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THE DETERMINATION OF THE NATURAL FREQUENCIES
AND MODE SHAPES FOR ANISOTROPIC LAMINATED
PLATES INCLUDING THE EFFECTS OF SHEAR
DEFORMATION AND ROTATORY INERTIA

THESIS

John A. Bowlus

AFIT/GA/AA/85S-1

DEPARTMENT OF THE AIR FORCE

AIR UNIVERSITY

# AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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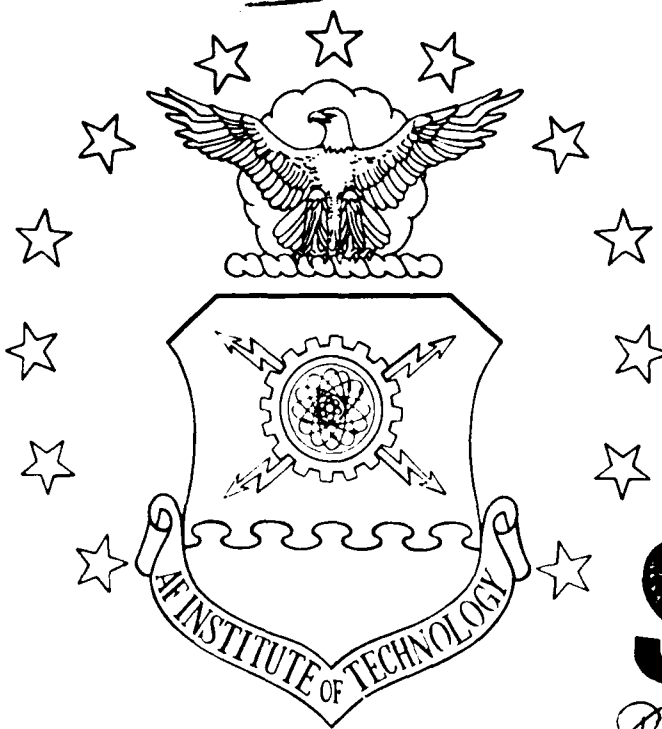
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THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Astronautical Engineering

John A. Bowlus, B.S., P.E.

September 1985

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## List of Figures

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## List of Symbols

| | |
|---|---|
| $a$ | Plate x dimension |
| $a_{ij}$ | Normalized extensional stiffness |
| $A_{ij}$ | Extensional stiffness |
| $A_{mn}$ | Undetermined admissible function coefficient for $\Psi_x$ |
| $b$ | Plate y dimension |
| $B_{mn}$ | Undetermined admissible function coefficient for $\Psi_y$ |
| $C_{mn}$ | Undetermined admissible function coefficient for $w$ |
| $d_{ij}$ | Normalized bending stiffness |
| $D_{ij}$ | Bending stiffness |
| $E_i$ | Young's Modulus in the i direction |
| $G_{ij}$ | Shear modulus in the i-j plane |
| $h$ | Plate thickness |
| $I$ | Mass moment of inertia |
| $k$ | Mindlin's shear deformation correction factor |
| $k_i$ | Dummy coefficients |
| $L$ | Differential operator |
| $m,n$ | Integers determining the number of terms in Galerkin's equations |
| $M_x$, $M_y$, $M_{xy}$ | Moments |
| $N_x$, $N_y$ | Axial forces |
| $N_{xy}$ | Shear force in the x-y plane |
| $p,q$ | Integers determining the number of Galerkin equations (equal to m and n) |
| $p$ | Product of lamina density and lamina thickness, summed thru the laminate |
| $Q_i$ | Shear forces on the ith face in the z direction |
| $Q_{ij}$ | Reduced stiffnesses |

| | |
|---|---|
| $\bar{Q}_{ij}$ | Transformed reduced stiffnesses |
| $R$ | Plate length to width ratio |
| $s$ | Plate length to thickness ratio |
| $S_{ij}$ | Compliance matrix terms |
| $t$ | Lamina thickness |
| $u,v,w$ | Displacements |
| $\bar{w}$ | Normalized displacement in the z direction |
| $x,y,z$ | Plate geometrical axes |
| $1,2,3$ | Lamina principal axes |
| $\epsilon_{1-3}$ | Normal strain components |
| $\epsilon_{4-6}$ | Shear strain components |
| $\eta$ | Normalized y coordinate |
| $\theta$ | Fiber orientation angle |
| $\lambda_1$ | Normalized axial force |
| $\lambda_2$ | Normalized natural frequency |
| $\nu_{ij}$ | Poisson's ratio between the i and j directions |
| $\xi$ | Normalized x coordinate |
| $\pi$ | 3.1415927 |
| $\rho$ | Density |
| $\sigma_{1-3}$ | Normal stress components |
| $\sigma_{4-6}$ | Shear stress components |
| $\psi_i$ | Rotations due to bending |
| $\omega$ | Natural frequency |
| $\bar{\omega}$ | Normalized natural frequency |

## Abstract

An analytical study was conducted to determine the natural frequencies and mode shapes for laminated anisotropic plates, including the effects of shear deformation and rotatory inertia, by using the Galerkin Technique. Three different boundary conditions, simply-supported, clamped, and two opposite sides clamped, two opposite sides simply-supported, were considered. Two different graphite-epoxy symmetric plates were used in the analysis. Convergence characteristics and the effects of length to thickness ratios were investigated. Comparison to classical results and contour plots for several mode shapes are provided.

It was found that as the length to thickness ratios were reduced, shear deformation effects significantly lowered the natural frequencies. Analysis also showed that rotatory inertia effects were very small. Convergence characteristics for all three boundary conditions were very good and excellent agreement with classical solutions was achieved.

# Introduction

## Background

In recent years interest in the use and application of composite materials has greatly increased. This is due in part to their high strength to weight ratios and the fact that they can be tailor made for specific applications. Because of their unique properties, they have opened up numerous fields of research and analysis.

Past developments have shown that the dynamic response of composite plates departs more from Classical Thin Plate Theory than isotropic ones do. Classical Thin Plate Theory is based on the assumption that plane sections remain plane after deformation occurs. It has been found that vibration analyses based on this theory yield frequencies that are too high. Therefore, to gain better agreement with reality, the theories used to analyze the response of composite plates need to include the effects of shear deformation and rotatory inertia.

A number of theories including shear deformation and rotatory inertia have been proposed to date. Mindlin [11] introduced a two dimensional theory of flexural motion for isotropic elastic plates. This theory is based on the fact that the change of displacement results from two rotations due to bending and two rotations due to shear deformation. This theory assumes that no warping of the plane section due to shear occurs, but does include a correction factor to account for this inconsistency.

Yang, Norris, and Stavsky [20] extended this theory (commonly referred to as the YNS theory) to laminates consisting of an arbitrary number of bonded anisotropic layers. They considered the frequency equations for the propagation of harmonic waves in a two- layer, infinite, isotropic plate.

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Whitney and Pagano [19] applied the YNS theory to laminated plates consisting of an arbitrary number of bonded anisotropic layers, each having one plane of symmetry parallel to the central plane of the plate. They employed this theory to study the cylindrical bending of antisymmetric cross-ply and angle-ply plate strips under sinusoidal loading and presented a closed form solution for the free vibrations of antisymmetric angle-ply plate strips. Following Whitney and Pagano, Bert and Chen [2] presented a closed-form solution for the free vibration of simply-supported rectangular plates of antisymmetric angle- ply laminates.

Finite element analysis of laminated plates including transverse shear effects began with Pryor and Barker [13]. Their model was based on Reissner's plate theory and was applied to the cylindrical bending of a symmetric cross-ply laminate. Reddy [14] developed a simple finite element model based on the YNS theory and applied it to the free vibration of antisymmetric, angle-ply plates. Reddy and others [15] also applied this technique to orthotropic laminates of bimodulus materials.

Up to this point, all of the methods discussed could not be applied to laminates which possessed bending/torsional stiffness parameters. This implies coupling of the equations of motion. Recently, Sathyamoorthy and Chia [16] used the nonlinear von Karman equations to develop the theory to study the large amplitude vibrations of anisotropic skew plates for simply-supported, clamped, and clamped simply-supported boundary conditions. This theory did include the bending/torsional stiffness parameters. They solved these equations using the Galerkin method and the Runge-Kutta procedure, however, their applications were limited to homogeneous laminated plates.

Thus, a need exists for a method that will take into account shear deformation and rotatory inertia effects that includes the bending/torsional

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stiffness parameters, yet is not burdened by large computational requirements.

## Objectives

The purpose of this thesis is threefold. First, it will present a method to determine the natural frequencies and mode shapes for anisotropic symmetric laminated plates, including the effects of shear deformation and rotatory inertia, using existing theories and techniques. This method will include the use of bending/torsional stiffness parameters. Second, the method will be used to analyze the effects of shear deformation and rotatory inertia on two different symmetric laminated plates. And third, comparison of this study with classical solutions, where available, will be accomplished to validate the method.

## Approach

The approach used for achieving the desired objectives is straightforward. The motion of an anisotropic symmetric laminated plate will be modelled using the YNS extension of the Mindlin Plate Theory [20] and the resulting differential equations of motion will be solved using the Galerkin Technique. To do this, assumed functions will be selected for each boundary condition and the Galerkin equations will be established. From these equations the eigenvalue problem will be formulated. A computer program(s) will then be written to compute the stiffness and mass (or inertia) matrix elements and to solve the eigenvalue problem. The solution of this problem will yield the desired natural frequencies and coefficients to determine the mode shapes. Comparison of these results with classical solutions, where avaliable, will be accomplished. Also, to insure valid results, a study of the convergence characteristics will be performed by increasing the number of terms in each Galerkin equation.

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Shear deformation and rotatory inertia will be studied by computing the natural frequencies over a range of length to thickness ratios. These frequencies will then be compared with classical results. This will first be done without rotatory inertia, thus determining when shear deformation becomes important. Then, rotatory inertia will be included and the frequencies will be recalculated over the same range of length to thickness ratios, thus ascertaining the effects of rotatory inertia.

## II. Theory and Modelling

We will begin our discussion of the theory used in this thesis by starting with some results from Classical Laminated Plate Theory. From there, we will discuss the equations of motion for a thick laminated plate using the Mindlin Plate Theory, and then discuss an approximate method of solution for differential equations called the Galerkin Technique. Galerkin's technique will then be applied to our differential equations of motion for three different boundary conditions.

### Anisotropic Thick Plate Theory with Rotatory Inertia

Classical Laminated Plate Theory (commonly referred to as CLPT) incorporates constitutive relationships for an orthotropic lamina through the plate thickness resulting in expressions which approximate force resulants in terms of displacement functions. The concepts from CLPT are essential for the later development of the equations of motion. We will begin by describing the basic constituitive relationships for an individual lamina. The reader should refer to References [1] and [8] for an in-depth development of these relations.

The basic constitutive relationships for a single orthotropic layer in the fiber oriented reference system, as shown in Figure 2.1, are:
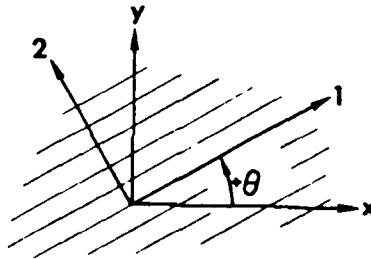


Figure 2.1  Definition of Coordinate System

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$$
\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \qquad (1)
$$

where $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ are the normal strains, $\epsilon_4$, $\epsilon_5$, and $\epsilon_6$ are the shear strains, $\sigma_1$, $\sigma_2$, and $\sigma_3$ are the normal stresses, and $\sigma_4$, $\sigma_5$, and $\sigma_6$ are the shear stresses. The $S_{ij}$ terms form the compliance matrix and, expressed in terms of the engineering constants, are:

$$S_{11} = 1/E_1$$

$$S_{12} = -\nu_{21}/E_2$$

$$S_{13} = -\nu_{31}/E_3$$

$$S_{22} = 1/E_2$$

$$S_{23} = -\nu_{23}/E_2 \qquad (2)$$

$$S_{33} = 1/E_3$$

$$S_{44} = 1/G_{23}$$

$$S_{55} = 1/G_{31}$$

$$S_{66} = 1/G_{12}$$

where $E_i$ are Young's moduli in the ith direction, $\nu_{ij}$ is Poisson's ratio for

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transverse strain in the jth direction when stressed in the ith direction, and $G_{ij}$ is the shear modulus in the i-j plane.

If we invert Eq. (1) to get stresses in terms of strains we have

$$\{\sigma\} = [Q'] \{\epsilon\} \tag{3}$$

where $[Q']$ is the reduced stiffness matrix and has the form

$$[Q'] = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{13} & 0 & 0 & 0 \\ Q'_{12} & Q'_{22} & Q'_{23} & 0 & 0 & 0 \\ Q'_{13} & Q'_{23} & Q'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q'_{66} \end{bmatrix} \tag{4}$$

where

$$Q'_{11} = (S_{22}S_{33} - S_{23}{}^2)/S$$

$$Q'_{12} = (S_{13}S_{23} - S_{12}S_{33})/S$$

$$Q'_{13} = (S_{12}S_{23} - S_{13}S_{22})/S$$

$$Q'_{22} = (S_{33}S_{11} - S_{13}{}^2)/S$$

$$Q'_{23} = (S_{12}S_{13} - S_{23}S_{11})/S \tag{5}$$

$$Q'_{33} = (S_{11}S_{22} - S_{12}{}^2)/S$$

$$Q'_{44} = 1/S_{44}$$

$$Q'_{55} = 1/S_{55}$$

$$Q'_{66} = 1/S_{66}$$

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13}$$

If we wish to rotate our principal lamina axis an amount $\Theta$ (in the 1-2 plane) with respect to some geometrical axis (refer to Figure 2.1) then our stiffness matrix must under go a tensor transformation. The transformation matrix would be defined as

$$[T] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix} \tag{6}$$

where $m = \cos(\Theta)$ and $n = \sin(\Theta)$. The transformed stiffness matrix would then be

$$[\bar{Q}'] = [T][Q'][T]^T \tag{7}$$

and

$$\{\sigma\} = [\bar{Q}']\{\epsilon\} \tag{8}$$

where

$$[\bar{Q}'] = \begin{bmatrix} \bar{Q}'_{11} & \bar{Q}'_{12} & \bar{Q}'_{13} & 0 & 0 & \bar{Q}'_{16} \\ \bar{Q}'_{12} & \bar{Q}'_{22} & \bar{Q}'_{23} & 0 & 0 & \bar{Q}'_{26} \\ \bar{Q}'_{13} & \bar{Q}'_{23} & \bar{Q}'_{33} & 0 & 0 & \bar{Q}'_{36} \\ 0 & 0 & 0 & \bar{Q}'_{44} & \bar{Q}'_{45} & 0 \\ 0 & 0 & 0 & \bar{Q}'_{45} & \bar{Q}'_{55} & 0 \\ \bar{Q}'_{16} & \bar{Q}'_{26} & \bar{Q}'_{36} & 0 & 0 & \bar{Q}'_{66} \end{bmatrix} \tag{9}$$

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At this point we shall assume for a thin laminae a state of plane stress. Therefore, $\sigma_z = 0$. From Eq. (8), we have

$$\sigma_z = 0 = \bar{Q}'_{13}\epsilon_x + \bar{Q}'_{23}\epsilon_y + \bar{Q}'_{33}\epsilon_z + \bar{Q}'_{36}\gamma_{xy} \tag{10}$$

or

$$\epsilon_z = (\bar{Q}'_{13}/\bar{Q}'_{33})\epsilon_x + (\bar{Q}'_{23}/\bar{Q}'_{33})\epsilon_y + (\bar{Q}'_{36}/\bar{Q}'_{33})\gamma_{xy} \tag{11}$$

Substituting Eq. (11) back into Eq. (8) the resulting $[\bar{Q}']$ matrix becomes a 5x5 matrix referred to as $[\bar{Q}]$.

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \tag{12}$$

where

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta)$$

$$\bar{Q}_{16} = (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{26} = (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta \tag{13}$$

$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$$

$$\bar{Q}_{45} = (Q_{44}-Q_{55})\cos\theta\sin\theta$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta$$

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$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\Theta\cos^2\Theta + Q_{66}(\sin^4\Theta + \cos^4\Theta)$$

and

$$Q_{11} = E_1/(1 - \nu_{12}\nu_{21})$$

$$Q_{12} = \nu_{12}E_2/(1 - \nu_{12}\nu_{21}) = \nu_{21}E_1/(1 - \nu_{12}\nu_{21})$$

$$Q_{22} = E_2/(1 - \nu_{12}\nu_{21}) \qquad\qquad (14)$$

$$Q_{44} = G_{23}$$

$$Q_{55} = G_{31}$$

$$Q_{12} = G_{12}$$

We can, therefore, write Eq. (8) as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (15)$$

and

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \qquad (16)$$

We now wish to build a laminate from N perfectly bonded lamina and express the forces and moments acting on this laminate in terms of displacement functions. Before we do this, we need to make an assumption concerning $\epsilon_z$ for the laminate and we need to discuss our sign convention.

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We shall assume that $\epsilon_z = 0$. This implies that a line perpendicular to the midplane will not stretch under deformation and is an accepted inconsistency in plate theory. In reality, $\epsilon_z$ is not zero, but is small compared to the other strains. For our laminate, it means there will be discontinuities in $\epsilon_z$ at the lamina boundaries (they too, will be small). With this assumption we can assume a displacement field of the form

$$u = u^o(x,y,t) + z\Psi_x(x,y,t)$$

$$v = v^o(x,y,t) + z\Psi_y(x,y,t) \qquad (17)$$

$$w = w(x,y,t)$$

where u,v, and w are the x,y, and z coordinate displacements respectively, $u^o$ and $v^o$ are the displacements of the midplane of the laminate, and $\Psi_x$ and $\Psi_y$ are rotations of a line perpendicular to the midplane due to bending. It is important to note here that in this thesis $\Psi_x$ and $\Psi_y$ will be thought of as rotations. Thus, for an axis system as defined in Figure 2.2, a right hand rotation about the positive y axis will give us a positive $\Psi_x$ rotation. On the other hand, we will think of $w_{,x}$ and $w_{,y}$ (the commas denote differentiation) as rates of change of displacement with respect to the appropriate coordinate. Some authors look at the rotations $\Psi_x$ and $\Psi_y$ as rates of change of displacements also (see Ref. [13]). Depending upon the point of view one takes, the sign on these rotations is different. However, if one is consistent throughout his development with his notation then either approach will yield the same result related to generalized functions such as eigenfunctions and eigenvalues.
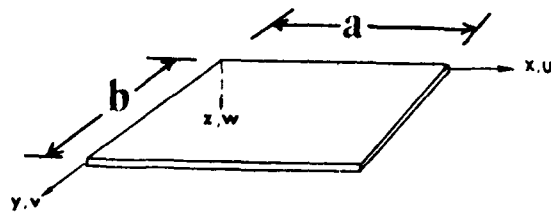
Figure 2.2    Plate Coordinate System

We can recall that for small strains

$$\epsilon_x = u_{,x}$$

$$\epsilon_y = v_{,y} \qquad (18)$$

$$\gamma_{xy} = u_{,y} + v_{,x}$$

Thus, for our displacement field in Eq. (17), our strains are

$$\epsilon_x = u^o_{,x} + z\Psi_{x,x}$$

$$\epsilon_y = v^o_{,y} + z\Psi_{y,y} \qquad (19)$$

$$\gamma_{xy} = u^o_{,y} + v^o_{,x} + \Psi_{x,y} + \Psi_{y,x}$$

or

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon^o_x \\ \epsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \qquad (20)$$

where the middle surface strains are

$$\begin{Bmatrix} \epsilon^o_x \\ \epsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} = \begin{Bmatrix} u^o_{,x} \\ v^o_{,y} \\ u^o_{,y} + v^o_{,x} \end{Bmatrix} \qquad (21)$$

and the middle surface curvatures due to bending are

$$\left\{ \begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \Psi_{x,x} \\ \Psi_{y,y} \\ \Psi_{x,y} + \Psi_{y,x} \end{array} \right\} \tag{22}$$

At this point we have not modelled shear deformation and one should take note that the above curvatures do not include their effects. Now Eq. (15) can be written (for the Kth layer of a laminate) as

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\}_k = \left[ \begin{array}{ccc} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{array} \right]_k \left\{ \left\{ \begin{array}{c} u^o{}_{,x} \\ v^o{}_{,y} \\ u^o{}_{,y} + v^o{}_{,x} \end{array} \right\} + \left\{ \begin{array}{c} \Psi_{x,x} \\ \Psi_{y,y} \\ \Psi_{x,y} + \Psi_{y,x} \end{array} \right\} \right\} \tag{23}$$

The resulant forces and moments acting on a laminate are obtained by integrating the stresses in each layer through the laminate thickness. Thus, we have (see Figure 2.3 for the geometry of an N layered laminate)
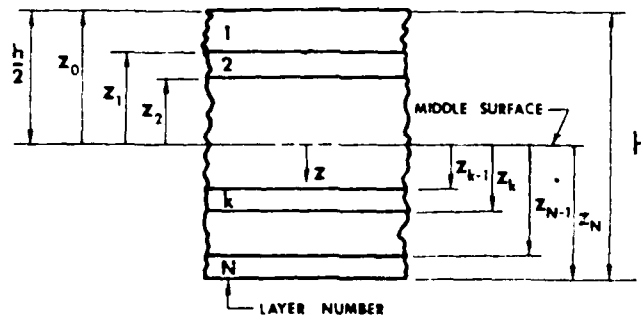


Figure 2.3  Geometry of an N Layered Laminate

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_x, \sigma_y, \tau_{xy}) \, dz \qquad (24)$$

and

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \, z \, dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_x, \sigma_y, \tau_{xy}) \, z \, dz \qquad (25)$$

The integrations in Eqs. (24) and (25) can be rearranged to take advantage of the fact that the stiffness matrix for a lamina is constant within the lamina. We can also recall that $\epsilon^o_x$, $\epsilon^o_y$, $\gamma^o_{xy}$, $\kappa_x$, $\kappa_y$, and $\kappa_{xy}$ are not functions of z but are middle surface values of the laminate and can be removed from under the summation signs. Therefore, Eqs. (24) and (25) can be written as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon^o_x \\ \epsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (26)$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon^o_x \\ \epsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (27)$$

where

$$A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k \ (z_k - z_{k-1}) \tag{28}$$

$$B_{ij} = \tfrac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{ij})_k \ (z_k^2 - z_{k-1}^2) \tag{29}$$

$$D_{ij} = 1/3 \sum_{k=1}^{N} (\bar{Q}_{ij})_k \ (z_k^3 - z_{k-1}^3) \tag{30}$$

In Eqs. (28), (29), and (30) the $A_{ij}$'s are called the extensional stiffnesses, the $B_{ij}$'s are called the coupling stiffnesses, and the $D_{ij}$'s are called the bending stiffnesses.

We now need to determine an expression for the transverse shear forces $Q_x$ and $Q_y$ in terms of displacement functions. To do this we need to discuss Mindlin Plate Theory.

