \overset{2}{S}_{1,1} \\ \overset{3}{S}_{1,1} \end{bmatrix} + \gamma \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \ . \tag{48}$$

The associated boundary conditions are on ∂V

$$n^{(\alpha)}\sigma_{ji}^{(\alpha a)}\nu_j = \overset{\nu}{T}_i^{(\alpha p)} \ . \qquad \text{or} \qquad \delta U_i^{(\alpha)} = 0 \qquad , \qquad i=1,2,3 \ , \tag{49}$$

$$\overset{2}{M}_{ji}\nu_j = \overset{2\nu}{T}_i \qquad \text{or} \qquad \delta\overset{2}{S}_i = 0 \qquad , \qquad i=1,2 \ , \tag{50a}$$

$$\overset{3}{M}_{ji}\nu_j = \overset{3\nu}{T}_i \qquad \text{or} \qquad \delta\overset{3}{S}_i = 0 \qquad , \qquad i=1,3 \ , \tag{50b}$$

$$(\overset{3}{M}_{j2} + \overset{2}{M}_{j3})\nu_j - \overset{3\nu}{T}_2 + \overset{2\nu}{T}_3 \quad \text{or} \quad \delta \overset{3}{S}_2 = 0 \ . \tag{50c}$$

where

$$\overset{\nu}{T}_i^{(\alpha\rho)} \equiv \frac{1}{A} \iint_{A^{(\alpha)}} \overset{\nu}{T}_i^{(\alpha)} \, dx_2^* \, dx_3^* \ .$$

$$\epsilon\left(\overset{2\nu}{T}_i, \overset{3\nu}{T}_i\right) = \frac{1}{A} \sum_{\alpha=1} \overset{\nu}{T}_i^{(\alpha)} g^{(\alpha)}(\cos\theta \ , \sin\theta) \, dx_2^* \, dx_3^* \ . \tag{51}$$

Equations (36)-(50) and the initial condition

$$U_i^{(\alpha)} \ , \ U_{i,t}^{(\alpha)} \ , \ \overset{j}{S}_i \ , \ \overset{j}{S}_{i,t} \quad \text{at} \quad t = 0 \quad \text{on} \quad V \tag{52}$$

define a well posed initial boundary value problem with respect time t and the macrocoordinates x_k.

6. Harmonic Wave Dispersion Spectra

In an attempt to test the accuracy of the mixture model, the phase velocity and group velocity spectra of the mixture theory have been compared with available experimental data for time harmonic waves. For the comparison harmonic waves which are propagating at an arbitrary angle of incidence in a full space of the following form are considered:

$$[U_1^{(1)} \ , \ U_1^{(2)} \ , \ U_2^{(1)} \ , \ U_2^{(2)} \ , \ U_3^{(1)} \ , \ U_3^{(2)} \ , \ \overset{2}{S_1}/ik \ , \ \overset{2}{S_2}/ik \ , \ 2\overset{3}{S_2}/ik \ , \ \overset{3}{S_3}/ik]^T$$

$$= \exp\left\{ik\,(x_1 \cos\phi + x_2 \sin\phi \cos\theta + x_3 \sin\phi \sin\theta) - i\omega t\right\} \underset{\sim}{U}^* \tag{53}$$

where

$$\underset{\sim}{U}^* = [\dot{U}_1^{(1)} \ , \ \dot{U}_1^{(2)} \ , \ \dot{U}_2^{(1)} \ , \ \dot{U}_2^{(2)} \ , \ \dot{U}_3^{(1)} \ , \ \dot{U}_3^{(2)} \ , \ \overset{2}{s_1} \ , \ \overset{2}{s_2} \ , \ 2\overset{3}{s_2} \ , \ \overset{3}{s_1} \ , \ \overset{3}{s_3}]^T \tag{54}$$

and $[\ ]^T$ denotes the transpose of $[\ ]$. In (53) $\dot{U}_i^{(\alpha)}$ and $\overset{j}{s_i}$ are constant amplitudes, k denotes the wave number, ω represents angular frequency, ϕ is the azimuth measured from the x_1 axis, and θ is the longitude; the direction of the wave propagation may be best represented by the wave vector $\underset{\sim}{k}$:

$$\underset{\sim}{k} = k\,[\cos\phi \ , \ \sin\phi \cos\theta \ , \ \sin\phi \sin\theta]^T \ . \tag{55}$$

