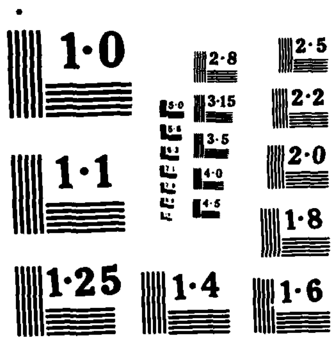
AD-A162 876 PEAKEDNESS OF WEIGHTED AVERAGES OF JOINTLY DISTRIBUTED RANDOM VARIABLES..(U) FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS W CHAN ET AL. OCT 85 1/1

UNCLASSIFIED FSU-STATISTICS-M712 AFOSR-TR-85-1123 F/G 12/1 NL

END
FILMED
DTIC


1·0
2·8
2·5
3·15
2·2
3·5
2·0
1·1
4·0
4·5
1·8
1·25
1·4
1·6

PEAKEDNESS OF WEIGHTED AVERAGES OF JOINTLY DISTRIBUTED RANDOM VARIABLES

1473 in Back

by

Wai Chan[1], Dong Ho Park[2], and Frank Proschan[3],

FSU Technical Report No. M712
AFOSR Technical Report No. 85-184

October, 1985

[1]Ohio State University
Department of Statistics
Columbus, Ohio 43210

and

[2]University of Nebraska
Department of Statistics
Lincoln, Nebraska 68588

and

[3]Florida State University
Department of Statistics
Tallahassee, Florida 32306

Research sponsored by the Air Force Office of Scientific Research under Contract Number [redacted] The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation theron.
F49620-82-K-0007

Key Words: Peakedness, convex combination, majorization, Schur-concave density, Cauchy distributions.

AMS (1981) subject classification No. 60E15.

SECURITY CLASSIFICATION OF THIS PAGE AD-A162 876

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER
FSU M 712
AFOSA No. 85-185

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and subtitle)
Peakedness of Weighted Averages of Jointly Distributed Random Varibles

5. TYPE OF REPORT & PERIOD COVERED
Technical

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Wai Chan, Dong Ho Park, and Frank Proschan

8. CONTRACT OR GRANT NUMBER(s)
AFOSR 82-K-0007

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Florida State University
Department of Statistics
Tallahassee, Florida 32306

Ohio State University
Department of Statistics
Columbus, Ohio 43210

University of Nebraska
Department of Statistics
Lincoln, Nebraska 68588

10. PROGRAM ELEMENT, PROJECT, TASK AREA, & WORK UNIT NUMBERS
61102F 2304 A5

11. CONTROLLING OFFICE NAME AND ADDRESS
U.S. Air Force
Air Force Office of Scientific Research
Bolling Air Force Base, DC 20332

12. REPORT DATE
October, 1985

13. NUMBER OF PAGES
8

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
(U)

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)
Unclassified

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)
Distribution Unlimited

18. SUPPLEMENTARY NOTES

19. KEY WORDS
Peakedness, Convex combination, majorization, Schur-concave density, Cauchy distributions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This note extends the Proschan (1965) result on peakedness comparison for a convex combination of i.i.d. random variables from a PF_2 density. Now the underlying random variables are jointly distributed from a Schur-concave density. The result permits a more refined discription of convergence in the Law of Large Numbers. Keywords: Cauchy distributions

PEAKEDNESS OF WEIGHTED AVERAGES OF JOINTLY DISTRIBUTED RANDOM VARIABLES

by

Wai Chan, Dong Ho Park, and Frank Proschan

ABSTRACT

This note extends the Proschan (1965) result on peakedness comparison for a convex combination of i.i.d. random variables from a PF_2 density. Now the underlying random variables are jointly distributed from a Schur-concave density. The result permits a more refined discription of convergence in the Law of Large Numbers.

1. Introduction

Proschan (1965) shows that:

1.1 Theorem. Let f be PF_2, $f(t) = f(-t)$ for all t, $X_1, \ldots, X_n$ independently distributed with density f, $\underset{\sim}{p} \overset{m}{>} \underset{\sim}{p}'$, $\underset{\sim}{p}$, $\underset{\sim}{p}'$ not identical, $\sum_1^n p_i = 1 = \sum_1^n p_i'$. Then $\sum_1^n p_i' X_i$ is strictly more peaked than $\sum_1^n p_i X_i$.

(Definitions of majorization ($\underset{\sim}{p} \overset{m}{>} \underset{\sim}{p}'$), PF_2 density, and peakedness are presented in Section 2.) Roughly speaking, Theorem 1.1 states that a weighted average of i.i.d. random variables converges more rapidly in the case in which weights are close together as compared with the case in which the weights are diverse.