In classical plate theory the Kirchhoff hypothesis states that straight lines perpendicular to the neutral axis in the undeformed state remain straight and perpendicular to the neutral axis after deformation. Mindlin discarded this assumption and assumed that these lines would remain straight (no warping) but would not remain perpendicular to the midplane after deformation. The assumption of no warping is not correct and he did introduce a factor, k, as a means of compensating for it. Therefore, the slope of the midplane ($w_{,x}$ and $w_{,y}$) now consists of a rotation due to bending ($\Psi_x$, $\Psi_y$) and a rotation due to shear deformation ($\gamma_{xz}$, $\gamma_{yz}$). See Figure 2.4.

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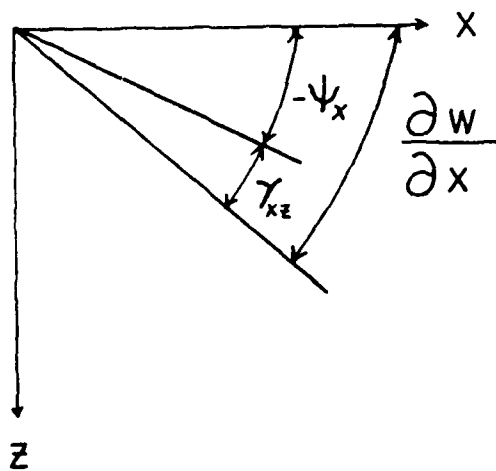
Figure 2.4  Shear Deformation Convention

From the definition of engineering strain we have

$$\gamma_{xz} = u_{,z} + w_{,x} \qquad (31)$$

Substituting the first expression in Eq. (17) into Eq. (31), we have

$$u_{,z} = \Psi_x \qquad (32)$$

Therefore

$$\gamma_{xz} = w_{,x} + \Psi_x \qquad (33)$$

and from similiar reasoning

$$\gamma_{yz} = w_{,y} + \Psi_y \qquad (34)$$

Here we can see that the curvature is a function of both bending and shear deformation. Substituting Eqs. (33) and (34) into Eq. (16) and introducing k, the correction factor for assuming the shear warping to be a straight line, we have

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = k \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix} \begin{Bmatrix} w_{,y} + \Psi_y \\ w_{,x} + \Psi_x \end{Bmatrix} \qquad (35)$$

The shear forces, $Q_x$ and $Q_y$ may now be determined by integrating the shear stresses $\gamma_{xz}$ and $\gamma_{yz}$ through the lamina thickness and summing over the laminate. We have

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} \, dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\tau_{xz})_k \, dz \qquad (36)$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} \, dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\tau_{yz})_k \, dz \qquad (37)$$

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As before the integration can be rearranged to take advantage of the fact that the stiffness matrix for a laminate is constant within the lamina. We can also note that $w_{,x}$, $w_{,y}$, $\Psi_x$, and $\Psi_y$ are not functions of z but are midplane values of the laminate. Therefore, we have

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} w_{,y} + \Psi_y \\ w_{,x} + \Psi_x \end{Bmatrix} \tag{38}$$

where the $A_{ij}$ terms are defined in Eq. (28). We now have the required forces and moments in terms of displacement functions as we desired and may proceed on to the governing equations of motion for a plate.

The governing equations of motion for a plate may be derived by formulating the Lagrangian function (in terms of plate variables) for the plate and applying Hamilton's principle to that Lagrangian. This approach will also yield the required boundary conditions. Reference [6] formulates the Lagrangian for a plate including the effects of transverse shear and rotatory inertia (modelled using the Mindlin Plate Theory). Following the derivation from that text, we have, after applying Hamilton's principle and Green's theorem, the following:

$$\int_{t_1}^{t_2} \iint_D \{ [-I\ddot{\Psi}_x + M_{x,x} + M_{xy,y} - Q_x] \delta\Psi_x + [-I\ddot{\Psi}_y + M_{y,y} + M_{xy,x} - Q_y] \delta\Psi_y$$

$$+ [-\rho\ddot{w} + Q_{x,x} + Q_{y,y} + q] \delta w\} \, dA \, dt$$

$$+ \int_{t_1}^{t_2} \int_\Gamma \{[M_{xy}dx - M_x dy]\delta\Psi_x + [M_y dx - M_{xy}dy]\delta\Psi_y + [Q_y dx - Q_x dy]\delta w\} dt = 0 \tag{39}$$

where the double integral over the domain represents the equations of motion for the plate and the line integral around the contour represents the required boundary conditions. Also

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$$I = \int_{-h/2}^{h/2} \rho\, z^2\, dz \tag{40}$$

To determine the equations of motion for the plate at any time t, we take the double integral expression over the domain and note that the variation cannot be zero over this region. Therefore, the coefficients for each variation coordinate must be equal to zero. Thus we have

$$M_{x,x} + M_{xy,y} - Q_x = I\ddot{\Psi}_x$$

$$M_{xy,x} + M_{y,y} - Q_y = I\ddot{\Psi}_y \tag{41}$$

$$Q_{x,x} + Q_{y,y} + q = \rho\ddot{w}$$

If we assume the time dependance to be harmonic, then we can seperate out the time variable in Eq. (41) and we have

$$M_{x,x} + M_{xy,y} - Q_x + \omega^2 I\Psi_x = 0 \tag{42a}$$

$$M_{xy,x} + M_{y,y} - Q_y + \omega^2 I\Psi_y = 0 \tag{42b}$$

$$Q_{x,x} + Q_{y,y} + N_x w_{,xx} + 2N_{xy}w_{,xy} + N_y w_{,yy} + \rho\omega^2 w = 0 \tag{42c}$$

where $\omega$ is the frequency of vibration and we have let $q = N_x w_{,xx} + 2N_{xy}w_{,xy} + N_y w_{,yy}$. This will allow the use of our equations to solve the linear bifurcation problem. We will retain this expression throughout our equation development but will not consider it when solving the vibration problem. Thus, Eqs. (42a) thru (42c) are our equations of motion.

Before we proceed we must determine the appropriate boundary conditions for the system of equations which are sufficient to assure a unique solution.

These can be determined from the line integral expression in Eq. (39). The boundary conditions associated with Eq. (42a) are represented by $\delta\Psi_x$ and its coefficient. Thus, we have

$$\int_\Gamma (-M_x dy + M_{xy} dx) \, \delta\Psi_x = 0 \qquad (43a)$$

Similiarly, the associated boundary conditions for Eqs. (42b) and (42c) are, respectively

$$\int_\Gamma (M_y dx - M_{xy} dy) \, \delta\Psi_y = 0 \qquad (43b)$$

$$\int_\Gamma (-Q_x dy + Q_y dx) \, \delta w = 0 \qquad (43c)$$

We now have the equations of motion and the required boundary conditions and can proceed with finding a solution for this system. Before we do this, however, *we need to apply a restriction.* This thesis will only be concerned with symmetric laminates. Therefore, all of the coupling stiffnesses ($B_{ij}$'s) are equal to zero and the extensional stiffness term, $A_{45}$, is also equal to zero. With this restriction Eqs. (27) and (38) become

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \Psi_{x,x} \\ \Psi_{y,y} \\ \Psi_{x,y} + \Psi_{y,x} \end{Bmatrix} \qquad (44)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \Psi_y + w_{,y} \\ \Psi_x + w_{,x} \end{Bmatrix} \qquad (45)$$

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## Galerkin's Technique

Since we have arrived at a set of partial differential equations that describe the motion of our plate, we now need to solve them. This thesis will use an approximate method called the Galerkin technique. As opposed to a technique like the Ritz method, this technique involves the direct use of the differential equation and does not require the existence of a functional. Thus, it has a broader range of application than the Ritz method. However, in the area of solid mechanics, the two are closely related.

The basic idea behind the Galerkin technique can be discussed quite briefly. Suppose we want to solve the equation

$$L(u) = 0 \qquad (46)$$

where L is some differential operator in two variables whose solution satisfies homogeneous boundary conditions. We will look for an approximate solution in the form

$$\bar{u}(x,y) = \sum_{i=1}^{N} c_i \phi_i (x,y) \qquad (47)$$

where $\phi_i(x,y)$ is a system of functions which satisfy the boundary conditions and $c_i$ are undetermined coefficients. We will assume the functions $\phi_i(x,y)$ to be linearly independant. It is important to note here that if we are to get an "accurate" answer, the system of functions $\phi_i(x,y)$, must be complete in the given region. If they are not, then we are excluding part of the solution and our results will be very misleading. If $\bar{u}(x,y)$ is to be an exact solution, then $L(\bar{u})$ will be identically equal to zero. If $L(\bar{u})$ is continuous then we are saying that $L(\bar{u})$ is orthogonal to all of the functions of the system $\phi_i(x,y)$. Mathematically speaking we have

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$$\int_D L(\overline{u}(x,y))\phi_i(x,y)\ dxdy = \int_D L(\sum_{j=1}^{N} c_j\phi_j(x,y))\phi_i(x,y)\ dxdy = 0 \quad (48)$$

where $i = 1,...,N$ individual equations. From this system of equations we can solve for the undetermined coefficients and therefore arrive at our solution.

The question now arises what happens if our assumed function does not satisfy the boundary conditions. If it doesn't, we want to force our approximate functions $\phi_i(x,y)$ to satisfy them also. Thus, for the case of two variables we would set up a Galerkin equation as a line integral around the boundary that would force the function to satisfy the boundary. For homogeneous boundary conditions this line integral in and of itself would also equal zero.

## Galerkin's Equations for Anisotropic Laminated Plates

At this point we are ready to formulate the Galerkin equations for the plate equations of motion. Our approach will be to formulate the Galerkin equations, make appropriate substitutions, and then normalize the equations.

In formulating the Galerkin equations we must be careful. If we could find approximate functions which would satisfy both the geometric and natural boundary conditions, then we could apply Eq. (48) to each of our three equations of motion and our Galerkin equations would be formulated. However, finding such functions is extremely difficult, if not impossible. We can, however, find functions which satisfy the geometric boundary conditions. This does impose the added condition that we now must formulate a Galerkin equation for the boundaries as well as the equations of motion. We'll start by formulating the Galerkin equations for Eqs. (42a) and (43a). If we assume that $\bar{\Psi}_x$ is the approximate rotation function about the y axis and use Eq. (48), we have for Eq. (42a)

$$\int_\theta^b \int_\theta^a (M_{x,x} + M_{xy,y} - Q_x + \omega^2 I \bar{\Psi}_x) \, \bar{\Psi}'_x \, dxdy = \theta \qquad (49)$$

where $\bar{\Psi}_x$ is made up of a series of functions (each which satisfy the geometric boundary conditions) where each term in the series is multiplied by an undetermined coefficient, $A_{mn}$. $\bar{\Psi}'_x$ on the other hand, is a single term of the $\bar{\Psi}_x$ approximate series and has no undetermined coefficient associated with it. This is a result of the Galerkin derivation and several examples of this may be found in References [5] and [17]. For each term in the approximate $\bar{\Psi}_x$ series, a new equation will be generated. We should note here that one of the conditions for solving for the undetermined coefficients is that there are as many equations as there are undetermined coefficients. Thus, if we have m*n undetermined coefficients in our approximate series, $\bar{\Psi}_x$, Eq. (49) will generate m*n equations.

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We will now turn our attention to the formulation of the Galerkin equation for the boundary, Eq. (43a). Eq. (43a) is different from Eq. (42a) in that it still contains the variation of $\Psi_x$. This poses no real problem in that the Galerkin method can be related to the functional expression in Eq. (39). Using Eq. (43a) and following the procedure from Reference [5] we have

$$\int_\Gamma (-M_x dy + M_{xy} dx)\, \overline{\Psi}'_x = 0 \qquad (50a)$$

where $\overline{\Psi}'_x$ is again a single term of the $\overline{\Psi}_x$ approximate series and does not contain an undetermined coefficient. The undetermined coefficients are hidden in the $M_x$ and $M_{xy}$ terms (see Eq. (44) where $\Psi_x$ is replaced with $\overline{\Psi}_x$). If we use the plate coordinates (see Figure 2.2) to define the contour for the integral in Eq. (50a), it becomes (see Ref. [18])

$$\int_0^b (M_x(0,y)\Psi'_x(0,y) - M_x(a,y)\Psi'_x(a,y))\, dy$$

$$+ \int_0^a (M_{xy}(x,0)\Psi'_x(x,0) - M_{xy}(x,b)\Psi'_x(x,b))\, dx = 0 \qquad (50b)$$

Eqs. (49) and (50b) form the Galerkin equations for the first coordinate $\Psi_x$. In order to reduce the size of the eigenvalue problem we will be solving later, we can approximate the two Eqs. (49) and (50b) by combining them into one equation. We have therefore

$$\int_0^b \int_0^a (M_{x,x} + M_{xy,y} - Q_x + \omega^2 I \overline{\Psi}_x)\, \overline{\Psi}'_x\, dxdy$$

$$+ \int_0^b (M_x(0,y)\, \overline{\Psi}'_x(0,y) - M_x(a,y)\, \overline{\Psi}'_x(a,y))\, dy$$

$$+ \int_0^a (M_{xy}(x,0)\, \overline{\Psi}'_x(x,0) - M_{xy}(x,b)\, \overline{\Psi}'_x(x,b))\, dx = 0 \qquad (51)$$

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Now substituting Eq.(44) and (45) into (51) and combining terms we have

$$\int_0^a \int_0^b \{ D_{11}\bar{\psi}_{x,xx} + 2D_{16}\bar{\psi}_{x,xy} + D_{66}\bar{\psi}_{x,yy} + D_{16}\bar{\psi}_{y,xx} + (D_{12}+D_{66})\bar{\psi}_{y,xy}$$

$$+ D_{26}\bar{\psi}_{y,yy} - kA_{55}\bar{\psi}_x - kA_{55}w_{,x} + I\omega^2\bar{\psi}_x \} \; \bar{\psi}'_x \; dx \; dy$$

$$+\int_0^b \{ D_{11}\bar{\psi}_{x,x}(0,y) + D_{12}\bar{\psi}_{y,y}(0,y) + D_{16}(\bar{\psi}_{x,y}(0,y)+\bar{\psi}_{y,x}(0,y))) \; \bar{\psi}'_x(0,y)$$

$$- (D_{11}\bar{\psi}_{x,x}(a,y) + D_{12}\bar{\psi}_{y,y}(a,y) + D_{16}(\bar{\psi}_{x,y}(a,y)+\bar{\psi}_{y,x}(a,y))) \; \bar{\psi}'_x(a,y) \quad dy$$

$$+\int_0^a \{ D_{16}\bar{\psi}_{x,x}(x,0) + D_{26}\bar{\psi}_{y,y}(x,0) + D_{66}(\bar{\psi}_{x,y}(x,0)+\bar{\psi}_{y,x}(x,0))) \; \bar{\psi}'_x(x,0)$$

$$-(D_{16}\bar{\psi}_{x,x}(x,b)+D_{26}\bar{\psi}_{y,y}(x,b)+D_{66}(\bar{\psi}_{x,y}(x,b)+\bar{\psi}_{y,x}(x,b)))\bar{\psi}'_x(x,b)dx = 0 \quad (52)$$

We choose to normalize Eq. (52) using the normalization scheme similiar to the one used in Reference [18].

$$d_{ij} = D_{ij}/E_2 h^3$$

$$a_{ij} = A_{ij}/E_2 h$$

$$R \; = a/b$$

$$s \; = a/h$$

$$\bar{\omega}^2 = pa^2\omega^2/E_2 h \qquad (53)$$

$$\bar{w} = w/h$$

$$\xi \; = x/a$$

$$\eta \; = y/b$$

where $d_{ij}$ and $a_{ij}$ are the normalized bending and extensional stiffnesses respectively, $E_2$ is Young's Modulus in the direction normal to the lamina fibers, R is the aspect ratio, s is the length to thickness ratio, $\bar{\omega}$ is the normalized

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frequency, p is the product of lamina density and lamina thickness summed over the laminate, $\xi$ is the normalized x coordinate, $\eta$ is the normalized y coordinate, and h is the plate thickness. We will also make the substitution

$$I = \int_{-h/2}^{h/2} \rho z^2 \, dz = \rho h^3/12 = ph^2/12 \qquad (54)$$

where

$$p = \sum_{k=1}^{N} \rho_k * t_k \qquad (55)$$

The reason for this substitution is that it will allow us to build up a laminate of different materials. Thus, we can use our homogeneous theory to handle hybrid composite plates. After these substitutions and subsequent simplifications, Eq. (52) becomes

$$\int_0^1 \int_0^1 (d_{11}\bar{v}_{\xi,\xi\xi} + 2Rd_{16}\bar{v}_{\xi,\xi\eta} + R^2 d_{66}\bar{v}_{\xi,\eta\eta} + d_{16}\bar{v}_{\eta,\xi\xi} + R(d_{12}+d_{66})\bar{v}_{\eta,\xi\eta}$$

$$+ R^2 d_{26}\bar{v}_{\eta,\eta\eta} - ks^2 a_{55}\bar{v}_\xi - ksa_{55}\bar{w}_{,\xi} + (\omega^2/12)v_\xi) \ \bar{v}'_\xi \ d\xi \ d\eta$$

$$+\int_0^1 (d_{11}\bar{v}_{\xi,\xi}(0,\eta) + Rd_{12}\bar{v}_{\eta,\eta}(0,\eta) + d_{16}(R\bar{v}_{\xi,\eta}(0,\eta)+\bar{v}_{\eta,\xi}(0,\eta))) \ \bar{v}'_\xi(0,\eta)$$

$$- (d_{11}\bar{v}_{\xi,\xi}(1,\eta) + Rd_{12}\bar{v}_{\eta,\eta}(1,\eta) + d_{16}(R\bar{v}_{\xi,\eta}(1,\eta)+\bar{v}_{\eta,\xi}(1,\eta))) \ \bar{v}'_\xi(1,\eta) \ d\eta$$

$$+\int_0^1 (d_{16}\bar{v}_{\xi,\xi}(\xi,0) + Rd_{26}\bar{v}_{\eta,\eta}(\xi,0) + d_{66}(R\bar{v}_{\xi,\eta}(\xi,0)+\bar{v}_{\eta,\xi}(\xi,0))) \ \bar{v}'_\xi(\xi,0)$$

$$-(d_{16}\bar{v}_{\xi,\xi}(\xi,1)+Rd_{26}\bar{v}_{\eta,\eta}(\xi,1)$$

$$+d_{66}(R\bar{v}_{\xi,\eta}(\xi,1)+\bar{v}_{\eta,\xi}(\xi,1)))\bar{v}'_\xi(\xi,1) \ d\xi = 0 \qquad (56)$$

We are now ready to formulate Galerkin's equation for Eqs. (42b) and (43b). Using the same procedure as before, we arrive at

$$\int_0^b \int_0^a (M_{xy,x} + M_{y,y} - Q_y + \omega^2 I \bar{\Psi}_y)\, \bar{\Psi}'_y \, dx\, dy$$

$$+ \int_0^b (M_{xy}(0,y)\, \bar{\Psi}'_y(0,y) - M_{xy}(a,y)\, \bar{\Psi}'_y(a,y))\, dy$$

$$+ \int_0^a (M_y(x,0)\, \bar{\Psi}'_y(x,0) - M_y(x,b)\, \bar{\Psi}'_y(x,b))\, dx \;=\; 0 \qquad (57)$$

where $\bar{\Psi}_y$ is the assumed rotation function about the x axis. As before the double integral will force our approximate rotation function to satisfy the plate domain while the two line integrals will force the function to satisfy the natural boundary conditions. Now substituting Eqs. (44) and (45) into (57) and combining terms we have

$$\int_0^a \int_0^b (D_{16}\bar{\Psi}_{x,xx} + (D_{12}+D_{66})\bar{\Psi}_{x,xy} + D_{26}\bar{\Psi}_{x,yy} + D_{66}\bar{\Psi}_{y,xx} + 2D_{26}\bar{\Psi}_{y,xy}$$

$$+ D_{22}\bar{\Psi}_{y,yy} - kA_{44}\bar{\Psi}_y - kA_{44}\bar{w}_{,y} + I\omega^2\bar{\Psi}_y)\, \bar{\Psi}'_y \, dx\, dy$$

$$+\int_0^b (D_{16}\bar{\Psi}_{x,x}(0,y) + D_{26}\bar{\Psi}_{y,y}(0,y) + D_{66}(\bar{\Psi}_{x,y}(0,y)+\bar{\Psi}_{y,x}(0,y)))\, \bar{\Psi}'_y(0,y)$$

$$- (D_{16}\bar{\Psi}_{x,x}(a,y) + D_{26}\bar{\Psi}_{y,y}(a,y) + D_{66}(\bar{\Psi}_{x,y}(a,y)+\bar{\Psi}_{y,x}(a,y)))\, \bar{\Psi}'_y(a,y)\quad dy$$

$$+\int_0^a (D_{12}\bar{\Psi}_{x,x}(x,0) + D_{22}\bar{\Psi}_{y,y}(x,0) + D_{26}(\bar{\Psi}_{x,y}(x,0)+\bar{\Psi}_{y,x}(x,0)))\, \bar{\Psi}'_y(x,0)$$

$$-(D_{12}\bar{\Psi}_{x,x}(x,b)+D_{22}\bar{\Psi}_{y,y}(x,b)+D_{26}(\bar{\Psi}_{x,y}(x,b)+\bar{\Psi}_{y,x}(x,b)))\bar{\Psi}'_y(x,b)dx = 0 \quad (58)$$

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If we normalize Eq.(58) using the same normalization scheme as before, we have

$$\int_0^1 \int_0^1 (d_{16}\bar{v}_{\xi,\xi\xi} + (d_{12}+d_{66})\bar{v}_{\xi,\xi\eta} + R^2 d_{26}\bar{v}_{\xi,\eta\eta} + d_{66}\bar{v}_{\eta,\xi\xi} + 2Rd_{26}\bar{v}_{\eta,\xi\eta}$$

$$+ R^2 d_{22}\bar{v}_{\eta,\eta\eta} - ks^2 a_{44}\bar{v}_\eta - ksa_{44}\bar{w}_{,\eta} + (\bar{\omega}^2/12)\bar{v}_\eta)\ \bar{v}_\eta'\ d\xi\ d\eta$$

$$+\int_0^1 (d_{16}\bar{v}_{\xi,\xi}(0,\eta) + Rd_{26}\bar{v}_{\eta,\eta}(0,\eta) + d_{66}(R\bar{v}_{\xi,\eta}(0,\eta)+\bar{v}_{\eta,\xi}(0,\eta)))\ \bar{v}_\eta'(0,\eta)$$

$$- (d_{16}\bar{v}_{\xi,\xi}(1,\eta) + Rd_{26}\bar{v}_{\eta,\eta}(1,\eta) + d_{66}(R\bar{v}_{\xi,\eta}(1,\eta)+\bar{v}_{\eta,\xi}(1,\eta)))\ \bar{v}_\eta'(1,\eta)\ d\eta$$

$$+\int_0^1 (d_{12}\bar{v}_{\xi,\xi}(\xi,0) + Rd_{22}\bar{v}_{\eta,\eta}(\xi,0) + d_{26}(R\bar{v}_{\xi,\eta}(\xi,0)+v_{\eta,\xi}(\xi,0)))\ \bar{v}_\eta'(\xi,0)$$

$$-(d_{12}\bar{v}_{\xi,\xi}(\xi,1)+Rd_{22}\bar{v}_{\eta,\eta}(\xi,1)$$

$$+d_{26}(R\bar{v}_{\xi,\eta}(\xi,1)+\bar{v}_{\eta,\xi}(\xi,1)))\bar{v}_\eta'(\xi,1)\ d\xi = 0 \tag{59}$$

We now need to formulate Galerkin's equation for the last equations of our system, Eqs. (42c) and (43c). We have

$$\int_0^b \int_0^a (Q_{x,x} + Q_{y,y} + N_x w_{,xx} + 2N_{xy}w_{,xy} + N_y w_{,yy} + \rho\omega^2 w)\ \bar{W}'dxdy$$

$$+ \int_0^b (Q_x(0,y)\ \bar{W}(0,y) - Q_x(a,y)\ \bar{W}(a,y))\ \ dy$$

$$+ \int_0^a (Q_y(x,0)\ \bar{W}'(x,0) - Q_y(x,b)\ \bar{W}'(x,b))\ dx\ =\ 0 \tag{60}$$

where $\bar{W}$ is the assumed displacement function. Again, the double integral is to satisfy the domain and the two line integrals are to satisfy the boundary. Substituting Eqs. (44) and (45) into (60) and combining terms we have

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$$\int_0^a \int_0^b (kA_{55}\overline{\Psi}_{x,x} + kA_{55}w_{,xx} + kA_{44}\overline{\Psi}_{y,y} + kA_{44}w_{,yy} - k_2 N_0 w_{,xx}$$

$$+ 2k_3 N_0 w_{,xy} - k_1 N_0 w_{,yy} + \rho\omega^2 w)\, \overline{W}' \ dxdy$$

$$+\int_0^b (kA_{55}\overline{\Psi}_x(0,y) + kA_{55}w_{,x}(0,y))\, \overline{W}'(0,y)$$

$$- (kA_{55}\overline{\Psi}_x(a,y) + kA_{55}w_{,x}(a,y))\, \overline{W}'(a,y)\ dy$$

$$+\int_0^a (kA_{44}\overline{\Psi}_y(x,0) + kA_{44}w_{,y}(x,0))\, \overline{W}'(x,0)$$

$$-(kA_{44}\overline{\Psi}_y(x,b)+kA_{44}w_{,y}(x,b))\, \overline{W}'(x,b)\ \ dx\ =\ 0 \qquad\qquad (61)$$

where $N_y = -k_1 N_0$, $N_x = -k_2 N_0$, and $N_{xy} = k_3 N_0$.

