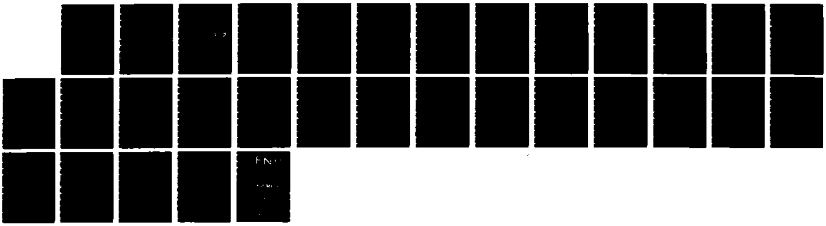
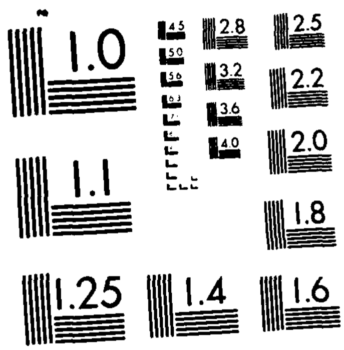
AD-A162 833 PRODUCT STOCHASTIC MEASURES(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROCESSES V PEREZ-ABREU OCT 85 TR-118 AFOSR-TR-85-1178 1/1

UNCLASSIFIED F49620-85-C-0144 F/G 12/1 NL





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SECURITY CLASSIFICATION OF THIS PAGE

AD-A162 833

(2)

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY

3. DISTRIBUTION/AVAILABILITY OF REPORT: Unlimited. Approved for public release; distribution unlimited.

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 118

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85 - 1178

6a. NAME OF PERFORMING ORGANIZATION: Center for Stochastic Processes

6b. OFFICE SYMBOL (If applicable)

7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research

6c. ADDRESS (City, State and ZIP Code): University of North Carolina, Statistics Dept., Phillips Hall 039-A, Chapel Hill, NC 27514

7b. ADDRESS (City, State and ZIP Code): Bolling Air Force Base, Washington, DC 20332

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR

8b. OFFICE SYMBOL (If applicable)

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144

8c. ADDRESS (City, State and ZIP Code): Bolling Air Force Base, Washington, DC 20332

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): "Product Stochastic Measures"

12. PERSONAL AUTHOR(S): Dr. Victor Perez-Abreu

13a. TYPE OF REPORT: technical

13b. TIME COVERED: FROM 9/1/85 TO ~~8/31/86~~

14. DATE OF REPORT (Yr., Mo., Day): October 1985

15. PAGE COUNT: 25

16. SUPPLEMENTARY NOTATION: product stochastic measures, multiple integrals, exponential space, vector valued measures, orthogonally scattered measures.

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: [redacted]

22b. TELEPHONE NUMBER (Include Area Code): 919-962-2307 (202) 767-5027

22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

CENTER FOR STOCHASTIC PROCESSES

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Product Stochastic Measures

by

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Technical Report No. 118

October 1985

PRODUCT STOCHASTIC MEASURES

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PRODUCT STOCHASTIC MEASURES*

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ABSTRACT. The concept of symmetric tensor product of a Hilbert space is used to construct a product measure of orthogonally scattered measures. The result is applied to the construction of an L^2-valued product stochastic measure (p.s.m.) of non-identically distributed L^2-valued independently scattered measures. Using the theory of vector valued measures we construct multiple integrals with respect to the p.s.m. A relationship between the theory of multiple stochastic integrals and the theory of vector valued measures is established.

AMS 1980 Subject Classification: Primary 60H05. Secondary 46G10, 28A35, 28B05.

Keywords and phrases: product stochastic measures, multiple integrals, exponential space, vector valued measures, orthogonally scattered measures.

*These results constitute part of the author's Ph.D. dissertation under the direction of Professor G. Kallianpur at the University of North Carolina at Chapel Hill.

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Research supported by AFOSR Grant No. F49620 85 C 0144.

1. INTRODUCTION

Let (T,A) be a measurable space and (Ω,F,P) be a probability space. The random variable valued set function X on (T,A) is said to be an independently scattered measure (i.s.m.) on (T,A) if for each sequence of pairwise disjoint sets $\{A_k\}_{k\geq 1}$ in A, $\{X(A_k)\}_{k\geq 1}$ is a sequence of independent random variables on (Ω,F,P) and

$$X\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} X(A_k) \quad \text{a.s.}$$

We say that X is an $L^2(\Omega)$-valued i.s.m. if $X(A)$ belongs to $L^2(\Omega,F,P)$ for each $A \in A$ and the above series converges in $L^2(\Omega)$.

Let $n \geq 1$ and $X_1,\ldots,X_n$ be $L^2(\Omega)$-valued i.s.m.'s on (T,A). Define the $L^0(\Omega)$-valued product set function $X_1\times\ldots\times X_n$ on the semifield of rectangles of T^n by

$$X_1\times\ldots\times X_n(A_1\times\ldots\times A_n) = X_1(A_1)\ldots X_n(A_n) \tag{1.1}$$

where $A_i \in A$ $i = 1,\ldots,n$. Extend $X_1\times\ldots\times X_n$ in the usual way to be an additive set function on the field generated by the rectangles of T^n. Engel and Kakutani ([4]) have considered the extension of $X_1\times\ldots\times X_n$ to an $L^2(\Omega)$-countably additive measure on (T^n,A^n). The case $n = 2$ has been studied by Rosinski and Szulga [15]. In both works additional higher moment conditions are imposed on the i.s.m.'s $X_1,\ldots,X_n$ to assure that the product stochastic measure $X_1\times\ldots\times X_n$ is $L^2(\Omega)$-valued.

In this work we use the symmetric tensor product rather than the usual product to obtain a vector valued measure on (T^n,A^n) denoted by $X_1\circledast\ldots\circledast X_n$, and called the symmetric tensor product measure (s.t.p.m.). It is such that for $A_i \in A$ $i = 1,\ldots,n$

$$X_1\circledast\ldots\circledast X_n(A_1\times\ldots\times A_n) = X_1(A_1)\circledast\ldots\circledast X_n(A_n) \tag{1.2}$$

where $\circledcirc$ means symmetric tensor product. By doing this we exploit the $L^2(\Omega)$-valued property of $X_1,\ldots,X_n$ using Hilbert space methods.

In general we consider the case when $X_1,\ldots,X_n$ are orthogonally scattered measures (o.s.m.) on (T,A) (not necessarily stochastic) with values in any real separable Hilbert space H (Masani [11]), of which zero mean $L^2(\Omega)$-valued i.s.m.'s are particular examples. Similar to the classical theory of real valued product measures one can construct (Chevet [1, Th. 2.1]) an $H^{\otimes n}$-valued o.s.m. $X_1\otimes\ldots\otimes X_n$ on (T^n,A^n), where $H^{\otimes n}$ is the n-fold tensor product of H and $\otimes$ denotes tensor product. Moreover, if we consider a sequence of o.s.m.'s $\{X_i\}_{i>1}$ on (T,A) with values in H, it is possible to obtain an infinite tensor product valued o.s.m. on (T^∞,A^∞), where $T^\infty = \overset{\infty}{\underset{i=1}{\times}} T$ and $A^\infty = \overset{\infty}{\underset{i=1}{\times}} A$ (Perez-Abreu [14, Th. 2.1.4]). However, in connection with stochastic processes and product stochastic measures the concepts of symmetric tensor product and the Exponential Hilbert space of H (EXP(H)) are more useful. For example, EXP(H) will be the common range of the different powers (in the symmetric tensor product sense) of the s.t.p.m.'s.

In Section 2 of this work we establish a link between the spaces EXP(H) and $L^2(\Omega,\sigma(H),P)$ where H is the Hilbert space direct sum of a Gaussian space and a generalized Poisson space. This result extends the work of Neveu [12] and Kallianpur [8] who established this link in the case H is a Gaussian space.

