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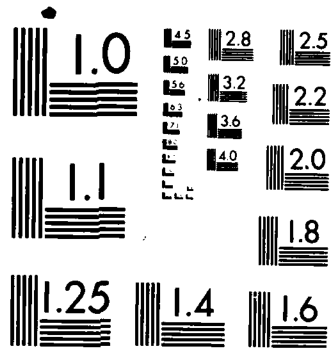
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1. INTRODUCTION AND SUMMARY

Consider the classical MANOVA model

$$Y = X\Theta + E \tag{1.1}$$

where $E \sim N(0, I_n \otimes \Sigma)$, $E = (E_1\ E_2)$, $Y = (Y_1\ Y_2)$, $X = (X_1\ X_2)$, and

$$\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{1.2}$$

Also, Σ_{ij} and X_i are of order $p_i \times p_j$ and $n \times r_i$ respectively, $p = p_1 + p_2$ and $r = r_1 + r_2$. In addition, Y_i and E_i are of order $n \times p_i$. Then, we are interested in testing the following hypotheses:

Problem [I] $H: \theta_{12} = 0$ and $\Sigma_{12} = 0$ vs K: not H

Problem [II] $H: \theta_{12} = 0$, $\theta_{21} = 0$ and $\Sigma_{12} = 0$ vs K: not H

Problem [III] $H: \Sigma_{12} = 0$ under $\theta_{12} = 0$ and $\theta_{21} = 0$

vs $K: \Sigma_{12} \neq 0$ under $\theta_{12} = 0$ and $\theta_{21} = 0$.

The motivation behind each problem is stated in Section 2 and some examples are also given there. In this section, some formal features of the problems are made clear and our results are briefly summarized together with some results in the literature. A basic feature in the problems treated here is that in each hypothesis the independence ($\Sigma_{12} = 0$) between Y_1 and Y_2 is included corresponding to the structure of the regression coefficient matrix Θ.

In Problem [I], the hypothesis will be regarded in Section 2 as a formulation of the hypothesis of no causality from X_1 to Y_2 where $X = [X_1, X_2]$ may be random but is fixed with full rank. Also $\theta_{12} = 0$ in the hypothesis may be viewed as a special case of the GMANOVA (general MANOVA) hypothesis

$$M_1 \Theta M_2 = 0 \tag{1.3}$$

where M_1 and M_2 are fixed matrices of full rank. In fact, $M_1 = [I, 0]$ and $M_2 = \binom{0}{I}$ implies $\theta_{12} = 0$. The problem of testing (1.3) in the GMANOVA model of the form $Y = Z_1 \Theta Z_2 + E$ is known as the GMANOVA problem and has been treated by Potthoff and Roy (1964), Rao (1965,1966), Khatri (1967), Krishnaiah (1969), Gleser and Olkin (1970), Fujikoshi (1973), Kariya (1978), Marden (1982a) and others.Potthoff and Roy (1964) proposed the GMANOVA model and considered adhoc procedures for testing the general linear hypothesis on the location parameters. Rao (1965,1966) reduced the problem to testing the general linear hypothesis under a conditional model. Khatri (1966) derived the LRT for testing the general linear hypothesis under Potthoff and Roy model. Later, Gleser and Olkin (1970) gave a canonical reduction of the problem and discussed the LRT procedure using the canonical form. However they have not treated Problem [I]. It should be noted that when (1.3) is dealt with, the presence of a general M_2 in (1.3) affects the covariance structure so that in the case of (1.3) $\Sigma_{12} = 0$ should be replaced by the hypothesis

$$\Sigma = M_2 \Gamma M_2' + P_2 \Delta P_2' \tag{1.4}$$

where P_2 is a matrix of full rank satisfying $P_2' M_2 = 0$ (see Kariya (1985), pp 175-176). That is, Problem [I] is considered equivalent to the problem of testing (1.3) and (1.4) simultaneously since both problems give the same canonical form. Hence, solving the former problem implies solving the latter problem and vice versa. Now for Problem [I], applying the invariance principle, we first expand the power function of an invariant test in the neighborhood of the hypothesis and based on it propose a test, in Section 3, which maximizes the power in a slightly restricted neighborhood of the null hypothesis. Of course due to the local optimality, it is admissible. The test statistic there is a linear combination of the LBI test statistic T_1 for testing $\theta_{12} = 0$ (Schwartz (1967)) and the LBI test statistic T_2 for testing $\Sigma_{12} = 0$ under $\theta_{12} = 0$. The latter test is equivalent to the LBI test of independence with some data missing in Eaton and Kariya (1983). Because of the form of T_2, T_1 and T_2 are correlated and so our test is not equal to a test combining the two independent LBI test statistics T_1 and T_3 for the two separate hypotheses $\theta_{12} = 0$ and $\Sigma_{12} = 0$ (without $\theta_{12} = 0$), though T_1 is the same. This is a feature of the joint treatment of the two hypotheses and it implies that a test combining the two independent statistics T_1 and T_3 which are LBI for each hypothesis does not maximize the local power in any direction except for the case that the test depends on T_1 only and that the alternative space is restricted to the space on which $\Sigma_{12} = 0$. The problem of how to combine independent tests is discussed in the literature (see e.g., Marden (1982b)), though we do not discuss it here. But the LRT statistic for Problem [I] gives a natural combination for two separate hypotheses. In fact, it is the product of the two independent LRT statistics. This might support the idea that we separately treat the hypotheses and then combine the two tests. However, as has been observed in Eaton and Kariya (1983), even when $\theta_{12} = 0$, the LRT for testing $\Sigma_{12} = 0$ ignores the additional information (data) available through $\theta_{12} = 0$. In this sense, the above fact may not be seriously taken into account. The asymptotic null distributions of the test based on T_1 and T_2 and the LRT are derived in Section 5 and the unbiasedness of the LRT is shown. It is noted that the group leaving the problem invariant is small so that the power function of an invariant test including the LRT depends on many parameters including the canonical correlations.

In Problem [II], the hypothesis will be regarded in Section 2 as the hypothesis of no additional information in canonical correlation analysis or a formulation of the hypothesis of no causality from X_1 to Y_2 and from X_2 to Y_1, where X may be random but is fixed with full rank. Here the restrictions $\theta_{12} = 0$ and $\theta_{21} = 0$ are special cases of

$$M_1\Theta M_2 = 0 \text{ and } M_3\Theta M_4 = 0. \tag{1.5}$$

However, the two GMANOVA type restrictions in (1.5) cannot be expressed as a single GMANOVA hypothesis of the form $\tilde{M}_1\theta\tilde{M}_2 = 0$. That is, the problem of testing (1.5) even in our MANOVA model $Y = X\theta + E$ is no longer the GMANOVA problem and difficult to treat unless M_1 and M_3 are nested relative to $X'X$ or orthogonal relative to $X'X$, i.e., $M_1X'XM_3' = 0$ (see Kariya (1985) p 143). Since $M_1 = [I,0]$ and $M_3 = [0,I]$ in our present case, $M_1X'XM_3' = X_1'X_2$ and hence without $X_1'X_2 = 0$ the problem of testing. Problem [IV]: $H:\theta_{12} = 0$ and $\theta_{21} = 0$ is difficult to treat. In fact, it is not only difficult to derive the LRT explicity but it is also difficult to find a similar test detecting both $\theta_{12} = 0$ and $\theta_{21} = 0$ in a meaningful manner (see Section 7). On the other hand, the hypothesis on the covariance structure which corresponds consistently to the hypothesis (1.5) is expressed as

$$\Sigma = M_2\Gamma_1M_2' + P_2\Delta_1P_2' \text{ and } \Sigma = M_4\Gamma_2M_4' + P_4\Delta_2P_4'. \tag{1.6}$$

where $P_2'M_2 = 0$ and $P_4'M_4 = 0$. Since in Problem [II] $M_2'M_4 = 0$, we can take $P_2 = M_4$ and $P_4 = M_2$ so that the two covariances in (1.6) become the same. In Section 4, first in the case of $X_1'X_2 \neq 0$ we analyze Problem [II] via invariance, but because the group leaving the problem invariant is quite small, no sufficient reduction is obtained and the space of a maximal invariant parameter is of high dimmension. Hence in the case of $X_1'X_2 \neq 0$ we simply show the unbiasedness of the LRT derived by Fujikoshi (1982). The LRT statistic here is the product of the three LRT statistics for the three separate hypotheses $\theta_{12} = 0$, $\theta_{21} = 0$ and $\Sigma_{12} = 0$ but the three are dependent. It is noted that it is difficult to consider the monotonicity of the power function of the LRT because of the high dimmensional parammeter space. Next in the case of $X_1'X_2 = 0$, we expand the power function of an invariant test in the neighborhood of the null hypothesis and based on it propose a test, which is a linear combination of three statistics R_1, R_2 and R_3. Here similar to Problem [I], R_1 and R_2 are respectively the LBI test statistics for the hypotheses $\theta_{12} = 0$ and $\theta_{21} = 0$ and R_3 is the LBI test statistic for $\Sigma_{12} = 0$ <u>under</u> $\theta_{12} = 0$ and $\theta_{21} = 0$. Hence R_1, R_2 and R_3 are dependent. This is a feature different from the separate treatment of the three hypotheses. The asymptotic null distributions of this test as well as the LRT are given in Section 5.

Problem [III] was treated by Kariya, Fujikoshi and Krishnaiah (1984) (abbreviated as KFK henceforth). In this model X is fixed but it may not be of full rank. The model (1.1) with $\theta_{12} = 0$ and $\theta_{21} = 0$ is regarded as a combined expression of two correlated multivariate regression models with different design matrices. When $p_1 = p_2 = 1$, Zellner (1962, 1963) called it a seemingly unrelated regression (SUR) equation model, while KFK called it a correlated regression equations (CRE) model. As has been discussed above, the information $\theta_{12} = 0$ and $\theta_{21} = 0$ on Θ cannot be expressed as a single GMANOVA restriction and when it is expressed as the form in (1.5), M_1 and M_3 are neither nested nor orthogonal relative to $X'X$ (see Kariya (1985) p 143, 159-163, 207). Therefore, the model itself is not put in the framework of the GMANOVA model and even the LRT for Problem [II] is difficult to derive (in the case that $X_1'X_2 \neq 0$ or X_1 is not nested to X_2). In Section 7, based on the idea of Mukherjee and Chandra (1984), we compare the power of the LRT-like test, the Pillai type test and the LBI test proposed in KFK. The comparison is made asymptotically in n with contiguous alternatives and asymptotically in small canonical correlations $\rho_1 \geq \ldots \geq \rho_{p_1}$ (as $\rho_1 \to 0$). There it is shown that the Pillai type test is asymptotically (in the same sense) equivalent to the LBI test and that for those small ρ_i's such that $\bar{\tau} = \Sigma\rho_i^4/(\Sigma\rho_i^2)^2 >$ some constant c, the LRT-like test is asymptotically better than the LBI test, while for those small ρ_i's such that $\bar{\tau} < c$, the LBI test is better. Further, because in the case of $X_1 = X_2 \equiv X_0$, the model in (1.1) is reduced to

$$[Y_1, Y_2] = X_0[\theta_{11}, \theta_{22}] + [E_1, E_2],$$

which is nothing but the MANOVA model. Therefore the comparison in Section 7 holds as it stands for the LRT and the Pillai (LBI) test of independence in the MANOVA model. This is a corollary of our result.