Substitution of (53) into (36) and (37), which are expressed by the displacement variables with (40)-(46), yields an eigenvalue problem for $\epsilon\omega$ of the form:

$$[K]\dot{U} = (\epsilon\omega)^2[M]\dot{U} \tag{56}$$

where $[K]$ and $[M]$ and are 11×11 real symmetric matrices, the elements of which are functions of the mixture constants and the wave vector. Furthermore, $[M]$ is a diagonal matrix. Upon calculation of the eigenvalue $\epsilon\omega$ for a given ϵk, one obtains the phase velocity C_p as

$$C_p = (\epsilon\omega)/(\epsilon k) \quad . \tag{57}$$

For each computed eigen pairs $(\epsilon\omega, \dot{U})_k$, $k = 1,2, \cdots 11$ the group velocity

$$C_g = \frac{d\omega}{dk} \tag{58}$$

can be obtained by taking the derivative of (56) with respect to ϵk:

$$[K']_k \dot{U}_k^* = \left\{ 2(\epsilon\omega)C_g[M] + (\epsilon\omega)^2[M'] \right\}_k \dot{U}_k \quad . \tag{59}$$

For the $k th$ eigenpair equation (59) yields

$$(C_g)_k = \frac{\dot{U}^T\{[K'] - (\epsilon\omega)^2[M']\}\dot{U}}{2(\epsilon\omega)_k(\dot{U}^T[M]\dot{U})_k} \quad . \tag{60}$$

In the subsequent simulation a typical cell dimension $\overline{\Delta}$ was chosen to be a cell radius by introducing the concentric cylinders approximation of the equal area. The reference elastic modulus and density used for the scaling are

$$\overline{E}_{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \overline{E}^{(\alpha)} \quad , \quad \overline{\rho}_{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \overline{\rho}^{(\alpha)} \tag{61}$$

where $\overline{E}^{(\alpha)}$ is Young's modulus. The dimensional frequency $\nu(H_z)$ can be computed from $\epsilon\omega$ by

$$\nu = (\epsilon\omega)\sqrt{\overline{E}_{(m}/\overline{\rho}_{(m}}/(2\pi\overline{\Delta}) \quad . \tag{62}$$

Numerical results are presented for a boron-epoxy composite, for which experimental results were presented by Tauchert and Guzelse (1972) for a waveguide case $\phi = 0°$ and a waveflect case $\phi = 90°$. The material properties are summarized in Table 1 in which the values for Poisson's ratio are estimated. In the simulation $\overline{\Delta}$ was computed from the fiber diameter $(= 2\sqrt{n^{(1)}\overline{\Delta}})$ which was 1.016×10^{-4} m. The group velocity spectra for a waveguide case $\phi = \theta = 0°$ are shown in Fig. 3 for two acoustic modes: a "gross" longitudinal mode and a "gross" shear mode. In the figure the same symbols as the reference of Tauchert and Guzelse are used for the experimental data points. It is noted that reasonable agreement is achieved for the waveguide case in which pronounced dispersion is observed. The group velocity spectra for a wavereflect case $\phi = 90°$, $\theta = 0°$ in which the wave vector is normal to the fiber axis are shown in Fig. 4 with the experimental data. The figure includes three acoustic modes: a "gross" longitudinal wave (P-mode), a "gross" vertically polarized shear wave (SV-mode), and a "gross" horizontally polarized shear wave (SH-mode). The sets of experimental data correspond to the "gross" P-mode and the "gross" SH-mode. It is noted that there are significant deviations from the "gross" SH-mode, but the overall agreement is not unsatisfactory if one admits the scarcity of the experimental data and the difficulties associated with the measurement of shear wave velocities. It was reported by Tauchert and Guzelse (1972) that a shear wave exhibited extremely high damping of the pulse. A similar observation and the scatter of shear wave data were reported by Sachse (1974) who conducted modulus measurements of boron/epoxy composites by using pulse-echo techniques. He concluded that "the measurements of the present investigation indicate that shear waves propagating along and across fibers in the composite materials tested do not always propagate at the same speed."

Sutherland and Lingle (1972) reported phase velocity measurements for tungsten/aluminum composites whose material properties are shown in Table 2. The equivalent cell radius $\overline{\Delta}$ was computed from the given fiber spacings which yield the area of a typical cell $\overline{A} (= \pi \overline{\Delta}^2)$ 0.579×10^{-6} m^2. Figure 5 shows the phase velocity vs. frequency relation for the "gross" longitudinal mode. A reasonable agreement is observed between the experimental data and the theoretical prediction.

7. Effective Moduli

The $O(1)$ homogenization theory which yields the effective moduli of composites can be obtained by taking the limit of $\epsilon \rightarrow 0$ and equating the constituents' displacements

$$U_i^{(1)} = U_i^{(1)} = U_i \tag{63}$$

By introducing the above constraints, equations (36) yield

$$\sigma_{ji,j}^{(m)} = \rho^{(m)} U_{i,tt} \tag{64}$$

where

$$\sigma_{ij}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \sigma_{ij}^{(\alpha\alpha)} \quad , \quad \rho^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \rho^{(\alpha)} \quad . \tag{65}$$