We will now normalize Eq.(61), however, we need to define two more normalization factors. They are

$$\lambda_1 = N_0 b^2 / E_2 h^3$$

$$\qquad\qquad\qquad (62)$$

$$\lambda_2 = \rho b^4 \omega^2 / E_2 h^3$$

Therefore, Eq.(61) will become

$$\int_0^1 \int_0^1 (ksa_{55}\bar{\Psi}_\xi{}_{,\xi} + ka_{55}\bar{w}_{,\xi\xi} + kRsa_{44}\bar{\Psi}_\eta{}_{,\eta} + kR^2a_{44}\bar{w}_{,\xi\xi} - k_2\lambda_1(R^2/s^2)\bar{w}_{,\xi\xi}$$

$$+ 2k_3(R^3/s^2)\lambda_1\bar{w}_{,\xi\eta} - k_1(R^4/s^2)\lambda_1\bar{w}_{,\eta\eta} + (R^4/s^2)\lambda_2\bar{w})\ \bar{W}'d\xi d\eta$$

$$+\int_0^1 (ksa_{55}\bar{\Psi}_\xi(0,\eta) + ka_{55}\bar{w}_{,\xi}(0,\eta))\ \bar{W}(0,\eta)$$

$$- (ksa_{55}\bar{\Psi}_\xi(1,\eta) + ka_{55}\bar{w}_{,\xi}(1,\eta))\ \bar{W}(1,\eta)\ \ d\eta$$

$$+\int_0^1 (ksa_{44}\bar{\Psi}_\eta(\xi,0) + Rka_{44}\bar{w}_{,\eta}(\xi,0))\ \bar{W}'(\xi,0)$$

$$- (ksa_{44}\bar{\Psi}_\eta(\xi,1) + Rka_{44}\bar{w}_{,\eta}(\xi,1))\ \bar{W}'(\xi,1)\ \ d\xi\ =\ 0 \tag{63}$$

At this point we are ready to pick a boundary condition, find approximate functions for $\Psi_x$, $\Psi_y$, and w for that condition, and substitute into Eqs. (56), (59), and (63).

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## Simply-Supported Boundary Condition

Since the Galerkin equations have been formulated it is possible to choose the displacement and rotation functions (w, $\Psi_x$, $\Psi_y$) which satisify the three equations for each boundary condition being considered. These differential equations can be integrated resulting in three algebraic equations (for each boundary condition). The eigenvalue problem can be formulated from which the natural frequencies and mode shapes can be obtained. Let us begin with the simply-supported boundary condition.

For a plate simply supported on all sides, we know from plate theory that:

at x = 0,a

$$w = \Psi_y = 0 \tag{64}$$

$$M_x = D_{11}\,\Psi_{x,x} + D_{12}\,\Psi_{y,y} + D_{16}\,(\Psi_{y,x} + \Psi_{x,y}) = 0$$

and

at y = 0,b

$$w = \Psi_x = 0 \tag{65}$$

$$M_y = D_{12}\,\Psi_{x,x} + D_{22}\,\Psi_{y,y} + D_{26}\,(\Psi_{y,x} + \Psi_{x,y}) = 0$$

Therefore, we shall choose for our admissible functions (functions which satisfy the geometric boundary conditions and are continuously differentiable one time)

$$\bar{\Psi}_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, \cos(m\pi x/a)\, \sin(n\pi y/b)$$

$$\bar{\Psi}_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\, \sin(m\pi x/a)\, \cos(n\pi y/b) \tag{66}$$

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, \sin(m\pi x/a)\, \sin(n\pi y/b)$$

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or in terms of normalized coordinates

$$\bar{v}_\xi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\bar{v}_\eta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi\xi) \cos(n\pi\eta) \qquad (67)$$

$$\bar{W} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$

In order to substitute into Eqs. (56), (59), and (63) the following derivatives are needed (note: all $\Sigma\Sigma$ are over $m=1$ to $\infty$ and $n=1$ to $\infty$).

$$\bar{v}_{\xi,\xi} = \Sigma\Sigma \ -A_{mn} \ m\pi \ \sin(m\pi\xi) \ \sin(n\pi\eta)$$

$$\bar{v}_{\xi,\xi\xi} = \Sigma\Sigma \ -A_{mn} \ m^2\pi^2 \ \cos(m\pi\xi) \ \sin(n\pi\eta)$$

$$\bar{v}_{\xi,\xi\eta} = \Sigma\Sigma \ -A_{mn} \ mn\pi^2 \ \sin(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{v}_{\xi,\eta} = \Sigma\Sigma \ \ A_{mn} \ n\pi \ \cos(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{v}_{\xi,\eta\eta} = \Sigma\Sigma \ -A_{mn} \ n^2\pi^2 \ \cos(m\pi\xi) \ \sin(n\pi\eta)$$

$$\bar{v}_{\eta,\xi} = \Sigma\Sigma \ \ B_{mn} \ m\pi \ \cos(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{v}_{\eta,\xi\xi} = \Sigma\Sigma \ -B_{mn} \ m^2\pi^2 \ \sin(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{v}_{\eta,\xi\eta} = \Sigma\Sigma \ -B_{mn} \ mn\pi^2 \ \cos(m\pi\xi) \ \sin(n\pi\eta) \qquad (68)$$

$$\bar{v}_{\eta,\eta} = \Sigma\Sigma \ -B_{mn} \ n\pi \ \sin(m\pi\xi) \ \sin(n\pi\eta)$$

$$\bar{v}_{\eta,\eta\eta} = \Sigma\Sigma \ -B_{mn} \ n^2\pi^2 \ \sin(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{w}_{,\xi} = \Sigma\Sigma \ \ C_{mn} \ m\pi \ \cos(m\pi\xi) \ \sin(n\pi\eta)$$

$$\bar{w}_{,\xi\xi} = \Sigma\Sigma \ -C_{mn} \ m^2\pi^2 \ \sin(m\pi\xi) \ \sin(n\pi\eta)$$

$$\bar{w}_{,\xi\eta} = \Sigma\Sigma \ \ C_{mn} \ mn\pi^2 \ \cos(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{w}_{,\eta} = \Sigma\Sigma \ \ C_{mn} \ n\pi \ \sin(m\pi\xi) \ \cos(n\pi\eta)$$

$$\bar{w}_{,\eta\eta} = \Sigma\Sigma \ -C_{mn} \ n^2\pi^2 \ \sin(m\pi\xi) \ \sin(n\pi\eta)$$

Thus from Eq. (56)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1}\int_{0}^{1} (-d_{11}A_{mn}m^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - 2Rd_{16}A_{mn}mn\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- R^2d_{66}A_{mn}n^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - d_{16}B_{mn}m^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- R(d_{12}+d_{66})B_{mn}mn\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - R^2d_{26}B_{mn}n^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- ks^2a_{55}A_{mn}\cos(m\pi\xi)\sin(n\pi\eta) - ksa_{55}C_{mn}m\pi\cos(m\pi\xi)\sin(n\pi\eta)$$

$$+ (\bar{\omega}^2/12)A_{mn}\cos(m\pi\xi)\sin(n\pi\eta))\,(\cos(p\pi\xi)\sin(q\pi\eta))\,d\xi\,d\eta$$

$$+ \int_{0}^{1} (-d_{11}A_{mn}m\pi\sin(0)\sin(n\pi\eta) - Rd_{12}B_{mn}n\pi\sin(0)\sin(n\pi\eta)$$

$$+ d_{16}[RA_{mn}n\pi\cos(0)\cos(n\pi\eta) + B_{mn}m\pi\cos(0)\cos(n\pi\eta)])\,(\cos(0)\sin(q\pi\eta))$$

$$- (-d_{11}A_{mn}m\pi\sin(m\pi)\sin(n\pi\eta) - Rd_{12}B_{mn}n\pi\sin(m\pi)\sin(n\pi\eta)$$

$$+ d_{16}[RA_{mn}n\pi\cos(m\pi)\cos(n\pi\eta) + B_{mn}m\pi\cos(m\pi)\cos(n\pi\eta)])$$

$$* (\cos(p\pi)\sin(q\pi\eta))\,d\eta$$

$$+ \int_{0}^{1} (-d_{16}A_{mn}m\pi\sin(m\pi\xi)\sin(0) - Rd_{26}B_{mn}n\pi\sin(m\pi\xi)\sin(0)$$

$$+ d_{66}[RA_{mn}n\pi\cos(m\pi\xi)\cos(0) + B_{mn}m\pi\cos(m\pi\xi)\cos(0)])\,(\cos(p\pi\xi)\sin(0))$$

$$- (-d_{16}A_{mn}m\pi\sin(m\pi\xi)\sin(n\pi) - Rd_{26}B_{mn}n\pi\sin(m\pi\xi)\sin(n\pi)$$

$$+ d_{66}[RA_{mn}n\pi\cos(m\pi\xi)\cos(n\pi) + B_{mn}m\pi\cos(m\pi\xi)\cos(n\pi)])$$

$$* (\cos(p\pi\xi)\sin(q\pi))\,d\xi = 0 \tag{69}$$

From Eq. (59)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{0}^{1}\int_{0}^{1}(-d_{16}A_{mn}m^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta)-R(d_{12}+d_{66})A_{mn}mn\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$-R^2d_{26}A_{mn}n^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - d_{66}B_{mn}m^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- 2Rd_{26}B_{mn}mn\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - R^2d_{22}B_{mn}n^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- ks^2 a_{44}B_{mn}\sin(m\pi\xi)\cos(n\pi\eta) - kRs a_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(n\pi\eta)$$

$$+ (\bar{\omega}^2/12)B_{mn}\sin(m\pi\xi)\cos(n\pi\eta))\,(\sin(p\pi\xi)\cos(q\pi\eta))\,d\xi\,d\eta$$

$$+ \int_{0}^{1}(-d_{16}A_{mn}m\pi\sin(0)\sin(n\pi\eta) - Rd_{26}B_{mn}n\pi\sin(0)\sin(n\pi\eta)$$

$$+ d_{66}[RA_{mn}n\pi\cos(0)\cos(n\pi\eta) + B_{mn}m\pi\cos(0)\cos(n\pi\eta)])\,(\sin(0)\cos(q\pi\eta))$$

$$- (-d_{16}A_{mn}m\pi\sin(m\pi)\sin(n\pi\eta) - Rd_{26}B_{mn}n\pi\sin(m\pi)\sin(n\pi\eta)$$

$$+ d_{66}[RA_{mn}n\pi\cos(m\pi)\cos(n\pi\eta) + B_{mn}m\pi\cos(m\pi)\cos(n\pi\eta)])$$

$$* (\sin(p\pi)\cos(q\pi\eta))\,d\eta$$

$$+ \int_{0}^{1}(-d_{12}A_{mn}m\pi\sin(m\pi\xi)\sin(0) - Rd_{22}B_{mn}n\pi\sin(m\pi\xi)\sin(0)$$

$$+ d_{26}[RA_{mn}n\pi\cos(m\pi\xi)\cos(0) + B_{mn}m\pi\cos(m\pi\xi)\cos(0)])\,(\sin(p\pi\xi)\cos(0))$$

$$- (-d_{12}A_{mn}m\pi\sin(m\pi\xi)\sin(n\pi) - Rd_{22}B_{mn}n\pi\sin(m\pi\xi)\sin(n\pi)$$

$$+ d_{26}[RA_{mn}n\pi\cos(m\pi\xi)\cos(n\pi) + B_{mn}m\pi\cos(m\pi\xi)\cos(n\pi)])$$

$$* (\sin(p\pi\xi)\cos(q\pi))\,d\xi = 0 \qquad (70)$$

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And from Eq. (63)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1}\int_{0}^{1} (-ksa_{55}A_{mn}m\pi\sin(m\pi\xi)\sin(n\pi\eta) - ka_{55}C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$- kRsa_{44}B_{mn}n\pi\sin(m\pi\xi)\sin(n\pi\eta) - kR^2a_{44}C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ k_2\lambda_1(R^2/s^2)C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ 2k_3(R^3/s^2)\lambda_1 C_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)$$

$$+ k_1(R^4/s^2)\lambda_1 C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ (R^4/s^2)\lambda_2 C_{mn}\sin(m\pi\xi)\sin(n\pi\eta)) \, (\sin(p\pi\xi)\sin(q\pi\eta)) \, d\xi \, d\eta$$

$$+ \int_{0}^{1} (ksa_{55}A_{mn}\cos(0)\sin(n\pi\eta) + ka_{55}C_{mn}m\pi\cos(0)\sin(n\pi\eta)) \, (\sin(0)\sin(q\pi\eta))$$

$$- (ksa_{55}A_{mn}\cos(m\pi)\sin(n\pi\eta) + ka_{55}C_{mn}m\pi\cos(m\pi)\sin(n\pi\eta))$$

$$* \, (\sin(p\pi)\sin(q\pi\eta)) \, d\eta$$

$$+ \int_{0}^{1} (ksa_{44}B_{mn}\sin(m\pi\xi)\cos(0) + Rka_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(0))(\sin(p\pi\xi)\sin(0))$$

$$- (ksa_{44}B_{mn}\sin(m\pi\xi)\cos(n\pi) + Rka_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(n\pi))$$

$$* \, (\sin(p\pi\xi)\sin(q\pi)) \, d\xi = 0 \tag{71}$$

The indicated multiplications are performed after noting that $\sin(0) = \sin(p\pi) = \sin(q\pi) = 0$ (p and q are integers) and $\cos(0) = 1$.

From Eq. (69)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 -d_{11}A_{mn}m^2\pi^2\cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- 2Rd_{16}A_{mn}mn\pi^2\sin(m\pi\xi)\cos(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2d_{66}A_{mn}n^2\pi^2\cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- d_{16}B_{mn}m^2\pi^2\sin(m\pi\xi)\cos(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R(d_{12}+d_{66})B_{mn}mn\pi^2\cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2d_{26}B_{mn}n^2\pi^2\sin(m\pi\xi)\cos(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- ks^2a_{55}A_{mn}\cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ksa_{55}C_{mn}m\pi\cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ (\bar{\omega}^2/12)A_{mn}\cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \quad d\xi \; d\eta$$

$$+ \int_0^1 d_{16}[RA_{mn}n\pi\cos(n\pi\eta)\sin(q\pi\eta) + B_{mn}m\pi\cos(n\pi\eta)\sin(q\pi\eta)]$$

$$- d_{16}[RA_{mn}n\pi\cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$+ B_{mn}m\pi\cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)] \; d\eta = 0 \qquad (72)$$

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From Eq. (70)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \int_{0}^{1} -d_{16}A_{mn}m^{2}\pi^{2}\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\cos(q\pi\eta)$$

$$- R(d_{12}+d_{66})A_{mn}mn\pi^{2}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- R^{2}d_{26}A_{mn}n^{2}\pi^{2}\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\cos(q\pi\eta)$$

$$- d_{66}B_{mn}m^{2}\pi^{2}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- 2Rd_{26}B_{mn}mn\pi^{2}\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\cos(q\pi\eta)$$

$$- R^{2}d_{22}B_{mn}n^{2}\pi^{2}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- ks^{2}a_{44}B_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- kRsa_{44}C_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$+ (\bar{\omega}^{2}/12)B_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta) \quad d\xi \; d\eta$$

$$+ \int_{0}^{1} d_{26}[RA_{mn}n\pi\cos(m\pi\xi)\sin(p\pi\xi) + B_{mn}m\pi\cos(m\pi\xi)\sin(p\pi\xi)]$$

$$- d_{26}[RA_{mn}n\pi\cos(n\pi)\cos(q\pi)\cos(m\pi\xi)\sin(p\pi\xi)$$

$$+ B_{mn}m\pi\cos(n\pi)\cos(q\pi)\cos(m\pi\xi)\sin(p\pi\xi)] \quad d\xi \; = \; 0 \qquad (73)$$

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And from Eq. (71)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 -ksa_{55}A_{mn}m\pi\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ka_{55}C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- kRsa_{44}B_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- kR^2a_{44}C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ k_2\lambda_1(R^2/s^2)C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ 2k_3(R^3/s^2)\lambda_1 C_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$+ k_1(R^4/s^2)\lambda_1 C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ (R^4/s^2)\lambda_2 C_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)\ d\xi d\eta\ =\ 0 \quad (74)$$

In order to integrate these equations, we shall use the following integrals taken from [3].

When $m = p$ (or $n = q$)

$$\int_0^1 \cos(m\pi x)\cos(p\pi x)\ dx = [\tfrac{1}{2}x + (1/4m\pi)\sin(2m\pi x)]\Big|_0^1 = \tfrac{1}{2} \quad (75)$$

$$\int_0^1 \sin(m\pi x)\sin(p\pi x)\ dx = [\tfrac{1}{2}x - (1/4m\pi)\sin(2m\pi x)]\Big|_0^1 = \tfrac{1}{2} \quad (76)$$

$$\int_0^1 \sin(m\pi x)\cos(p\pi x)\ dx = [(1/2m\pi)\sin^2(m\pi x)]\Big|_0^1 = 0 \quad (77)$$

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When m ≠ p  (or n ≠ q)

$$\int_0^1 \cos(m\pi x) \cos(p\pi x) \, dx = \frac{\sin(m-p)\pi x}{2\pi(m-p)} + \frac{\sin(m+p)\pi x}{2\pi(m+p)}\bigg|_0^1 = 0 \qquad (78)$$

$$\int_0^1 \sin(m\pi x) \sin(p\pi x) \, dx = \frac{\sin(m-p)\pi x}{2\pi(m-p)} - \frac{\sin(m+p)\pi x}{2\pi(m+p)}\bigg|_0^1 = 0 \qquad (79)$$

$$\int_0^1 \sin(m\pi x) \cos(p\pi x) \, dx = -\frac{\cos(m-p)\pi x}{2\pi(m-p)} - \frac{\cos(m+p)\pi x}{2\pi(m+p)}\bigg|_0^1$$

$$= -\frac{\cos(m-p)\pi - 1}{2\pi(m-p)} - \frac{\cos(m+p)\pi - 1}{2\pi(m+p)}$$

$$= \frac{(-\cos(m-p)\pi + 1)(m+p)}{2\pi(m-p)(m+p)} + \frac{(-\cos(m+p)\pi + 1)(m-p)}{2\pi(m-p)(m+p)}$$

$$= \frac{(-\cos(m-p)\pi + 1)(m+p) + (-\cos(m+p)\pi + 1)(m-p)}{2(m^2 - p^2)}$$

$$= 0 \quad \text{for } (m+p) \text{ an even integer} \qquad (80)$$

and

$$= 2m/\pi(m^2 - p^2) \quad \text{for } (m+p) \text{ an odd integer} \qquad (81)$$

Similarly for

$$\int_0^1 \sin(p\pi x) \cos(m\pi x) \, dx = 0 \quad \text{for } (p+m) \text{ an even integer} \qquad (82)$$

$$= 2p/\pi(p^2 - m^2) \quad \text{for } (p+m) \text{ an odd integer} \qquad (83)$$

We shall now introduce the notation

$$\int_0^1 \cos(m\pi x) \cos(p\pi x) \, dx = (\tfrac{1}{2} \text{ or } 0) \qquad (84)$$

where the value on the left side of the parenthetical expression is the value of the integral when m=p and the value on the right side of the expression is when m≠p.