In the present note, we extend the basic univariate result to the multivariate situation in which the underlying random variables have a joint Schur-concave density. Theorem 2.3 presents the precise statement of the multivariate extension.

2. Peakedness comparisons

The theory of majorization is exploited in this section to obtain more general versions of the result of Proschan (1965). We begin with some definitions

Definition 2.1. Let $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ be decreasing rearrangements of the components of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. We say that the vector $\underset{\sim}{b}$ is majorized by $\underset{\sim}{a}$, written $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$ if

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$$

and $$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \quad \text{for } k = 1,\ldots, n-1.$$

Definition 2.2. A real valued function f defined on R^n is said to be a Schur-concave function if $f(\underset{\sim}{a}) \leq f(\underset{\sim}{b})$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$.

A function f defined on R^n is said to be sign invariant if $f(x_1,\ldots, x_n) = f(|x_1|,\ldots, |x_n|)$. In the following Lemma, we give a peakedness comparison for random variables with a sign invariant and Schur-concave density.

Theorem 2.3. Suppose the random vector $\underset{\sim}{X} = (X_1,\ldots, X_n)$ has a Schur-concave density f. If f is sign-invariant and satisfies

$$\int_{-\infty}^{\infty} uf(u, u, x_3,\ldots, x_n)\, du < \infty \text{ for all } x_3,\ldots, x_n.$$

Then for all $t \geq 0$,

$$\Psi(a_1,\ldots, a_n) = P(\sum a_i X_i \leq t)$$

is a Schur-concave function of $\underset{\sim}{a} = (a_1,\ldots, a_n)$, $a_i \geq 0$ for all i. Equivalently, $\sum b_i X_i$ is more peaked than $\sum a_i X_i$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$.

Proof.

Without loss of generality, we may assume that $\sum a_i = 1$. We first consider the case $n = 2$.

Let $0 \leq a \leq \frac{1}{2}$ and $\bar{a} = 1 - a$. Let $h(a) = P(\bar{a}X_1 + \bar{a}X_2 \leq t) = \int_{-\infty}^{\infty} G_{X_2|X_1 = u}(\frac{t - au}{\bar{a}})\, g_1(u)\, du$ where g_1 is the marginal density of X_1 and $G_{X_2|X_1 = u}$ is the conditional distribution function of X_2 given that $X_1 = u$.

Differentiation under the integral sign is permissible here, so that

$$\bar{a}^2\, h'(a) = \int_{-\infty}^{\infty} g_{X_2|X_1=u}\left(\frac{t-au}{\bar{a}}\right) g_1(u)(t-u)\; du$$

$$= \int_{-\infty}^{\infty} f\left(u, \frac{t-au}{\bar{a}}\right)(t-u)\; du.$$

$$= \int_{-\infty}^{t} f\left(u, \frac{t-au}{\bar{a}}\right)(t-u)\; du$$

$$+ \int_{t}^{\infty} f\left(u, \frac{t-au}{\bar{a}}\right)(t-u)\; du.$$

Now let $v = t - u$ in the first integral and $v = u - t$ in the second integral. We obtain

$$\bar{a}^2\, h'(a) = \int_0^{\infty} v\, [f(t-v,\, t+\tfrac{a}{\bar{a}}v) - f(t+v,\, t-\tfrac{a}{\bar{a}}v)]\; dv$$

$$= \int_0^{\infty} v\, [f(v-t,\, \tfrac{a}{\bar{a}}v+t) - f(v+t,\, \tfrac{a}{\bar{a}}v-t)]\; dv,$$

since f is sign invariant. But this is nonpositive because

$$(v+t,\, \tfrac{a}{\bar{a}}v-t) \overset{m}{\geq} (v-t,\, \tfrac{a}{\bar{a}}v+t)$$

and f is Schur-concave. Thus h(a) is increasing in a, $0 \leq a \leq \frac{1}{2}$.

The result for $n \geq 3$ now follows since

$$P(\sum a_i X_i \leq t)$$

$$= E\, [P(a_1X_1 + a_2X_2 \leq t - \sum_3^n a_iX_i \,|\, X_3, \ldots, X_n)]$$

and the conditional density $f(x_1, x_2 | x_3, \ldots, x_n)$ is also Schur-concave and sign invariant. □

Remark 2.4. To justify differentiation under the integral sign, we note that

$$\int_{-\infty}^{\infty} |f(u, \frac{t-au}{\bar{a}})(t-u)|\; du$$

$$\leq \int_{-\infty}^{\infty} |t-u|\; f\left(\frac{|u-t|}{\bar{a}}, \frac{|u-t|}{\bar{a}}\right)\; du < \infty,$$

which follows from (2.1).