2. MOTIVATION OF THE WORK

The motivation behind Problem [I]: $\theta_{12} = 0$ and $\theta_{21} = 0$ is associated with the problem of no causality from X_1 to Y_2 and total exogneity of X_1 for Y_2. We will first write the model (1.1) as two correlated classical multivariate regression models.

$$
\begin{aligned}
Y_1 &= X_1\theta_{11} + X_2\theta_{21} + E_1 \\
Y_2 &= X_1\theta_{12} + X_2\theta_{22} + E_2.
\end{aligned}
\tag{2.1}
$$

Then the hypothesis $\theta_{12} = 0$ is equivalent to no effect of X_1 on Y_2 as in the usual case. However, since E_1 and E_2 are correlated, the regression equation of Y_2 under $\theta_{12} = 0$, conditional on Y_1 is expressed as

$$Y_2 = X_2\theta_{22} + (Y_1 - X_1\theta_{11} - X_2\theta_{21})\Sigma_{11}^{-1}\Sigma_{12} + E_3.$$

In this sense the effect of X_1 on Y_2 still remains unless $\Sigma_{12} = 0$. Therefore the hypothesis $\theta_{12} = 0$ and $\Sigma_{12} = 0$ in Problem [I] is considered as a formulation of no causality from X_1 to Y_2 or total exogeneity of X_1 for Y_2. An example for Problem [I] is found in a problem of economic policy evaluation. Suppose the model (2.1) is a reduced form of an econometric simultaneous equations model which describes the interaction of economic variables, and X_1 is a matrix of policy variables (tax rate, government investment etc.). Then the hypothesis in Problem [I] is interpreted as no effect of the policy on some economic variables such as inflation rate, sales, consumption etc.

The motivation behind Problem [II]: $\theta_{12} = 0$, $\theta_{21} = 0$ and $\Sigma_{12} = 0$ is similarly given in association with no causality from X_1 to Y_2 and no causality from X_2 to Y_1. In addition, the problem is also considered as a formulation of no additional information hypothesis in canonical correlation analysis, which was given by Fujikoshi (1982) based on Mckay (1977). To see this, suppose there are two groups of measurements (variables), say x_1 and x_2, where x_i's are $r_i \times 1$ random vectors with means μ_i and joint covariance matrix $\Psi = (\Psi_{ij}):(r_1+r_2)\times(r_1+r_2)$ with Ψ_i: $r_i \times r_j$ $(i,j = 1,2)$. Let $\delta^2(x_1,x_2)$ denote the sum of squares of the canonical correlations between x_1 and x_2

$$\delta^2(x_1,x_2) = \mathrm{tr}\Psi_{11}^{-1}\Psi_{12}\Psi_{22}^{-1}\Psi_{21}, \tag{2.2}$$

which is regarded as a measure of total correlation between x_1 and x_2. Sometimes for each group, there are some other measurements available, say y_1 and y_2, which appear to be of some relevance for the correlation between the two groups where $y_i:p_i \times 1$ $(i=1,2)$. Then adding these variables to x_1 and x_2, the total correlation is measured by the sum of the canonical correlations between $z_1 = (x_1',y_1')'$ and $z_2 = (x_2',y_2')'$, say $\delta^2(z_1,z_2)$ as in the case of (2.2). But the real or significant relevance of including the additional variables y_1 and y_2 may be in question relative to the original variables. This question gives the following testing problem

$$H:\delta^2(z_1,z_2) = \delta^2(x_1,x_2) \text{ vs } K:\delta^2(z_1,z_2) > \delta^2(x_1,x_2). \tag{2.3}$$

Using a conditional argument, Fujikoshi (1982) showed that this problem is equivalent to Problem [II] and he derived the LRT where X_i's and Y_j's in (1.1) are the sample matrices of x_i's and y_j's. Mckay (1977a) treated the hypothesis $\delta^2(z_1,x_2) = \delta^2(x_1,x_2)$(i.e., no additional information in y_1 relative to (x_1,x_2)) and showed that this hypothesis is equivalent to Σ_{12} in the model (1.1). Some related topics are also found in Mckay (1977b) and Rao (1970).

The motivation for Problem [III] is stated in KFK (1984), Zellner (1962, 1963) and the articles therein.

3. TESTS FOR $\theta_{12} = 0$ AND $\Sigma_{12} = 0$

Based on the motivation stated in Section 2, we here consider the problem of testing the hypothesis H against K where

$$H: \theta_{12} = 0, \ \Sigma_{12} = 0, \quad K: \text{not } H. \tag{3.1}$$

First we make an invariance consideration into the problem and obtain an expression for the local behavior of the power function of an invariant test in the neighborhood of the null hypothesis. Based on the expression, an invariant test together with the LRT will be proposed and then the null distributions of these test statistics will be given in Section 5. To begin with, a canonical reduction of the problem is performed. Write

$$X = P\begin{bmatrix} A \\ 0 \end{bmatrix} \text{ with } P\varepsilon\mathcal{O}(n) \text{ and } A = (X'X)^{\frac{1}{2}} \ \varepsilon G\ell(r)$$

and express $\theta_{12} = 0$ as

$$M_1\Theta M_2 = 0 \text{ with } M_1 = [I_{p_1}, 0] \text{ and } M_2 = \begin{pmatrix} 0 \\ I_{p_2} \end{pmatrix}, \tag{3.2}$$

where $\mathcal{O}(n)$ denotes the group of n×n orthogonal matrices and $G\ell(r)$ the group of r×r nonsingular matrices. Further let

$$M_1A^{-1} = F(I,0)\Psi \text{ with } F\varepsilon G\ell(r_1) \text{ and } \Psi \ \varepsilon\mathcal{O}(r)$$

and

$$Z = \begin{pmatrix} \Psi & 0 \\ 0 & I_{n-r} \end{pmatrix} P'Y \quad \text{and } \eta = \begin{array}{c} \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix} \\ \begin{matrix} p_1 & p_2 \end{matrix} \end{array} \equiv \Psi A\Theta. \tag{3.3}$$

Then the problem is to test $(I,0)\,\eta\begin{pmatrix} 0 \\ I \end{pmatrix} = \eta_{12} = 0$ and $\Sigma_{12} = 0$, i.e.,

$$H: \eta_{12} = 0, \ \Sigma_{12} = 0 \tag{3.4}$$

in the canonical model

$$Z = \begin{array}{cc} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ Z_{31} & Z_{32} \end{pmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} \;\; p_1 & \;\; p_2 \end{matrix} & \end{array} \sim N\left(\begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \\ 0 & 0 \end{pmatrix}, I \otimes \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$$

where $r_3 = n - r$. This problem is clearly left invariant under the group $G = \tilde{O}(n)\times\mathcal{B}(p)\times F$ acting on Z and (η,Σ) by

$$g(Z) = PZB + F \tag{3.6}$$

$$g(\eta,\Sigma) = \left(\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\eta B + \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix}, B'\Sigma B\right) \tag{3.7}$$

where

$$g = (P,B,F)\varepsilon G$$

$$\tilde{O}(n) = \{P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & & P_3 \end{pmatrix} \mid P_i \varepsilon \mathcal{O}(r_i)\},$$

$$\mathcal{B}(p) = \{B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \mid B_i \varepsilon G\ell(p_i)\}$$

$$F = \{ F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \\ 0 & 0 \end{pmatrix} \mid F_{ij} \varepsilon R^{r_i p_j}\}$$

It follows from (3.7) that the power function of an invariant test is a function of $(\xi_{12}'\xi_{12},\Omega)$ where

$$\xi_{12} = \eta_{12}\Sigma_{22}^{-\frac{1}{2}}\Phi, \ \Omega \equiv \Omega(\rho) = \begin{pmatrix} I & \Delta \\ \Delta' & I \end{pmatrix} \text{ and} \tag{3.8}$$

$$\Delta \equiv \Delta(\rho) = \mathrm{diag}\{\rho_1,\ldots,\rho_t\}: \ p_1 \times p_2 \tag{3.9}$$

with $t = \min(p_1, p_2)$. Here $\rho_1^2 \geq \cdots \geq \rho_t^2$ are the characteristic roots of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and Φ is an orthogonal matrix which diagonalizes $\Sigma_{22}^{-\frac{1}{2}}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-\frac{1}{2}}$. Hence without loss of generality we assume $\eta_{12} = \xi_{12}$ and $\Sigma = \Omega$. Further, it also follows from (3.6) that any invariant test is a function of (Z_{12}, U) with $U = (Z_{31}, Z_{32})$, on which G acts by

$$g(Z_{12}, U) = (P_1 Z_{12} B_2,\ P_3 U B) \text{ for } g = (P,B,F)\varepsilon G. \tag{3.10}$$

Now to state one of our main results in this section, let $\mathcal{D}_\alpha^I$ be the set of all invariant tests of size α (i.e. $\phi \varepsilon \mathcal{D}_\alpha^I \Longleftrightarrow \phi(g(Z)) = \phi(Z)$),

$$S = U'U \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \ S_{ij} = Z_{3i}' Z_{3j} \tag{3.11}$$

and let $\delta = \delta_1 + \delta_2$ with

$$\begin{cases} \delta_1 = \mathrm{tr}\,\xi_{12}\xi_{12}' = \mathrm{tr}\,\eta_{12}\Sigma_{22}^{-1}\eta_{12}' \\ \delta_2 = \sum_{i=1}^{t}\rho_i^2 = \mathrm{tr}\,\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \end{cases} \text{ and} \tag{3.12}$$

Clearly $\eta_{12} = 0$ and $\Sigma_{12} = 0$ if and only if $\delta = 0$, or $\delta_1 = 0$ and $\delta_2 = 0$.