Equations (37) yield

$$\sigma_{2i}^{(2a)} - \sigma_{2i}^{(1a)} + R_{2i}^{(2)} = 0 \quad , \quad i=1,2,3 \tag{66a}$$

$$\sigma_{3i}^{(2a)} - \sigma_{3i}^{(1a)} + R_{3i}^{(2)} = 0 \quad , \quad i=1,3 \quad . \tag{66b}$$

By eliminating $\overset{j}{S}_i$ by (66), equations (65a), (40) and (47) with (63) furnish

$$\underset{\sim}{\sigma}^{(m)} = [E^{(m)}] \underset{\sim}{e}^{(m)} \tag{67}$$

where

$$\underset{\sim}{\sigma}^{(m)} = [\sigma_{11}^{(m)}, \sigma_{22}^{(m)}, \sigma_{33}^{(m)}, \sigma_{23}^{(m)}, \sigma_{31}^{(m)}, \sigma_{12}^{(m)}]^T \quad ,$$

$$\underset{\sim}{e}^{(m)} = [U_{1,1}, U_{2,2}, U_{3,3}, U_{2,3} + U_{3,2}, U_{3,1} + U_{1,3}, U_{1,2} + U_{2,1}]^T \quad , \tag{68}$$

and $[E^{(m)}]$ is the effective modulus matrix with transverse isotropy due to the concentric cylinders approximation and is defined in the Appendix.

The formulas for the effective moduli (B2) are assessed by comparing the results with the experimental data reported by Datta and Ledbetter (1983) for boron/aluminum composites. The results are shown in Table 3 in which the moduli computed from the effective stiffness theories for the square cell by Achenback (1976) and for the hexagonal cell by Hlavacek (1976) are included by using the formulas

reported by Datta and Ledbetter (1983). The comparison has revealed that all high-order theories yield almost similar results. It can be easily shown that the formulas for the effective moduli yield values which fall between the upper and the lower bounds obtained by Hashin and Rosen (1964) for fiber-reinforced composites.

8. Concluding Remarks

An asymptotic mixture theory with multiple scales was applied to unidirectionally fiber-reinforced elastic composites with periodic microstructure. In the model construction, Reissner's new mixed variational principle was applied to the synthesized fields with multivariable field representations. In order to assess the accuracy of the model the mixture dispersion spectra were compared with the experimental data obtained for the boron/epoxy composite by Tauchert and Guzelse (1972) and for the tungusten/aluminum composite by Sutherland and Lingle (1972).

A satisfactory correlation with the experimental data indicates that the proposed mixture model furnishes a basic tool by which dynamic responses of the composite structures can be investigated.

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Appendix

A. Exact $u_i^{(\alpha)}$ of the O(1) MBVP's:

$$u_{1(1)}^{(\alpha)} = G\, g^{(\alpha)}(r)\, \{(U_{1(0),2} + U_{2(0),1})\cos\theta + (U_{3(0),1} + U_{1(0),3})\sin\theta\} \quad,$$

$$u_{2(1)}^{(\alpha)} = b_o\,(U_{2(0),2} + U_{3(0),3} + \hat\lambda\, U_{1(0),1})g^{(\alpha)}(r)\cos\theta$$

$$+ a_2^{(\alpha)}\Big[\{g^{(\alpha)}(r)\cos\theta + \hat\kappa^{(\alpha)}\, g_{II}^{(\alpha)}(r)\cos 3\theta\}\,(U_{2(0),2} - U_{3(0),3})$$

$$+ \{g^{(\alpha)}(r)\sin\theta + \hat\kappa^{(\alpha)}\, g_{II}^{(\alpha)}(r)\sin 3\theta\}\,(U_{2(0),3} + U_{3(0),2})\Big]$$

$$+ b_2^{(\alpha)}\Big[\{3(1 - \hat\kappa^{(\alpha)})g_I^{(\alpha)}(r)\cos\theta + (1 + \hat\kappa^{(\alpha)})g_{III}^{(\alpha)}(r)\cos 3\theta\}\,(U_{2(0),2} - U_{3(0),3})$$

$$+ \{3(1 - \kappa^{(\alpha)})g_I^{(\alpha)}(r)\sin\theta + (1 + \hat\kappa^{(\alpha)})g_{III}^{(\alpha)}(r)\sin 3\theta\}\,(U_{2(0),3} + U_{3(0),2})\Big] \quad,$$

$$u_{3(1)}^{(\alpha)} = b_o\,(U_{2(0),2} + U_{3(0),3} + \hat\lambda\, U_{1(0),1})g^{(\alpha)}(r)\sin\theta$$

$$+ a_2^{(\alpha)}\Big[\{g^{(\alpha)}(r)\sin\theta - \hat\kappa^{(\alpha)}g_{II}^{(\alpha)}\sin 3\theta\}\,(- U_{2(0),2} + U_{3(0),3})$$

$$+ \{g^{(\alpha)}(r)\cos\theta - \hat\kappa^{(\alpha)}g_{II}^{(\alpha)}(r)\cos 3\theta\}\,(U_{2(0),3} + U_{3(0),2})\Big]$$