This condition is clearly not a necessary condition, but it can be easily varified for most Schur-concave multivariate distributions. For example, the multivariate Cauchy density:

$$f(x_1,\ldots, x_n) = \pi^{-(n+1)/2}\,\Gamma\,((n+1)/2)(1+\sum_{i=1}^{n} x_i^2)^{-(n+1)/2}$$

has this property.

The following result is an immediate application of Theorem 2.3.

Corollary 2.5. Let $X_1,\ldots, X_n$ be random variables with joint Schur-concave density f. Let f be sign invariant and satisfy

$$\int_{-\infty}^{\infty} uf(u, u, x_2,\ldots, x_n)\,du < \infty \text{ for all } x_3,\ldots, x_n.$$

Then $\frac{1}{k}\sum_{i=1}^{k} X_i$ is increasing in peakedness as k increases from 1 to n.

Proof.

Let $\underset{\sim}{a}_1 = (1, 0,\ldots 0)$, $\underset{\sim}{a}_2 = (\frac{1}{2}, \frac{1}{2}, 0,\ldots 0),\ldots$ and $\underset{\sim}{a}_n = (\frac{1}{n},\ldots, \frac{1}{n})$ where each vector contains n components. Then $\underset{\sim}{a}_1 \overset{m}{\geq} \underset{\sim}{a}_2 \overset{m}{\geq} \ldots \overset{m}{\geq} \underset{\sim}{a}_n$. The result follows from Theorem 2.3. □

Suppose $\underset{\sim}{X} = (X_1,\ldots, X_n)$ and $\underset{\sim}{Y} = (Y_1,\ldots, Y_n)$ are independently distributed with respective densities f and g where both f and g are Schur-concave and sign invariant, Theorem 2.3 implies that $\sum b_i(X_i + Y_i)$ is more peaked than $\sum a_i(X_i + Y_i)$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$. This is true because the convolution of Schur-concave functions is Schur-concave. However, if $Y_1,\ldots, Y_n$ are i.i.d. Cauchy, then the joint density given by

$$g(X_1,\ldots, X_n) = (\frac{a}{\pi})^n \prod_{i=1}^{n} (1 + a^2X_i^2)^{-1},\quad a > 0,$$

is not Schur-concave. Theorem 2.7 below, we give conditions on f for which (2.2) holds. First we prove the following Lemma.

Lemma 2.6. Let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ be independently distributed with respective densities f_1 and f_2. Suppose $f_i(t_1, \ldots, t_n)$ is symmetric with respect to zero and nonincreasing in each argument for $t_k > 0$, $k = 1, \ldots, n$, and for $i = 1,2$.

Let $\sum_1^n b_i X_i$ be more peaked than $\sum_1^n a_i X_i$ and

$\sum_1^n b_i Y_i$ be more peaked than $\sum_1^n a_i Y_i$ where $a_i \geq 0$ and $b_i \geq 0$ for $i = 1, \ldots, n$. Then

$$\sum_1^n b_i(X_i + Y_i) \text{ is more peaked than is } \sum_1^n a_i(X_i + Y_i).$$

Proof.

This result follows immediately from the Lemma of Birnbaum (1948) by noting that the random variables $\sum_1^n a_i X_i$, $\sum_1^n a_i Y_i$, $\sum_1^n b_i X_i$, and $\sum_1^n b_i Y_i$ have symmetric and unimodal densities. $\square$

The following theorem identifies a different class of densities for which the conclusion of Theorem 2.3 holds.

Theorem 2.7. Suppose that the random vector $\underset{\sim}{X} = (X_1, \ldots, X_n)$ has a Schur-concave sign-invariant density f. Let f be nonincreasing in each argument over the positive values and satisfy (2.1). Let $Y_1, \ldots, Y_n$ be i.i.d. Cauchy with joint density g as given in (2.3). Let $\underset{\sim}{X}$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ be independent, and $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}_i$ where $a_i \geq 0$, $b_i \geq 0$ for all i and $1 = \sum_1^n a_i = \sum_1^n b_i$. Then

$$\sum_1^n b_i(X_i + Y_i) \text{ is more peaked than is } \sum_1^n a_i(X_i + Y_i).$$

Proof.

We use the fact that $\sum_1^n a_i Y_i$, $\sum_1^n b_i Y_i$ have the same distribution as does Y_1. From Theorem 2.3, $\sum_1^n b_i X_i$ is more peaked than is $\sum_1^n a_i X_i$. The result now follows from Lemma 2.6. $\square$

REFERENCES

[1] Birnbaum, Z. W. (1948). On random variables with comparable peakedness. Ann. Math. Statist. 19. 76-81.

[2] Proschan, F. (1965). Peakedness of distributions of convex combinations. Ann. Math. Statist. 36. 1703-1706.

END

FILMED

2-86

DTIC