In Section 3 we take H to be a real separable Hilbert space and construct the symmetric tensor product measure $X_1\circledcirc\ldots\circledcirc X_n$ of the o.s.m. $X_1,\ldots,X_n$ with values in H. This is a vector valued measure with values in EXP(H). Then we use the vector valued measure approach to construct multiple integrals, i.e. the latter are constructed as integrals w.r.t. the s.t.p.m.'s using the theory of integration w.r.t. vector valued measures.

In Section 4 we take $X_1,\ldots,X_n$ to be $L^2(\Omega)$-valued i.s.m.'s and apply the results of the last two sections to construct an $L^2(\Omega)$-valued product stochastic measure also denoted by $X_1 \odot \ldots \odot X_n$. In the construction of this measure we do not need to assume additional higher moment conditions on $X_1,\ldots,X_n$. Further, using the method of Section 3, we construct integrals w.r.t. $X_1 \odot \ldots \odot X_n$, showing that this approach includes the known centered multiple stochastic integrals of [5] [7] [13] and [16]. The measure $X_1 \odot \ldots \odot X_n$ is different from the product stochastic measure $X_1 \times \ldots \times X_n$ in [4]. Comparisons between the two measures is the subject of Section 5.

2. EXPONENTIAL SPACES

Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$. For $h_i \in H$ $i = 1,\ldots,n$ let $h_1\otimes\ldots\otimes h_n$ denote their n-fold tensor product and define

$$S_n(h_1\otimes\ldots\otimes h_n) = \frac{1}{n!}\sum_{\Pi} h_{\Pi(1)}\otimes\ldots\otimes h_{\Pi(n)} \tag{2.1}$$

where $\Pi = (\Pi(1),\ldots,\Pi(n))$ is a permutation of $(1,2,\ldots,n)$ and the sum goes over all such permutations. We recall that the n-fold symmetric tensor product Hilbert space $H^{\odot n}$ is the subspace of $H^{\otimes n}$ which is the closure of the finite linear combinations of elements of the form (2.1), and that the operator S_n can be extended to an orthogonal projection operator on $H^{\otimes n}$ whose range is $H^{\odot n}$. We write $h_1\odot\ldots\odot h_n = S_n(h_1\otimes\ldots\otimes h_n)$ and note that for $h_i, g_i \in H$ $i = 1,\ldots,n$

$$\langle h_1\odot\ldots\odot h_n,\ g_1\odot\ldots\odot g_n\rangle_{H^{\odot n}} = \frac{1}{n!}\sum_{\Pi}\langle h_1, g_{\Pi(1)}\rangle_H\cdots\langle h_n, g_{\Pi(n)}\rangle_H. \tag{2.2}$$

Let us denote by $H^{\odot 0}$ the one dimensional space of real constants and by EXP(H) the orthogonal direct sum of the subspaces $H^{\odot n}$ $n \geq 0$. This space is called the Exponential space of H and in the mathematical physics literature it is known as the Fock space. For the sake of completeness and further reference we review some facts about this Hilbert space (see Guichardet [6]):

i) The elements of EXP(H) are interpreted as sequences $\underline{h} = (h_0, h_1, \ldots)$ $h_n \in H^{\odot n}$ $n \geq 0$ with inner product

$$\langle \underline{h}, \underline{k}\rangle_e = \sum_{n\geq 0}\langle h_i, k_i\rangle_{H^{\odot n}}. \tag{2.3}$$

ii) Of special interest are the "exponential" elements

$$\exp^{\odot}(h) = (1, h, (2)^{-1/2}h^{\odot 2}, \ldots) \qquad h \in H \tag{2.4}$$

which generate EXP(H) and whose inner product is given by

$$\langle \exp^{\circledast}(h), \exp^{\circledast}(k)\rangle_e = \exp\,(\langle h,k\rangle_H). \tag{2.5}$$

iii) For each $n \geq 0$ $H^{\circledast n}$ is seen as a subspace of EXP(H) and for $n \neq m$ $H^{\circledast n}$ and $H^{\circledast m}$ are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_e$. Then we can write

$$\exp^{\circledast}(h) = \sum_{n\geq 0} (n!)^{-1/2} h^{\circledast n}.$$

Let (Ω,F,P) be a complete probability space and H_g be a real Gaussian space of random variables defined on (Ω,F,P). It is well known (Neveu [12], Kallianpur [8]) that

$$\mathrm{EXP}(H_g) \overset{\psi}{\cong} L^2(\Omega,F^g,P) \tag{2.6}$$

where $F^g = \sigma(H_g)$ and for $h \in H_g$

$$\psi(\exp^{\circledast}(h)) = \exp(h - (1/2)E(h^2)) \tag{2.7}$$

and that $\{\psi(\exp^{\circledast}(h)) : h \in H_g\}$ generates $L^2(\Omega,F^g,P)$, where $E(\cdot)$ denotes expected value.

A similar result is possible for the Poisson case: Let q be a centered Poisson random measure on an arbitrary measurable space (S,E) with control measure ν and let

$$H_q = \{I_q(f) : f \in L^2(S,E,\nu)\} \tag{2.8}$$

where $I_q(\cdot)$ denotes the isometric integral w.r.t. the o.s.m. q. The Hilbert space of random variables H_q is called the generalized Poisson space associated with q. The following result is Proposition 7.13 in Neveu [12] (see also Kreé [9]).

PROPOSITION 2.1. Let ν be a finite measure on (S,E). Then

$$\mathrm{EXP}(H_q) \overset{\eta}{\cong} L^2(\Omega,\sigma(H_q),P) \tag{2.9}$$

where $F^q = \sigma(H_q)$ and for $f \in L^2(S,E,\nu)$

$$\eta(\exp \circledast (I_q(f)) = \{\prod_{j=1}^{N(S)} (1 + f(Z_j))\}\exp(-\int_S f(s)d\nu(s)) \tag{2.10}$$

where $\{Z_j\}_{j\geq 1}$ is a sequence of independent random elements, independent of $N(S) = q(S) + \nu(S)$, each Z_j taking values in S and having distribution $\{\nu(S)\}^{-1}\nu(\cdot)$.

The next theorem extends the above result to the case when ν is a σ-finite measure on (S,E). For this situation Surgailis [16] has shown an isometry between $\mathrm{EXP}(H_q)$ and $L^2(\Omega,F^q,P)$. However we cannot use Surgailis' result since he uses multiple Poisson integrals techniques to prove it, and we need to proceed in the opposite direction, namely, we first have to identify $\mathrm{EXP}(H_q)$ and then construct product stochastic measures and multiple stochastic integrals of q(see Section 4).

THEOREM 2.1. Let ν be a σ-finite measure on (S,E). Then

$$\mathrm{EXP}(H_q) \overset{\phi}{\cong} L^2(\Omega,F^q,P) \tag{2.11}$$

where $F^q = \sigma(H_q)$ and for $f \in L^2(S,E,\nu)$

$$\phi(\exp \circledast I_q(f)) = \{\prod_{i=1}^{\infty}\prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp(-\int_{S_i} f(s)d\nu(s))\} \tag{2.12}$$

where: (i) S_i $i \geq 1$ are disjoint sets in E, $0 < \nu(S_i) < \infty$ and $\bigcup_{i\geq 1} S_i = S$; (ii) for each $i = 1,2,\ldots$ and $j = 1,2,\ldots$ $Z_j^{(i)}$ is an S_i-valued random element with distribution given by the measure $\nu(S_i)^{-1}\nu(\cdot)$, and for each $i = 1,2,\ldots$ $N(S_i)$ follows a Poisson distribution with parameter $\nu(S_i)$; (iii) $Z_j^{(i)}$, $N(S_i)$ $i = 1,2,\ldots$, $j = 1,2,\ldots$ are mutually independent.