<u>Theorem 3.1</u> . There is $\varepsilon > 0$ such that on the set $\{(\eta,\Sigma) | \delta < \varepsilon\}$, the power function of any test ϕ in $\mathcal{D}_\alpha^I$ is evaluated as

$$\pi(\phi,(\eta,\Sigma)) = \alpha + \delta_1 C_1(\phi) + \delta_2 C_2(\phi) + o(\delta) \tag{3.13}$$

where

$$C_1(\phi) = \frac{r_1+r_3}{2p_1p_2}\{\frac{p_1}{r_1} E_0[\phi \mathrm{tr} Z_{12}'Z_{12}(Z_{12}'Z_{12}+S_{22})^{-1}]\}, \tag{3.14}$$

$$C_2(\phi) = \frac{r_1+r_3}{2p_1p_2} E_0\{\phi[r_3 \mathrm{tr} S_{11}^{-1}S_{12}(S_{22}+Z_{12}'Z_{12})^{-1}S_{21} - p_1 \mathrm{tr} S_{22}(S_{22}+Z_{12}'Z_{12})^{-1}\} - \frac{r_3}{2}, \tag{3.15}$$

$$\lim_{\delta\to 0} \sup_{\phi} |o(\delta)/\delta| = 0.$$

Here $E_0(\cdot)$ denotes the expectation of $(\cdot)$ under the null hypothesis. The proof is given at the end of this section. The expression (3.13) shows the local behavior of the power function of an invariant test ϕ, according to which the power function is approximated by $\alpha + \delta_1 C_1(\phi) + \delta_2 C_2(\phi)$ in the neighborhood

$$N_\varepsilon = \{(\eta,\Sigma)|\mathrm{tr}\eta_{12}\Sigma_{22}^{-1}\eta_{12}' + \mathrm{tr}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} < \varepsilon\}$$

and the approximation is uniform in ϕ. Since a test maximizing the local power $\alpha + \delta_1 C_1(\phi) + \delta_2 C_2(\phi)$ depends on at least the ratio of δ_1 and δ_2, no LBI test exists. This is natural in the sense that the quantities δ_1 and δ_2 indicate are different and the deviation of δ_1(or η_{12}) from 0 is independent of the deviation of δ_2 (or Σ_{12}) from 0. On the contrary, the form of the power function in (3.13) reflects how an invariant test can detect each local deviation from each null case. That is, $C_1(\phi)$ basically measures the local power of ϕ against the local deviation of δ_1 from 0, while $C_2(\phi)$ the local power of ϕ against the deviation of δ_2 from 0.

However, the statistics which define the expectations of $C_1(\phi)$ and $C_2(\phi)$

in (3.14) and (3.15), are not independent even under the null hypothesis. In fact, the statistics

$$\begin{cases} T_1 \equiv \dfrac{p_1}{r_1} \operatorname{tr} Z_{12}'Z_{12}(Z_{12}'Z_{12}+S_{22})^{-1} \quad \text{and} \\ T_2 \equiv r_3 \operatorname{tr} S_{11}^{-1}S_{12}(S_{22}+Z_{12}'Z_{12})^{-1}S_{21}-p_1 \operatorname{tr} S_{22}(S_{22}+Z_{12}'Z_{12})^{-1} \end{cases} \tag{3.16}$$

are dependent on each other. This is a feature of the simultaneous treatment of the two seperate hypotheses. To investigate this point further, observe that the test, say ψ_1, which maximizes $C_1(\phi)$ is given by the critical region

$$T_1 > c$$

and it is the LBI test for testing $\eta_{12} = 0$ without $\Sigma_{12} = 0$. More specifically it is LBI for testing the General MANOVA hypothesis

$$H' : \tilde{M}_1 \Theta \tilde{M}_2 = 0 \tag{3.17}$$

in the MANOVA model $Y = X\Theta + E$ with $E \sim N(0, I \otimes \Sigma)$ where X: $n \times r$, $\tilde{M}_1$: $r_1 \times r$ and $\tilde{M}_2$: $p \times p_2$ are arbitrarily fixed matrices of full rank. This test is even UMPI (uniformly most powerful invariant) when $r_1 = 1$ or $p_2 = 1$ and the power function of an invariant test ψ for the hypothesis in (3.17) is locally expressed as

$$\pi_1(\psi,(\eta,\Sigma)) = \alpha + \delta_1 C_1(\psi) + o(\delta_1) \tag{3.18}$$

where (η,Σ) is the parameter of a canonical form corresponding to $(\Theta,\tilde{\Sigma})$(see Kariya(1985), p 109). On the other hand, the test, say ψ_2, which mazimizes $C_2(\phi)$ is given by the critical region

$$T_2 > c$$

with T_2 in (3.16) and it is the LBI test for testing $\Sigma_{12} = 0$ <u>in the case of</u> $\eta_{12} = 0$. More specifically it is LBI for testing independence $\Sigma_{12} = 0$ in the missing data model

$$
\begin{aligned}
&Z_{12} \sim N(\eta_{12}, I_{r_1} \otimes \Sigma_{22}) \quad \text{with } \eta_{12} = 0 \\
&U \sim N(0, I_{r_3} \otimes \Sigma) \text{ and } Z_{12} \text{ and } U \text{ are independent}
\end{aligned}
\tag{3.19}
$$

where the counterpart of Z_{12} is missing (see Eaton and Kariya (1983)) and the power function of any invariant test in the model (3.19) is expressed as

$$\pi_2(\psi,(0,\Sigma)) = \alpha + \delta_2 C_2(\psi) + o(\delta_2) \tag{3.20}$$

However, when $\eta_{12} \neq 0$, the test based on the critical region $T_2 > c$ is not easy to interpret as a test for testing the single hypothesis of the independence $\Sigma_{12} = 0$ alone because in the case of $\eta_{12} \neq 0$, Z_{12} would not be involved in a test statistic. In other words, this is a difference between treating the two hypotheses simultaneously and treating them separately.

Now by taking this point into account, we propose the test maximizing a linear combination of $C_1(\phi)$ and $C_2(\phi)$;

$$C_\beta(\phi) = \beta r_3 C_1(\phi) + C_2(\phi) \quad (0 < \beta < \infty) \tag{3.21}$$

where β is a constant independent of n. Using the Generalized Neyman-Pearson Lemma, the critical region is given by

$$T(\beta) \equiv \beta r_3 T_1 + T_2 > k. \tag{3.22}$$

Here T_1 is multiplied by $r_3 = n-r$ because from (3.3) $\eta = O(n^{\frac{1}{2}})$ so that $\delta_1 = O(n)$ provided $X'X = O(n)$. The test ϕ_β with critical region $T(\beta) > c$ maximizes the power $\pi(\phi,(\eta,\Sigma))$ locally in the neighborhood

$$N_{\varepsilon\beta} = \{(\eta,\Sigma) \mid \delta_1 = \beta r_3 \delta_2\} \cap N_\varepsilon$$

since it maximizes $(\delta_1/r_3)\beta r_3 C_1(\phi) + \delta_2 C_2(\phi) = \delta_2[\beta r_3 C_1(\phi) + C_2(\phi)]$ on $N_{\varepsilon\beta}$. The constant β may be regarded as a weight for the importance of the hypothesis $\eta_{12} = 0$ relative to the hypothesis $\Sigma_{12} = 0$, and it is chosen in advance. It is noted that the test based on $T(\beta)$ is not a linear combination of the two LBI tests ψ_1 and ψ_2 stated above.

We remark that for a given $\phi \in \mathcal{D}_\alpha^I$, the local sensitivity of ϕ against (δ_1,δ_2) is measured by the two coordinate $(C_1(\phi),C_2(\phi))$. Second, when the information $\eta_{12} = 0$ is ignored in the missing data model (3.22), the LBI test for independence $\Sigma_{12} = 0$ is given by the critical region

$$T_3 \equiv \mathrm{tr} S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} > k \qquad (3.23)$$

(see Schwartz (1967)). That is, this is the LBI test for independence in the MANOVA model without $\eta_{12} = 0$. Since the test with critical region $T_1 > k$ is LBI for testing $\eta_{12} = 0$ without $\Sigma_{12} = 0$, we may combine these two tests. Here while the simultaneous treatment of the hypotheses yielded the dependent statistics T_1 and T_2, the separate treatment yields the independent test statistics T_1 and T_3. The problem of how to combine independent test statistics is discussed in the literature (see, e.q., Marden (1982)). In this paper we do not get involved in this problem. But as is shown next, the LRT for the simultaneous hypotheses gives a natural combination of the LRT statistics for two separate hypotheses.

The LRT for our problem is easily obtained by using the canonical model in (3.5) as follows:

$$\frac{|S|}{|S_{11}||S_{22}+Z_{12}'Z_{12}|} = \frac{|S|}{|S_{11}||S_{22}|} \cdot \frac{|S_{22}|}{|S_{22}+Z_{12}'Z_{12}|} = L_1 L_2 \qquad (3.24)$$

Of course, when L_1L_2 is small, the joint hypothesis is rejected. As is well known, L_1 is the LRT statistic for $\Sigma_{12} = 0$ without $\eta_{12} = 0$ while L_2 is the LRT statistic for $\eta_{12} = 0$ without $\Sigma_{12} = 0$. Further L_1 and L_2 are independent so that the two independent LRT statistics are combined in (3.24). It is noted that even when $\eta_{12} = 0$, the LRT statistic for $\Sigma_{12} = 0$ is given by L_1 so that the LRT ignores the additional data Z_{12} (see Eaton and Kariya (1984)). The unbiasedness property of the power function is considered as a special case of Problem (II) in the next section.

Proof of Theorem 3.1. The proof is similar to KFK (1984). To derive the distribution of a maximal invariant $T = T = T(Z_{12},U)$ under the action (3.10) of G on (Z_{12},U), we apply the Wijsman's representation theorem. Let $P^T_{(\xi_k,\Omega)}$ be the distribution of a maximal invariant T. Then the density of T with respect to $P^T_{(0,I)}$ evaluated at $T = T(Z_{12},U)$ is given by

$$R \equiv dP^T_{(\xi_{12},\Omega)}/dP^T_{(0,I)} = H(Z_{12},U|\xi_{12},\Omega)/H(Z_{12},U|0,I) \tag{3.25}$$

where

$$H(Z_{12},U|\xi_{12},\Omega) = \int_H f(P_1Z_{12}B_{22}, P_3UB|\xi_{12},\Omega)X(B)\nu_1(dP_1)\nu_3(dP_3)\mu_1(dB_1)\mu_2(dB_2)$$

$$X(B) = |B_1'B_1|^{r_3/2}|B_2'B_2|^{(r_1+r_3)/2}, \quad \mu_j(dBj) = |B_j'B_j|^{-p_j/2}dB_j$$