$$+ \{\dot g^{(\alpha)}(r)\cos\theta - \hat\kappa^{(\alpha)}g_{II}^{(\alpha)}(r)\cos 3\theta\}\,(U_{2(0),3} + U_{3(0),2})\Big]$$

$$+ b_2^{(\alpha)}\Big[\{3(1 - \kappa^{(\alpha)})g_I^{(\alpha)}(r)\sin\theta - (1 + \kappa^{(\alpha)})g_{III}^{(\alpha)}(r)\sin 3\theta\}\,(- U_{2(0),2} + U_{3(0),3})$$

$$+ \{3(1 - \kappa^{(\alpha)})g_I^{(\alpha)}(r)\cos\theta - (1 + \kappa^{(\alpha)})g_{III}^{(\alpha)}(r)\cos 3\theta\}\,(U_{2(0),3} + U_{3(0),2})\Big] \qquad (A1)$$

where

$$b_o \equiv \{(\lambda + \mu)^{(2)} - (\lambda + \mu)^{(1)}\}/(2d_1) \quad,\quad d_1 \equiv \sum_{\alpha=1}^{2}(\lambda+\mu)^{(\alpha)}/n^{(\alpha)} + \mu^{(2)}/(n^{(1)}n^{(2)})$$

$$G \equiv -\,(\mu^{(1)} - \mu^{(2)})/d_3 \quad,\quad d_3 \equiv \sum_{\alpha=1}^{2}\mu^{(\alpha)}/n^{(\alpha)} + \mu^{(2)}/(n^{(1)}n^{(2)}) \quad,$$

$$\hat\lambda \equiv (\lambda^{(1)} - \lambda^{(2)})/\{(\lambda+\mu)^{(1)} - (\lambda+\mu)^{(2)}\}$$

$$\kappa^{(\alpha)} \equiv (\lambda+2\mu)^{(\alpha)}/\mu^{(\alpha)} \quad,\quad \hat\kappa^{(\alpha)} \equiv (1-\kappa^{(\alpha)})/(1+\kappa^{(\alpha)}) \qquad (A2)$$

$$g_I^{(1)}(r) = g_{III}^{(1)}(r) = r^3 \quad , \quad g_{II}^{(1)}(r) = 0$$

$$g_I^{(2)}(r) = -r^3 + r^{-1} \quad , \quad g_{II}^{(2)}(r) = (-r^{-1} + r^{-3})/n^{(2)} \quad ,$$

$$g_{III}^{(2)}(r) = r^3 - (1+\kappa^{(2)})^{-2}\{4(1-\kappa^{(2)}+\kappa^{(2)2})r^{-3} - 3(1-\kappa^{(2)})^2 r^{-1}\} \quad , \tag{A3}$$

In Eqs. (A1) $a_2^{(\alpha)}$ and $b_2^{(\alpha)}$ are obtained by solving the linear equations for $\underline{x} = [a_2^{(1)}, b_2^{(1)}, a_2^{(2)},$ $b_2^{(2)}]^T$:

$$\underset{4 \times 4}{[A]} \underset{4 \times 1}{\underline{x}} = \underset{4 \times 1}{\underline{B}} \tag{A4}$$

where

$$A_{11} = 1 , A_{21} = 0 , A_{31} = \mu^{(1)}/n^{(1)} , A_{41} = 0$$

$$A_{12} = 3(1-\kappa^{(1)})n^{(1)2} \quad , \quad A_{22} = (1+\kappa^{(1)})n^{(1)2}$$

$$A_{32} = -A_{42} = 3n^{(1)}\mu^{(1)}(1-\kappa^{(1)}) \quad , \quad A_{13} = -1 \quad , \quad A_{23} = -\hat{\kappa}^{(\alpha)}/n^{(\alpha)} \quad ,$$

$$A_{33} = \mu^{(2)}/n^{(2)}\{1 - \hat{\kappa}^{(\alpha)}/n^{(1)}\} \quad , \quad A_{43} = 3\hat{\kappa}^{(2)}\mu^{(2)}/n^{(1)2}$$

$$A_{14} = -3n^{(2)}(1 + n^{(1)})(1 - \kappa^{(2)})$$

$$A_{24} = (1 + \kappa^{(2)})n^{(1)2} - (4 - 3n^{(1)})(1 - \kappa^{(2)} + \kappa^{(2)2})/\{n^{(1)}(1 + \kappa^{(2)})\}$$

$$A_{34} = 3\mu^{(2)}(1 - \kappa^{(2)})(n^{(1)} - \hat{\kappa}^{(2)}/n^{(1)})$$

$$A_{44} = -3\mu^{(2)}(1 - \kappa^{(2)})(n^{(1)} + 3\hat{\kappa}^{(2)}/n^{(1)}) + 12\mu^{(2)}(1 - \kappa^{(2)} + \kappa^{(2)2})/\{n^{(1)2}(1 + \kappa^{(2)})\} \quad , \tag{A5}$$

and

$$B_1 = B_2 = B_4 = 0 \quad , \quad B_3 = -(\mu^{(1)} - \mu^{(2)})/2 \quad . \tag{A6}$$

It is interesting to note that for most of practical composites $b_2^{(\alpha)}$, $\alpha = 1,2$ are small compared with $a_2^{(\alpha)}$.