In order to prove this theorem we will use the following technical result which proof follows easily.

LEMMA 2.1. Let ν and S_i, $N(S_i)$, $Z_j^{(i)}$ $j = 1,2,\ldots$, $i = 1,2,\ldots$ be as in (i) - (iii) of the above theorem. If for some $i \geq 1$ $g \in L^1(S_i, E \cap S_i, \nu)$ then

$$E\left(\prod_{j=1}^{N(S_i)} g(Z_j^{(i)})\right) = \exp\left(\int_{S_i} (g-1)\,d\nu\right).$$

Proof of Theorem 2.1. We have to show the following three conditions:

a) for each $f \in L^2(S,E,\nu)$ $\phi(\exp \circledcirc (I_q(f))) \in L^2(\Omega, F^q, P)$.

b) for f_1, $f_2 \in L^2(S,E,\nu)$

$$E(\phi(\exp \circledcirc (I_q(f_1)))\phi(\exp \circledcirc (I_q(f_2)))) = \exp\left(\int_S f_1 f_2\, d\nu\right).$$

c) $\{\phi(\exp \circledcirc (I_q(f))) : f \in L^2(S,E,\nu)\}$ generates $L^2(\Omega, F^q, P)$.

Since ν is a σ-finite measure on (S,E) there exists a sequence of sets $\{S_i\}_{i \geq 1}$ in E such that $0 < \nu(S_i) < \infty$ and $\bigcup_{i=1}^{\infty} S_i = S$. The existence of the random elements $Z_j^{(i)}$ $j=1,2,\ldots$, $i=1,2,\ldots$ satisfying (ii) and (iii) follows from the construction of a Poisson random measure N with control measure ν.

Let $f \in L^2(S,E,\nu)$, then for each $i \geq 1$ f belongs to $L^2(S_i, E \cap S_i, \nu)$ and $L^1(S_i, E \cap S_i, \nu)$. Then by taking $g = (1+f)$ in Lemma 2.1 we obtain

$$E\left[\prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp\left(-\int_{S_i} f\,d\nu\right)\right] = 1 \quad i = 1,2,\ldots \quad .$$

Then using (iii) $G_i = \prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp(-\int_{S_i} f\,d\nu)$ is a sequence of independent random variables with $E(G_i) = 1$ $i \geq 1$ and therefore $D_n = \prod_{i=1}^{n} G_i$ is a martingale. Next, using Lemma 2.1 with $g = (1+f)^2$ and the independence of $Z_j^{(i)}$, $N(S_i)$

$j=1,2,\ldots,\ i=1,2,\ldots$

$$E\,D_n^2 = \prod_{i=1}^{n} E\left(\prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp(-\textstyle\int_{S_i} f d\nu)\right)^2 = \exp(\textstyle\int_{\bigcup_{i=1}^{n} S_i} f^2 d\nu) \le \exp(\textstyle\int_S f^2 d\nu) < \infty.$$

Then by the martingale convergence theorem D_n converges a.s. and in mean square to $\phi(\exp \circledast (I_q(f)))$. Therefore

$$E\left(\prod_{i=1}^{\infty} \prod_{j=1}^{N(S_i)} (1+f(Z_k^{(i)}))\exp(-\textstyle\int_{S_i} f d\nu)\right)^2 = \lim_{n\to\infty} E\,D_n^2 = \lim_{n\to\infty} \exp(\textstyle\int_{\bigcup_{i=1}^{n} S_i} f^2 d\nu) = \exp(\textstyle\int_S f^2 d\nu) < \infty$$

which shows (a).

Let $f_1, f_2 \in L^2(S,E,\nu)$, then applying Lemma 2.1 to $g = (1+f_1)(1+f_2)$ one shows in a similar way as above that

$$E(\exp\circledast(I_q(f_1))\exp\circledast(I_q(f_2))) = \lim_{n\to\infty} \prod_{i=1}^{n} E\left(\prod_{j=1}^{N(S_i)} (1+f_1)(1+f_2)(Z_j^{(i)})\exp(-\textstyle\int_{S_i} (f_1^2+f_2^2) d\nu)\right)$$

$$= \lim_{n\to\infty} \prod_{i=1}^{n} \exp(\textstyle\int_{S_i} f_1 f_2 d\nu) = \exp(\textstyle\int_{S_i} f_1 f_2 d\nu)$$

proving (b).

Finally, to prove (c) let $G \in L^2(\Omega, F^q, P)$ and suppose that

$$E(\exp\circledast(I_q(f))G) = 0 \text{ for all } f \in L^2(S,E,\nu).$$

We have to show that $G=0$ a.e. dP_{F^q}. Using (2.12) we have that for all $f \in L^2(S,E,\nu)$

$$E\left(\prod_{i=1}^{\infty}\left\{\prod_{j=1}^{N(S_i)} (1+f(Z_j^{(i)}))\exp(-\textstyle\int_{S_i} f d\nu)\right\}G\right) = 0.$$

Next let $i \ge 1$ be fixed and for $g \in L^2(S_i, E\cap S_i, \nu)$ define $f: S \to \mathbb{R}$ by $f(t) = g(t)$ $t \in S_i$ and zero if $t \notin S_i$. Then $f \in L^2(S,E,\nu)$ and

$$E\left[\prod_{j=1}^{N(S_i)} (1+g(Z_j^{(i)}))\exp(-\int_{S_i} g d\nu) G\right] = 0 \quad \text{all } g \in L^2(S_i, E\cap S_i, \nu)$$

Hence by Proposition 2.1 $E(G|F_i^q) = 0$ a.s. where $F_i^q = \sigma(I_q(g): g\in L^2(S_i, E\cap S_i,\nu))$, and $F_i^q \subset F^q$ all $i\geq 1$. Thus for all $n\geq 1$ $E(G| \bigvee_{i=1}^{n} F_i^q)=0$ a.s. since $F_1^q,\ldots,F_n^q$ are independent σ-fields. Let $F_n = \bigvee_{i=1}^{n} F_i^q$ then $F^q = \bigvee_{n=1}^{\infty} F_n$.

Thus since $E(G^2) < \infty$ it follows by the martingale convergence theorem that $G=0$ a.s. dP_{F^q} and the theorem is proved.

The last result of this section is a general one, in the sense that it identifies the exponential space of any Hilbert space H which is a direct sum of an arbitrary Gaussian space H_g and an arbitrary Poisson space H_q, where H_g and H_q are stochastically independent.

THEOREM 2.2. Let (Ω,F,P) be a complete probability space and q be a centered Poisson random measure on a measurable space (S,E) defined on (Ω,F,P), with σ-finite control measure ν and generating the Poisson space (2.8). Let H_g be a Gaussian space on (Ω,F,P) stochastically independent of the system of random variables H_q. Define the σ-fields $F^g = \sigma(H_g)$, $F^q = \sigma(H_q)$ and the Hilbert space $H = H_g \oplus H_q$. Then

$$\mathrm{EXP}(H) \overset{\gamma}{\cong} L^2(\Omega, F^g \vee F^q, P) \tag{2.13}$$

where for $h \in H$, $h = h_g + h_q$, $h_g \in H_g$, $h_q \in H_q$, $\gamma: \mathrm{EXP}(H) \to L^2(\Omega, F^g\vee F^q, P)$ is defined by

$$\gamma(\exp \odot (h)) = \psi(\exp \odot (h_g))\phi(\exp \odot (h_q)) \tag{2.14}$$

where ψ and φ are the isometries given in (2.7) and (2.12) respectively.

Proof. It follows by the independence of H_g and H_q that for all $h \in H$ $\psi(\exp \circledast (h))$ is an element of $L^2(\Omega, F^g \vee F^q, P)$ and that $E(\psi(\exp\circledast(h)))^2 = \exp(\|h\|^2)$.