$H = O(r_1) \times O(r_3) \times G\ell(p_1) \times G\ell(p_2)$, $f(Z_{12},U|\xi_{12},\Omega)$ is the density of Z_{12} and U with $Z_{12} \sim N(\xi_{12},I \otimes I)$ and $U \sim N(0,I_{r_2} \otimes \Omega)$, and $\nu_i(dP_i)$ is the invariant probability measure on $O(r_i)$ (i=1,3: j=1,2). The condition for which (3.25) holds is satisfied (see Wijsman (1967)). In order to obtain the local behavior of the power function of an invariant test in a neighborhood of $\xi_{12} = 0$ and $\Sigma_{12} = 0$, we evaluate R in (3.25) locally. After cancellation of some constants the numerator in (3.25) is expressed as

$$H(Z_{12},U|\xi_{12},\Omega) = \int_{\mathcal{O}(r_1)\times G\ell(p_1)\times G\ell(p_2)} |\Omega|^{-r_3/2} \exp[-\tfrac{1}{2}K]X(B)\nu_1(dP_1)\mu_1(dB_1)\mu_2(dB_2)$$

where

$$K = \mathrm{tr}B_2'Z_{12}'Z_{12}B_2 - 2\mathrm{tr}P_1Z_{12}B_2\xi_{12}' + \mathrm{tr}\xi_{12}'\xi_{12}$$
$$+ \mathrm{tr}B_1'S_{11}B_1F_{11} + 2\mathrm{tr}B_1'S_{12}B_2F_{12}' + \mathrm{tr}B_2'S_{22}B_2F_{22} \qquad (3.26)$$

with

$$\Omega^{-1} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}. \qquad (3.27)$$

Here replacing B_1 by $S_{11}^{-\frac{1}{2}}B_1$ and B_2 by $(Z_{12}'Z_{12}+S_{22})^{-\frac{1}{2}}B_2$ leaves the ratio R remain the same since the Jacobin factors coming out are cancelled out with those of the denominator. Further writing F_{ii} as

$$F_{ii} = (I-\Delta_{ii})^{-1} = I + \Delta_{ii} + \Delta_{ii}^2(I-\Delta_{ii})^{-1}$$

with $\Delta_{11} = \Delta\Delta'$ and $\Delta_{22} = \Delta'\Delta$. Then K in (3.26) becomes

$$K = \mathrm{tr}B_1B_1' + \mathrm{tr}B_2B_2' - 2\mathrm{tr}P_1\tilde{Z}_{12}B_2\xi_{12} + \mathrm{tr}\xi_{12}\xi_{12}' + 2\mathrm{tr}B_1\tilde{S}_{12}F_{12} + K_1 + K_2$$

where

$$K_1 = \mathrm{tr}B_1'B_1[\Delta_1+\Delta_1^2(I-\Delta_1)^{-1}];\ K_2 = \mathrm{tr}B_2'\tilde{S}_{22}B_2[\Delta_2+\Delta_2^2(I-\Delta_2)^{-1}]$$

$$\tilde{Z}_{12} = Z_{12}(Z_{12}'Z_{12}+S_{22})^{-\frac{1}{2}},\quad \tilde{S}_{12} = S_{11}^{-\frac{1}{2}}S_{12}(Z_{12}'Z_{12}+S_{22})^{-\frac{1}{2}}$$ and

$$\tilde{S}_{22} = (Z_{12}'Z_{12} + S_{22})^{-\frac{1}{2}} S_{22} (Z_{12}'Z_{12} + S_{22})^{-\frac{1}{2}}.$$

Now expand the components of $\exp[-\frac{1}{2}K]$ as

$$\exp[\mathrm{tr}P_1\tilde{Z}_{12}B_2\xi_{12}'] = 1 + \mathrm{tr}P_1\tilde{Z}_{12}B_2\xi_{12}' + \tfrac{1}{2}(\mathrm{tr}P_1\tilde{Z}_{12}B_2\xi_{12}')^2 + o_1$$

$$\exp[-\mathrm{tr}B_1'\tilde{S}_{12}B_2F_{12}'] = 1 - \mathrm{tr}B_1'\tilde{S}_{12}B_2F_{12}' + \tfrac{1}{2}(\mathrm{tr}B_1'\tilde{S}_{12}B_2F_{12}')^2 + o_2$$

$$\exp[-\tfrac{1}{2}K_1] = 1 - \tfrac{1}{2}\mathrm{tr}B_1'B_1\Delta_1 + o_3 \quad \text{and}$$

$$\exp[-\tfrac{1}{2}K_2] = 1 - \tfrac{1}{2}\mathrm{tr}B_2'\tilde{S}_{22}B_2\Delta_2 + o_4.$$

Then in the same way as in KFK(1984), the remainder terms o_i's are shown to be $o(\delta)$ with δ in (3.12), which is uniform in (Z_{12},S) because of the boundedness of $\tilde{Z}_{12}$ and $\tilde{S}_{ij}$'s . Further when o_i's are integrated over P_1 and B_i's, they are shown to be $o(\delta)$ uniformly in (Z_{12},S). Now taking the product of these terms and ignoring the odd functions of either P_1 or B_1 or B_2 because they are zero when integrated, we obtain

$$R = |\Omega|^{-r_3/2}\int J\,\nu_1(dP_1)h_1(B_1)h_2(B_2)dB_1dB_2$$

where

$$h_i(B_i) = \exp(-\tfrac{1}{2}\mathrm{tr}B_iB_i')|B_iB_i'|^{M_i/2}dB_i/D_i$$

$$J = 1 + \tfrac{1}{2}[(\mathrm{tr}P_1\tilde{Z}_{12}B_2\xi_{12}')^2 + (\mathrm{tr}B_1'\tilde{S}_{12}B_2F_{12})^2 - \mathrm{tr}B_1'B_1\Delta\Delta'$$

$$- \mathrm{tr}B_2'\tilde{S}_{22}B_2\Delta'\Delta] + o_5 \doteq 1 + \tfrac{1}{2}[I+II+III+IV] + o_5,\ \text{say}$$

$$D_i = \int\exp(-\tfrac{1}{2}\mathrm{tr}B_iB_i')|B_iB_i'|^{M_i/2}dB_i,$$

$M_1 = r_3 - p_1$ and $M_2 = r_1 + r_3 - p_2$. Here $H(Z_k, U(0,I)) = D_1 D_2$ and $|\Omega|^{-k_{3/2}} = 1 + o(\delta)$ were used. The remainder o_5 is shown to be $o(\delta)$ for δ small uniformly in (Z_{12}, S), for $F_{12} = (I - \Delta\Delta')^{-1}\Delta$. We now need to integrate J. But arguing as in KFK (1984) pp 391-392, we obtain

$$I = \int (\mathrm{tr} P_1 \tilde{Z}_{12} B_2 \xi_{12}')^2 \nu_1(dP_1) h_2(B_2) dB_2 = \frac{r_1 + r_3}{r_1 P_2} \mathrm{tr}\xi_{12}'\xi_{12} \mathrm{tr}\tilde{Z}_{12}'\tilde{Z}_{12},$$

$$II = \int (\mathrm{tr} B_1' \tilde{S}_{12} B_2 F_{12})^2 h_1(B_1) h_2(B_2) dB_1 dB_2 = \frac{r_3(r_1 + r_3)}{P_1 P_2} \mathrm{tr}\tilde{S}_{12}'\tilde{S}_{12} \mathrm{tr}\Delta\Delta' + o(\delta),$$

$$III = \int (\mathrm{tr} B_1' B_1 \Delta\Delta') h_1(B_1) dB_1 = -\frac{r_3 P_1}{P_1} \mathrm{tr}\Delta\Delta',$$

$$IV = \int (\mathrm{tr} B_2' \tilde{S}_{22} B_2 \Delta'\Delta) h_2(B_2) dB_2 = -\frac{r_1 + r_3}{P_2} \mathrm{tr}\tilde{S}_{22} \mathrm{tr}\Delta\Delta',$$

and $\int O_{5\ 1}(dP_1) h_1(B_1) h_2(B_2) dB_1 dB_2 = o(\delta)$. Thus, observing that the power function of an invariant test ϕ is given by

$$\pi(\phi, (\eta, \Sigma)) = \int \phi dP^T_{(\xi_{12}\Omega)} = \int \phi R dP^T_{(0,I)},$$

the result in (3.13) is finally obtained, completing the proof.

4. TESTS FOR THE HYPOTHESIS $\theta_{12}=0$, $\theta_{21}=0$, AND $\Sigma_{12}=0$

We shall first consider via invariance the Problem (II):

$$H: \theta_{12} = 0,\ \theta_{21} = 0,\ \Sigma_{12} = 0 \text{ vs } K: \text{not } H. \tag{4.1}$$

Since $\theta_{12} = 0$ and $\theta_{21} = 0$ are expressed as

$$M_1\Theta M_2 = 0 \text{ and } M_3\Theta M_4 = 0$$

and since $M_1X'XM_3 = X_1'X_2 \neq 0$ in general, the group leaving this problem invariant is smaller than the group leaving the problem (I) invariant, where M_1 and M_2 are defined by (3.2), $M_3 = (0,I)$ and $M_4 = \binom{I}{0}$. The special case $X_1'X_2 = 0$ will be briefly treated later. Here we use the following canonical form

$$\begin{cases} W = K(X'X)^{-1}X'Y \sim N(\eta, A\otimes\Sigma) \text{ with } \eta = K\Theta \\ S = Y'(I-P_0)Y \sim W(\Sigma, r_3) \text{ with } r_3 = n - r \\ W \text{ and } S \text{ are independent} \end{cases} \tag{4.2}$$

where $P_0 = X(X'X)^{-1}X'$,

$$K = \begin{pmatrix} Q_{11}^{-\frac{1}{2}} & 0 \\ 0 & Q_{22}^{-\frac{1}{2}} \end{pmatrix} \text{ with } (X'X)^{-1} = Q \equiv \begin{array}{cc} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \\ \begin{array}{cc} r_1 & r_2 \end{array} & \end{array} \tag{4.3}$$

and

$$A = \begin{pmatrix} I & \bar{Q}_{12} \\ \bar{Q}_{12}' & I \end{pmatrix} \text{ with } \bar{Q}_{12} = Q_{11}^{-\frac{1}{2}}Q_{12}Q_{22}^{-\frac{1}{2}}\ . \tag{4.4}$$

Partition W and η as

$$W = \begin{array}{cc} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \\ \begin{array}{cc} p_1 & p_2 \end{array} & \end{array} \quad \text{and} \quad \eta = \begin{array}{cc} \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{21} \end{pmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \\ \begin{array}{cc} p_1 & p_2 \end{array} & \end{array}$$

respectively. Then in the model (4.2) the problem is to test

$$H: \eta_{12} = 0, \eta_{21} = 0, \Sigma_{12} = 0.$$

The problem is clearly left invariant under the group $G = G\ell(p_1)\times G\ell(p_2)\times R^{r_1p_1}\times R^{r_2p_2}$ acting on (W,S) by

$$g(W,S) = (WB+F, B'SB) \text{ with } B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ and } F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \qquad (4.5)$$

where $g = (B_1,B_2,F_1,F_2)\varepsilon G$. From this action of G, it is easy to see that an invariant test is a function of (W_{12},W_{21},S) only and that the power function of an invariant test is a function of

$$\Omega = \begin{pmatrix} I & \Delta \\ \Delta' & I \end{pmatrix}, \ \xi_{12} = Q_{11}^{-\frac{1}{2}}\theta_{12}\Sigma_{22}^{-\frac{1}{2}}P_2 \text{ and } \xi_{21} = Q_{22}^{-\frac{1}{2}}\theta_{21}\Sigma_{11}^{-\frac{1}{2}}P_1 \qquad (4.6)$$

where $P_i\varepsilon\mathcal{O}(p_i)$'s $(i=1,2)$ satisfy

$$P_1\Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-\frac{1}{2}}P_2' = \Delta = \text{diag}(\rho_1,\ldots,\rho_t,0,\ldots,0): p_1\times p_2 \text{ with } t = \min(p_1,p_2) \qquad (4.7)$$

and $\rho_1 \geq \quad \geq \rho_t$ are the canonical correlations. So we can assume $\eta_{12} = \xi_{12}$, $\eta_{21} = \xi_{21}$ and $\Sigma = \Omega$ without loss of generality. In this set-up, we may proceed in the same way as we did in Section 3 to obtain an expression for the locally approximate power of an invariant test. However by doing so we end up with a very complicated expression of local power which depends on many parameters in an intractrable way. This implies that a further invariance consideration into the problem will not help to propose a new test which possibly takes into account the simultaneous occurance of the three hypotheses $\eta_{12} = 0$, $\eta_{21} = 0$, and $\Sigma_{12} = 0$. Of course no LBI test exists.