B. The Definition of $[E^{(m)}]$ in (67)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}^{(m)} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{12} & E_{23} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{55} \end{bmatrix}^{(m)} \begin{bmatrix} U_{1,1} \\ U_{2,2} \\ U_{3,3} \\ U_{2,3} + U_{3,2} \\ U_{3,1} + U_{1,3} \\ U_{1,2} + U_{2,1} \end{bmatrix} \qquad (B1)$$

where

$$E_{11}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} (\lambda + 2\mu)^{(\alpha)} - (\lambda^{(1)} - \lambda^{(2)})^2 / d_1 \quad,$$

$$E_{12}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \lambda^{(\alpha)} - (\lambda^{(1)} - \lambda^{(2)})\{(\lambda + \mu)^{(1)} - (\lambda + \mu)^{(2)}\} / d_1 \quad,$$

$$E_{22}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} (\lambda + 2\mu)^{(\alpha)} - \{(\lambda + \mu)^{(1)} - (\lambda + \mu)^{(2)}\}^2 / d_1 - (\mu^{(1)} - \mu^{(2)})^2 / d_2 \quad,$$

$$E_{23}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \lambda^{(\alpha)} - \{(\lambda + \mu)^{(1)} - (\lambda + \mu)^{(2)}\}^2 / d_1 + (\mu^{(1)} - \mu^{(2)})^2 / d_2 \quad,$$

$$E_{44}^{(m)} = (E_{22}^{(m)} - E_{23}^{(m)}) / 2 \quad,$$

$$E_{55}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \mu^{(\alpha)} - (\mu^{(1)} - \mu^{(2)})^2 / d_3 \quad, \qquad (B2)$$

and where

$$d_2 = \sum_{\alpha=1}^{2} \mu^{(\alpha)} / n^{(\alpha)} + (\lambda + \mu)^{(2)} / (2n^{(1)} n^{(2)}) \quad. \qquad (B3)$$

LIST OF TABLES

Table 1.   Material Properties of the Boron/Epoxy Composite Tested by Tauchert and Guzelse (1972)

	Volume Fraction $n^{(\alpha)}$	Young's Modulus $\bar{E}^{(\alpha)}$	Poisson's Ratio $\nu^{(\alpha)}$	Mass Density $\bar{\rho}^{(\alpha)}$
(1)Boron	0.54	379.2 GPa (55×10^6 psi)	0.18	2682 kg/m^3 (251×10^{-6} lb sec^2/in^4)
(2)Epoxy	0.46	5.033 GPa (0.73×10^6 psi)	0.40	1261 kg/m^3 (118×10^{-6} lb sec^2/in^4)

Table 2.   Material Properties of the Tungsten/Aluminum Composite Tested by Sutherland and Lingle (1972)

	Volume Fraction $n^{(\alpha)}$	Young's Modulus $\bar{E}^{(\alpha)}$	Poisson's Ratio $\nu^{(\alpha)}$	Mass Density $\bar{\rho}^{(\alpha)}$
(1)Tungsten	0.022	398 GPa	0.28	19194 kg/m^3
(2)Aluminum	0.978	71.0 GPa	0.34	2700 kg/m^3

Table 3. Comparison of Effective Moduli of a Boron/Aluminum Unidirectionally Fiber-Reinforced Composite in Units of 10^{11} N/m^2

	Data[a]	Mixture Model	Square Cell[a] Model	Hexagonal Cell[a] Model
$E_{11}^{(m)}$	2.450	2.551	2.480	2.551
$E_{22}^{(m)}$	1.825	1.868	1.856	1.872
$E_{23}^{(m)}$	0.779	0.661	-----	0.661
$E_{12}^{(m)}$	0.604	0.578	-----	0.578
$E_{44}^{(m)}$	0.526	0.604	-----	0.606
$E_{55}^{(m)}$	0.566	0.559	0.451	0.561

[a] After Datta and Ledbetter (1983)