Next we shall prove that $\{\psi(\exp\circledast(h)) : h \in H\}$ generates $L^2(\Omega, F^g \vee F^q, P)$. Let $Z \in L^2(\Omega, F^g \vee F^q, P)$ and suppose that

$$E(Z\psi(\exp \circledast (h))) = 0 \quad \text{for each } h \in H.$$

Then for all $h_g \in H_g$ and $h_q \in H_q$ $E(Z\psi(\exp \circledast (h_g))\psi(\exp\circledast(h_q))) = 0$.

But $\{\psi(\exp \circledast (h_g)) : h_g \in H_g\}$ and $\{\psi(\exp \circledast (h_q)) : h_q \in H_q\}$ generate $L^2(\Omega, F^g, P)$ and $L^2(\Omega, F^q, P)$ respectively. Then for all $A_1 \in F^g$ and $A_2 \in F^q$ $\int_{A_1 \cap A_2} Z dP = 0$. But since the σ-fields F^g and F^q are independent then $F^g \vee F^q$ is generated by the field C_0 of all finite disjoint unions of sets $A_1 \cap A_2$ $A_1 \in F^g$, $A_2 \in F^q$. Thus since Z is P-integrable $C = \{A \in F : \int_A Z dP = 0\}$ is a monotone class, and by the monotone class theorem

$$\int_A Z dP = 0 \quad \forall\, A \in F^g \vee F^q$$

since $C_0 \subset C$. That is, $Z = 0$ a.e. $dP_{F^g \vee F^q}$ and the theorem is proved.

3. THE SYMMETRIC TENSOR PRODUCT MEASURE

Throughout this section we assume, unless otherwise stated, that (T,A) is a measurable space, H is a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $||\cdot||_H$. Furthermore for $n \geq 1$ let X_i $i=1,\ldots,n$ be o.s.m.'s on A, taking values in H and with corresponding finite control measures μ_i $i=1,\ldots,n$. Define the signed measures μ_{ij} on A such that for $i,j=1,\ldots,n$

$$\mu_{ij}(A \cap B) = \langle X_i(A), X_j(B)\rangle_H. \tag{3.1}$$

If $i_1,j_1,\ldots,i_n,j_n \in \{1,\ldots,n\}$ let $\mu_{i_1j_1}\otimes\ldots\otimes\mu_{i_nj_n}$ denote the real valued product measure on (T^n,A^n) of $\mu_{i_1j_1},\ldots,\mu_{i_nj_n}$.

For $A_i \in A$ $i=1,\ldots,n$ define the $H^{\circledast n}$-valued set function $X_1\circledast\ldots\circledast X_n$ on the semifield of rectangles of T^n by

$$X_1\circledast\ldots\circledast X_n(A_1\times\ldots\times A_n) = X_1(A_1)\circledast\ldots\circledast X_n(A_n). \tag{3.2}$$

This vector valued function can be extended in an additive manner to the field generated by the rectangles in A^n. The next result gives the extension of $X_1\circledast\ldots\circledast X_n$ to the σ-field A^n. This $H^{\circledast n}$-valued measure is called the *symmetric tensor product measure*. Observe that for each $n \geq 1$ it is an EXP(H)-valued measure.

THEOREM 3.1. Under the above hypotheses and notation there exists a unique $H^{\circledast n}$-valued measure $X_1\circledast\ldots\circledast X_n$ on (T^n,A^n) such that (3.2) is satisfied for $A_i \in A$ $i=1,\ldots,n$ and for $A \in A^n$

$$||X_1\circledast\ldots\circledast X_n(A)||^2_{H^{\circledast n}} \leq \mu_1\otimes\ldots\otimes\mu_n(A) \tag{3.3}$$

Proof. Let $X_1\otimes\dots\otimes X_n$ be the $H^{\otimes n}$-valued o.s.m. on (T^n, A^n) such that $X_1\otimes\dots\otimes X_n(A_1\times\dots\times A_n) = X_1(A_1)\otimes\dots\otimes X_n(A_n)$ and

$$||X_1\otimes\dots\otimes X_n(A)||^2_{H^{\otimes n}} = \mu_1\otimes\dots\otimes\mu_n(A) \tag{3.4}$$

(see [1, Th. 2.1] or [14, Th. 2.1.3]). For $A \in A^n$ define

$$X_1\odot\dots\odot X_n(A) = S_n(X_1\otimes\dots\otimes X_n(A))$$

where S_n is the projection operator on $H^{\otimes n}$ with range $H^{\odot n}$ defined in (2.1). Then

$$||X_1\odot\dots\odot X_n(A)||_{H^{\odot n}} \leq ||X_1\otimes\dots\otimes X_n(A)||_{H^{\otimes n}}.$$

and the σ-additive property of $X_1\odot\dots\odot X_n$ in $H^{\odot n}$ follows by the linearity and continuity of S_n. Finally (3.3) follows from (3.4) and the last inequality. Q.E.D.

Expression (3.3) gives an upper bound for the norm of $X_1\odot\dots\odot X_n(A)$. We shall now obtain an exact expression for this norm which uses the signed measures μ_{ij} defined in (3.1). For $A \in A^n$ and $\Pi = (\Pi(1),\dots,\Pi(n))$ a permutation of $(1,\dots,n)$ define

$$A^{\Pi} = \{(t_1,\dots,t_n) \in T^n\colon (t_{\Pi(1)},\dots,t_{\Pi(n)}) \in A\}. \tag{3.5}$$

LEMMA 3.1. a) For $A \in A^n$ and $B \in A^m$

$$\langle X_1\odot\dots\odot X_n(A), X_1\odot\dots\odot X_m(B)\rangle_e = \delta_{nm}\langle X_1\odot\dots\odot X_n(A), X_1\odot\dots\odot X_n(B)\rangle_{H^{\odot n}}$$

$$= (\delta_{nm}/n!)\sum_{\Pi}\mu_{1\Pi(1)}\otimes\dots\otimes\mu_{n\Pi(n)}(A\cap B^{\Pi}) = (\delta_{nm}/n!)\sum_{\Pi}\mu_{1\Pi(1)}\otimes\dots\otimes\mu_{n\Pi(n)}(A^{\Pi}\cap B).$$

b) For $A \in A^n$

$$||X_1\odot\dots\odot X_n(A)||^2_{H^{\odot n}} = (n!)^{-1}\sum_{\Pi}\mu_{1\Pi(1)}\otimes\dots\otimes\mu_{n\Pi(n)}(A\cap A^{\Pi}).$$

Proof. Since $X_1\circledcirc\ldots\circledcirc X_n$ and $X_1\circledcirc\ldots\circledcirc X_m$ are $H^{\circledcirc n}$ and $H^{\circledcirc m}$ valued respectively and $H^{\circledcirc n}$ and $H^{\circledcirc m}$ are orthogonal subspaces of EXP(H), then $X_1\circledcirc\ldots\circledcirc X_n$ and $X_1\circledcirc\ldots\circledcirc X_m$ are orthogonal for $n\neq m$. Hence assume $n=m$ and let $A = A_1\times\ldots\times A_n$, $B = B_1\times\ldots\times B_n$ where $A_i, B_i \in \mathcal{A}$ $i=1,\ldots,n$. Then from Theorem 3.1, (3.2) and (2.2)

$$<X_1\circledcirc\ldots\circledcirc X_n(A), X_1\circledcirc\ldots\circledcirc X_n(B)>_{H^{\circledcirc n}} = <X_1(A_1)\circledcirc\ldots\circledcirc X_n(A_n), X_1(B_1)\circledcirc\ldots\circledcirc X_n(B_n)>_{H^{\circledcirc n}}$$

$$= (n!)^{-1}\sum_{\Pi} <X_1(A_1), X_{\Pi(1)}(B_{\Pi(1)})>_H\cdots<X_n(A_n), X_{\Pi(n)}(B_{\Pi(n)})>_H$$

$$= (n!)^{-1}\sum_{\Pi} \mu_{1\Pi(1)}(A_1\cap B_{\Pi(1)})\cdots\mu_{n\Pi(n)}(A_n\cap B_{\Pi(n)})$$

$$= (n!)^{-1}\sum_{\Pi} \mu_{1\Pi(1)}\otimes\ldots\otimes\mu_{n\Pi(n)}((A_1\times\ldots\times A_n) \cap (B_1\times\ldots\times B_n)^{\Pi})$$

$$= (n!)^{-1}\sum_{\Pi} \mu_{1\Pi(1)}\otimes\ldots\otimes\mu_{n\Pi(n)}(A\cap B^{\Pi}).$$

which shows (a) if A,B are rectangles. The result is extended in an obvious manner to the field generated by the rectangles. Finally an approximation argument shows (a) for any $A,B \in \mathcal{A}^n$ (see details in [14]). The proof of (b) follows from (a) by taking B=A.