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Thus, upon integrating, simplifying, and taking only a finite number of terms we have (note: even and odd tests are for the sum of the indices):

From Eq. (72)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [\pi^2(\text{½ or } \theta)(\text{½ or } \theta)(-m^2 d_{11} - n^2 R^2 d_{66} - ks^2 a_{55}/\pi^2 + \bar{\omega}^2/12\pi^2)$$

$$- (\theta \text{ or } \theta \text{ even,2m odd})(\theta \text{ or } \theta \text{ even,2q odd})(2Rmnd_{16})/((m^2-p^2)(q^2-n^2))$$

$$+ (\theta \text{ or } \theta \text{ even,2q odd})(nRd_{16}(1-\cos(m\pi)\cos(p\pi)))/(q^2-n^2)] \; A_{mn}$$

$$+ [\pi^2(\text{½ or } \theta)(\text{½ or } \theta)(-mnR(d_{12} + d_{66}))$$

$$- (\theta \text{or} \theta \text{ even,2m odd})(\theta \text{or} \theta \text{ even,2q odd})(m^2 d_{16}+n^2 R^2 d_{26})/((m^2-p^2)(q^2-n^2))$$

$$+ (\theta \text{ or } \theta \text{ even,2q odd})(md_{16}(1-\cos(m\pi)\cos(p\pi)))/(q^2-n^2)] \; B_{mn}$$

$$- [\pi(\text{½ or } \theta)(\text{½ or } \theta)(mksa_{55})] \; C_{mn} \quad = \quad \theta \qquad\qquad (85)$$

From Eq. (73)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [\pi^2(\text{½ or } \theta)(\text{½ or } \theta)(-mnR(d_{12} + d_{66}))$$

$$- (\theta \text{or} \theta \text{ even,2p odd})(\theta \text{or} \theta \text{ even,2n odd})(m^2 d_{16}+n^2 R^2 d_{26})/((p^2-m^2)(n^2-q^2))$$

$$+ (\theta \text{ or } \theta \text{ even,2p odd})(nRd_{26}(1-\cos(n\pi)\cos(q\pi)))/(p^2-m^2)] \; A_{mn}$$

$$+ [\pi^2(\text{½ or } \theta)(\text{½ or } \theta)(-m^2 d_{66} - n^2 R^2 d_{22} - (ks^2 a_{44})/\pi^2 + \bar{\omega}^2/12\pi^2)$$

$$- (\theta \text{ or } \theta \text{ even,2p odd})(\theta \text{ or } \theta \text{ even,2n odd})(2Rmnd_{26})/((p^2-m^2)(n^2-q^2))$$

$$+ (\theta \text{ or } \theta \text{ even,2p odd})(md_{26}(1-\cos(n\pi)\cos(q\pi)))/(p^2-m^2)] \; B_{mn}$$

$$- [\pi(\text{½ or } \theta)(\text{½ or } \theta)(nRksa_{44})] \; C_{mn} \quad = \quad \theta \qquad\qquad (86)$$

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And from Eq. (74)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} - \pi(\text{½ or }\theta)(\text{½ or }\theta)(mksa_{55}) \ A_{mn}$$

$$- \pi(\text{½ or }\theta)(\text{½ or }\theta)(nRksa_{44}) \ B_{mn}$$

$$+ [\pi^2 \ (\text{½ or }\theta)(\text{½ or }\theta)(-m^2ka_{55}-n^2R^2ka_{44}$$

$$+ k_2\lambda_1 m^2(R^2/s^2) + k_1\lambda_1 n^2(R^4/s^2 + \lambda_2(R^4/\pi^2 s^2))$$

$$+ (\theta \text{ or }\theta \text{ even},2p \text{ odd})(\theta \text{ or }\theta \text{ even},2q \text{ odd})$$

$$* \ (2k_3 mnR^3\lambda_1)/(s^2(p^2-m^2)(q^2-n^2))] \ C_{mn} = \theta \qquad (87)$$

Equations (85), (86), and (87) are now ready to be programmed. The process
to generate the Galerkin equations is to pick an M and N (they must be equal), then
cycle through p and q (which are equal to M and N). Thus, M and N determine the
number of terms in each Galerkin equation and p and q determine the number of
Galerkin equations.

## Clamped Boundary Condition

For a plate clamped on all sides, one knows from plate theory that:

$$\text{at } x = 0,a$$

$$w = \Psi_x = \Psi_y = 0 \qquad (88)$$

and

$$\text{at } y = 0,b$$

$$w = \Psi_x = \Psi_y = 0 \qquad (89)$$

Therefore, we shall choose for our admissible functions

$$\bar{\Psi}_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

$$\bar{\Psi}_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \qquad (90)$$

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

or in terms of normalized coordinates

$$\bar{\Psi}_\xi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\bar{\Psi}_\eta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi\xi) \sin(n\pi\eta) \qquad (91)$$

$$\bar{W} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$

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In order to substitute into Eqs. (56), (59), and (63) one needs the following derivatives (note: all $\Sigma\Sigma$ are over $m=1$ to $\infty$ and $n=1$ to $\infty$).

$$\bar{v}_{\xi,\xi} = \Sigma\,\Sigma \quad A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{v}_{\xi,\xi\xi} = \Sigma\,\Sigma \quad -A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{v}_{\xi,\xi\eta} = \Sigma\,\Sigma \quad A_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$$

$$\bar{v}_{\xi,\eta} = \Sigma\,\Sigma \quad A_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi\eta)$$

$$\bar{v}_{\xi,\eta\eta} = \Sigma\,\Sigma \quad -A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{v}_{\eta,\xi} = \Sigma\,\Sigma \quad B_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{v}_{\eta,\xi\xi} = \Sigma\,\Sigma \quad -B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{v}_{\eta,\xi\eta} = \Sigma\,\Sigma \quad B_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) \qquad (92)$$

$$\bar{v}_{\eta,\eta} = \Sigma\,\Sigma \quad B_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi\eta)$$

$$\bar{v}_{\eta,\eta\eta} = \Sigma\,\Sigma \quad -B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{w}_{,\xi} = \Sigma\,\Sigma \quad C_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{w}_{,\xi\xi} = \Sigma\,\Sigma \quad -C_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$\bar{w}_{,\xi\eta} = \Sigma\,\Sigma \quad C_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$$

$$\bar{w}_{,\eta} = \Sigma\,\Sigma \quad C_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi\eta)$$

$$\bar{w}_{,\eta\eta} = \Sigma\,\Sigma \quad -C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

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Now, substituting into Eq. (56) we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 (-d_{11}A_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) + 2Rd_{16}A_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)$$

$$- R^2 d_{66}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) - d_{16}B_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ R(d_{12}+d_{66})B_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta) - R^2 d_{26}B_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$- ks^2 a_{55}A_{mn}\sin(m\pi\xi)\sin(n\pi\eta) - ksa_{55}C_{mn}m\pi\cos(m\pi\xi)\sin(n\pi\eta)$$

$$+ (\bar{\omega}^2/12)A_{mn}\sin(m\pi\xi)\sin(n\pi\eta)) \ (\sin(p\pi\xi)\sin(q\pi\eta)) \ d\xi \ d\eta$$

$$+ \int_0^1 ( \ d_{11}A_{mn}m\pi\cos(\theta)\sin(n\pi\eta) + Rd_{12}B_{mn}n\pi\sin(\theta)\cos(n\pi\eta)$$

$$+ d_{16}[RA_{mn}n\pi\sin(\theta)\cos(n\pi\eta) + B_{mn}m\pi\cos(\theta)\sin(n\pi\eta)]) \ (\sin(\theta)\sin(q\pi\eta))$$

$$- ( \ d_{11}A_{mn}m\pi\cos(m\pi)\sin(n\pi\eta) + Rd_{12}B_{mn}n\pi\sin(m\pi)\cos(n\pi\eta)$$

$$+ d_{16}[RA_{mn}n\pi\sin(m\pi)\cos(n\pi\eta) + B_{mn}m\pi\cos(m\pi)\sin(n\pi\eta)])$$

$$* \ (\sin(p\pi)\sin(q\pi\eta)) \ d\eta$$

$$+ \int_0^1 ( \ d_{16}A_{mn}m\pi\cos(m\pi\xi)\sin(\theta) + Rd_{26}B_{mn}n\pi\sin(m\pi\xi)\cos(\theta)$$

$$+ d_{66}[RA_{mn}n\pi\sin(m\pi\xi)\cos(\theta) + B_{mn}m\pi\cos(m\pi\xi)\sin(\theta)]) \ (\sin(p\pi\xi)\sin(\theta))$$

$$- ( \ d_{16}A_{mn}m\pi\cos(m\pi\xi)\sin(n\pi) + Rd_{26}B_{mn}n\pi\sin(m\pi\xi)\cos(n\pi)$$

$$+ d_{66}[RA_{mn}n\pi\sin(m\pi\xi)\cos(n\pi) + B_{mn}m\pi\cos(m\pi\xi)\sin(n\pi)])$$

$$* \ (\sin(p\pi\xi)\sin(q\pi)) \ d\xi = 0 \tag{93}$$

Substituting into Eq. (59) we have

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 (-d_{16}A_{mn}m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) + R(d_{12}+d_{66})A_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)
$$

$$
- R^2 d_{26}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) - d_{66}B_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)
$$

$$
+ 2Rd_{26}B_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta) - R^2 d_{22}B_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)
$$

$$
- ks^2 a_{44}B_{mn}\sin(m\pi\xi)\sin(n\pi\eta) - kRs a_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(n\pi\eta)
$$

$$
+ (\bar{w}^2/12)B_{mn}\sin(m\pi\xi)\sin(n\pi\eta)) \ (\sin(p\pi\xi)\sin(q\pi\eta)) \ d\xi \ d\eta
$$

$$
+ \int_0^1 \{ \ d_{16}A_{mn}m\pi\cos(\theta)\sin(n\pi\eta) + Rd_{26}B_{mn}n\pi\sin(\theta)\cos(n\pi\eta)
$$

$$
+ d_{66}[RA_{mn}n\pi\sin(\theta)\cos(n\pi\eta) + B_{mn}m\pi\cos(\theta)\sin(n\pi\eta)]\} \ (\sin(\theta)\sin(q\pi\eta))
$$

$$
- \{ \ d_{16}A_{mn}m\pi\cos(m\pi)\sin(n\pi\eta) + Rd_{26}B_{mn}n\pi\sin(m\pi)\cos(n\pi\eta)
$$

$$
+ d_{66}[RA_{mn}n\pi\sin(m\pi)\cos(n\pi\eta) + B_{mn}m\pi\cos(m\pi)\sin(n\pi\eta)]\}
$$

$$
* \ (\sin(p\pi)\sin(q\pi\eta)) \ d\eta
$$

$$
+ \int_0^1 \{ \ d_{12}A_{mn}m\pi\cos(m\pi\xi)\sin(\theta) + Rd_{22}B_{mn}n\pi\sin(m\pi\xi)\cos(\theta)
$$

$$
+ d_{26}[RA_{mn}n\pi\sin(m\pi\xi)\cos(\theta) + B_{mn}m\pi\cos(m\pi\xi)\sin(\theta)]\} \ (\sin(p\pi\xi)\sin(\theta))
$$

$$
- \{ \ d_{12}A_{mn}m\pi\cos(m\pi\xi)\sin(n\pi) + Rd_{22}B_{mn}n\pi\sin(m\pi\xi)\cos(n\pi)
$$

$$
+ d_{26}[RA_{mn}n\pi\sin(m\pi\xi)\cos(n\pi) + B_{mn}m\pi\cos(m\pi\xi)\sin(n\pi)]\}
$$

$$
* \ (\sin(p\pi\xi)\sin(q\pi)) \ d\xi = 0 \tag{94}
$$

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And finally substituting into Eq. (63) we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1}\int_{0}^{1} ( ksa_{55}A_{mn}m\pi\cos(m\pi\xi)\sin(n\pi\eta) - ka_{55}C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$- kRsa_{44}B_{mn}n\pi\sin(m\pi\xi)\cos(n\pi\eta) - kR^2a_{44}C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ k_2\lambda_1(R^2/s^2)C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ 2k_3(R^3/s^2)\lambda_1 C_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)$$

$$+ k_1(R^4/s^2)\lambda_1 C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ (R^4/s^2)\lambda_2 C_{mn}\sin(m\pi\xi)\sin(n\pi\eta)) \ (\sin(p\pi\xi)\sin(q\pi\eta)) \ d\xi \ d\eta$$

$$+ \int_{0}^{1} (ksa_{55}A_{mn}\sin(0)\sin(n\pi\eta) + ka_{55}C_{mn}m\pi\cos(0)\sin(n\pi\eta)) \ (\sin(0)\sin(q\pi\eta))$$

$$- (ksa_{55}A_{mn}\sin(m\pi)\sin(n\pi\eta) + ka_{55}C_{mn}m\pi\cos(m\pi)\sin(n\pi\eta))$$

$$* \ (\sin(p\pi)\sin(q\pi\eta)) \ d\eta$$

$$+ \int_{0}^{1} (ksa_{44}B_{mn}\sin(m\pi\xi)\sin(0) + Rka_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(0))(\sin(p\pi\xi)\sin(0))$$

$$- (ksa_{44}B_{mn}\sin(m\pi\xi)\sin(n\pi) + Rka_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(n\pi))$$

$$* \ (\sin(p\pi\xi)\sin(q\pi)) \ d\xi = 0 \tag{95}$$

The indicated multiplications are performed after noting that $\sin(0) = \sin(p\pi) = \sin(q\pi) = 0$ (p and q are integers) and $\cos(0) = 1$.

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From Eq. (93)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 -d_{11}A_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ 2Rd_{16}A_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2 d_{66}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- d_{16}B_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ R(d_{12}+d_{66})B_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2 d_{26}B_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ks^2 a_{55}A_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ksa_{55}C_{mn}m\pi\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ (\bar{\omega}^2/12)A_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \quad d\xi \ d\eta = 0 \qquad (96)$$

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From Eq. (94)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \int_{0}^{1} -d_{16}A_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ R(d_{12}+d_{66})A_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2 d_{26}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- d_{66}B_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ 2Rd_{26}B_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2 d_{22}B_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ks^2 a_{44}B_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- kRsa_{44}C_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$+ (\bar{\omega}^2/12)B_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \quad d\xi\, d\eta = 0 \qquad (97)$$

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From Eq. (95)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{\theta}^{1}\int_{\theta}^{1} ksa_{55}A_{mn}m\pi\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ka_{55}C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ kRsa_{44}B_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- kR^2a_{44}C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+k_2\lambda_1(R^2/s^2)C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ 2k_3(R^3/s^2)\lambda_1 C_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$+ k_1(R^4/s^2)\lambda_1 C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ (R^4/s^2)\lambda_2 C_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)\ d\xi d\eta\ =\ \theta \qquad (98)$$

We can now integrate Eqs. (96), (97), and (98) using the same integrals from the
previous section. Thus, upon integrating , simplifying, and taking only a finite
number of terms we have

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From Eq. (96)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [\pi^2(\tfrac{1}{2} \text{ or } \theta)(\tfrac{1}{2} \text{ or } \theta)(-m^2 d_{11} - n^2 R^2 d_{66} - ks^2 a_{55}/\pi^2 + \bar{\omega}^2/12\pi^2)$$

$$- (\theta \text{ or } \theta \text{ even}, 2p \text{ odd})(\theta \text{ or } \theta \text{ even}, 2q \text{ odd})$$

$$* (2Rmnd_{16})/((p^2-m^2)(q^2-n^2))] \ A_{mn}$$

$$+ [\pi^2(\tfrac{1}{2} \text{ or } \theta)(\tfrac{1}{2} \text{ or } \theta)(-m^2 d_{16} - n^2 R^2 d_{26})$$

$$+ (\theta \text{ or } \theta \text{ even}, 2p \text{ odd})(\theta \text{ or } \theta \text{ even}, 2q \text{ odd})$$

$$* (mnR(d_{12}+d_{66}))/((p^2-m^2)(q^2-n^2))] \ B_{mn}$$

$$- [\pi(\theta \text{ or } \theta \text{ even}, 2p \text{ odd})(\tfrac{1}{2} \text{ or } \theta)(mksa_{55})/(p^2-m^2)] \ C_{mn} \ = \ \theta \qquad (99)$$


From Eq. (97)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [\pi^2(\tfrac{1}{2} \text{ or } \theta)(\tfrac{1}{2} \text{ or } \theta)(-m^2 d_{16} - n^2 R^2 d_{26})$$

$$- (\theta \text{ or } \theta \text{ even}, 2p \text{ odd})(\theta \text{ or } \theta \text{ even}, 2q \text{ odd})$$

$$* (mnR(d_{12}+d_{66}))/((p^2-m^2)(q^2-n^2))] \ A_{mn}$$

$$+ [\pi^2(\tfrac{1}{2} \text{ or } \theta)(\tfrac{1}{2} \text{ or } \theta)(-m^2 d_{66} - n^2 R^2 d_{22} - (ks^2 a_{44})/\pi^2 + \bar{\omega}^2/12\pi^2)$$

$$+ (\theta \text{ or } \theta \text{ even}, 2p \text{ odd})(\theta \text{ or } \theta \text{ even}, 2q \text{ odd})$$

$$* (2Rmnd_{26})/((p^2-m^2)(q^2-n^2))] \ B_{mn}$$

$$- [\pi(\tfrac{1}{2} \text{ or } \theta)(\theta \text{ or } \theta \text{ even}, 2q \text{ odd})(nRksa_{44})/(q^2-n^2)] \ C_{mn} \ = \ \theta \qquad (100)$$


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And from Eq. (98)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \ -\pi(\theta \text{ or } \theta \text{ even},2p \text{ odd})(\%\text{ or }\theta)(mksa_{55})/(p^2-m^2) \ A_{mn}$$

$$+ \ \pi(\%\text{ or }\theta)(\theta\text{ or }\theta\text{ even},2q\text{ odd})(nRksa_{44})/(q^2-n^2) \ B_{mn}$$

$$+ \ [\pi^2 \ (\%\text{ or }\theta)(\%\text{ or }\theta)(-m^2ka_{55}-n^2R^2ka_{44}$$

$$+ \ k_2\lambda_1 m^2(R^2/s^2)+k_1\lambda_1 n^2(R^4/s^2+\lambda_2(R^4/\pi^2 s^2))$$

$$+ \ (\theta\text{ or }\theta\text{ even},2p\text{ odd})(\theta\text{ or }\theta\text{ even},2q\text{ odd})$$

$$* \ (2k_3 mnR^3\lambda_1)/(s^2(p^2-m^2)(q^2-n^2))] \ C_{mn} \ = \ \theta \qquad (101)$$

Equations (99), (100), and (101) are now ready to be programmed to generate
the Galerkin equations as before.

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## Clamped - Simply-Supported Boundary Condition

For a plate clamped on two opposite sides and simply-supported on two opposite sides, one has the following boundary conditions.

at $x = 0, a$

$$w = \Psi_x = \Psi_y = 0 \tag{102}$$

and

at $y = 0, b$

$$w = \Psi_x = 0 \tag{103}$$

$$M_y = D_{12} \Psi_{x,x} + D_{22} \Psi_{y,y} + D_{26} (\Psi_{y,x} + \Psi_{x,y}) = 0$$

Therefore, we shall choose for our admissible functions

$$\bar{\Psi}_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

$$\bar{\Psi}_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \tag{104}$$

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

or in terms of normalized coordinates

$$\bar{\Psi}_\xi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\bar{\Psi}_\eta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi\xi) \cos(n\pi\eta) \tag{105}$$

$$\bar{W} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$

To substitute into Eqs. (56), (59), and (63) we need the following derivatives (note: all $\Sigma\Sigma$ are over m=1 to $\infty$ and n=1 to $\infty$).

$$\bar{v}_{\xi,\xi} = \Sigma\Sigma \quad A_{mn} \ m\pi \cos(m\pi\xi)\sin(n\pi\eta)$$

$$\bar{v}_{\xi,\xi\xi} = \Sigma\Sigma \ -A_{mn} \ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$\bar{v}_{\xi,\xi\eta} = \Sigma\Sigma \quad A_{mn} \ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{v}_{\xi,\eta} = \Sigma\Sigma \quad A_{mn} \ n\pi \sin(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{v}_{\xi,\eta\eta} = \Sigma\Sigma \ -A_{mn} \ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$\bar{v}_{\eta,\xi} = \Sigma\Sigma \quad B_{mn} \ m\pi \cos(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{v}_{\eta,\xi\xi} = \Sigma\Sigma \ -B_{mn} \ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{v}_{\eta,\xi\eta} = \Sigma\Sigma \ -B_{mn} \ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) \qquad (106)$$

$$\bar{v}_{\eta,\eta} = \Sigma\Sigma \ -B_{mn} \ n\pi \sin(m\pi\xi)\sin(n\pi\eta)$$

$$\bar{v}_{\eta,\eta\eta} = \Sigma\Sigma \ -B_{mn} \ n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{w}_{,\xi} = \Sigma\Sigma \quad C_{mn} \ m\pi \cos(m\pi\xi)\sin(n\pi\eta)$$

$$\bar{w}_{,\xi\xi} = \Sigma\Sigma \ -C_{mn} \ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$\bar{w}_{,\xi\eta} = \Sigma\Sigma \quad C_{mn} \ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{w}_{,\eta} = \Sigma\Sigma \quad C_{mn} \ n\pi \sin(m\pi\xi)\cos(n\pi\eta)$$

$$\bar{w}_{,\eta\eta} = \Sigma\Sigma \ -C_{mn} \ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

Now substituting into Eq. (56) we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 (-d_{11}A_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) + 2Rd_{16}A_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)$$

$$- R^2 d_{66}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) - d_{16}B_{mn}m^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- R(d_{12}+d_{66})B_{mn}mn\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - R^2 d_{26}B_{mn}n^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- ks^2 a_{55}A_{mn}\sin(m\pi\xi)\sin(n\pi\eta) - ksa_{55}C_{mn}m\pi\cos(m\pi\xi)\sin(n\pi\eta)$$

$$+ (\bar{\omega}^2/12)A_{mn}\sin(m\pi\xi)\sin(n\pi\eta)) \ (\sin(p\pi\xi)\sin(q\pi\eta)) \ d\xi \ d\eta$$

$$+ \int_0^1 \{ \ d_{11}A_{mn}m\pi\cos(0)\sin(n\pi\eta) - Rd_{12}B_{mn}n\pi\sin(0)\sin(n\pi\eta)$$

$$+ d_{16}[RA_{mn}n\pi\sin(0)\cos(n\pi\eta) + B_{mn}m\pi\cos(0)\cos(n\pi\eta)]\} \ (\sin(0)\sin(q\pi\eta))$$

$$- \{ \ d_{11}A_{mn}m\pi\cos(m\pi)\sin(n\pi\eta) - Rd_{12}B_{mn}n\pi\sin(m\pi)\sin(n\pi\eta)$$

$$+ d_{16}[RA_{mn}n\pi\sin(m\pi)\cos(n\pi\eta) + B_{mn}m\pi\cos(m\pi)\cos(n\pi\eta)]\}$$

$$* \ (\sin(p\pi)\sin(q\pi\eta)) \ d\eta$$

$$+ \int_0^1 \{ \ d_{16}A_{mn}m\pi\cos(m\pi\xi)\sin(0) - Rd_{26}B_{mn}n\pi\sin(m\pi\xi)\sin(0)$$

$$+ d_{66}[RA_{mn}n\pi\sin(m\pi\xi)\cos(0) + B_{mn}m\pi\cos(m\pi\xi)\cos(0)]\} \ (\sin(p\pi\xi)\sin(0))$$

$$- \{ \ d_{16}A_{mn}m\pi\cos(m\pi\xi)\sin(n\pi) - Rd_{26}B_{mn}n\pi\sin(m\pi\xi)\sin(n\pi)$$

$$+ d_{66}[RA_{mn}n\pi\sin(m\pi\xi)\cos(n\pi) + B_{mn}m\pi\cos(m\pi\xi)\cos(n\pi)]\}$$

$$* \ (\sin(p\pi\xi)\sin(q\pi)) \ d\xi = 0 \tag{107}$$

Substituting into Eq. (59) we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\int_0^1 (-d_{16}A_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)+R(d_{12}+d_{66})A_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)$$

$$- R^2d_{26}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) - d_{66}B_{mn}m^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- 2Rd_{26}B_{mn}mn\pi^2\cos(m\pi\xi)\sin(n\pi\eta) - R^2d_{22}B_{mn}n^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$$

$$- ks^2a_{44}B_{mn}\sin(m\pi\xi)\cos(n\pi\eta) - kRsa_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(n\pi\eta)$$

$$+ (\bar{v}^2/12)B_{mn}\sin(m\pi\xi)\cos(n\pi\eta)) (\sin(p\pi\xi)\cos(q\pi\eta)) \, d\xi \, d\eta$$

$$+ \int_0^1 ( d_{16}A_{mn}m\pi\cos(0)\sin(n\pi\eta) - Rd_{26}B_{mn}n\pi\sin(0)\sin(n\pi\eta)$$