On the other hand, the LRT is given by the critical region $L < c$, which was originally derived by Fujikoshi (1982), where

$$L \equiv \frac{|S|}{|S_{22}+W_{12}'W_{12}||S_{11}+W_{21}'W_{21}|} = \frac{|S|}{|S_{11}||S_{22}|} \cdot \frac{|S_{22}|}{|S_{22}+W_{12}'W_{12}|} \cdot \frac{|S_{11}|}{|S_{11}+W_{21}'W_{21}|} = L_1L_2L_3, \text{ say} \tag{4.8}$$

It is also directly obtained from the distribution of (W_{12},W_{21},S) since the LRT is always invariant (under a very mild condition). The statistics L_1, L_2 and L_3 in (4.8) are respectively the LRT statistics for the three separate hypotheses $\Sigma_{12} = 0$, $\theta_{12} = 0$, and $\theta_{21} = 0$, and the LRT statistic $L_1L_2L_3$ for our simultaneous hypothesis in (4.8) may be viewed as showing how to combine the three LRT statistics for the three separate hypotheses. But here it is noted that L_2 and L_3 are correlated because of the correlation of W_{12} and W_{21} unless $X_1'X_2 = 0$ or $\Sigma_{12} = 0$. In fact, if u_i' and v_j' are respectively the i-th and j-th rows of W_{12} and W_{21}, the covariance matrix of $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_j$ is shown to be

$$\mathrm{Cov}\ (u_i,v_j) = q_{ij}\Delta' : p_2 \times p_1$$

where q_{ij} is the (i,j)th element of $\bar{Q}_{12}$ in (4.4) and $\bar{Q}_{12} = 0$ and $\Delta = 0$ are respectively equivalent to $X_1'X_2 = 0$ and $\Sigma_{12} = 0$. This correlation between L_2 and L_3 makes it difficult to investigate optimality properties of the LRT. Now to show the unbiasedness of the LRT, note that from $W_{12} \sim N(\xi_{12}, I \otimes I)$ and $W_{21} \sim N(\xi_{21}, I \otimes I)$

$$\begin{cases} V_{22} \equiv W_{12}'W_{12} \sim W_{p_2}(I,\bar{r}_2:\tau_2) \text{ with } \bar{r}_2 = r_1 \text{ and } \tau_2 = \xi_{12}'\xi_{12} \\ V_{11} \equiv W_{21}'W_{21} \sim W_{p_1}(I,\bar{r}_1:\tau_1) \text{ with } \bar{r}_1 = r_2 \text{ and } \tau_1 = \xi_{21}'\xi_{21} \end{cases} \tag{4.9}$$

where $W(\Psi,m:\nu)$ denotes the noncentral Wishart distribution with mean $m\Psi$, degrees of freedom m and noncentrality parameter ν. In the original term, V_{ii} is easily shown to be equal to

$$V_{ii} = Y_i'[P_0-P_i]Y_i \quad \text{with } P_i = X_i(X_i'X_i)^{-1}X_i'. \tag{4.10}$$

Further, from (4.2), S and $\{V_{11},V_{22}\}$ are independent, and when $\Delta = 0$, V_{11} and V_{22} are independent. Here we use a conditional argument. First write L_1L_2 in (4.8) as

$$L_1L_2 = |S_{22\cdot 1}|/|S_{22\cdot 1} + V_{22} + S_{21}S_{11}^{-1}S_{12}| = |\tilde{S}_{22\cdot 1}|/|\tilde{S}_{22\cdot 1} + \tilde{U}_{22}| \tag{4.11}$$

where

$$S_{22\cdot 1} = S_{22} - S_{21}S_{11}^{-1}S_{12}, \quad \tilde{S}_{22\cdot 1} = \gamma^{-\frac{1}{2}}S_{22\cdot 1}\gamma^{-\frac{1}{2}}$$

$$\tilde{U}_{22} = \gamma^{-\frac{1}{2}}(V_{22}+S_{21}S_{11}^{-1}S_{12})\gamma^{-\frac{1}{2}} \text{ and } \gamma = I - \Delta'\Delta.$$

Lemma 4.1 (1) $S_{22\cdot 1} \sim W_{p_2}(I,n-r-p_1)$

(2) $\tilde{S}_{22\cdot 1}$ and $\tilde{U}_{22}$ are independent

(3) Conditional on Y_1, $\tilde{U}_{22} \sim W_{p_2}(I,r_1+p_1:\Psi)$ where

$$\begin{cases} \Psi = \gamma^{-\frac{1}{2}}[\Xi+\Delta'S_{11}\Delta]\gamma^{-\frac{1}{2}} \\ \Xi = [X_1\bar{\theta}_{12}+Y_1\Delta]'(P_0-P_2)[X_1\bar{\theta}_{12}+Y_1\Delta] \text{ with } \bar{\theta}_{12} = \theta_{12}\Sigma_{22}^{-\frac{1}{2}}P_2. \end{cases} \tag{4.12}$$

Proof. Our original model may be viewed as

$$[Y_1,Y_2] \sim N\left([X_1,X_2]\begin{bmatrix} 0 & \bar{\theta}_{12} \\ \bar{\theta}_{21} & 0 \end{bmatrix}, I_n \otimes \Omega\right)$$

with Ω in (4.6), where $\bar{\theta}_{21} = \theta_{21}\Sigma^{-\frac{1}{2}}P_1$. Hence conditional on Y_1,

$$Y_2 \sim N(X_1\bar{\theta}_{12}+(Y_1-X_2\bar{\theta}_{21})\Delta,\ I \otimes \gamma),$$

from which conditional on Y_1, we obtain

$$S_{21}S_{11}^{-1}S_{12} = Y_2'P_0Y_2 \sim W_{p_2}(\gamma, p_1 : \Delta' S_{11}\Delta),$$

$$V_{22} = Y_2'[P_0-P_1]Y_1 \sim W_{p_2}(\gamma, r_1 : \Xi) \text{ and}$$

$$S_{22\cdot 1} \sim W_{p_2}(\gamma, n-r-p_1).$$

Further conditional on Y_1, $S_{22\cdot 1}$, $S_{21}S_{11}^{-1}S_{12}$ and V_{22} are independent and $S_{22\cdot 1}$ does not depend on Y_1. Thus all the result follow.

Theorem 4.1. The LRT with critical region $L < c$ is unbiased.

Proof. Let

$$C = \{(S_{22\cdot 1},\ V_{22}+S_{21}S_{11}^{-1}S_{12}, V_{11}, S_{11}) : L_1L_2L_3 < c\}$$

be the critical region of size α in the space of $(S_{22\cdot 1},\ V_{22}+S_{21}S_{11}^{-1}S_{12}, V_{11}, S_{11})$. Then since S_{11} and V_{11} are functions of Y_1 only, from (4.10) the power function of the LRT ϕ_L is expressed as

$$\pi(\phi_L, (\bar{\theta}_{12}, \bar{\theta}_{21}, \Delta)) = P(C|\bar{\theta}_{12}, \bar{\theta}_{21}, \Delta)$$

$$= E_{Y_1}[P(\{(\tilde{S}_{22\cdot 1}, \tilde{U}_{22})|\ |\tilde{S}_{22\cdot 1}|/|\tilde{S}_{22\cdot 1} + \tilde{U}_{22}| < cL_3^{-1}\}|(S_{11}, V_{11}, \bar{\theta}_{12}, \bar{\theta}_{21}, \Delta)] \quad (4.13)$$

where E_{Y_1} denotes the expectation with respect to Y_1. Since

$$|\tilde{S}_{22\cdot 1}|/|\tilde{S}_{22\cdot 1} + \tilde{U}_{22}| < c'$$

is regarded as the LRT for testing $\Psi = 0$ in the MANOVA set-up with $(\tilde{U}_{22},\tilde{S}_{22\cdot1})$, as is shown in Anderson et.al. (1964), it is an increasing function of each characteristic root of Ψ. Since $\bar{\theta}_{12} = 0$ and $\Delta = 0$ imply $\Psi = 0$, from (4.13)

$$\pi(\phi_L,(\bar{\theta}_{12}, \bar{\theta}_{21},\Delta)) \geq \pi(\phi_L,(0,\bar{\theta}_{21},0)) = P(C|0,\bar{\theta}_{21},0). \quad (4.14)$$

But under $\Delta = 0$, L_1L_2 and L_3 are independent because V_{11} and V_{22} are independent. Further $L_3 = |S_{11}|/|S_{11} + V_{11}|$ is regarded as the LRT statistic for testing $\bar{\theta}_{21} = 0$ in the MANOVA set-up with (V_{11},S_{11}). Hence the inside of the conditional expectation

$$P(C|0,\bar{\theta}_{21},0) = E_{L_1L_2}[P(\{(S_{11},V_{11})|L_3 < c(L_1L_2)^{-1}\}|(0,\bar{\theta}_{21},0)]$$

is an increasing function of each characteristic root of $\bar{\theta}_{21}\bar{\theta}_{21}'$ implies

$$P(C|0,\bar{\theta}_{21},0) \geq P(C|0,0,0) \geq \alpha$$

Combining this with (4.13) yields the result.

Under $\Delta \neq 0$ or equivalently $\Sigma_{12} \neq 0$, because L_1L_2 and L_3 are correlated unless $X_1'X_2 = 0$, it seems difficult not only to establish a monotonicity property of the power function of the LRT but also to find a naturel parameter space on which the monotonicity is considered.