COROLLARY 3.1. Let $\mathcal{A}^{\circledcirc n} = \{A \in \mathcal{A}^n : A^{\Pi} = A \text{ for all } \Pi\}$. Then the vector measure $X_1\circledcirc\ldots\circledcirc X_n$ is an $H^{\circledcirc n}$-valued orthogonally scattered measure on $(T^n, \mathcal{A}^{\circledcirc n})$ with control measure $\mu^{\circledcirc n}$ given by

$$\mu^{\circledcirc n}(A) = (n!)^{-1}\sum_{\Pi} \mu_{1\Pi(1)}\otimes\ldots\otimes\mu_{n\Pi(n)}(A) \qquad A \in \mathcal{A}^{\circledcirc n}.$$

Proof. Since $A \in \mathcal{A}^{\circledcirc n}$ implies $A^{\Pi} = A$ for all Π, then $A \cap A^{\Pi} = A$ for all Π and therefore using Lemma 3.1

$$||X_1 \circledast \ldots \circledast X_n(A)||^2_{H^{\circledast n}} = (n!)^{-1} \sum_{\Pi} \mu_{1\Pi(1)} \otimes \ldots \otimes \mu_{n\Pi(n)}(A)$$

from which the corollary follows.

The above results can be obtained, with the obvious modifications, if the measures $\mu_1, \ldots, \mu_n$ are assumed σ-finite. However we have restricted ourselves to the case where each o.s.m. is bounded and defined on a σ-field. The reason for these requirements is that we are primarily interested in using the well established theory of vector valued measures in order to construct multiple integrals w.r.t. $X_1 \circledast \ldots \circledast X_n$, and in this theory these requirements are needed. Moreover from now on we will assume that $T = [0,1]$, $A = \mathcal{B}(T)$ and $\mu_1, \ldots, \mu_n$ are finite non-atomic measures. For terminology and concepts from the theory of vector valued measures we refer to Diestel and Uhl [2] and Kussmaul [10]. An important notion in this theory is that of semivariation of a vector valued measure. It plays a key role in constructing integrals. The next lemma gives useful upper and lower bounds for the semivariation of the EXP(H)-valued measure $X_1 \circledast \ldots \circledast X_n$ on a set $A \in A^n$, which we denote by $sv(X_1 \circledast \ldots \circledast X_n; A)$.

<u>LEMMA 3.2.</u> For $A \in A^n$

$$(n!)^{-1}\{\mu_1 \otimes \ldots \otimes \mu_n(A)\}^{1/2} \leq sv(X_1 \circledast \ldots \circledast X_n; A) \leq \{\mu_1 \otimes \ldots \otimes \mu_n(A)\}^{1/2}. \tag{3.6}$$

<u>Proof.</u> For each permutation $\Pi = (\Pi(1), \ldots, \Pi(n))$ of $(1, \ldots, n)$ let

$$T^n_{\Pi} = \{(t_1, \ldots, t_n) \in T^n;\ t_{\Pi(1)} < \ldots < t_{\Pi(n)}\}. \tag{3.7}$$

Note that if Π and Π^* are two distinct permutations of $(1, \ldots, n)$, then T^n_{Π} and $T^n_{\Pi^*}$ are disjoint. Hence since the measures $\mu_1, \ldots, \mu_n$ are non-atomic

$$\mu_1\otimes\ldots\otimes\mu_n(A) = \sum_{\Pi} \mu_1\otimes\ldots\otimes\mu_n(A\cap T^n_\Pi). \tag{3.8}$$

Next for each Π $(T^n_\Pi)^{\Pi^*} \cap T^n_\Pi = \phi$ for each Π^* distinct from the identity permutation. Thus using Lemma 3.1(b)

$$\sum_{\Pi} ||X_1\otimes\ldots\otimes X_n(A \cap T^n_\Pi)||^2_{H^{\otimes n}} = (n!)^{-1}\mu_1\otimes\ldots\otimes\mu_n(A)$$

and from Proposition 11 in Diestel and Uhl [2]

$$n! \sum_{\Pi} ||X_1\otimes\ldots\otimes X_n(A \cap T^n_\Pi)||^2_{H^{\otimes n}} \leq n! \sum_{\Pi} \{sv(X_1\otimes\ldots\otimes X_n;A\}^2$$

$$= (n!)^2\{sv(X_1\otimes\ldots\otimes X_n;A\}^2$$

from which the first inequality follows. The proof of the second inequality in (3.6) follows using the definition of $sv(X_1\otimes\ldots\otimes X_n;A)$ and the fact that for all finite collection of real numbers (α_j) $j=1,\ldots,m$ such that $|\alpha_j| \leq 1$ and $A_1,\ldots,A_m$ disjoint elements in A^n such that $\bigcup_{j=1}^{m} A_j=A$, the following inequality holds

$$||\sum_{j=1}^{m} {}_jX_1\otimes\ldots\otimes X_n(A_j)||^2_{H^{\otimes n}} \leq \mu_1\otimes\ldots\otimes\mu_n(A). \qquad \text{Q.E.D.}$$

The next two consequences of the above lemma characterize convergence in $X_1\otimes\ldots\otimes X_n$-measure (Kussmaul ([10]) in terms of the control measures $\mu_1,\ldots,\mu_n$.

<u>COROLLARY 3.2</u>. For a sequences $\{A_m\}_{m\geq 1}$ in A^n $sv(X_1\otimes\ldots\otimes X_{nj};A_m)$ goes to zero if and only if $\mu_1\otimes\ldots\otimes\mu_n(A_m)$ goes to zero as $m \to \infty$.

<u>COROLLARY 3.3</u>. A sequence of real valued A^n-measurable functions $(f_m)_{m\geq 1}$ converges in $X_1\otimes\ldots\otimes X_n$-measure to a real function f if and only if f_m converges to f in $\mu_1\otimes\ldots\otimes\mu_n$-measure.

We now present the definition of an $X_1\otimes\ldots\otimes X_n$-integrable function in the sense of the theory of integration with respect to bounded vector valued measures ([10, Def. 10.3]).

DEFINITION 3.1. Let $f(t)$ be an A^n-measurable simple function on T^n, that is

$$f(\underline{t}) = \sum_{j=1}^{k} \alpha_j 1_{A_j}(\underline{t}) \qquad \underline{t} = (t_1,\ldots,t_n) \tag{3.9}$$

where $\alpha_j \in \mathbb{R}$ $j=1,\ldots,k$ and $A_1,\ldots,A_k$ are disjoint elements in A^n. The integral of f with respect to $X_1 \circledast \ldots \circledast X_n$ denoted by $\int_{T^n} f(\underline{t}) \; d(X_1 \circledast \ldots \circledast X_n)(\underline{t})$, is the element of $H^{\circledast n}$ given by

$$\int_{T^n} f(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t}) = \sum_{j=1}^{k} \alpha_j X_1 \circledast \ldots \circledast X_n(A_j). \tag{3.10}$$

A real valued function f on T^n is said to be $X_1 \circledast \ldots \circledast X_n$-integrable if there exists a sequence $\{f_m\}_{m \geq 1}$ of A^n-measurable simple functions on T^n such that

i) f_m converges to f in $X_1 \circledast \ldots \circledast X_n$-measure and

ii) $\lim \int_{T^n} (1_A f_m)(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t}) = 0$ as $sv(X_1 \circledast \ldots \circledast X_n; A) \to 0$

uniformly in $m = 1,2,\ldots$.