$$+ d_{66}[RA_{mn}n\pi\sin(0)\cos(n\pi\eta) + B_{mn}m\pi\cos(0)\cos(n\pi\eta)]) (\sin(0)\cos(q\pi\eta))$$

$$- ( d_{16}A_{mn}m\pi\cos(m\pi)\sin(n\pi\eta) - Rd_{26}B_{mn}n\pi\sin(m\pi)\sin(n\pi\eta)$$

$$+ d_{66}[RA_{mn}n\pi\sin(m\pi)\cos(n\pi\eta) + B_{mn}m\pi\cos(m\pi)\cos(n\pi\eta)])$$

$$* (\sin(p\pi)\cos(q\pi\eta)) \, d\eta$$

$$+ \int_0^1 ( d_{12}A_{mn}m\pi\cos(m\pi\xi)\sin(0) - Rd_{22}B_{mn}n\pi\sin(m\pi\xi)\sin(0)$$

$$+ d_{26}[RA_{mn}n\pi\sin(m\pi\xi)\cos(0) + B_{mn}m\pi\cos(m\pi\xi)\cos(0)]) (\sin(p\pi\xi)\cos(0))$$

$$- ( d_{12}A_{mn}m\pi\cos(m\pi\xi)\sin(n\pi) - Rd_{22}B_{mn}n\pi\sin(m\pi\xi)\sin(n\pi)$$

$$+ d_{26}[RA_{mn}n\pi\sin(m\pi\xi)\cos(n\pi) + B_{mn}m\pi\cos(m\pi\xi)\cos(n\pi)])$$

$$* (\sin(p\pi\xi)\cos(q\pi)) \, d\xi = 0 \qquad\qquad (108)$$

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And finally substituting into Eq. (63) we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{0}^{1}\int_{0}^{1}(\; ksa_{55}A_{mn}m\pi\cos(m\pi\xi)\sin(n\pi\eta) - ka_{55}C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$- kRsa_{44}B_{mn}n\pi\sin(m\pi\xi)\sin(n\pi\eta) - kR^2a_{44}C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ k_2\lambda_1(R^2/s^2)C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ 2k_3(R^3/s^2)\lambda_1 C_{mn}mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta)$$

$$+ k_1(R^4/s^2)\lambda_1 C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ (R^4/s^2)\lambda_2 C_{mn}\sin(m\pi\xi)\sin(n\pi\eta))\;(\sin(p\pi\xi)\sin(q\pi\eta))\; d\xi\; d\eta$$

$$+ \int_{0}^{1}(ksa_{55}A_{mn}\sin(0)\sin(n\pi\eta) + ka_{55}C_{mn}m\pi\cos(0)\sin(n\pi\eta))\;(\sin(0)\sin(q\pi\eta))$$

$$- (ksa_{55}A_{mn}\sin(m\pi)\sin(n\pi\eta) + ka_{55}C_{mn}m\pi\cos(m\pi)\sin(n\pi\eta))$$

$$* (\sin(p\pi)\sin(q\pi\eta))\; d\eta$$

$$+ \int_{0}^{1}(ksa_{44}B_{mn}\sin(m\pi\xi)\cos(0) + Rka_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(0))(\sin(p\pi\xi)\sin(0))$$

$$- (ksa_{44}B_{mn}\sin(m\pi\xi)\cos(n\pi) + Rka_{44}C_{mn}n\pi\sin(m\pi\xi)\cos(n\pi))$$

$$* (\sin(p\pi\xi)\sin(q\pi))\; d\xi = 0 \tag{109}$$

Again, performing the indicated multiplications and noting that $\sin(0) = \sin(p\pi) = \sin(q\pi) = 0$ (p and q are integers) and $\cos(0) = 1$ we have

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From Eq. (107)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \int_{0}^{1} -d_{11}A_{mn}m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ 2Rd_{16}A_{mn}mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2 d_{66}A_{mn}n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- d_{16}B_{mn}m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- R(d_{12}+d_{66})B_{mn}mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- R^2 d_{26}B_{mn}n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$- ks^2 a_{55}A_{mn} \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ksa_{55}C_{mn}m\pi \cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ (\bar{\omega}^2/12)A_{mn} \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \quad d\xi \, d\eta = 0 \qquad (118)$$

From Eq. (108)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \int_{0}^{1} -d_{16}A_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\cos(q\pi\eta)$$

$$+ R(d_{12}+d_{66})A_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- R^2d_{26}A_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\cos(q\pi\eta)$$

$$- d_{66}B_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- 2Rd_{26}B_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\cos(q\pi\eta)$$

$$- R^2d_{22}B_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- ks^2a_{44}B_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$- kRsa_{44}C_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)$$

$$+ (\bar{\omega}^2/12)B_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta) \quad d\xi \ d\eta$$

$$+ \int_{0}^{1} d_{26}[RA_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi) + B_{mn}m\pi\cos(m\pi\xi)\sin(p\pi\xi)]$$

$$- d_{26}[RA_{mn}n\pi\cos(n\pi)\cos(q\pi)\sin(m\pi\xi)\sin(p\pi\xi)$$

$$+ B_{mn}m\pi\cos(n\pi)\cos(q\pi)\cos(m\pi\xi)\sin(p\pi\xi)] \quad d\xi \ = \ 0 \qquad (111)$$

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From Eq. (109)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \int_{0}^{1} ksa_{55}A_{mn}m\pi\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- ka_{55}C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- kRsa_{44}B_{mn}n\pi\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$- kR^2a_{44}C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ k_2\lambda_1(R^2/s^2)C_{mn}m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ 2k_3(R^3/s^2)\lambda_1 C_{mn}mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$

$$+ k_1(R^4/s^2)\lambda_1 C_{mn}n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$

$$+ (R^4/s^2)\lambda_2 C_{mn}\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \; d\xi d\eta = 0 \qquad (112)$$

We can now integrate Eqs. (110), (111), and (112) using the same integrals from the previous sections. Thus upon integrating, simplifying, and taking only a finite number of terms we have

From Eq. (110)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [\pi^2 (\text{½ or } \theta)(\text{½ or } \theta)(-m^2 d_{11} - n^2 R^2 d_{66} - ks^2 a_{55}/\pi^2 + \bar{\omega}^2/12\pi^2)$$

$$+ (\theta \text{ or } \theta \text{ even},2p \text{ odd})(\theta \text{ or } \theta \text{ even},2q \text{ odd})$$

$$* (2Rmnd_{16})/((p^2-m^2)(q^2-n^2))] A_{mn}$$

$$- [\pi(\theta \text{ or } \theta \text{ even},2p \text{ odd})(\text{½ or } \theta)(mnR(d_{12} + d_{66}))/(p^2-m^2)$$

$$- \pi(\text{½ or } \theta)(\theta \text{ or } \theta \text{ even},2q \text{ odd})(m^2 d_{16}+n^2 R^2 d_{26})/(q^2-n^2)] B_{mn}$$

$$- [(\theta \text{ or } \theta \text{ even},2p \text{ odd})(\text{½ or } \theta)(mksa_{55})/(p^2-m^2)] C_{mn} = \theta \qquad (113)$$

From Eq. (111)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [\pi(\theta \text{ or } \theta \text{ even},2p \text{ odd})(\text{½ or } \theta)(mnR(d_{12} + d_{66}))/(p^2-m^2)$$

$$- \pi(\text{½ or } \theta)(\theta \text{ or } \theta \text{ even},2n \text{ odd})(m^2 d_{16}+n^2 R^2 d_{26})/(n^2-q^2)$$

$$+ \pi(\text{½ or } \theta)(nRd_{26}(1-\cos(n\pi)\cos(q\pi)))] A_{mn}$$

$$+ [\pi^2 (\text{½ or } \theta)(\text{½ or } \theta)(-m^2 d_{66} - n^2 R^2 d_{22} - (ks^2 a_{44})/\pi^2 + \bar{\omega}^2/12\pi^2)$$

$$- (\theta \text{ or } \theta \text{ even},2p \text{ odd})(\theta \text{ or } \theta \text{ even},2n \text{ odd})(2Rmnd_{26})/((p^2-m^2)(n^2-q^2))$$

$$+ (\theta \text{ or } \theta \text{ even},2p \text{ odd})(md_{26}(1-\cos(n\pi)\cos(q\pi)))/(p^2-m^2)] B_{mn}$$

$$- [\pi(\text{½ or } \theta)(\text{½ or } \theta)(nRksa_{44})] C_{mn} = \theta \qquad (114)$$

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And from Eq. (112)

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \quad (\theta \text{ or } \theta \text{ even},2p \text{ odd})(\text{\% or } \theta)(mksa_{55})/(p^2-m^2) \ A_{mn}$$

$$- \text{\%}(\text{\% or } \theta)(\text{\% or } \theta)(nRksa_{44}) \ B_{mn}$$

$$+ [\pi^2 \ (\text{\% or } \theta)(\text{\% or } \theta)(-m^2 ka_{55}-n^2 R^2 ka_{44}$$

$$+ k_2\lambda_1 m^2(R^2/s^2)+k_1\lambda_1 n^2(R^4/s^2+\lambda_2(R^4/\pi^2 s^2))$$

$$+ (\theta \text{ or } \theta \text{ even},2p \text{ odd})(\theta \text{ or } \theta \text{ even},2q \text{ odd})$$

$$* \ (2k_3 mnR^3\lambda_1)/(s^2(p^2-m^2)(q^2-n^2))] \ C_{mn} \ = \ \theta \qquad (115)$$

Equations (113), (114), and (115) are now ready to be programmed to generate the Galerkin equations as before.

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# III. Discussion and Results

This chapter will describe the computer programs used to solve the eigenvalue problem formulated in the last chapter. It will also discuss the two types of laminated plates used in the programs and the subsequent analysis performed with those plates. We will begin by describing the computer algorithms.

## Computer Algorithms

Four computer programs were written to set up and solve for the approximate natural frequencies and mode shapes for the three boundary conditions considered. The first program determines the nondimensional bending and extensional stiffness elements for a symmetric laminate. The second program formulates the eigenvalue problem $[A]x = \lambda [B]x$ where $[A]$ is the stiffness matrix and $[B]$ is the combined mass and inertia matrix. The third program solves the eigenvalue problem. The last program generates the data for the mode shape plots. A description of each program follows.

Program One determines the nondimensional stiffness elements and the density of the plate. It is divided into three basic sections and allows for a laminate to be built up from different materials and orientations. The three basic sections are 1) the input, 2) the computation of the dimensional bending and extensional stiffnesses, and 3) the computation of the nondimensional stiffnesses. The input section gathers the following data:

1) Orientation angle, $\theta$

2) $E_1$

3) $E_2$

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4) $\nu_{12}$

5) $G_{12}$

Poisson's ratio, $\nu_{21}$, is calculated from $\nu_{12}$ by the equation $\nu_{21} = \nu_{12} \times (E_2/E_1)$. The shear moduli terms, $G_{23}$ and $G_{13}$, are computed from $G_{12}$. $G_{13}$ is set equal to $G_{12}$ and $G_{23}$ is set equal to 80% of $G_{12}$.

The second section of this program is the longest. It calculates the dimensional bending and extensional stiffnesses for a lamina and then loops back to the beginning for more input data. If the material properties are the same, it will only ask for the new orientation angle. It will continue to sum these stiffnesses together until the laminate lay-up is completed. The stiffness elements are determined by computing the reduced stiffness terms ($Q_{ij}$'s) from Eq. (14) and then computing the transformed reduced stiffness terms ($\bar{Q}_{ij}$'s) using Eq. (13). The extensional and bending stiffnesses are then determined using Eqs. (28) and (30). After the laminate buildup is complete, the program moves on to the third section.

The third section takes the dimensional stiffnesses and computes the nondimensional stiffnesses using Eq. (53) from the previous chapter. These stiffnesses are then printed. A listing of this program may be found in Appendix A.

The second program computes the mass and stiffness matrices for the eigenvalue problem. Like the first, it too is divided into three sections. The first section contains the plate data generated from Program One. The second section builds the stiffness and mass matrices and the third section outputs these matrices to a file.

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Section one is the input section. It contains all of the required material properties, geometrical plate data, and integers which determine the number of terms in Galerkin's equations and the number of Galerkin equations.

Section two generates the stiffness and mass matrices. It takes the algebraic equations (Eqs. 85-87 for the first boundary condition, Eqs. 99-101 for the second boundary condition, and Eqs. 113-115 for the third boundary condition), determines the value of the integrated sine and cosine terms, and then builds the family of Galerkin equations. The stiffness and mass (or inertia) terms are separated and the matrices are constructed. It should be noted here that the buckling terms (those multiplied by $k_1$, $k_2$, and $k_3$) were not used in this analysis and will not appear in the computer listing.

Section three outputs the two matrices to a file. A listing of this program can be found in Appendix B.

The third program solves the eigenvalue problem. It is also divided into the input, process, output format. The input section reads in the stiffness and mass matrices generated from Program Two. The process section solves the eigenvalue problem by calling the EIGZS routine from the IMSL library [7]. The output section writes the eigenvalues and the eigenvectors to a file. A listing of this program may be found in Appendix C.

The fourth program generates the data base which is used to produce a contour plot of the mode shape. It can do this as a result of solving the eigenvalue problem. Our admissible functions for each boundary condition were made up of an infinite series of sine functions (each term the product of a sine function of x and a sine function of y) multiplied by an infinite number of undetermined coefficients (the $C_{mn}$'s). When we solved the eigenvalue problem, the

bottom third of our eigenvector was the solution for the $C_{mn}$'s. Therefore to generate the displacement w, as a function of x and y we only need this portion of the eigenvector multiplied by its appropriate sine functions (see Eqs. (67), (91), and (105)). That's exactly what the last program does. It reads in the eigenvector for the desired mode, strips off the last third and then normalizes that portion of it. The program then generates a value for w as a function of x and y over the domain of the plate. This datafile is then used by the SUPERPROC file on the CYBER (developed by Captain Hinrichsen and appended to by Major Hodge) to plot the contours of the mode shape. A listing of the last program may be found in Appendix D.

## Analysis Performed

Several different characteristics of both the Galerkin Method and shear deformation and rotatory inertia effects were investigated. To explore the Galerkin method, convergence characteristics and comparison to closed-form solutions were researched. To study the effects of shear deformation and rotatory inertia, several cases varying the length to thickness ratio were investigated. We will begin our discussion by describing the type of plates used in the subsequent analysis.

## Laminated Plate Properties

Two different types of laminated plates were used in this thesis. Both consisted of a graphite-epoxy material (AS/3501) with the following properties:

$$E_1 = 21.0E+06 \text{ psi.}$$

$$E_2 = 1.40E+06 \text{ psi.}$$

$$\nu_{12} = 0.3 \qquad\qquad (116)$$

$$G_{12} = 0.60E+06 \text{ psi.}$$

$$\rho = 0.055 \text{ lb./in.}^3$$

where $\rho$ is the mass density and the other material properties have been previously defined. One plate had a ply layup of $[0/90]_{2s}$ and the second had a layup of $[\pm45]_{2s}$. The thickness of the plates was held constant at one inch but the length and width dimensions varied (between 5 and 200 inches) depending on the type of analysis being performed. Tables 3.1 and 3.2 contain the stiffness elements computed from Program One for these two plate layups.

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Graphite-epoxy $[\theta/90]_{2s}$

One inch thick

| Element | Dimensional Value | Nondimensional Value |
|---------|-------------------|----------------------|
| $A_{44}$ | 540,000 | 0.385714 |
| $A_{45}$ | 0 | 0 |
| $A_{55}$ | 540,000 | 0.385714 |
| $D_{11}$ | 1,555,160 | 1.11083 |
| $D_{12}$ | 35,211.3 | 0.0251509 |
| $D_{16}$ | 0 | 0 |
| $D_{22}$ | 322,770 | 0.23055 |
| $D_{26}$ | 0 | 0 |
| $D_{66}$ | 50,000 | 0.0357143 |

Units for the dimensional $A_{ij}$ terms are lb./in.

Units for the dimensional $D_{ij}$ terms are in.·lbs.

Table 3.1  Stiffness Elements for Plate 1

Graphite-epoxy $[\pm 45]_{2s}$

One inch thick

| Element | Dimensional Value | Nondimensional Value |
|---|---|---|
| $A_{44}$ | 540,000 | 0.385714 |
| $A_{45}$ | 0 | 0 |
| $A_{55}$ | 540,000 | 0.385714 |
| $D_{11}$ | 537,089 | 0.383635 |
| $D_{12}$ | 437,089 | 0.312207 |
| $D_{16}$ | 308,099 | 0.22007 |
| $D_{22}$ | 537,089 | 0.383635 |
| $D_{26}$ | 308,099 | 0.22007 |
| $D_{66}$ | 451,878 | 0.32277 |

Units for the dimensional $A_{ij}$ terms are lb./in.

Units for the dimensional $D_{ij}$ terms are in.·lbs.

Table 3.2  Stiffness Elements for Plate 2

## Galerkin Method Characteristics

In using the Galerkin Method for solving our set of three coupled partial differential equations, we have to ask ourselves some questions. First, does the technique converge to a solution and second, how does the solution compare with accepted theory? We will now look at these two questions.

A discussion of the proof of convergence for the Galerkin Method may be found in Reference [9]. A criteria based on the method for convergence is whether the assumed functions form a complete set of functions. We will not attempt to prove convergence for the work completed herein. We will, however, show the necessary (but not sufficient) condition that the frequency drops by smaller and smaller amounts as the values of M and N are increased. Tables 3.3 and 3.4 show the values of normalized frequency $[\omega a^2 (\rho / E_2 h^3)]$ for the three boundary conditions considered for the two plates with increasing values of M and N. The tables are based on a length to thickness ratio (s) of 20. Boundary condition #1 is the simply-supported case. Boundary condition #2 is the clamped case and boundary condition #3 is the clamped (two opposite sides) simply-supported (two opposite sides) case. The author could not compute frequencies greater than M and N = 8 due to computer memory limitations.

As can be seen in Table 3.3, the simply-supported case for the $[0/90]_{2s}$ layup converges from the start. This is to be expected as the boundaries are completely satisfied. All of the other cases have not converged but display characteristics that makes one believe they will converge as M and N get larger. That is, for every increase in M and N, the normalized frequency drops by a smaller and smaller amount.

A plot of the normalized frequency vs M and N for BC #2 for the $[\pm 45]_{2s}$ plate (third mode) is presented in Figure 3.1. This condition tends to have one of

Non-dimensional Natural Frequencies $[0/90]_{2s}$
graphite-epoxy, a/b=1, s=20

| M | N | 1st Mode | 3rd Mode | 5th Mode |
|---|---|---|---|---|
| | | | Boundary Condition #1 | |
| 2 | 2 | 11.758 | 36.866 | ----- |
| 4 | 4 | 11.758 | 36.866 | 42.573 |
| 6 | 6 | 11.758 | 36.866 | 42.573 |
| 8 | 8 | 11.758 | 36.866 | 42.573 |
| | | | Boundary Condition #2 | |
| 2 | 2 | 31.751 | 69.038 | ----- |
| 4 | 4 | 23.734 | 49.403 | 67.560 |
| 6 | 6 | 22.992 | 48.569 | 57.062 |
| 8 | 8 | 22.776 | 48.328 | 55.665 |
| | | | Boundary Condition #3 | |
| 2 | 2 | 24.870 | 65.306 | ----- |
| 4 | 4 | 21.235 | 45.463 | 52.438 |
| 6 | 6 | 20.814 | 45.256 | 51.832 |
| 8 | 8 | 20.697 | 45.225 | 51.664 |

Table 3.3  Normalized Frequencies for Plate 1

Non-dimensional Natural Frequencies $[\pm 45]_{2s}$
graphite-epoxy, a/b=1, s=20

| M | N | 1st Mode | 3rd Mode | 5th Mode |
|---|---|----------|----------|----------|
|   |   |          | Boundary Condition #1 | |
| 2 | 2 | 14.699 | 36.164 | ----- |
| 4 | 4 | 14.418 | 35.444 | 57.883 |
| 6 | 6 | 14.283 | 34.734 | 55.082 |
| 8 | 8 | 14.205 | 34.613 | 54.856 |
|   |   |          | Boundary Condition #2 | |
| 2 | 2 | 31.091 | 69.925 | ----- |
| 4 | 4 | 22.363 | 44.553 | 72.156 |
| 6 | 6 | 21.412 | 42.964 | 64.264 |
| 8 | 8 | 21.110 | 42.449 | 63.095 |
|   |   |          | Boundary Condition #3 | |
| 2 | 2 | 24.300 | 66.437 | ----- |
| 4 | 4 | 18.956 | 41.549 | 59.675 |
| 6 | 6 | 18.337 | 40.438 | 58.116 |
| 8 | 8 | 18.126 | 40.061 | 57.673 |

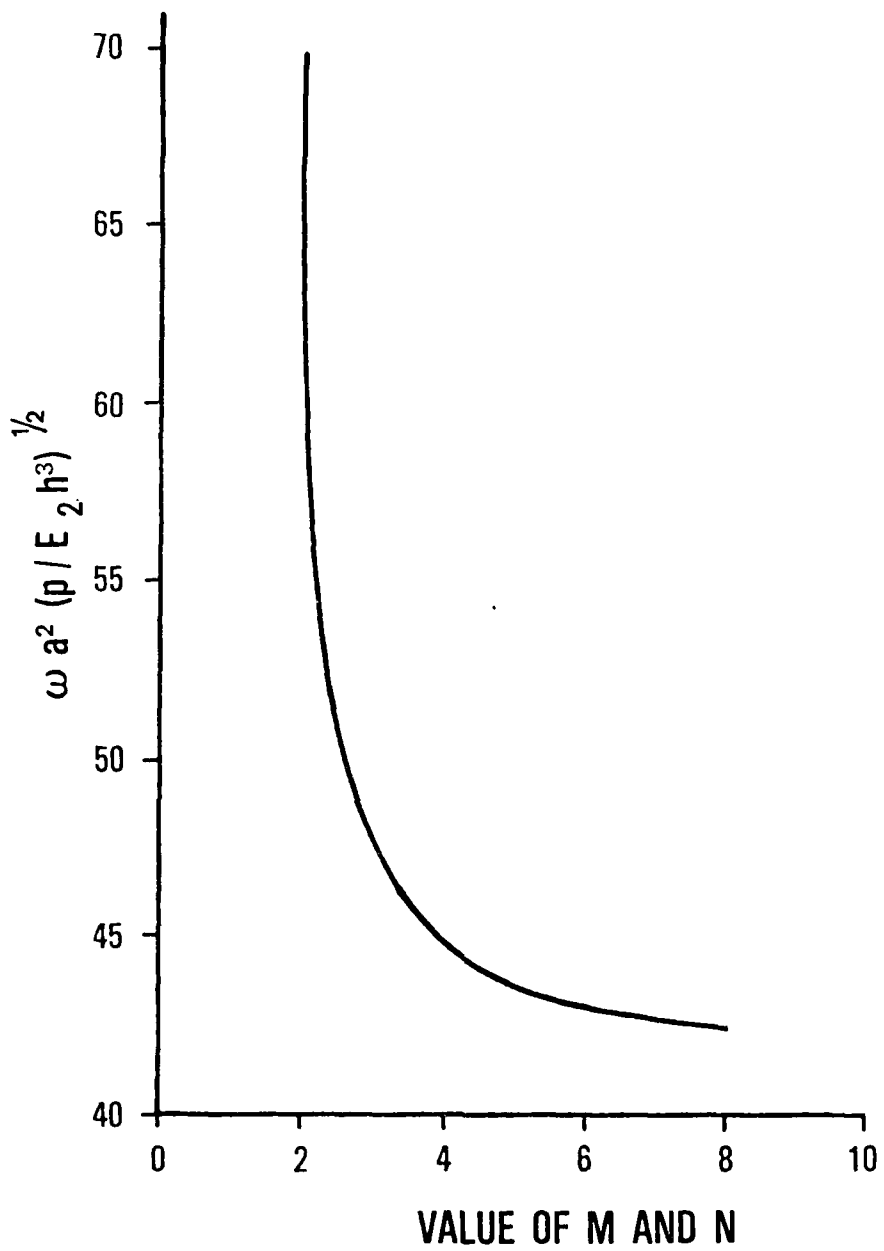Table 3.4  Normalized Frequencies for Plate 2

Figure 3.1  Plot of Normalized Frequency vs M and N  Plate #2

Clamped Boundary    Third Mode

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the "worst" convergence tendencies of the group. However, even it does show very good converging behavior.