LIST OF FIGURES

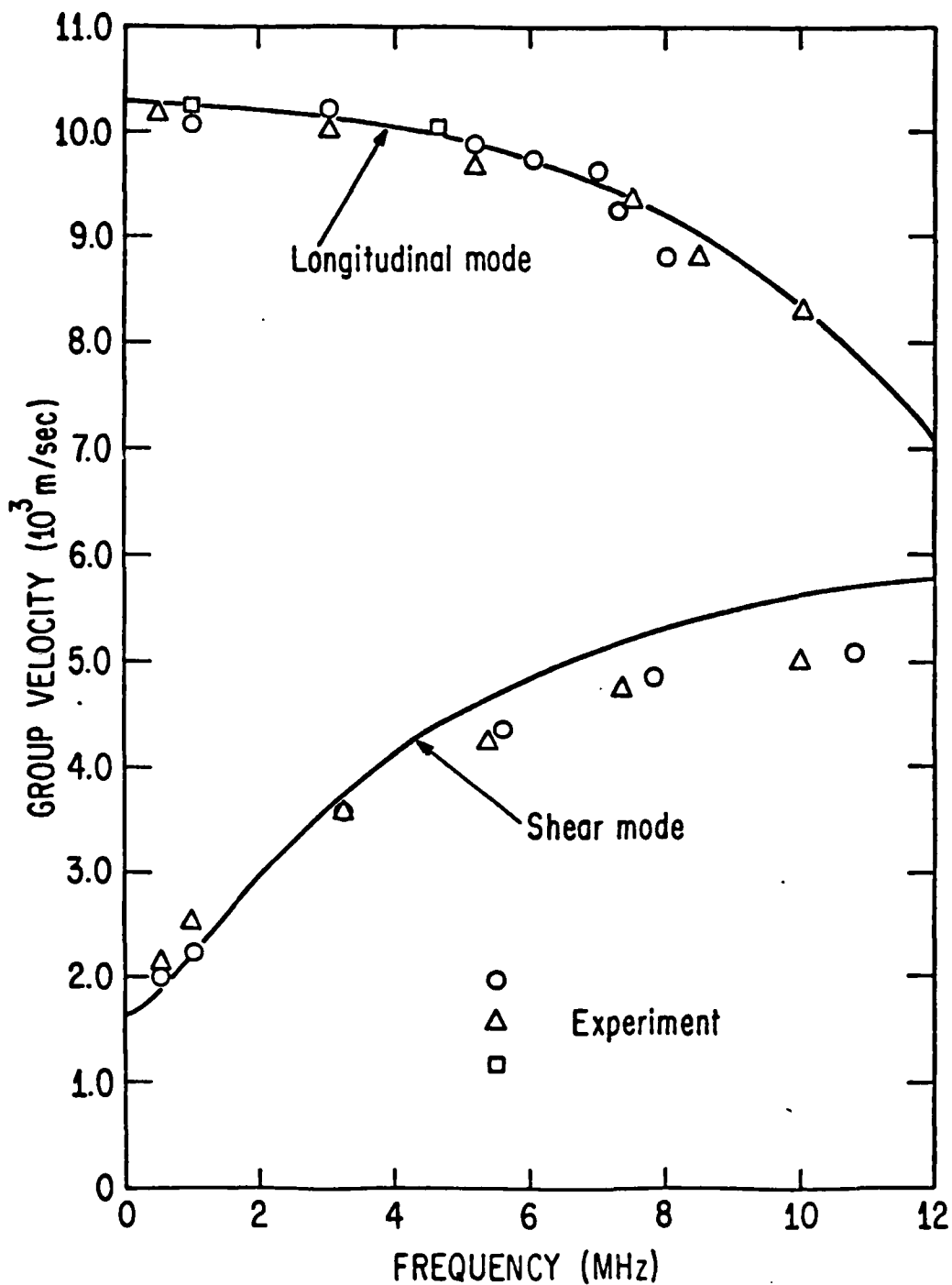Figure 1. A unidirectionally fiber-reinforced composite
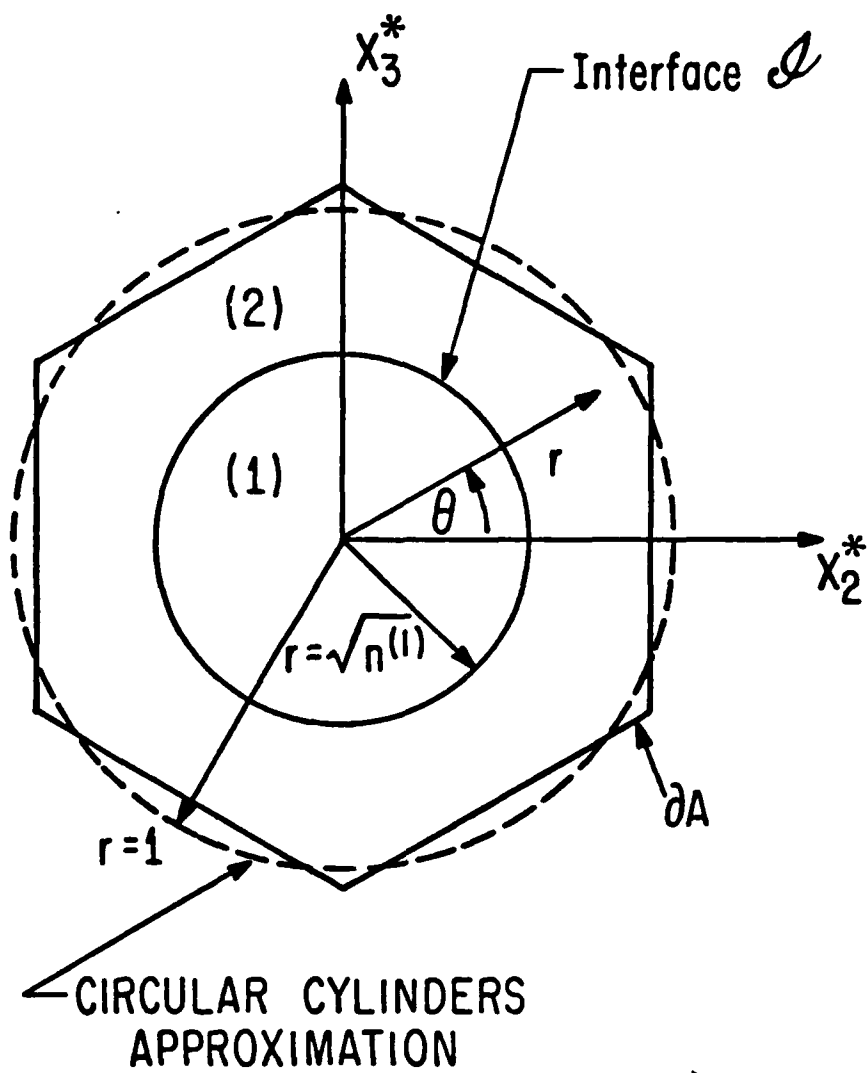
Figure 2. A typical cell