PROPOSITION 3.1. Let $f(\underline{t})$ $\underline{t} \in T^n$ be an $X_1 \circledast \ldots \circledast X_n$-integrable function and $\{f_m\}_{m \geq 1}$ be a sequence of A^n-simple functions satisfying (i) and (ii) above. Then for every $A \in A^n$ $(1_A f)(\underline{t})$ is $X_1 \circledast \ldots \circledast X_n$-integrable and the sequence

$$\{\int_{T^n} (1_A f_m)(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t})\}_{m \geq 1}$$

converges to an element in $H^{\circledast n}$ uniformly in $A \in A^n$. The element

$$\int_{T^n} (1_A f)(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t}) = \lim_{m \to \infty} \int_{T^n} (1_A f_m)(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t})$$

is called the integral of f with respect to $X_1 \circledast \ldots \circledast X_n$ over the set A.

Proof. See ([10]) who proves it for any bounded vector valued measure.

The next result characterizes $X_1\otimes\cdots\otimes X_n$-integrability in terms of the measures $\mu_1,\ldots,\mu_n$.

THEOREM 3.2. A real function f is $X_1\otimes\cdots\otimes X_n$-integrable if and only if $f \in L^2(T^n, A^n, \mu_1\otimes\cdots\otimes\mu_n)$.

Proof. Assume $f \in L^2(\mu_1\otimes\cdots\otimes\mu_n)$, then there exists a sequence $\{f_m\}_{m\geq 1}$ of A^n-measurable simple functions such that $|f_m| \leq |f|$ a.e. $d(\mu_1\otimes\cdots\otimes\mu_n)$ for $m \geq 1$ and f_m converges to f in $L^2(\mu_1\otimes\cdots\otimes\mu_n)$. Then by Corollary 3.3 f_m converges to f in $X_1\otimes\cdots\otimes X_n$-measure. Thus condition (i) in Definition 3.1 is satisfied.

Next for $A \in A^n$ $1_A(\underline{t})(f_m(\underline{t}) - f_k(\underline{t})) = \sum_{j=1}^{\ell} \alpha_j 1_{A_j}(\underline{t})$ for some $\alpha_j \in \mathbb{R}$ and $A_1,\ldots,A_\ell$ disjoint elements in A^n. Then

$$||\int_{T^n}(1_A(f_m-f_k))(\underline{t})\, d(X_1\otimes\cdots\otimes X_n)(\underline{t})||^2_{H^{\otimes n}} = ||\sum_{j=1}^{\ell} \alpha_j (X_1\otimes\cdots\otimes X_n)(A_j)||^2_{H^{\otimes n}}$$

$$\leq \sum_{j=1}^{\ell} \alpha_j^2 \mu_1\otimes\cdots\otimes\mu_n(A_j) = \int_{T^n} 1_A(\underline{t})|f_m(\underline{t}) - f_k(\underline{t})|^2 d(\mu_1\otimes\cdots\otimes)(\underline{t})$$

which goes to zero as $m,k \to \infty$ for $A \in A^n$. Therefore for each $A \in A^n$ $\lim_{m\to\infty} \int_A f_m(\underline{t})d(X_1\otimes\cdots\otimes X_n)(\underline{t})$ exists. Next, since for each $A \in A^n$ and $m \geq 1$ $1_A f_m$ is a simple function, i.e. there exist α_j^m $j=1,\ldots,\ell$ and disjoint elements $A_1,\ldots,A_\ell$ of A^n such that

$$1_A f_m(\underline{t}) = \sum_{j=1}^{\ell} \alpha_j^m 1_{A_j}(\underline{t}),$$

then from (3.10) and the definition of $sv(X_1\otimes\cdots\otimes X_n);A)$

$$||\int_A f_m(\underline{t})d(X_1\circledast\ldots\circledast X_n)(\underline{t})||_{H^{\circledast n}} = ||\sum_{j=1}^{\ell}\alpha_j^m X_1\circledast\ldots\circledast X_n\ (A)||_{H^{\circledast n}}$$

$$= ||f_m||_\infty\ ||\sum_{j=1}^{\ell}\frac{\alpha_j^m}{||f_m||_\infty}\ X_1\circledast\ldots\circledast X_n\ (A)||_{H^{\circledast n}} \le ||f_m||_\infty sv(X_1\circledast\ldots\circledast X_n;A)$$

where

$$||f_m||_\infty = \sup_{\underline{t}\in T^n} ||f_m(\underline{t})|| = \max(|\alpha_1^m|,\ldots,|\alpha_\ell^m|).$$

Hence we have that for each $m\geq 1$ $\int_{(\cdot)} f_m(\underline{t})d(X_1\circledast\ldots\circledast X_n)(\underline{t})$ is $sv(X_1\circledast\ldots\circledast X_n,\cdot)$-continuous. Then by the Vitali-Hahn-Saks Theorem (Dunford and Schwartz [3])

$$||\int_{T^n}(1_A f_m)(\underline{t})d(X_1\circledast\ldots\circledast X_n)(\underline{t})||_{H^{\circledast n}} \to 0 \text{ as } sv(X_1\circledast\ldots\circledast X_n;A)\to 0$$

uniformly in $m=1,2,\ldots$. Then condition (ii) in Definition 3.1 is satisfied and hence f is $X_1\circledast\ldots\circledast X_n$-integrable.

Now assume that f is $X_1\circledast\ldots\circledast X_n$-integrable, i.e. there exists a sequence $\{f_m\}_{m\geq 1}$ of A^n-measurable simple functions that satisfies conditions (i) and (ii) of Definition 3.1. Next, since for each k,m f_m-f_k is a simple function, $f_m(\underline{t})-f_k(\underline{t}) = \sum_{j=1}^{\ell}\alpha_j 1_{A_j}(\underline{t})$ say where $\alpha_j \in \mathbb{R}$ $j=1,\ldots,\ell$ and $A_1,\ldots,A_\ell$ are disjoint elements in A^n, then for each $A\in A^n$ using Lemma 3.1(a)

$$||\int_{T^n}1_A(f_m-f_k)(\underline{t})\ d(X_1\circledast\ldots\circledast X_n)(\underline{t})||^2_{H^{\circledast n}} =$$

$$<\int_{T^n}1_A(f_m-f_k)(\underline{t})d(X_1\circledast\ldots\circledast X_n)(\underline{t}),\ \int_{T^n}1_A(f_m-f_k)(\underline{t})d\ (X_1\circledast\ldots\circledast X_n)(\underline{t})>_{H^{\circledast n}}$$

$$= \sum_{j_1=1}^{\ell}\sum_{j_2=1}^{\ell}\alpha_{j_1}\alpha_{j_2} <X_1\circledast\ldots\circledast X_n(A_{j_1}\cap A),\ X_1\circledast\ldots\circledast X_n(A_{j_2}\cap A)>_{H^{\circledast n}}$$

$$= \frac{1}{n!}\sum_{j_1=1}^{\ell}\sum_{j_2=1}^{\ell}\alpha_{j_1}\alpha_{j_2}\sum_{\Pi}\mu_{1\Pi(1)}\circledast\ldots\circledast\mu_{n\Pi(n)}(A_{j_1}\cap A\cap(A_{j_2}\cap A)^{\Pi})$$

$$= \frac{1}{n!} \sum_{\Pi} \int_{T^n} 1_{A \cap A^{\Pi}}(\underline{t})(f_m - f_k)(\underline{t})(f_m - f_k)_{\Pi}(\underline{t}) d\mu_{1\Pi(1)} \otimes \ldots \otimes \mu_{n\Pi(n)}(\underline{t}).$$