These tables are by no means offered as proof of convergence, but they do display good convergence characteristics. They also point out a drawback with this method. As M and N increase, the required computer memory space to generate the mass and stiffness matrices and to solve the eigenvalue problem becomes quite large. Thus, if one wants a very accurate answer, particularly for a higher mode, they will need the appropriate computer resources.

The next question that must be addressed is the "How well do the solutions generated here compare to commonly accepted theory?" We will address that question by investigating a very long, narrow plate. There are several reasons for this. First, because of the nature of the system of equations for our plate, we can zero out the rotatory inertia but we cannot zero out the shear deformation effect. When shear deformation is zeroed, the equations become very ill-conditioned and the problem cannot be solved. Therefore, we must look to commonly accepted theory that includes either shear deformation or shear deformation and rotatory inertia effects. We would also like to find a simply-supported case for an orthotropic plate because our Galerkin Method converges quickly there. Reference [19] did contain a closed-form solution for an infinitely long, simply-supported, orthotropic plate including shear deformation effects, and this is what the author used to validate his program.

From Reference [19] we have

$$\omega_m = \omega'_m [1-(D_{11} m^2 \pi^2)/(D_{11} m^2 \pi^2 + k A_{55} a^2)] \qquad (117)$$

where

$$\omega'_m = (D_{11} m^4 \pi^4 / p a^4)^{\frac{1}{2}} \qquad (118)$$

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Here $\omega'_m$ is the natural frequency calculated from the classical theory based on the Kirchhoff hypothesis. If we use the properties for Plate 1 (Table 3.1) and compute the fundamental frequency for an infinitely long, 10" wide plate we have

$$\omega'_m = 1641.61 \text{ Hz}$$
$$\omega_m = 1417.56 \text{ Hz}$$ (119)

This author cannot place an "infinitely" long plate into his Galerkin algorithm. However, he can use a very long plate such as one 10 inches wide and 200 inches long. From Reference [1], we can compute the natural frequency without shear deformation and rotatory inertia effects from

$$\omega^2 \rho = D_{11}(\alpha_1^4/a^4) + 2(D_{12}+2D_{66})/(\alpha_2/a^2b^2) + D_{22}(\alpha_3^4/b^4) \quad (120)$$

where

$$\alpha_1 = m\pi$$

$$\alpha_2 = n\pi \quad (121)$$

$$\alpha_3 = m^2 n^2 \pi^4$$

for all m and n (m and n determine the mode).

Again using the properties from Table 3.1 for our 10" by 200" plate we have

$$\omega = 1,641.26 \text{ Hz} \quad (122)$$

This compares to 1,641.61 Hz for Whitney's infinitely long plate so we can conclude that our 200" plate reasonably approximates an infinitely long one and we can therefore use Eq. (117) to compare with our Galerkin program output. Table 3.5 shows the comparison between the closed formed solution and the output from the

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Galerkin Method.  As can be seen, excellent agreement was obtained.  Table 3.5 also shows the effect of rotatory inertia for this condition.

The author could not find information to compare with the other two boundary conditions.

| Mode | Closed-Form Solution with SD | Galerkin Method with SD  no RI  M = N = 5 | Galerkin Method with SD and RI |
|------|------------------------------|-------------------------------------------|--------------------------------|
| 1 | 1417.56 | 1417.56 | 1414.32 |
| 2 | 1418.5 | 1418.38 | 1415.09 |
| 3 | 1420.1 | 1419.97 | 1416.60 |
| 4 | 1422.41 | 1422.42 | 1418.96 |
| 5 | 1425.51 | 1425.68 | 1422.00 |

Note:  All frequencies are in Hz.

Table 3.5  Galerkin Method and Closed-Form Solution
Comparison     Plate #1

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## Shear Deformation and Rotatory Inertia Effects

The second area of investigation was to determine the effect of shear deformation and rotatory inertia for the boundary conditions considered. This was accomplished by varying the length to thickness ratio and comparing those results to classical laminated plate theory. Only square plates were considered.

For the $[0/90]_{2s}$ plate, simply-supported, the length to thickness ratio was varied from 5 to 50. The results are plotted in Figures 3.2 and 3.3 as normalized frequency vs thickness ratio for the first and fifth bending modes. It can be seen that shear deformation effects can be significant (up to 33% lower for the first mode and 52% lower for the fifth mode for a length to thickness ratio of 5) while rotatory inertia effects account for only a 2% lower frequency at the worst point. Notice that as the length to thickness ratio approaches 50, we asymptotically approach the classical laminated solution.

These same trends are seen for the $[\pm45]_{2s}$ simply-supported plate also. Figures 3.4 and 3.5 plot the normalized frequency vs length to thickness ratio for the first and fifth modes for this plate. Although the frequencies are slightly higher, the same behavior occurs as before. The shear deformation effect becomes significant for length to thickness ratios of less than 35 and rotatory inertia effects are very small. Table 3.6 presents this data. There were no classical solutions avaliable to compare these results with, but, based on the behavior of the first plate, we could estimate a first mode normalized frequency of about 15 and a fifth mode normalized frequency of about 63.

Tables 3.7 and 3.8 show the comparisons for the second and third boundary conditions for the $[0/90]_{2s}$ plate. They too show the same type of trends as the previous two cases. Shear deformation is a significant effect below length to

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Figure 3.2  Plot of Normalized Frequency vs Thickness Ratio

Plate #1  Simply-Supported Boundary First Mode
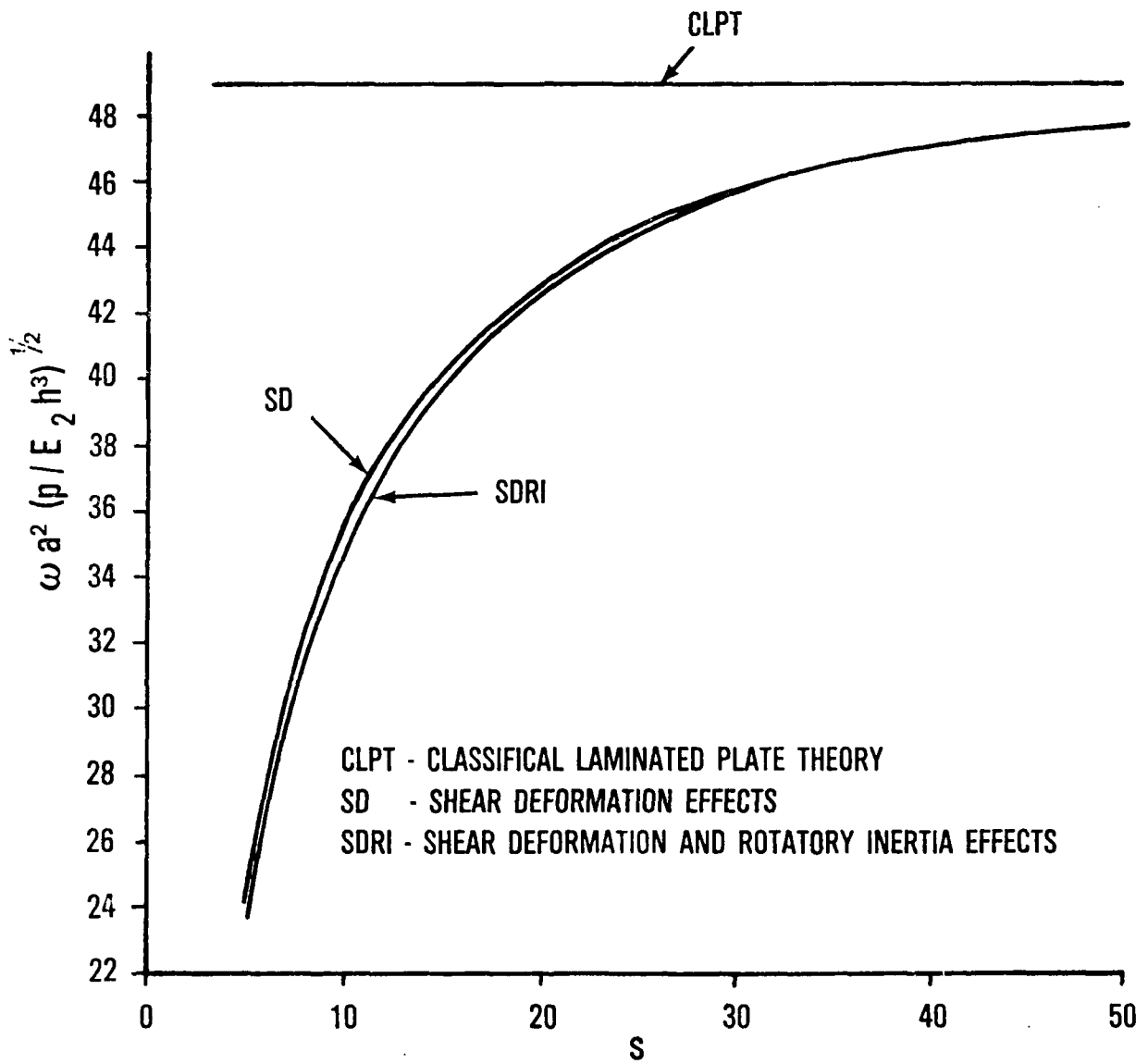
Figure 3.3 Plot of Normalized Frequency vs Thickness Ratio

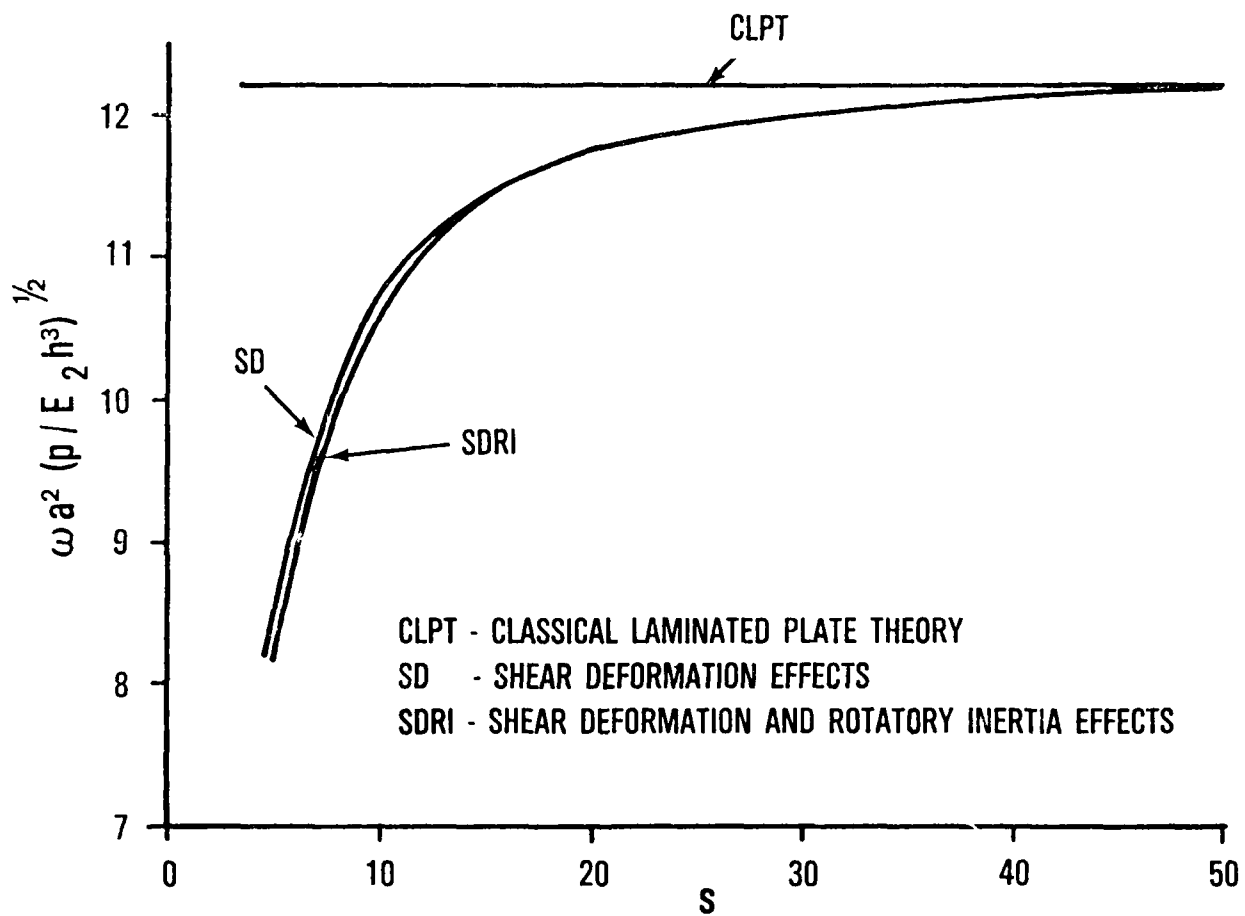Plate #1 Simply-Supported Boundary Fifth Mode
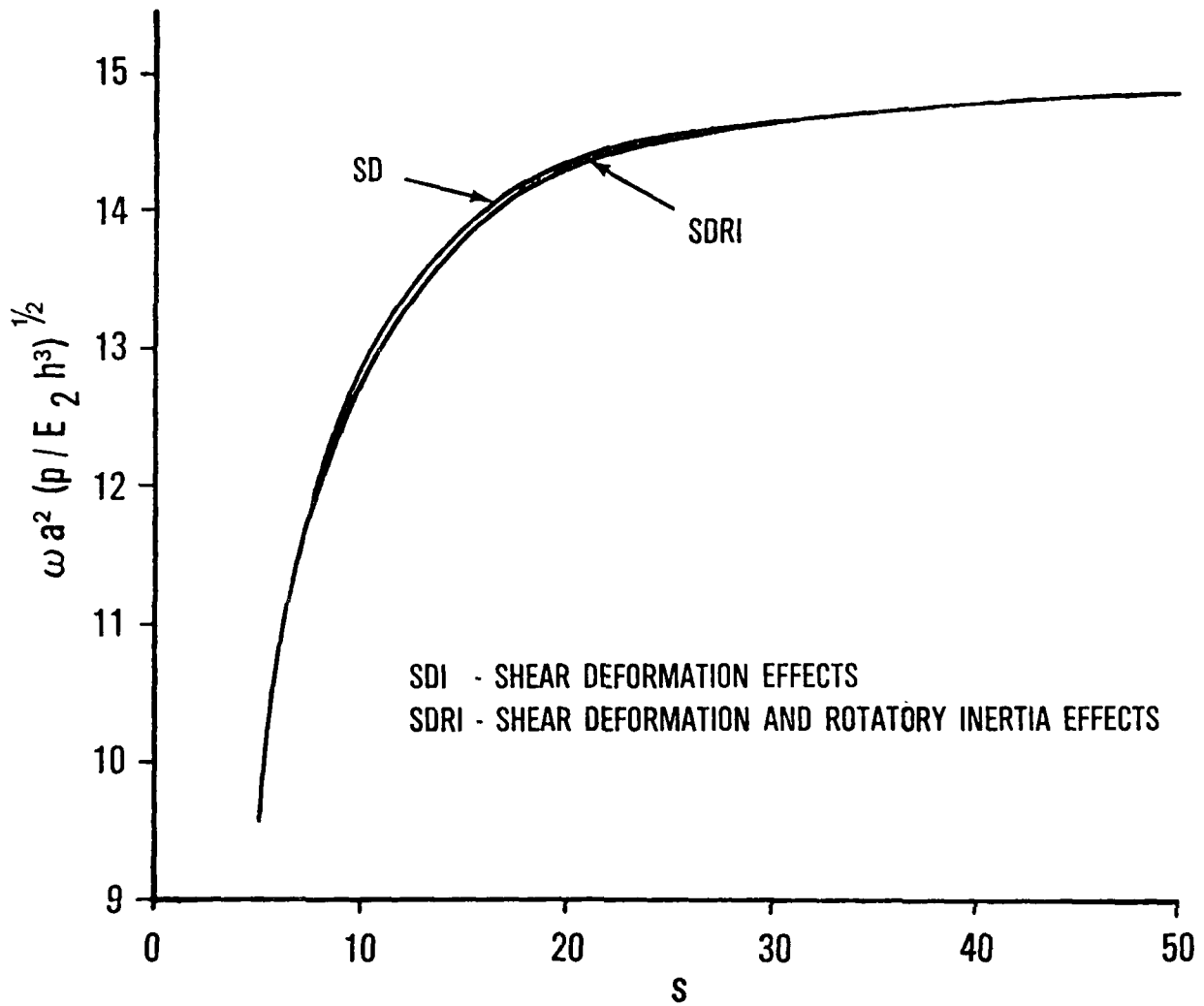
Figure 3.4  Plot of Normalized Frequency vs Thickness Ratio

Plate #2  Simply-Supported Boundary First Mode
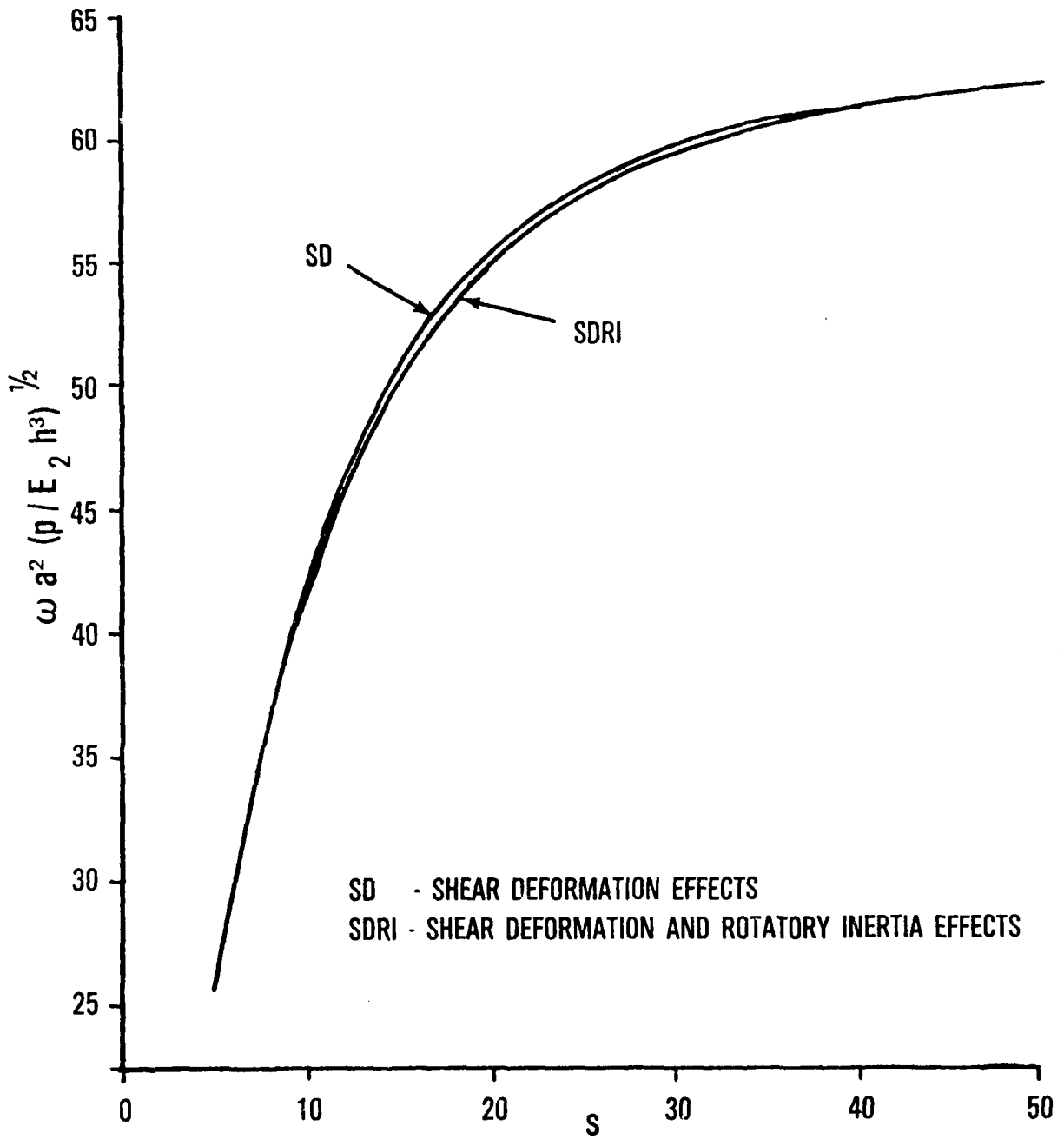
Figure 3.5  Plot of Normalized Frequency vs Thickness Ratio

Plate #2  Simply-Supported Boundary Fifth Mode

Nondimensional Frequency $\omega a^2 (p/E_2 h^3)^{\frac{1}{4}}$

| s | 1st Mode | | 5th Mode | |
|---|---|---|---|---|
| | SD no RI | SD and RI | SD no RI | SD and RI |
| 5 | 9.62 | 9.57 | 25.66 | 25.51 |
| 10 | 12.83 | 12.77 | 42.06 | 41.69 |
| 15 | 13.88 | 13.84 | 50.79 | 50.39 |
| 20 | 14.31 | 14.28 | 55.42 | 55.08 |
| 25 | 14.53 | 14.51 | 58.05 | 57.77 |
| 30 | 14.65 | 14.63 | 59.65 | 59.43 |
| 35 | 14.72 | 14.71 | 60.67 | 60.49 |
| 40 | 14.76 | 14.75 | 61.37 | 61.23 |
| 50 | 14.88 | 14.87 | 62.23 | 62.13 |

Classical Laminated Plate Frequency:   None available

Simply-Supported Boundary Condition   $[\pm 45]_{2s}$   M=N=6

Table 3.6  Shear Deformation and Rotatory Effects for Plate 2

$$\text{Nondimensional Frequency} \quad \omega a^2 (\rho / E_2 h^3)^{\frac{1}{4}}$$

| s | 1st Mode | | 5th Mode | |
|---|---|---|---|---|
| | SD no RI | SD and RI | SD no RI | SD and RI |
| 5 | 10.89 | 10.85 | 25.10 | 24.82 |
| 10 | 17.27 | 17.21 | 41.44 | 41.01 |
| 15 | 20.89 | 20.84 | 51.30 | 50.86 |
| 20 | 23.03 | 23.00 | 57.45 | 57.06 |
| 25 | 24.39 | 24.36 | 61.53 | 61.21 |
| 30 | 25.32 | 25.30 | 65.23 | 65.00 |
| 35 | 26.01 | 25.99 | 68.09 | 67.93 |

Classical Laminated Plate Frequency   1st Mode: 26.47   5th Mode: 74.25

Clamped Boundary Condition   $[0/90]_{2s}$   M=N=6

Table 3.7  Comparison of Shear Deformation and Rotatory
Inertia Effects to Classical Laminated Plate Theory.