We remark that the above result holds even when the model $Y = X\Theta + E$ is defined for X conditioned and the marginal distribution of X does not depend on (Θ,Σ). Hence the LRT for testing no additional information hypothesis in canonical correlation model, which is nothing but our LRT though X is random, is unbiased (see Section 2).

In the case of $X_1'X_2 \neq 0$, we consider two special cases. First consider the case $\Delta = 0$ or equivalenty $\Sigma_{12} = 0$. In this case the problem of testing joint hypothesis $\theta_{12} = 0$ and $\theta_{21} = 0$ is simply split into two independent problems in the two independent models $Y_i = X_1\theta_{1i} + X_2\theta_{2i} + E_i (i=1,2)$. However, the LRT for the joint hypothesis is given by $\phi_L = X(L_2L_3 < c)$ and it is not the product of the two LRT $X(L_2 < c_2)X(L_3 < c_3)$ where $X(A)$ denotes the indicator function of a set A. The power function of ϕ_L for $\theta_{12} = 0$ and $\theta_{21} = 0$ is a function of the characteristic roots.

$$\lambda_i = ch_i(\Sigma_{22}^{-\frac{1}{2}}\theta_{12}'X_1'[P_0-P_2]X_1\theta_{12}\Sigma_{22}^{-\frac{1}{2}}) \ (i=1,\ldots,p_2)$$

and the characteristic roots

$$\gamma_j = ch_j(\Sigma_{11}^{-\frac{1}{2}}\theta_{21}'X_2'[P_0-P_1]X_2\theta_{21}\Sigma_{11}^{-\frac{1}{2}}) \ (j=1,\ldots,p_1),$$

and by similar argument as in the proof of Theorem 4.1, we obtain

<u>Corollary 4.1</u>. Under $\Sigma_{12} = 0$, the power function of the LRT with critical region $L_2L_3 < c$ is increasing in each λ_i or γ_j.

Next we consider the unbiasedness of the LRT with critical region $L_1L_2 < c$. Since the distribution property of L_1L_2 is simply obtained in the proof of Theorem 4.1 by setting $V_{11} = 0$, we obtain

<u>Corollary 4.2</u>. The LRT with critical region $L_1L_2 < c$ for Problem [I] is unbiased.

In fact, from (4.10) and Lemma 4.1, the power function of the LRT is easily seen to be an increasing function of each characteristic root of Ψ conditional on Y. Hence the unbiasedness immediately follows from $\Psi \geq 0$ and the fact that $\theta_{12} = 0$ and $\Sigma_{12} = 0$ imply $\Psi = 0$.

Finally we consider Problem [II] under $X_1'X_2 = 0$. An example for this case is found in time series models (see Anderson (1971 p.92)). Under $X_1'X_2 = 0$, we can take $A = I$ in the canonical form in (4.2) as well as $Q_{ii} = (X_i'X_i)^{-1}$. Then the group $\tilde{G} = \mathcal{O}(r)\times\mathcal{O}(r_2)\times G$ leaves the problem invariant by the action

$$\tilde{g}(W,S) = (\Gamma WB+F, B'SB) \text{ with } \Gamma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}$$

where $\tilde{g} = (P_1, P_2, g)\epsilon\tilde{G}$. Hence in the same way as above, the problem is reduced to the problem of testing $\xi_{12} = 0$, $\xi_{21} = 0$ and $\Delta = 0$ in the canonical form

$$\begin{cases} W_{12} \sim N(\xi_{12}, I \otimes I) \text{ with } \xi_{12} = Q_{11}^{-\frac{1}{2}}\theta_{12}\Sigma_{22}^{-\frac{1}{2}}\Gamma_2 \\ W_{21} \sim N(\xi_{21}, I \otimes I) \text{ with } \xi_{21} = Q_{22}^{-\frac{1}{2}}\theta_{21}\Sigma_{11}^{-\frac{1}{2}}\Gamma_1 \\ S \sim W(\Omega, r_3) \text{ with } \Omega \text{ in } (4.6) \end{cases}$$

where W_{12}, W_{21} and S here are independent. Also $\tilde{G}$ acts on (W_{12}, W_{21}, S) by

$$\tilde{g}(W_{12}, W_{21}, S) = (P_1W_{12}B_2, P_2W_{21}B_1, B'SB).$$

Since the group $\tilde{G}$ is bigger than G, a result corresponding to Theorem 4.1 can be derived. To see this, let $\nu = \nu_1 + \nu_2 + \nu_3$ with

$$\nu_1 = \mathrm{tr}\xi_{12}\xi_{12}' = \mathrm{tr}X_1'X_1\theta_{12}\Sigma_{22}^{-1}\theta_{12}'$$

$$\nu_2 = \mathrm{tr}\xi_{21}\xi_{21}' = \mathrm{tr}X_2'X_2\theta_{21}\Sigma_{11}^{-1}\theta_{21}' \quad \text{and}$$

$$\nu_3 = \sum_{i=1}^{t}\rho_i^2 = \mathrm{tr}\Delta\Delta' = \mathrm{tr}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

<u>Theorem 4.2</u>. There is an $\varepsilon > 0$ such that on the set $\{(\Theta,\Sigma | \nu < \varepsilon\}$ the power function of an invariant test ϕ of size α under $\tilde{G}$ is evaluated as

$$\pi(\phi,(\Theta,\Sigma)) = \alpha + \tfrac{1}{2}[\nu_1 D_1(\phi) + \nu_2 D_2(\phi) + \nu_3 D_3(\phi)] + o(\nu)$$

where

$$D_i(\phi) = E_0[\phi R_i]$$

$$R_1 = \frac{r_1+r_3}{p_1 p_2}\frac{p_1}{r_1}\, \mathrm{tr} W_{12}' W_{12}(W_{12}' W_{12} + S_{22})^{-1}$$

$$R_2 = \frac{r_2+r_3}{p_1 p_2}\frac{p_2}{r_2}\, \mathrm{tr} W_{21}' W_{21}(W_{21}' W_{21} + S_{11})^{-1}$$

$$R_3 = \frac{(r_1+r_3)(r_2+r_3)}{p_1 p_2}\, \mathrm{tr}(W_{21}' W_{21} + S_{11})^{-1} S_{12}(W_{12}' W_{12} + S_{22})^{-1} S_{21}$$

$$- \frac{r_1+r_3}{p_1}\, \mathrm{tr} S_{22}(W_{12}' W_{12} + S_{22})^{-1} - \frac{r_2+r_3}{p_2}\, \mathrm{tr} S_{11}(W_{21}' W_{21} + S_{11})^{-1}$$

and $\lim_{\nu\to 0} \sup_\phi |o(\nu)/\nu| = 0$.

Proof. The proof is completely similar to that of Theorem 4.1. Regarding Z_{12} as W_{12}, replace B_1 by $W_{21}(W_{21}' W_{21} + S_{11})^{-\frac{1}{2}} B_1$. Then every step goes through and the result is obtained.

In this expression, a symmetry which is lacking in Theorem 4.1 is secured, and the statistics R_1, R_2 and R_3 are respectively the LBI tests for testing the separate hypotheses $H_1:\xi_{12} = 0$, $H_2:\xi_{21} = 0$ and $H_3:\Sigma_{12} = 0$ under $\xi_{12} = 0$ and $\xi_{21} = 0$ (see Eaton and Kariya (1983) for H_3) Also for each hypothesis H_i, the power function of an invariant test is expressed as $\alpha + \frac{1}{2}\nu_i D_i(\psi) + o(\nu_i)(i=1,2,3)$. Here again it is noted that the statistics R_1, R_2 and R_3 are not independent under the null hypothesis. R_1 and R_2 are independent under the null hypothesis. Here following the discussion in Section 3, we may propose a combined test of these statistics with critical region

$$R(\beta_1,\beta_2) = \beta_1 r_3 R_1 + \beta_2 r_3 R_2 + R_3 > c \tag{4.15}$$

where r_3 is put on R_i's because $\nu_i = O(n)$ $(i=1,2)$. Here the constants β_i's may be considered indicating relative weights for the three hypotheses $\Theta_{12} = 0$, $\Theta_{21} = 0$ and $\Sigma_{12} = 0$. It will be often the case $\beta_1 = \beta_2$. This test clearly maximizes the local power $\alpha + \sum_{i=1}^{3}\nu_i D_i(\phi)$ in the neighborhood

$$\{(\Theta,\Sigma)|\ \nu_i = r_3\beta_i\nu_3(i=1,2)\}\cap\{(\Theta,\Sigma)|\ \nu < \varepsilon\}$$

Also the local sensitivity of an invariant test ϕ against (ν_1,ν_2,ν_3) is described by the coordinates $(D_1(\phi),D_2(\phi),\ D_3(\phi))$. Further, from the observations above, we may use the test based on $R(\beta_1,\beta_2)$ in (4.15) even if $X_1'X_2 \neq 0$.

It is remarked that the LRT statistic in this case is of course the same as $L = L_1L_2L_3$. But here because $X_1'X_2 = 0$, in addition to the independence of L_1 and $L_i(i=2,3)$, L_2 and L_3 are independent though L_1, L_2 and L_3 are jointly dependent.

5. ASYMPTOTIC NULL DISTRIBUTIONS OF THE TEST STATISTICS

The asymptotic null distribution of the LRT for Problem [II]: $\theta_{12} = 0$, $\theta_{21} = 0$ and $\Sigma_{12} = 0$ has been derived by Fujikoshi (1982) in the context of the problem of testing no additional information hypothesis in canonical correlation anaylsis. From a more general viewpoint, we here briefly treat it in a systematic way and then consider the asymptotic null distribution of the LRT for Problem [I]:$\theta_{12} = 0$ and $\Sigma_{12} = 0$. The notation

$$\lambda \sim \Lambda_p(q,n)$$

denotes that for independent Wishart matrices A and B,

$$\lambda = |A|/|A + B| \text{ with } A \sim W_p(\Sigma,n)\ B \sim W_p(\Sigma,q)$$

<u>Lemma 5.1</u> Let $\lambda_i \sim \Lambda_{p_i}(q_i, n-d_i)$ and λ_i's be *independent* $(i=1,2)$. Then

$$P(-m \log \lambda_1\lambda_2 \leq x) = G_f(x) + \frac{\mu}{m^2}[G_{f+4}(x) - G_f(x)] + O(m^{-3}), \tag{5.1}$$

where $G_f(x)$ is the cdf of $\chi^2(f)$,

$$\begin{aligned} & f_1 = p_1q_1,\ f_2 = p_2q_2,\ f = f_1 + f_2,\ m = n-\rho \\ & \rho = \frac{1}{f}\{f_1[d_1 - \tfrac{1}{2}(q_1-p_1-1)] + f_2[d_2 - \tfrac{1}{2}(q_2-p_2-1)]\} \\ & \mu = s + \frac{f_1f_2}{4f}[d_1-d_2+\tfrac{1}{2}(p_1-p_2-q_1+q_2)]^2 \\ & s = s_1 + s_2,\ s_1 = \frac{f_1}{48}(p_1^2+q_1^2-5) \text{ and } s_2 = \frac{f_2}{48}(p_2^2+q_2^2-5). \end{aligned} \tag{5.2}$$

<u>Proof</u>. The result follows directly by the usual method based on characteristic function.