Next for each permutation Π T^n_{Π} defined in (3.7) is such that $T^n_{\Pi} \cap (T^n_{\Pi})^{\Pi^*} = \phi$ for each permutation Π^* distinct from the identity permutation. Then for each Π the above expression simplifies as follows

$$\left|\left|\int_{T^n} 1_{T^n_{\Pi}}(f_m - f_k)(\underline{t})\ d(X_1 \otimes \ldots \otimes X_n)(\underline{t})\right|\right|^2_{H^{\otimes n}} =$$

$$(n!)^{-1} \int_{T^n} 1_{T^n_{\Pi}}(\underline{t})(f_m(\underline{t}) - f_k(\underline{t}))^2 d(\mu_1 \otimes \ldots \otimes \mu_n)(\underline{t}).$$

But from Proposition 3.1 if $m,k \to \infty$ $\left|\left|\int_{T^n}(1_A(f_m - f_k)(\underline{t}) d(X_1 \otimes \ldots \otimes X_n)(\underline{t})\right|\right|^2_{H^{\otimes n}}$ converges to zero uniformly in $A \in A^n$. Then if $S^n = \bigcup_{\Pi} T^n_{\Pi}$

$$\int_{T^n} 1_{S^n}(\underline{t})(f_m(\underline{t}) - f_k(\underline{t}))^2 d(\mu_1 \otimes \ldots \otimes \mu_n)(\underline{t}) \to 0 \quad \text{as } m,k \to \infty$$

and since the measures $\mu_1, \ldots, \mu_n$ are non-atomic $\mu_1 \otimes \ldots \otimes \mu_n((S^n)^c) = 0$ which implies that $\int_{T^n} |f_m(\underline{t}) - f_k(\underline{t})|^2 d(\mu_1 \otimes \ldots \otimes \mu_n)(\underline{t}) \to 0$ as $m,k \to \infty$.

Thus $\{f_m\}_{m \geq 1}$ is a Cauchy sequence in $L^2(\mu_1 \otimes \ldots \otimes \mu_n)$ and since by Corollary 3.3 $f_m \to f$ in $\mu_1 \otimes \ldots \otimes \mu_n$-measure then f belongs to $L^2(T^n, A^n, \mu_1 \otimes \ldots \otimes \mu_n)$. The proof of the theorem is complete.

Since the multiple integral $\int_{T^n} f(\underline{t}) d(X_1 \otimes \ldots \otimes X_n)(\underline{t})$ has been constructed using the theory of integration w.r.t. vector valued measures this integral inherits all properties from that theory including a dominated convergence theorem (see [10] and [14]). Additional properties are also available for this integral which uses the special structure of symmetric tensor product of

$X_1 \circledast \ldots \circledast X_n$ (see [14]). In particular we have the important properties of $\int_{T^n} f(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t})$ being $H^{\circledast n}$-valued (and hence EXP(H)-valued) and that for $n \neq m$

$$\langle \int_{T^n} f(\underline{t}) d(X_1 \circledast \ldots \circledast X_n)(\underline{t}), \int_{T^m} g(\underline{t}) d(X_1 \circledast \ldots \circledast X_m)(\underline{t}) \rangle_e = 0.$$

4. PRODUCT STOCHASTIC MEASURES

We now apply the results of Sections 2 and 3 to the construction of a (symmetric tensor) product stochastic measure.

THEOREM 4.1. Let (Ω, F, P) be a probability space, $T = [0,1]$, $A = B(T)$ and $X_1, \ldots, X_n$ be a system of $n \geq 1$ zero mean independently scattered measures on (T, A) such that $Y_t = (X_1(t), \ldots, X_n(t))$ is an n-dimensional stochastic process with independent increments, where $X_i(t) = X_i((0,t])$ for $i = 1, \ldots, n$. Then there exists a unique $L^2(\Omega)$-valued *Product Stochastic Measure* $X_1 \circledast \ldots \circledast X_n$ on (T^n, A^n) such that for $A_i \in A$ $i=1,\ldots,n$

$$X_1 \circledast \ldots \circledast X_n (A_1 \times \ldots \times A_n) = X_1(A_1) \circledast \ldots \circledast X_n(A_n). \tag{4.1}$$

The symmetric tensor product in RHS of (4.1) is an element of the space $H^{\circledast n}$ for $H = H_g \oplus H_q$ where H_g and H_q are the Gaussian and generalized Poisson spaces associated with the $L^2(\Omega)$-independent increments process Y_t.

Proof. Let g_t $t \in T$ be the n-dimensional Gaussian process with independent increments in the Levy-Itô decomposition of Y_t. Let q be the centered Poisson random measure on $(T \times \mathbb{R}_o^n, A \times B(\mathbb{R}_o^n))$ $(\mathbb{R}_o^n = \mathbb{R}^n \setminus \{0\})$ associated with the jumps of Y_t and having σ-finite control measure μ. Define the Gaussian Hilbert space subspace of $L^2(\Omega)$

$$H_g = \overline{sp}\{\underline{a}' Y(A) : \underline{a} \in \mathbb{R}^n, A \in A\} \tag{4.2}$$

where $Y(A) = (X_1(A),\ldots,X_n(A))$, and let H_q be the generalized Poisson Hilbert space defined in (2.8) for $(S,E,\nu) = (T\times \mathbb{R}_0^n, A \times B(\mathbb{R}_0^n),\mu)$. Define the Hilbert space of random variables $H = H_g \oplus H_q$ which has inner product $\langle\cdot,\cdot\rangle_H = E(\cdot\cdot)$. Then for each $i=1,\ldots,n$ X_i is an orthogonally scattered measure on (T,A) with values in H and control measure

$$\mu_i(A) = E(X_i(A))^2. \tag{4.3}$$

Then the theorem follows applying Theorem 2.2 and Theorem 3.1.

The basic properties of the product stochastic measure $X_1 \circledast \ldots \circledast X_n$ are presented in the following result.

LEMMA 4.1. Let $X_1,\ldots X_n$ and $X_1\circledast\ldots\circledast X_n$ be as in Theorem 4.1. For $i,j=1,\ldots,n$ and $A,B \in A$ define

$$\mu_{ij}(A \cap B) = E(X_i(A)X_j(B)). \tag{4.4}$$

Then:

(i) $$E(X_1\circledast\ldots\circledast X_n(A)) = 0 \qquad \forall\ A \in A^n.$$

(ii) $$VAR(X_1\circledast\ldots\circledast X_n(A)) = (n!)^{-1} \sum_{\Pi} \mu_{1\Pi(1)}\otimes\ldots\otimes\mu_{n\Pi(n)}(A \cap A^{\Pi}) \quad \forall\ A \in A^n.$$

(iii) For $A \in A^n$ and $B \in A^m$ $(m\leq n)$

$$COV(X_1\circledast\ldots\circledast X_n(A),\ X_1\circledast\ldots\circledast X_m(B)) = \delta_{nm}(n!)^{-1}\sum_{\Pi} \mu_{1\Pi(1)}\otimes\ldots\otimes\mu_{n\Pi(n)}(A\cap B^{\Pi}).$$

(iv) $X_1\circledast\ldots\circledast X_n$ is an $L^2(\Omega)$-valued o.s.m. on $(T^n,A^{\circledast n})$ with control measure given by

$$\mu^{\circledast n}(A) = (n!)^{-1}\sum_{\Pi} \mu_{1\Pi(1)}\otimes\ldots\otimes\mu_{n\Pi(n)}(A) \qquad A \in A^{\circledast n}.$$

(v) If $A_1,\ldots,A_n$ are pairwise disjoint sets in A

$$X_1(A_1)\circledast\ldots\circledast X_n(A_n) = (n!)^{-1/2}X_1(A_1)\ldots X_n(A_n)$$

(vi) A real valued function f on T^n is $X_1\circledast\ldots\circledast X_n$-integrable if and only if $f \in L^2(T^n,A^n,\mu_1\otimes\ldots\otimes\mu_n)$.