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Nondimensional Frequency $\omega a^2 (p/E_2 h^3)^{\frac{1}{4}}$

| s | 1st Mode | | 5th Mode | |
|---|---|---|---|---|
| | SD no RI | SD and RI | SD no RI | SD and RI |
| 5 | 9.50 | 9.41 | 24.26 | 23.96 |
| 10 | 15.25 | 15.20 | 36.80 | 36.28 |
| 15 | 18.76 | 18.72 | 45.12 | 44.86 |
| 20 | 20.85 | 20.81 | 52.09 | 51.83 |
| 25 | 22.16 | 22.13 | 56.97 | 56.77 |
| 30 | 23.05 | 23.03 | 60.48 | 60.31 |
| 35 | 23.70 | 23.68 | 63.08 | 62.94 |

Classical Laminated Plate Frequency   1st Mode: 24.53   5th Mode: 70.26

Clamped Simply-Supported Boundary Condition        $[0/90]_{2s}$      M=N=6

Table 3.8  Comparison of Shear Deformation and Rotatory
Inertia Effects to Classical Laminated Plate Theory.

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thickness ratios of 30 and rotatory inertia has little effect on these bending modes. These were not plotted because a value of M=N=6 was used to generate the data. This was done to conserve time and computer resources. However, the frequencies did not have enough terms to converge and at s=50 they overshot the classical solution by about 3%. For the purpose of showing trends between rotatory inertia and shear deformation the author feels this approach was justified.

END

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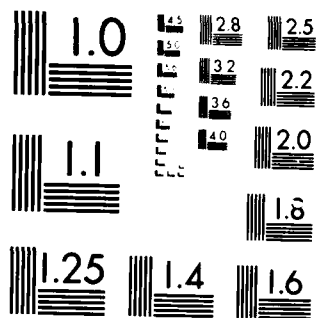
MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

## Mode Shape Determination

As stated in the discussion on the last computer program, when we solve the eigenvalue problem we also get back the values of the undetermined coefficients to determine the mode shape for a particular frequency (this includes the torsional modes also). Figures 3.6 thru 3.11 are contour plots of the first modes for both plates for all three boundary conditions. A length to thickness ratio of 10 was used when generating these plots.
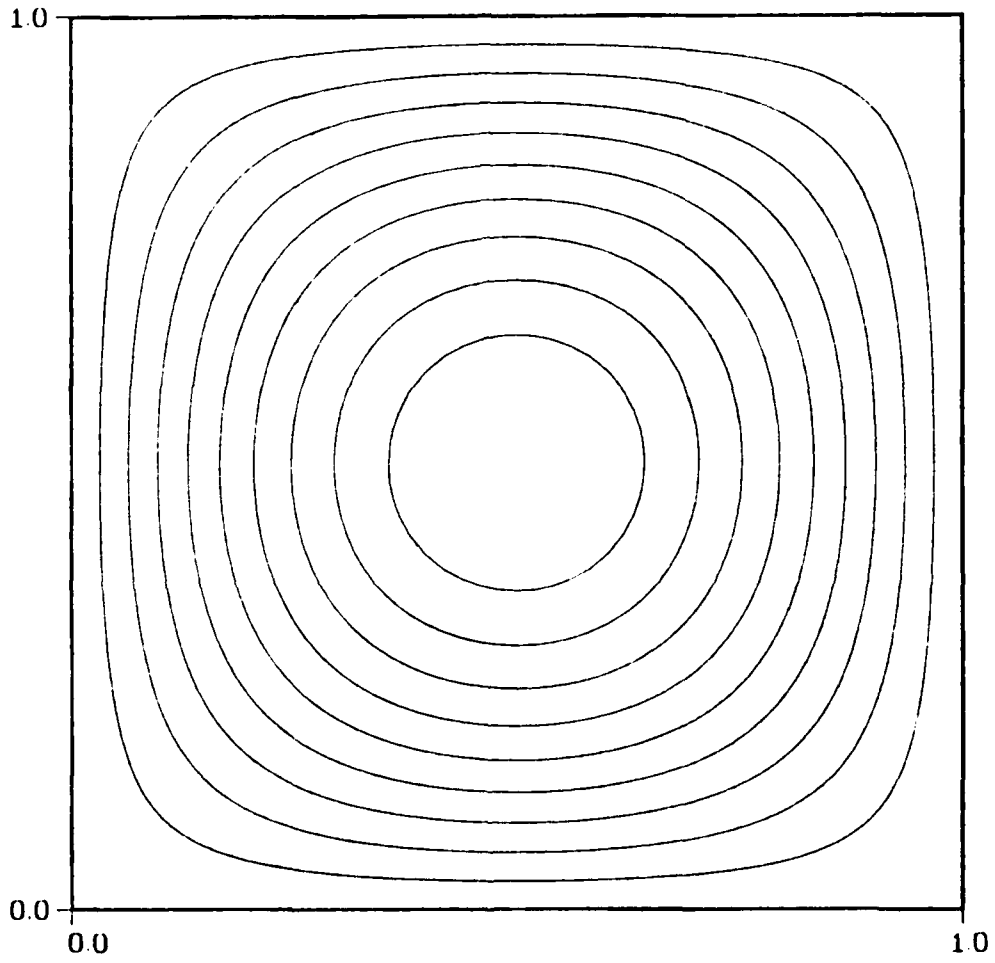
Figure 3.6  Mode Shape Plot  Simply-Supported Boundary
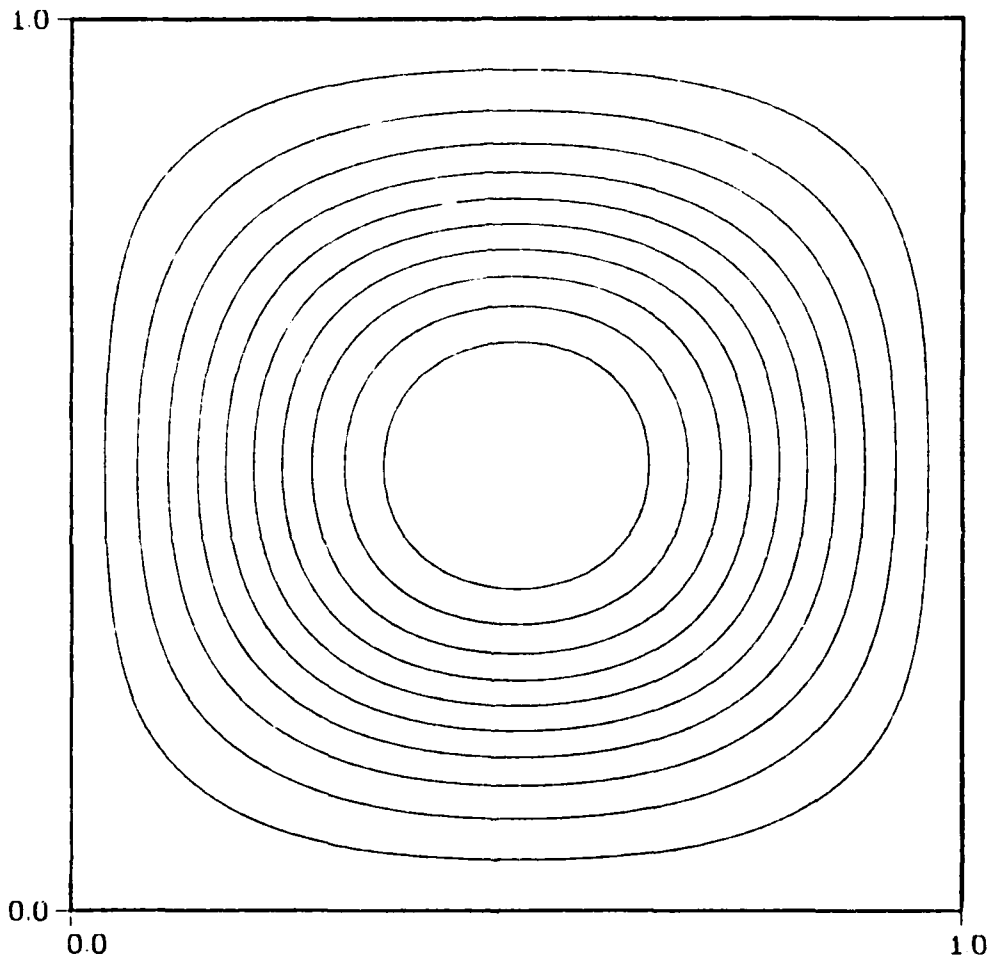
Plate #1  First Mode

Figure 3.7  Mode Shape Plot  Clamped Boundary
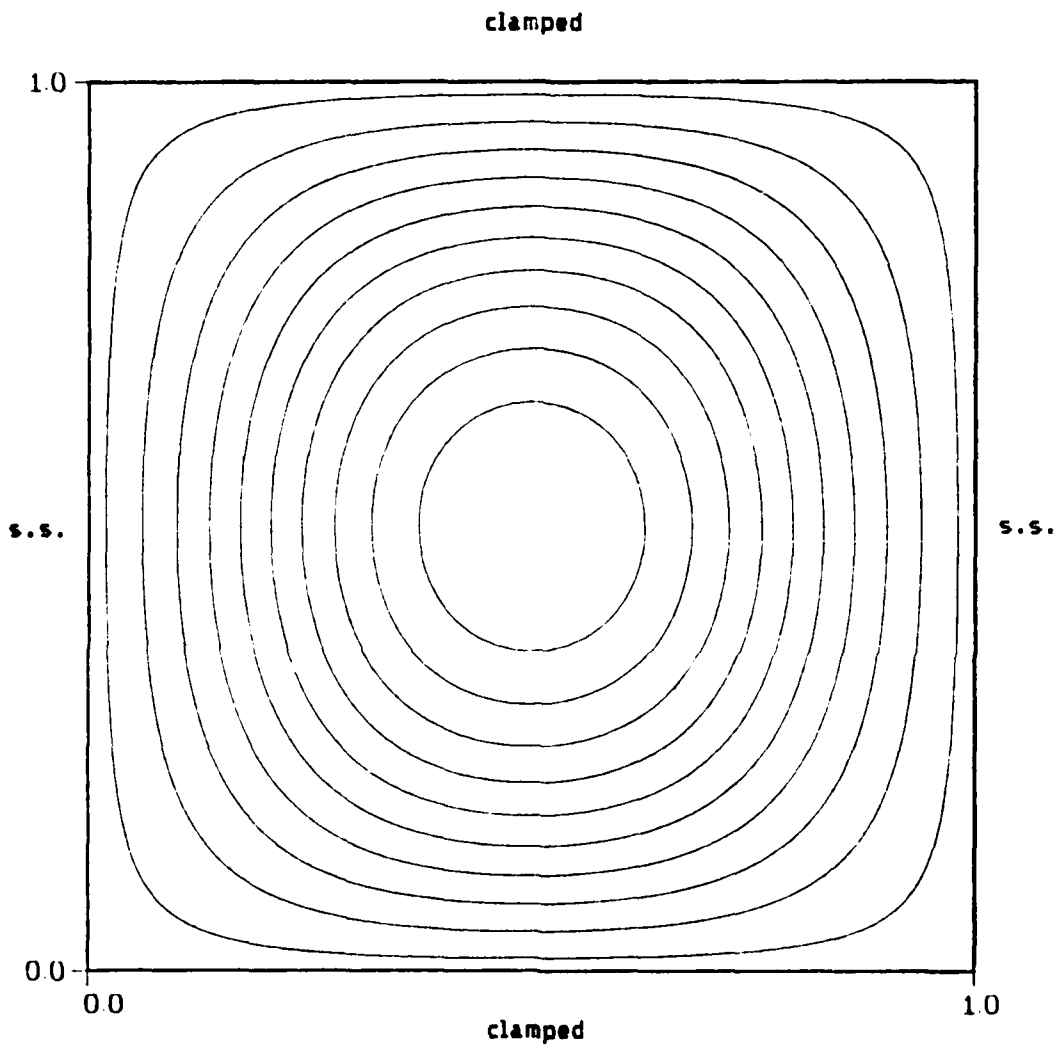
Plate #1  First Mode

Figure 3.8  Clamped Simply-Supported Boundary
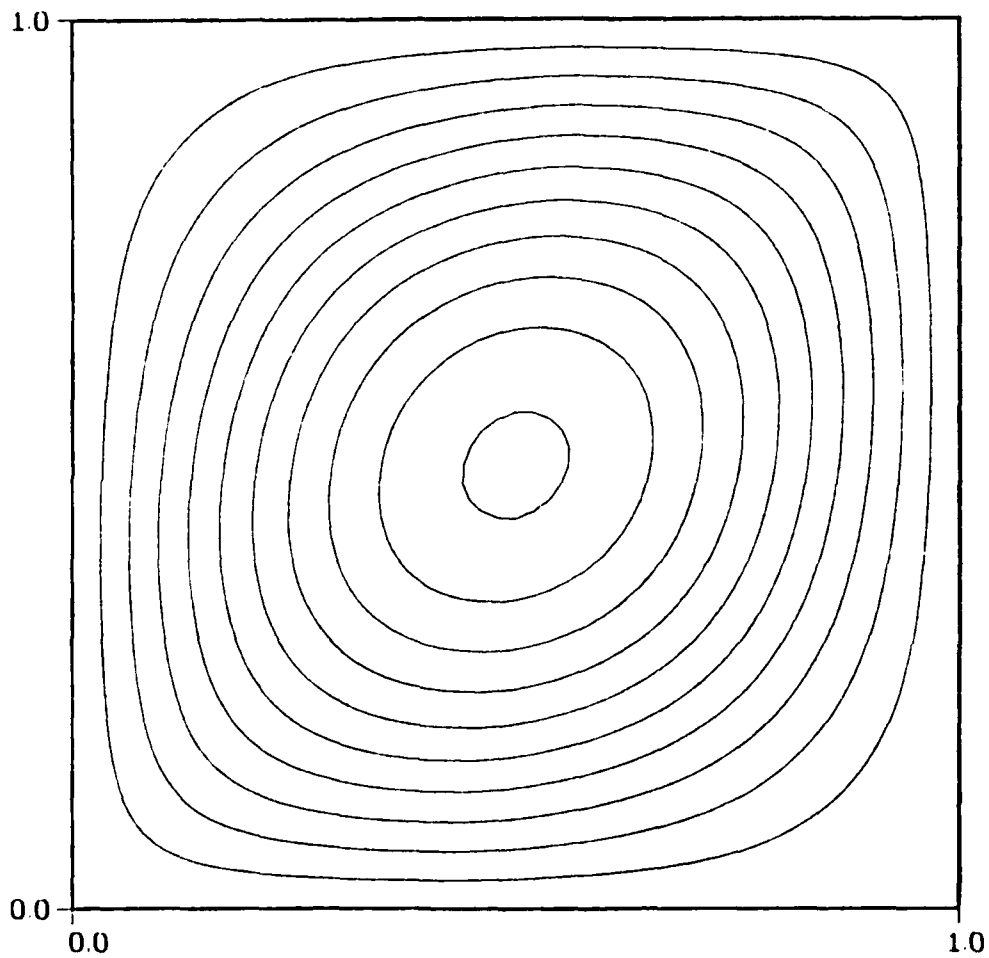
Plate #1  First Mode

Figure 3.9  Mode Shape Plot  Simply-Supported Boundary

Plate #2  First Mode

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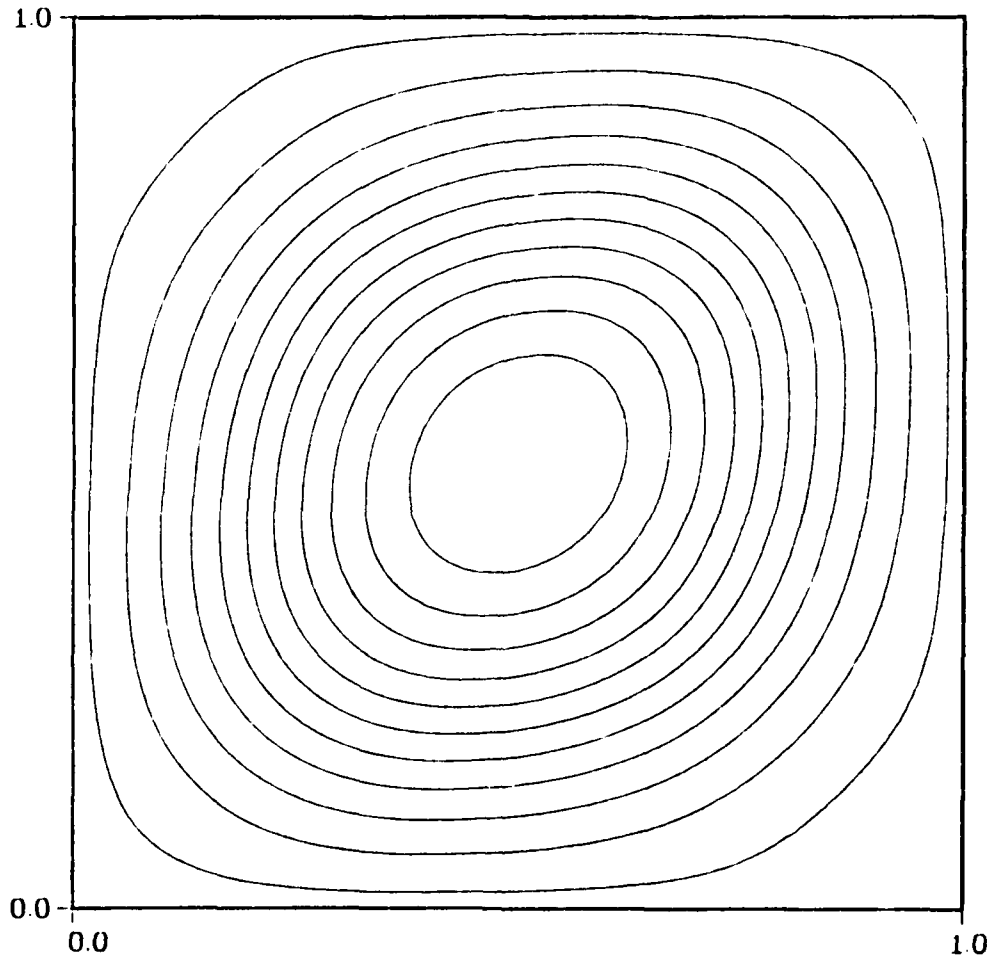
Figure 3.10  Mode Shape Plot  Clamped Boundary

Plate #2  First Mode

91
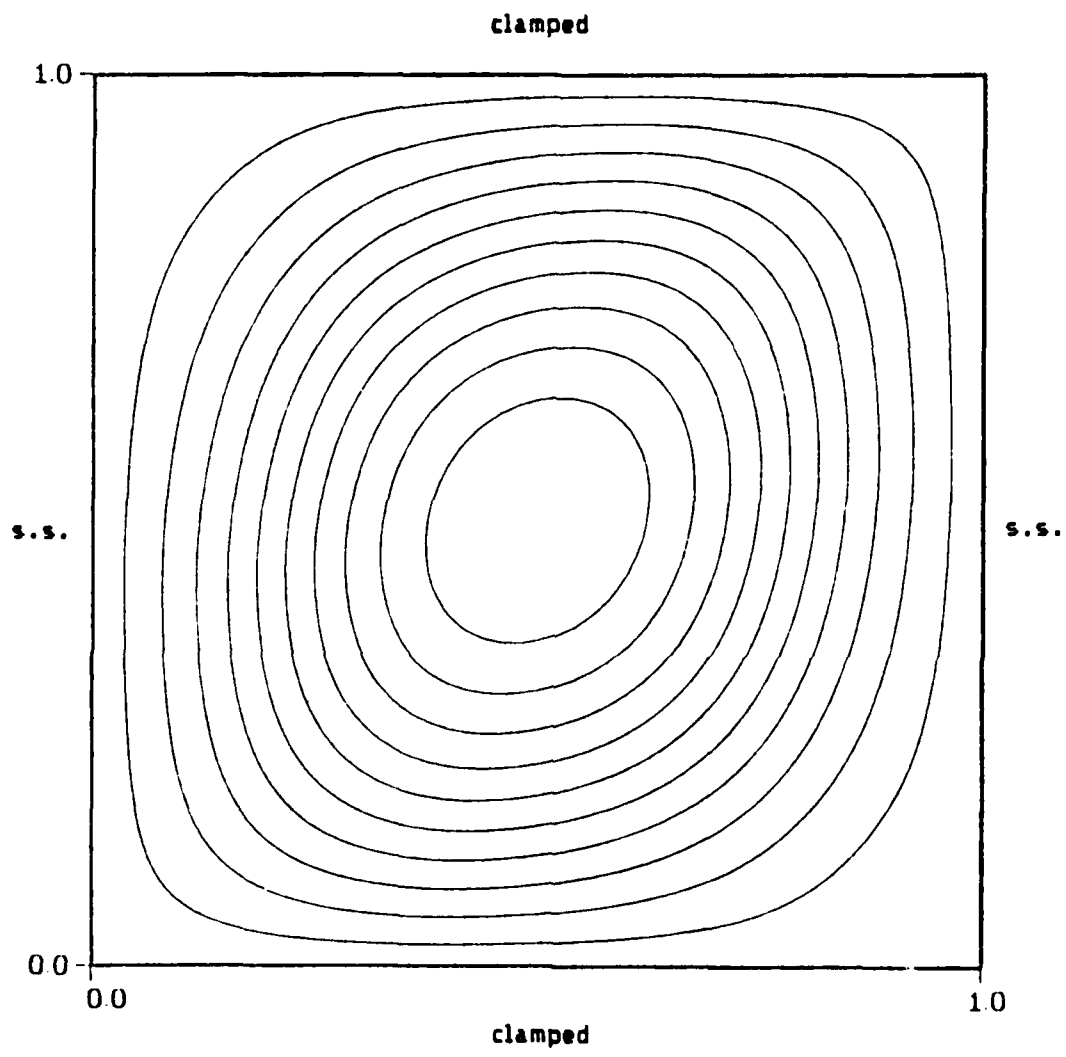
clamped



Figure 3.11  Clamped Simply-Supported Boundary

Plate #2  First Mode

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## IV. Conclusions

Based on the analysis presented in this thesis, the following conclusions are presented. They are organized in terms of some general comments on the Galerkin Method, some specific comments on the three boundary conditions, and then some general comments comparing the boundaries.

### Galerkin Method Comments

1. The Galerkin Method is a valid approach to solving the plate equations of motion and yields excellent results for the simply-supported boundary condition.

2. The convergence tendencies of the solution are easily checked with this method by generating more terms, and therefore more equations (provided the assumed functions are a complete set of functions).