Figure 3. Group velocity spectra of waveguide modes
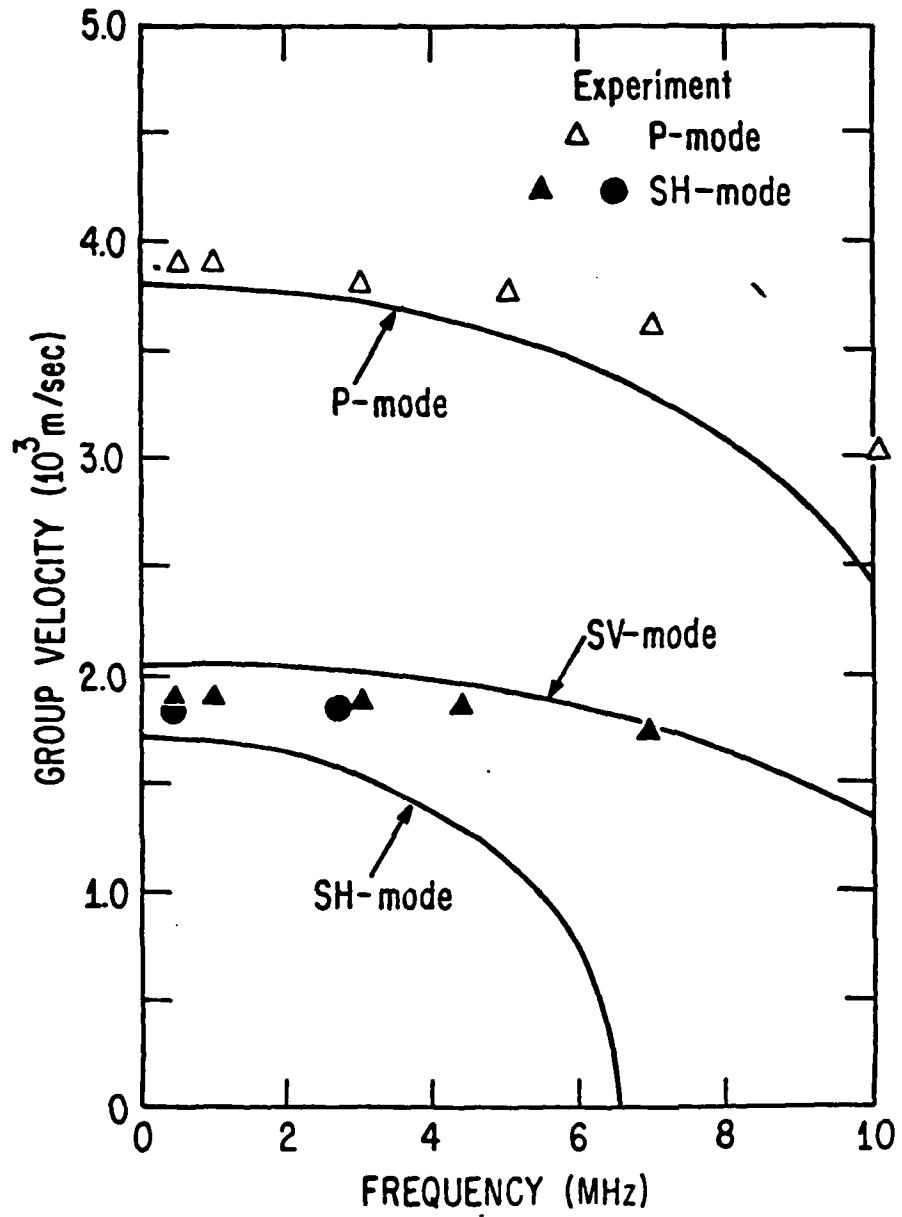for a boron/epoxy composite(Tauchert and Guzelse,
1972)