Proof. (i) - (iv) follow by Theorem 4.1, Lemma 3.1 and Corollary 3.1. (vi) follows by Theorem 3.2. To prove (v) use the facts that $X_1(A_1),\ldots,X_n(A_n)$ are mutually orthogonal,

$$X_1(A_1)\circledast\ldots\circledast X_n(A_n) = (n!)^{-1/2}\left(\frac{\partial^n}{\partial z_1\ldots\partial z_n}\right)_{\underline{z}=\underline{0}} \exp\circledast\left(\sum_{i=1}^{n} z_i X_i(A_i)\right)$$

and apply (2.14) together with (2.7) and (2.12). The proof is complete.

Using the notation of Definition 3.1, from Lemma 4.1(vi) and Theorem 10.8 in [10] (see also [14, Prop. 2.3.3])

$$I_n(f;X_1,\ldots,X_n) = \int_{T^n} f(\underline{t})d(X_1\circledast\ldots\circledast X_n)(\underline{t})$$

is a continuous linear operator from $L^2(\mu_1\otimes\ldots\otimes\mu_n)$ to $L^2(\Omega)$. We will see how this integral includes most of the known multiple stochastic integrals.

Consider the case of "dependent integrators", i.e.

$$\mu_{ij}(A) = s_{ij}\mu(A) \qquad A \in \mathcal{A}$$

where (s_{ij}) is an $n\times n$ non-negative definite matrix and μ is a measure on $(T,\mathcal{A})$ satisfying the continuity property (Itô [7]). Let f be a special elementary function on T^n i.e.

$$f(\underline{t}) = \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}(\underline{t})$$

where $A_1,\ldots,A_p$ is a collection of disjoint sets in $\mathcal{A}$ and $a_{i_1\ldots i_n} \in \mathbb{R}$ are zero unless $i_1,\ldots,i_n$ are all distinct. The linear manifold E_n of special elementary functions is dense in $L^2(T^n,\mathcal{A}^n,\mu^{\otimes n})$ ([7]). For $f \in E_n$ define

$$J_n(f;X_1,\ldots,X_n) = \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} X_1(A_{i_1})\ldots X_n(A_{i_n}).$$

Following Itô [7] (see also Fox and Taqqu [5]) we can extend this to a continuous linear functional J_n: $L^2(\mu^{\otimes n}) \to L^2(\Omega)$. This is the manner the multiple stochastic integrals in [5], [7], [13] and [16] are constructed.

On the other hand, using Definition 3.1 and Lemma 4.1 (v), if $f \in E_n$

$$I_n(f;X_1,\ldots,X_n) = \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} X_1(A_{i_1})\circledast\ldots\circledast X_n(A_{i_n})$$

$$= (n!)^{-1/2} \sum_{i_1\ldots i_n=1}^{p} a_{i_1\ldots i_n} X_1(A_{i_1})\ldots X_n(A_{i_n}) = (n!)^{-1/2} J_n(f;X_1,\ldots,X_n).$$

Thus I_n and J_n are $L^2(\Omega)$-valued continuous linear functionals which agree (up to a constant) on the dense linear manifold E_n and therefore they agree on all $L^2(T^n,A^n,\mu^{\otimes n})$ Then the method of constructing $\int_{T^n} f(\underline{t})d(X_1\circledast\ldots\circledast X_n)(\underline{t})$ gives rise to the known "centered" multiple stochastic integrals, including the multiple Wiener integrals of Itô [7], the multiple Poisson integrals of Ogura [13] and Surgailis [16], and the multiple Wiener integrals with dependent integrators of Fox and Taqqu [5]. This fact establishes a clear relationship between the theory of vector valued measures and the theory of multiple stochastic integrals.

REMARK. In Theorem 4.1 we have assumed that T is an interval of the real line. This has been done in order to apply the Levy-Itô decomposition of the process Y_t. If we only consider the Gaussian or the Poisson case, then it will not be necessary to impose this restriction on T. In these cases T can be taken quite general (see [14]).

5. COMPARISONS WITH OTHER PRODUCT STOCHASTIC MEASURES

Let (Ω,F,P) and (T,A) be as in the last section. Let $\{X_1,\ldots,X_n\}$ be a system of $n\geq 1$ stochastic process satisfying the regularity conditions in [4]. Engel and Kakutani (Engel [4]) have considered the extension of $X_1\times\ldots\times X_n$ (defined in (1.1)) to an $L^2(\Omega)$-valued product stochastic measure on the product σ-field A^n. Note that the condition (R4) in [4] imposes higher moments on the X_i's. The idea of the proof of this extension (Theorem 4.5 in [4]) is to partition the set T^n into disjoint pieces on which an appropriate $L^2(\Omega)$-valued measure can be defined and then show that the sum of all these measures is the required measure. This procedure uses a complicated double induction and involves prior knowledge of what the measure $X_1\times\ldots\times X_n$ should look like, even when the mean of each X_i is zero. On the other hand the construction of the symmetric tensor product stochastic measure $X_1\circledast\ldots\circledast X_n$ follows the more natural ideas from the theory of product real valued measures and does not assume higher moment conditions on the X_i's. Moreover, some of the combinatorial problems involved in the construction of $X_1\times\ldots\times X_n$ are absorbed in the definition of symmetric tensor product. In [4] it is not assumed that the mean of each X_i is zero as we have done in the construction of $X_1\circledast\ldots\circledast X_n$. This zero mean condition is satisfied in many interesting cases and enables us to use Hilbert space methods in the construction of the (symmetric tensor) product stochastic measure $X_1\circledast\ldots\circledast X_n$.

As well as differences in the assumptions and in the techniques used there are also important differences between the resulting product stochastic measures $X_1\times\ldots\times X_n$ and $X_1\circledast\ldots\circledast X_n$. While the latter is always a centered measure (Lemma 4.1(i)) the measure $X_1\times\ldots\times X_n$ is not, even in the case when the mean of each X_i is zero. For each permutation Π of $(1,\ldots,n)$ let T^n_Π be as in (3.7). If

$A \in \mathcal{A}^n \cap T^n_\Pi$ for some Π then $X_1 \times \ldots \times X_n(A) = X_1 \circledast \ldots \circledast X_n(A)$ but in general they are not equal (see [14]). As one can expect the $X_1 \times \ldots \times X_n$ measure of the "diagonals" is not zero while the $X_1 \circledast \ldots \circledast X_n$ measure is (see (3.3)). Then it is natural to expect that integrals w.r.t. $X_1 \circledast \ldots \circledast X_n$ will be centered while in the case of $X_1 \times \ldots \times X_n$ they will be uncentered. Integrals w.r.t. $X_1 \circledast \ldots \circledast X_n$ are easily constructed and the kind of $X_1 \circledast \ldots \circledast X_n$-integrable functions can be characterized. Engel [4] does not characterize the class of $X_1 \times \ldots \times X_n$-integrable functions. This last problem is not an easy one. A description of the class of $X \times X$-integrable functions was obtained by Rosinski and Szulga in [15]. The last named authors have considered the product stochastic measure of an i.s.m. X with itself in such a way that $X \times X(A_1 \times A_2) = X(A_1)X(A_2)$ $(A_1, A_2 \in \mathcal{A})$ can be extended to an $L^2(\Omega)$-valued vector measure on $\mathcal{A}^2$, under the assumption that $X(A) \in L^4(\Omega)$ for $A \in \mathcal{A}$. This corresponds to the case n=2 in [4].

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