3. More terms were required to achieve a converged solution for problems considered in this study whose assumed functions did not identically satisfy the natural boundary conditions than those whose assumed functions did identically satisfy them.

4. For a problem that requires a large number of terms to reach a converged solution, the eigenvalue problem becomes quite large. Because of this, the author could not use more than 8 terms in the Galerkin equations. This effected the quality of results for the clamped and clamped simply-supported boundary conditions.

### Simply-Supported Boundary Comments

1. The natural frequency for a square orthotropic plate, simply-supported, for the first mode is 10% lower than the Classical Laminated Plate Theory (CLPT)

frequency at a length to thickness ratio (represented by s) of 11 and drops to 33% lower for an s of 5. For the fifth mode, the frequency is 10% lower at an s of 23 and drops to 52% lower for an s of 5.

2. For a cross-ply laminate with the same conditions as above, the same type of shear deformation effects were seen. No CLPT solution was found in the current literature, therefore no comparison could be made.

3. For both types of plates above, the analysis of rotatory inertia effects showed a lowering of the natural frequency from the shear deformation frequency by one to two percent for s's below 25 for the first five modes.

Clamped Boundary Comments

1. The natural frequency for a square orthotropic plate, clamped on all sides, for the first mode, is 10% lower than the CLPT frequency for an s of 21 and drops to 59% lower for an s of 5. For the fifth mode, the frequency is 10% lower at an s of 33 and drops to 66% lower for an s of 5.

2. For the same case above, the value of the asymptotic limit for the natural frequency was 3% higher than the CLPT solution. This was due to the fact that an M and N of six was used to generate the data. To achieve closer results, a larger value of M and N would have to be used.

3. The analysis of rotatory inertia effects showed a lowering of the natural frequency from the shear deformation frequency to be at most 1%, occuring at an s of 5.

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Clamped Simply-Supported Boundary

1. The natural frequency for a square orthotropic plate, clamped on two opposite sides, simply-supported on two opposite sides, for the first mode, is 10% lower than the CLPT frequency for an s of 25 and drops to 61% lower for an s of 5. For the fifth mode, the frequency is 10% lower at an s of 35 and drops to 65% lower for an s of 5.

2. For the same case above, the value of the asymptotic limit for the natural frequency was 1.5% higher than the CLPT solution. This is again due to the fact that a value of M and N equal to six was used to generate the data.

3. The analysis of rotatory inertia effects showed a lowering of the natural frequency from the shear deformation frequency to be at most 1%, occuring at an s of 5.

General Comments

1. The effects of shear deformation are more significant for the two clamped boundary conditions than for the simply-supported boundary.

2. The effects of shear deformation increase with increasing mode for all three boundary conditions.

3. Analysis shows that rotatory inertia has very little effect for all three modes.

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VITA

John A. Bowlus was born on September 9, 1952 in Columbus, Ohio. He graduated from Fredericktown Senior High School in June 1970 and attended the University of Cincinnati, from which he received the degree of Bachelor of Science in Aerospace Engineering in June 1975. Upon graduation he worked for the National Aeronautics and Space Administration's Johnson Space Center in Houston, Texas. While employed there, he served as a flight test engineer on the Shuttle Training Aircraft and logged more than 200 hours of simulated Shuttle approaches. He also served as the guidance modeling engineer for that same aircraft. In June 1980, he went to work for the U. S. Air Force at Wright-Patterson AFB in Dayton, Ohio. While there, he was in charge of the Flight Dynamics Labortory's effort to modernize their real-time, man-in-the-loop simulation facility. He entered the School of Engineering, Air Force Institute of Technology, in October 1984.

Permanent address: 8029 Wescott Ave.

Fairborn, Ohio 45324

Appendix A

```
'9 REM THIS PROGRAM COMPUTES THE EXTENSIONAL AND BENDING
   REM STIFFNESS ELEMENTS FOR A SYMMETRIC LAMINATE BUILD
30 REM UP, GIVEN LAMINA PROPERTIES.
40 REM
50 A1=0
60 A2=0
70 A3=0
80 D1=0
90 D2=0
100 D3=0
110 D4=0
120 D5=0
130 D6=0
140 PS=0
150 INPUT"DO YOU WISH TO ADD ANOTHER LAYER";N$
160 IF N$="N" THEN GOTO 860
170 REM
180 REM THIS SECTION GETS THE LAMINA DATA.
190 REM
200 INPUT"ORIENTATION ANGLE ";TD
210 TH=TD*3.1415927/180
220 INPUT"ZK DIMENSION ";ZK
230 INPUT"ZK-1 DIMENSION ";Z1
240 INPUT"ARE THE REST OF THE LAMINA DATA THE SAME AS THE LAST TIME ";M$
250 IF M$="Y" THEN GOTO 440
260 INPUT"E1 ";E1
 '0 INPUT"E2 ";E2
280 INPUT"V12 ";V1
290 V2=V1*(E2/E1)
300 INPUT"G12 ";G1
310 G3=G1
320 G2=0.8*G1
330 INPUT"MASS DENSITY ";RHO
340 REM
350 REM COMPUTE THE A AND D ELEMENTS FOR THIS LAYER.
360 REM
370 Q1=E1/(1-(V1*V2))
380 Q2=(V1*E2)/(1-(V1*V2))
390 Q3=E2/(1-(V1*V2))
400 Q4=G2
410 Q5=G3
420 Q6=G1
435 REM
440 REM COMPUTE THE QBARS.
445 REM
450 B1=Q1*(COS(TH))[4 + 2*(Q2+2*Q6)*(SIN(TH))[2 *(COS(TH))[2 + Q3*(SIN(TH))[4
460 B2=(Q1+Q3-4*Q6)*(SIN(TH))[2 * (COS(TH))[2 + Q2*((SIN(TH))[4 + (COS(TH))[4)
470 B3=Q1*(SIN(TH))[4 + 2*(Q2+2*Q6)*(SIN(TH))[2 * (COS(TH))[2 + Q3*(COS(TH))[4
480 B4=(Q1-Q2-2*Q6)*SIN(TH)*(COS(TH))[3 + (Q2-Q3+2*Q6)*(SIN(TH))[3*COS(TH)
490 B5=(Q1-Q2-2*Q6)*(SIN(TH))[3*COS(TH) + (Q2-Q3+2*Q6)*SIN(TH)*(COS(TH))[3
500 B6=(Q1+Q3-2*Q2-2*Q6)*(SIN(TH))[2*(COS(TH))[2 + Q6*((SIN(TH))[4 + (COS(TH))[4
)
  '0 B7=Q4*(COS(TH))[2 + Q5*(SIN(TH))[2
520 B8=(Q4-Q5)*COS(TH)*SIN(TH)
```

```
530  B9=Q5*(COS(TH))[2 + Q4*(SIN(TH))[2
550  REM
560  REM COMPUTE THE A ELEMENTS.
570  REM
580  A4=B7*(ZK-Z1)
590  A5=B8*(ZK-Z1)
600  A6=B9*(ZK-Z1)
610  P=RHO*(ZK-Z1)
620  REM
630  REM COMPUTE THE D ELEMENTS.
640  REM
650  DZ=(ZK[3-Z1[3)
660  F1=(1/3)*B1*DZ
670  F2=(1/3)*B2*DZ
680  F3=(1/3)*B4*DZ
690  F4=(1/3)*B3*DZ
700  F5=(1/3)*B5*DZ
710  F6=(1/3)*B6*DZ
720  REM
730  REM SUM THE A'S AND D'S WITH THE PREVIOUS LAYERS.
740  REM
750  A1=A1+A4
760  A2=A2+A5
770  A3=A3+A6
780  D1=D1+F1
790  D2=D2+F2
800  D3=D3+F3
810  D4=D4+F4
820  D5=D5+F5
830  D6=D6+F6
840  PS=PS+P
850  GOTO 150
854  REM
855  REM PRINT OUT THE A AND D ELEMENTS.
856  REM
860  LPRINT"A44 = ";A1
870  LPRINT"A45 = ";A2
880  LPRINT"A55 = ";A3
890  LPRINT"D11 = ";D1
900  LPRINT"D12 = ";D2
910  LPRINT"D16 = ";D3
920  LPRINT"D22 = ";D4
930  LPRINT"D26 = ";D5
940  LPRINT"D66 = ";D6
950  LPRINT"P   = ";PS
960  REM NOW COMPUTE THE NORMALIZED STIFFNESSES.
970  INPUT"INPUT THE PLATE THICKNESS";H
980  A1=A1/(E2*H)
990  A2=A2/(E2*H)
1000 A3=A3/(E2*H)
1010 D1=D1/(E2*H[3)
1020 D2=D2/(E2*H[3)
1030 D3=D3/(E2*H[3)
1040 D4=D4/(E2*H[3)
```

101

```
1050 D5=D5/(E2*H[3)
1060 D6=D6/(E2*H[3)
1070 LPRINT
1080 LPRINT"a44 = ";A1
1090 LPRINT"a45 = ";A2
1100 LPRINT"a55 = ";A3
1110 LPRINT"d11 = ";D1
1120 LPRINT"d12 = ";D2
1130 LPRINT"d16 = ";D3
1140 LPRINT"d22 = ";D4
1150 LPRINT"d26 = ";D5
1160 LPRINT"d66 = ";D6
1170 LPRINT
1180 END
```

Appendix B

```fortran
      PROGRAM BCTWO(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=CUTPUT)
      REAL F1,F2,F3,F4,F5,F6,F7,F8,F9,F10,K,K4,K5,P1
      INTEGER M,N,P,Q,ITEST1,ITEST2,MMAX,NMAX
      REAL A1(10,10),A2(10,10),A3(10,10),B1(10,10),B2(10,10)
      REAL B3(10,10),C1(10,10),C2(10,10),C3(10,10)
      REAL IM(108,108),MC(108),ST(108,108),MA(108,108)
C
C     INPUT DATA FOR THE COMPCSITE PLATE.
C
      MMAX=6
      NMAX=MMAX
      PI=3.1415927
      R=1.0
      A=10.0
      B=A
      H=1.0
      S=A/H
      P1=1.4245E-04
      K=5./6.
      K4=1.0
      K5=1.0
      A44=3.8571E-01
      A55=3.8571E-01
      D11=1.11083
      D12=0.0251509
      D16=0.0
      D22=0.23055
      D26=0.0
      D66=0.0357143
      ET=1.45+06
C
C     DETERMINE THE VALUE OF THE INTEGRATED TERMS.
C
      I=1
      J=1
      DO 100 P=1,MMAX
      DO 100 Q=1,NMAX
      DO 50  M=1,MMAX
      DO 50  N=1,NMAX
      ITEST1= MOD(M+P,2)
      ITEST2= MCD(N+Q,2)
      IF (M.EQ.P) THEN
         F1=0.5
         F3=0.0
         F4=0.0
      ELSE
         F1=0.0
         IF (ITEST1.EQ.0) THEN
            F3=0.0
            F4=0.0
         ELSE
            F3=2*M
            F4=2*P
         ENDIF
      ENDIF
      IF (N.EQ.Q) THEN
         F2=0.5
         F5=0.0
         F6=0.0
      ELSE
```

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```
              F2=).0
              IF ((ITEST2.EQ.C) THEN
                  F5=0.0
                  F6=0.C
              ELSE
                  F5=2*N
                  F6=2*G
              ENDIF
          ENDIF
          F7=(M*M)-(P*P)
          F8= -F7
          IF (F7.EQ.0.) THEN
              F7=1.
              F8=1.
          ENDIF
          F9=(N*N)-(Q*Q)
          F10=-F9
          IF (F9.EQ.0.) THEN
              F9=1.
              F10=1.
          ENDIF
      C
      C   COMPUTE THE STIFFNESS MATRIX.
      C
          A1(M,N)=PI*PI*F1*F2*(-M*M*D11-N*N*R*R*C66-K*S*S*A55/(PI*PI))
         1+F4*F6*(2*M*N*R*D16)/(F8*F10)
          B1(M,N)=PI*PI*F1*F2*(-M*M*D16-N*N*R*R*C26)
         1+F4*F6*(M*N*R*(D12+C66))/(F8*F10)
          C1(M,N)=-F4*F2*(M*K*S*A55)/F8
          A2(M,N)=PI*PI*F1*F2*(-M*M*D16-N*N*R*R*D26)
         1+F4*F6*(M*N*R*(D12+C66))/(F8*F10)
          B2(M,N)=PI*PI*F1*F2*(-M*M*D66-N*N*R*R*D22-K*S*S*A44/(PI*PI))
         1+F4*F6*(2*M*N*R*D26/(F8*F10))
          C2(M,N)=-F1*F6*(N*R*K*S*A44/F10)
          A3(M,N)=F4*F2*(M*K*S*A55)/F8
          B3(M,N)=F1*F6*(N*R*N*S*A44)/F10
          C3(M,N)=PI*PI*F1*F2*(-M*M*K*A55-N*N*R*R*K*A44)
          ST(I,J)=-A1(M,N)
          ST(I,J+MMAX*NMAX)=-B1(M,N)
          ST(I,J+2*MMAX*NMAX)=-C1(M,N)
          ST(I+MMAX*NMAX,J)=-A2(M,N)
          ST(I+MMAX*NMAX,J+MMAX*NMAX)=-B2(M,N)
          ST(I+MMAX*NMAX,J+2*MMAX*NMAX)=-C2(M,N)
          ST(I+2*MMAX*NMAX,J)=-A3(M,N)
          ST(I+2*MMAX*NMAX,J+MMAX*NMAX)=-B3(M,N)
          ST(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=-C3(M,N)
          J=J+1
   50     CONTINUE
          I=I+1
          J=1
  100     CONTINUE
      C
      C   COMPUTE THE MASS MATRIX.
      C
          DO 20 I=1,(3*MMAX*NMAX)
          DO 20 J=1,(3*MMAX*NMAX)
          IF (I.EQ.J) THEN
              IM(I,J)=1.0
          ELSE
              IM(I,J)=0.0
```

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```
      ENDIF
  20  CONTINUE
      DO 30 I=1,(2*MMAX*NMAX)
      MC(I)=K5*(P1*A*A)/(48.0*H*ET)
  30  CONTINUE
      DO 40 I=(2*MMAX*NMAX+1),(3*MMAX*NMAX)
      MC(I)=K4*((R**4)*(B**4)*P1)/(4.0*S*S*ET*(H**3))
  40  CONTINUE
      DO 60 I=1,(3*MMAX*NMAX)
      DO 60 J=1,(3*MMAX*NMAX)
      MA(I,J)=IM(I,J)*MC(J)
  60  CONTINUE
C
C     WRITE THE STIFFNESS AND MASS MATRICES TO A DATA FILE.
C
      DO 151 I=1,3*MMAX*NMAX
      DO 151 J=1,3*MMAX*NMAX
      WRITE(6,150) ST(I,J),MA(I,J)
 150  FORMAT(2E13.5)
 151  CONTINUE
      STOP
      END
```

# Equations for Boundary Condition #1

COMPUTE THE STIFFNESS MATRIX.

```
A1(M,N)=PI*PI*F1*F2*(-M*M*D11-N*N*R*R*D66-K*S*S*A55/(PI*PI))
1-F3*F6*(2*R*M*N*D16)/(F7*F10)
2+F6*(N*R*D16*(1-COS(M*PI)*COS(P*PI)))/F10
 B1(M,N)=PI*PI*F1*F2*(-M*N*R*(D12+D66))
1-F3*F6*(M*M*D16+N*N*R*R*D26)/(F7*F10)
2+F6*(M*D16*(1-COS(M*PI)*COS(P*PI)))/F10
 C1(M,N)=-PI*F1*F2*M*K*S*A55
 A2(M,N)=PI*PI*F1*F2*(-M*N*R*(D12+D66))
1-F4*F5*(M*M*D16+N*N*R*R*D26)/(F8*F9)
2+F4*(N*R*D26*(1-COS(N*PI)*COS(Q*PI)))/F8
 B2(M,N)=PI*PI*F1*F2*(-M*M*D66-N*N*R*R*D22-K*S*S*A44/(PI*PI))
1-F4*F5*(2*M*N*R*D26/(F8*F9))
2+F4*M*D26*(1-COS(N*FI)*COS(Q*PI))/F8
 C2(M,N)=-PI*F1*F2*(N*R*K*S*A44)
 A3(M,N)=-PI*F1*F2*M*K*S*A55
 B3(M,N)=-PI*F1*F2*N*R*K*S*A44
 C3(M,N)=PI*PI*F1*F2*(-M*M*K*A55-N*N*R*R*K*A44)
 ST(I,J)=-A1(M,N)
 ST(I,J+MMAX*NMAX)=-B1(M,N)
 ST(I,J+2*MMAX*NMAX)=-C1(M,N)
 ST(I+MMAX*NMAX,J)=-A2(M,N)
 ST(I+MMAX*NMAX,J+MMAX*NMAX)=-B2(M,N)
 ST(I+MMAX*NMAX,J+2*MMAX*NMAX)=-C2(M,N)
 ST(I+2*MMAX*NMAX,J)=-A3(M,N)
 ST(I+2*MMAX*NMAX,J+MMAX*NMAX)=-B3(M,N)
 ST(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=-C3(M,N)
```

## Equations for Boundary Condition #3

```
C
C     COMPUTE THE STIFFNESS MATRIX.
C
   A1(M,N)=PI*PI*F1*F2*(-M*M*D11-N*N*R*R*D66-K*S*S*A55/(PI*P
  1*F4*F6*(2*R*M*N*D16)/(F8*F10)
   B1(M,N)=-F1*F6*PI*(M*M*D16+N*N*R*R*D26)/F10
  1-F4*F2*PI*(M*N*R*(D12+D66))/F8
   C1(M,N)=-F4*F2*M*K*S*A55/F8
   A2(M,N)=-F1*F5*PI*(M*M*D16+N*N*R*R*D26)/F9
  1+F4*F2*PI*(M*N*R*(D12+D66))/F8
  2+F1*PI*N*R*D26*(1-CCS(N*PI)*COS(Q*PI))
   B2(M,N)=PI*PI*F1*F2*(-M*M*D66-N*N*R*R*D22-K*S*S*A44/(PI*P
  1-F4*F5*(2*M*N*R*D26/(F8*F9))
  2+F4*M*D26*(1-COS(N*FI)*COS(Q*PI))/F8
   C2(M,N)=-PI*F1*F2*(N*R*K*S*A44)
   A3(M,N)=F4*F2*M*K*S*A55/F8
   B3(M,N)=-PI*F1*F2*N*R*K*S*A44
   C3(M,N)=PI*PI*F1*F2*(-M*M*K*A55-N*N*R*R*K*A44)
   ST(I,J)=-A1(M,N)
   ST(I,J+MMAX*NMAX)=-B1(M,N)
   ST(I,J+2*MMAX*NMAX)=-C1(M,N)
   ST(I+MMAX*NMAX,J)=-A2(M,N)
   ST(I+MMAX*NMAX,J+MMAX*NMAX)=-B2(M,N)
   ST(I+MMAX*NMAX,J+2*MMAX*NMAX)=-C2(M,N)
   ST(I+2*MMAX*NMAX,J)=-A3(M,N)
   ST(I+2*MMAX*NMAX,J+MMAX*NMAX)=-B3(M,N)
   ST(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=-C3(M,N)
```

Appendix C

```fortran
      PROGRAM EIGFIN(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      EXTERNAL EIGZS
      REAL A(78),B(78),Z(12,12),D(12),WK(198)
      REAL ST(13,98),MA(98,98)
      INTEGER I,J,R,C,MMAX,N,IJOB,IZ,IER
C
      MMAX=2
      N=3+MMAX*MMAX
C
C     LOAD IN THE MASS AND STIFFNESS MATRICES.
C
      DO 10 I=1,N
      DO 10 J=1,N
      READ(5,20) ST(I,J),MA(I,J)
   20 FORMAT(2E13.5)
   10 CONTINUE
C
C     PUT THE STIFFNESS AND MASS MATRICES IN SYMMETRIC FORM.
C
      T=1
      DO 30 R=1,N
      DO 30 C=1,R
      A(T)=ST(R,C)
      B(T)=MA(R,C)
      T=T+1
   30 CONTINUE
C
C     COMPUTE THE EIGENVALUES AND EIGENVECTORS.
C
      IZ=N
      IJOB=2
      CALL EIGZS(A,B,N,IJOB,D,Z,IZ,WK,IER)
      WRITE(6,40) IER
   40 FORMAT(I4)
      DO 50 I=1,N
      WRITE(6,50) I,D(I)
   50 FORMAT('D(',I2,') =',E13.5)
   50 CONTINUE
      DO 70 I=1,N
      DO 70 J=1,N
      WRITE(6,80) I,J,Z(I,J)
   80 FORMAT('Z',I2,1X,I2,' = ',E13.5)
   70 CONTINUE
      STOP
      END
```

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Appendix D

```
      PROGRAM LAST(INPUT,OUTPUT,TAPE1=INPUT,TAPE2=OUTPUT)
      REAL X,Y,Z,PI,K(108),C(6,6),NC(6,6),LARGE
      INTEGER I,J,L,M,N
      PI=3.1415927
      X=-1.1
      Y=-1.1
C     THIS SECTION GETS THE C(M,N) COEFFICIENTS
      DO 10 L=1,108
      READ(1,15) K(L)
   15 FORMAT(E15.5)
   10 CONTINUE
      I=77
      DO 20 M=1,6
      DO 20 N=1,6
      C(M,N)=K(I)
      I=I+1
   20 CONTINUE
C     NOW NORMALIZE THE C(M,N) COEFFICIENTS
      LARGE=ABS(C(1,1))
      DO 25 M=1,6
      DO 25 N=1,6
      IF(ABS(C(M,N)).GT.LARGE) THEN
          LARGE=ABS(C(M,N))
      ENDIF
   25 CONTINUE
      DO 30 M=1,6
      DO 30 N=1,6
      NC(M,N)=C(M,N)/LARGE
   30 CONTINUE
C     THIS SECTION COMPUTES THE PLATE DISPLACEMENT w
      DO 40 I=1,11
      X=X+.1
      DO 40 J=1,11
      Y=Y+.1
      Z=0.
      DO 50 M=1,6
      DO 50 N=1,6
      Z=Z+(NC(M,N)*SIN(M*PI*X)*SIN(N*PI*Y))
   50 CONTINUE
      WRITE(2,70) X,Y,Z
   70 FORMAT(3E15.5)
   50 CONTINUE
      Y=-1.1
   40 CONTINUE
      STOP
      END
```

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Thesis Advisor: Anthony N. Palazotto

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An analytical study was conducted to determine the natural frequencies and mode shapes for laminated anisotropic plates, including the effects of shear deformation and rotatory inertia, by using the Galerkin Technique. Three different boundary conditions, simply-supported, clamped, and two opposite sides clamped, two opposite sides simply-supported, were considered. Two different graphite-epoxy symmetric plates were used in the analysis. Convergence characteristics and the effects of length to thickness ratios were investigated. Comparison to classical results and contour plots for several mode shapes are provided.

It was found that as the length to thickness ratios were reduced, shear deformation effects significantly lowered the natural frequencies. Analysis also showed that rotatory inertia effects were very small. Convergence characteristics for all three boundary conditions were very good and excellent agreement with classical solutions was achieved.

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