Figure 4. Group velocity spectra of wavereflect modes
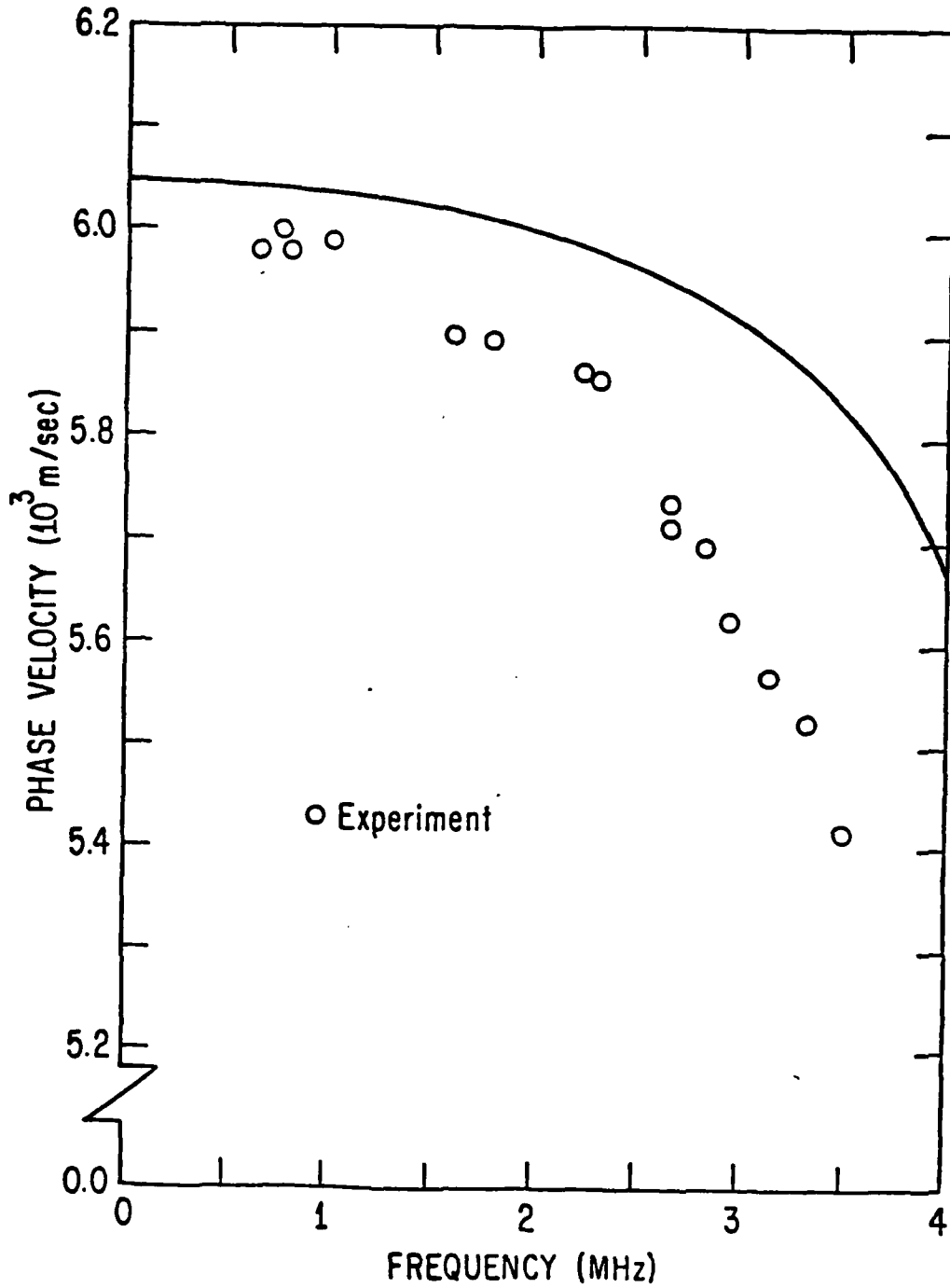for a boron/epoxy composite(Tauchert and
Guzelse,1972)

Figure 5. A phase velocity spectrum of a longitudinal
wavereflect mode for a tungusten/aluminum
composite (Sutherland and Lingle,1972)

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