Now for Problem [II], the LRT statistic is given by $L_1L_2L_3$ in (4.8) and from Lemma 4.1 and (4.11), it is easy to see that under the null hypothesis

$$\lambda_2 \equiv L_1 L_2 \sim \Lambda_{p_2}(r_1+p_1,\ n-r-p_1) \tag{5.3}$$

$$\lambda_1 = L_3 \sim \Lambda_{p_1}(r_2+p_2,\ n-r) \tag{5.4}$$

and λ_1 and λ_2 are independent. Therefore the following result follows from Lemma 5.1.

<u>Proposition 5.1</u>. For Problem [II]: $\theta_{12} = 0$, $\theta_{21} = 0$ and $\Sigma_{12} = 0$, the asymptotic null distribution of LRT based on $-m \log(L_1 L_2)L_3$ is given by (5.1) where in (5.2) $q_1 = r_2 + p_2$, $q_2 = r_1 + p_1$, $d_1 = r$ and $d_2 = r + p$.

On the other hand, for Problem [I], the LRT statistic is given by L_1L_2 in (3.24), which is the same as the L_1L_2 in Problem [II] and hence from (5.3) under the null hypothesis

$$\lambda_2 = L_1 L_2 \sim \Lambda_{p_2}(r_1+p_1, n-r-p_1)$$

The asymptotic null distribution of $-2m \log \lambda_2$ is well known.

Next we consider the asymptotic null distribution of $R(\beta_1,\beta_2)$ in (4.15) for Problem [II] and as a special case, obtain the null distribution of $T(\beta)$ in (3.22). But here for simplicity, the case $\beta_i p_i r_i = 1 (i=1,2)$ will be treated. A more general case can be obtained in a similar manner. From (4.15), let the test statistic be

$$\tilde{R} = \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 + 2p_1p_2[1+ \frac{1}{2m}(r_1+r_2)] \tag{5.5}$$

where $m = n-r$

$$\tilde{R}_1 = (m+r_2)\mathrm{tr}B_{11}(B_{11}+S_{11})^{-1} \text{ with } B_{11} = W_{21}'W_{21}$$

$$\tilde{R}_2 = (m+r_1)\mathrm{tr}B_{22}(B_{22}+S_{22})^{-1} \text{ with } B_{22} = W_{12}'W_{12}$$

$$\tilde{R}_3 = \frac{1}{m}(m+r_1)(m+r_2)\mathrm{tr}(B_{11}+S_{11})^{-1}S_{12}(B_{22}+S_{22})^{-1}S_{21}$$

$$-\frac{p_2}{m}(m+r_2)\mathrm{tr}S_{11}(B_{11}+S_{11})^{-1} - \frac{p_1}{m}(m+r_1)\mathrm{tr}S_{22}(B_{22}+S_{22})^{-1}.$$

Theorem 5.1. For Problem [II]: $\theta_{12} = 0$, $\theta_{21} = 0$, and $\Sigma_{12} = 0$, the asymptotic null distribution of $\tilde{R}$ in (5.5) is given by

$$\begin{aligned} P(\tilde{R} \le x) = G_f(x) &- \frac{1}{4m}(f_1s_1+f_2s_2+f_3s_3)[G_f(x) - 2G_{f+2}(x) + G_{f+4}(x)] \\ &- \frac{1}{2m}p_1p_2(r_1+r_2)[2G_f(x) - 3G_{f+2}(x) + G_{f+4}(x)] + O(m^{-3/2}) \end{aligned} \tag{5.6}$$

where $f = f_1 + f_2 + f_3$, $f_1 = p_1r_2$, $f_2 = p_2r_1$, $f_3 = p_1p_2$, $s_1 = p_1 + r_2 + 1$ $s_2 = p_2 + r_1 + 1$ and $s_3 = p_1 + p_2 + 1$.

The case of Problem [I] follows directly from this theorem.

Corollary 5.1. For Problem [I]: $\theta_{12} = 0$ and $\Sigma_{12} = 0$, the asymptotic null distribution of

$$\tilde{T} = T(r_1/p_1) + 2p_1p_2[1 + \frac{1}{2m}r_1] \tag{5.7}$$

with $T(\beta)$ in (3.22) is given by (5.6) with $r_2 = 0$.

Outline of the Proof of Theorem 5.1. Under the null hypothesis, assuming $\Sigma = I$ without loss of generality, we have the three independent Wishart variates $B_{11} \sim W_{p_1}(I,r_2)$, $B_{22} \sim W_{p_2}(I,r_1)$ and $S \sim W_p(I,m)$. As usual let

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \sqrt{m}\left(\frac{1}{m}S-I\right) \tag{5.8}$$

and expand R_i's in terms of V, B_{11} and B_{22} as

$$R_j = \mathrm{tr}B_{j1} + \frac{1}{\sqrt{m}} q_{j1} + \frac{1}{m} q_{j2} + O_p(m^{-3/2})(j=1,2)$$

$$R_3 = \mathrm{tr}V_{12}V_{21} - 2p_1p_2 + \frac{1}{\sqrt{m}} q_{31} + \frac{1}{m} [q_{32} + \tilde{q}_{32} - p_1p_2(r_1+r_2)] + O_p(m^{-3/2})$$

where $q_{j1} = -\mathrm{tr}B_{jj}V_{jj}$, $q_{j2} = \mathrm{tr}B_{jj}V_{jj}^2 - \mathrm{tr}B_{jj}^2 + \bar{r}_j\mathrm{tr}B_{jj}$ with $\bar{r}_1 = r_2$ and $\bar{r}_2 = r_1$ $(j=1,2)$, $q_{31} = -\mathrm{tr}V_{11}V_{12}V_{21} - \mathrm{tr}V_{12}V_{22}V_{21}$, $q_{32} = \mathrm{tr}V_{11}^2V_{12}V_{21} + \mathrm{tr}V_{12}V_{22}^2V_{21} + \mathrm{tr}V_{11}V_{12}V_{22}V_{21}$ and $\tilde{q}_{32} = (r_1+r_2)\mathrm{tr}V_{12}V_{21} - \mathrm{tr}B_{11}V_{12}V_{21} - \mathrm{tr}B_{22}V_{21}V_{12} + p_2\mathrm{tr}B_{11} + p_2\mathrm{tr}B_{22}$. Then $\tilde{R}$ is expressed as

$$\begin{aligned}\tilde{R} &= \mathrm{tr}B_{11} + \mathrm{tr}B_{22} + \mathrm{tr}V_{12}V_{21} + \frac{1}{\sqrt{m}} (q_{11}+q_{21}+q_{31}) \\ &\quad + \frac{1}{m} (q_{12}+q_{22}+q_{32}+\tilde{q}_{32}) + 2p_1p_2 \left[1+ \frac{1}{2m} (r_1+r_2)\right] + O_p(m^{-3/2}) .\end{aligned} \tag{5.9}$$

Then the characteristic function $C(t)$ of $\tilde{R}$ is expanded as

$$C(t) = E\{H(t)[1 + \frac{1}{\sqrt{m}} A_1 + \frac{1}{m} A_2]\} + O(m^{-3/2}). \tag{5.10}$$

where $H(t) = \exp[it\ \mathrm{tr}B_{11} + it\ \mathrm{tr}B_{22} + it\ \mathrm{tr}V_{12}V_{21}]$, and A_1 and A_2 are functions of B_{ii}'s and V_{ij}'s. Since V_{ij}'s are not independent, in evaluation of the expectation in (5.10), we may use the following lemma.

<u>Lemma 5.1</u>. (1) The pdf of V is expanded as

$$f(V) = c \exp(-\frac{1}{4} \mathrm{tr}V^2)\{1+ \frac{1}{\sqrt{m}} [-\frac{1}{2} (p+1)\mathrm{tr}V + \frac{1}{6} \mathrm{tr}V^3]\} + O(m^{-1}).$$

(2) The conditional pdf of V_{12} given V_{11} and V_{22} is expanded as

$$\begin{aligned}f(V_{12}|V_{11},V_{22}) &= c \exp(-\frac{1}{2} \mathrm{tr}V_{12}V_{21})\{1+ \frac{1}{2\sqrt{m}} [- p_2\mathrm{tr}V_{11} - p_1\mathrm{tr}V_{11} \\ &\quad + \mathrm{tr}V_{11}V_{12}V_{21} + \mathrm{tr}V_{12}V_{22}V_{21}]\} + O(m^{-1}).\end{aligned}$$

Proof. (1) is well known. (2) is obtained by $f(V)/f_1(V_{11})f_2(V_{22})$, where the marginal pdf's of V_{ii}'s are first expanded as in (1).

Lemma 5.2. The characteristic function of $\tilde{R}$ is evaluated as

$$C(t) = (1-2it)^{-f/2}\{1 - \frac{1}{4m}(f_1s_1+f_2s_2+f_3s_3)[(1-2it)^{-1}-1]^2 - \frac{1}{2m}p_1p_2(r_1+r_2)[2 - 3(1-2it)^{-1} + (1-2it)^{-2}]\} + O(m^{-3/2}). \tag{5.11}$$

Proof. The proof is straightforward although it involves a lot of computation. The result is obtained by using Lemma 5.1 and the following well known results:

$$\begin{aligned} &E[\exp(it\operatorname{tr}V_{12}V_{21})\{1+\frac{it}{\sqrt{m}}q_{31} + \frac{1}{m}(it)q_{32} + \frac{(it)^2}{2}q_{31}^2]\}] \\ &= E\exp(itm\operatorname{tr}S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}) + O(m^{-3/2}) \\ &= (1-2it)^{-f_3/2}[1 - \frac{1}{4m}f_3s_3\{(1-2it)^{-1} - 1\}^2 \\ &\quad + O(m^{-3/2})] \end{aligned} \tag{5.12}$$

(Fujikoshi (1970), Muirhead (1970))

$$\begin{aligned} &E[\exp\{it(m+r_2)\operatorname{tr}B_{11}(B_{11}+S_{11})^{-1} \\ &\quad + it(m+r_1)\operatorname{tr}B_{22}(B_{22}+S_{22})^{-1}\}] \end{aligned} \tag{5.13}$$

$$= (1-2it)^{-(f_ps_1+f_2s_2)/2}[1 - \frac{1}{4m}(f_1s_1+f_2s_2) + O(m^{-3/2})]$$

(Fujikoshi (1970), Murihead (1970))

Inverting C(t) in Lemma 5.2, we obtain the result in Theorem 5.1.

6. TESTS FOR INDEPENDENCE WHEN $\theta_{12} = 0$ AND $\theta_{21} = 0$

As has been stated in Section 2, for problem [III] $H:\Sigma_{12} = 0$ given $\theta_{12} = 0$ and $\theta_{21} = 0$ KFK (1984) proposed the LBI test, a LRT-like test and a trace test, and considered the asymptotic null and nonnull distributions of these tests. In this section, adopting what we call the method of small-small asymptotics due to Mukerjee and Chandra (1984), we compare the power functions of those tests. Since the model in Problem [III] is

$$\underset{n}{[\underset{p_1}{Y_1}, \underset{p_2}{Y_2}]} = [\underset{r_1}{X_1}, \underset{r_2}{X_2}] \begin{bmatrix} \theta_{11} & 0 \\ 0 & \theta_{22} \end{bmatrix} + [E_1, E_2] \tag{6.1}$$

$$[E_1, E_2] \sim N\left(0, I_n \otimes \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right) \tag{6.2}$$

it contains the MANOVA model as a special case with $X_1 = X_2$. The comparison we make here deals with the comparison of the Pillai test and the LRT for independence in the MANOVA model. In fact, as will be shown, when $X_1 = X_2$, the LBI test and the trace test in Problem [III] are both reduced to the Pillai test of independence in the MANOVA model, which is LBI, while the LRT-like test is reduced to the LRT of independence in the MANOVA model. Following KFK (1984), assume $p_1 \geq p_2$ without loss of generality and let

$$\begin{cases} Q_0 = I - X(X'X)^+X' = Z_0Z_0' \text{ with } Z_0'Z_0 = I_{n_0} \\ Q_i = I - X_i(X_i'X_i)^{-1}X_i \quad (i=1,2) \end{cases} \tag{6.3}$$

where $Z_0: n\times n_0$, $n_0 = n-r_0$ and $r_0 = \text{rank } [X_1, X_2]$ and A^+ is Penrose inverse of A. Further let

$$X(X'X)^+X' - X_i(X_i'X_i)^{-1} = \bar{Z}_i\bar{Z}_i' \text{ with } \bar{Z}_i'\bar{Z}_i = I_{r_0-r_i}$$

where $\bar{Z}_i : n\times(r_0-r_i)$, and let

$$Z_i = [\bar{Z}_i, Z_0] : n\times(n_0+r_0-r_i), \tag{6.5}$$

$$\begin{array}{c} \begin{bmatrix} M_i \\ U_i \end{bmatrix} \begin{array}{l} (r_0-r_i) \\ (n-r_0) \end{array} \\ p_i \end{array} = Z_i'Y_i = \begin{bmatrix} \bar{Z}_i' Y_i \\ Z_0' Y_i \end{bmatrix}, \tag{6.6}$$

$$S = G + B \text{ and } R = S_{12}S_{22}^{-1}S_{21}S_{11}^{-1} \tag{6.7}$$

where $S = (S_{ij})$ with $S_{ij} : p_i\times p_j$, $G = (G_{ij})$ with $G_{ij} = U_i'U_j : p_i\times p_j$ and $B = (B_{ij})$ for i, j=1,2 with

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} M_1'M_1 & M_1'KM_2 \\ M_2'K'M_1 & M_2'M_2 \end{pmatrix},\ K = \bar{Z}_1'\bar{Z}_2 \tag{6.8}$$

Here we note that when $X_1 = X_2$, $Q_0 = Q_i$ (i=1,2) so that $B = 0$ and $S = G$ with $G \sim W(\Sigma, n_0)$, which is nothing but the canonical form for the problem of testing $\Sigma_{12} = 0$ in the MANOVA model. Now based on the notation introduced above, our problem is to test $\Sigma_{12} = 0$ based on S in (5.7) and then the LRT-like test statistic, the tract test statistic and the LBI test statistic considered in KFK are respectively expressed as

$$T_1 = -\,n_0 \log|I - R| \tag{6.9}$$

$$T_2 = n_0 \operatorname{tr} R \tag{6.10}$$

$$\begin{aligned} T_3 = \frac{1}{n_0}\{&n_1n_2\operatorname{tr}R - n_1p_2\operatorname{tr}S_{11}^{-1}Y_1'Q_1Q_2Q_1Y_1 \\ &- n_2p_1\operatorname{tr}S_{22}^{-1}Y_2'Q_2Q_1Q_2Y_2\} + p_1p_2(2-\frac{1}{n_0}\,t) \end{aligned} \tag{6.11}$$

where

$$n_i = n - r_i \ (i=0,1,2) \text{ and } t = r_1 + r_2 - 2r. \tag{6.12}$$

As has been shown in KFK, the power functions of these invariant tests depend on (Θ,Σ) only through the canonical correlations $\rho_1^2 \geq \cdots \geq \rho_{p_2}^2$ of Σ or the characteristic roots of $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}$. Here to consider the small-small asymptotics for the power functions, first fix $\omega_1 \geq \cdots \geq \omega_{p_2} \geq 0$ with $\omega_1 \neq 0$, which are chosen to be small later, and take

$$\frac{1}{2}\rho_j^2 = \frac{1}{n_0}\omega_j \ (j=1,\ldots,p_2) \tag{6.13}$$

where $\frac{1}{n_0}$ is supposedly small. This implies $\rho_j = (2\omega_j/n_0)^{\frac{1}{2}}$ is eventually small-small. Further let

$$\delta = \omega_1 + \cdots + \omega_{p_2} = \mathrm{tr}\Phi \tag{6.14}$$

$$\Phi = \mathrm{diag}\{\omega_1,\ldots,\omega_{p_2}\} \tag{6.15}$$

$$f = p_1p_2 \text{ and } s = p_1 + p_2 + 1. \tag{6.16}$$

Then from the results in KFK, the asymptotic power functions of the tests based on T_i's in (5.9), (5.10) and (5.11) under the local alternative $\{\rho_j\}$ in (5.13) (as $n_0 \to \infty$) are given by

$$P(T_i > x_i) = \bar{G}_f(x_i\delta) + \frac{1}{n_0}\Sigma_{j=0}^{4} a_{ij}\bar{G}_{f+2j}(x_i:\delta) + O(n_0^{-2}), \tag{6.17}$$

$(i=1,2,3)$ where $\bar{G}_k(x:\delta) = 1 - G_k(x:\delta)$, $G_k(x:\delta)$ is the distribution function of X^2 distribution with degrees of freedom k and noncentrality parameter δ,

$$a_{10} = -\frac{1}{4}fs - \frac{1}{2}f(t+trKK') - (trKK')\delta - tr\Phi^2$$

$$a_{11} = \frac{1}{4}fs + \frac{1}{2}f(t+trKK') - t\delta$$

$$a_{12} = (t+trKK')\delta + 2tr\Phi^2,\ a_{13} = -tr\Phi^2,\ a_{14} = 0$$

$$a_{20} = -\frac{1}{4}fs - \frac{1}{2}f(t+trKK') - (trKK')\delta - tr\Phi^2$$

$$a_{21} = \frac{1}{2}fs + \frac{1}{2}f(t+trKK') - t\delta$$

$$a_{22} = -\frac{1}{4}fs + (s+t+trKK')\delta + 2tr\Phi^2$$

$$a_{23} = -s\delta,\quad a_{24} = -tr\Phi^2$$

$$a_{30} = -\frac{1}{4}fs - \frac{1}{2}ftrKK' - (trKK')\delta - tr\Phi^2$$

$$a_{31} = \frac{1}{2}fs + \frac{1}{2}ftrKK'$$

$$a_{32} = -fs + (s+trKK')\delta + 2tr\Phi^2$$

$$a_{33} = -s\delta \text{ and } a_{34} = -tr\Phi^2.$$

Note that $\Sigma_{j=0}^{4} a_{ij} = 0$ $(i=1,2,3)$. To evaluate the power functions in (6.17) further, let $g_f(x:\delta)$ be the pdf of $G_f(x:\delta)$ and let $G_f(x) = G_f(x:0)$ and $g_f(x) = g_f(x:0)$. Then it is easy to see that

$$\bar{G}_{f+2}(x:\delta) = 2q_{f+2}(x:\delta) + \bar{G}_f(x:\delta) \tag{6.18}$$

$$g_{f+2}(x) = xg_f(x)/f \tag{6.19}$$

$$\bar{G}_f(x+\frac{c}{m}:\delta) = \bar{G}_f(x:\delta) - \frac{c}{m}g_f(x:\delta) + O(m^{-2}). \tag{6.20}$$

Using (6.18), the power functions are evaluated as

$$P(T_i > x_i) = \bar{G}_f(x_i:\delta) + \frac{1}{n_0}\Sigma_{j=1}^{4} b_{ij}g_{f+2j}(x_i:\delta) + O(n_0^{-2}) \qquad (6.21)$$

where

$$b_{11} = \frac{1}{2}fs + f(t+trKK') + 2(trKK')\delta + 2tr\Phi^2$$

$$b_{12} = 2(t+trKK')\delta + 2tr\Phi^2$$

$$b_{13} = -2tr\Phi^2, \quad b_{14} = 0$$

$$b_{21} = \frac{1}{2}fs + f(t+trKK') + 2(trKK')\delta + 2tr\Phi^2$$

$$b_{22} = -\frac{1}{2}fs + 2(t+trKK')\delta + 2tr\Phi^2$$

$$b_{23} = -2s\delta - 2tr\Phi^2, \quad b_{24} = -2tr\Phi^2$$

$$b_{31} = \frac{1}{2}fs + f\,trKK' + 2(trKK')\delta + 2tr\Phi^2$$

$$b_{32} = -\frac{1}{2}fs + 2(trKK')\delta + 2tr\Phi^2$$

$$b_{33} = -2s\delta - 2tr\Phi^2, \quad b_{34} = -2tr\Phi^2.$$

Using (6.21) and (6.19), under the null hypothesis $\delta = 0$

$$\alpha = P(T_i > x_i|H_0) = G_f(x_i) - \frac{1}{n_0}\tilde{b}_{i0}g_f(x_i) + O(n_0^{-2}) \qquad (6.22)$$

$(i=1,2,3)$ where

$$\begin{cases} \tilde{b}_{10} = -x_1(\frac{1}{2}s+t+trKK') \equiv -x_1C_1 \\ \tilde{b}_{20} = -x_2(\frac{1}{2}s+t+trKK' - \frac{sx_2}{2(f+2)}) \equiv -x_2C_2 \\ \tilde{b}_{30} = -x_3(\frac{1}{2}s+trKK' - \frac{sx_3}{2(f+2)}) \equiv -x_3C_3 \end{cases} \qquad (6.23)$$

From (6.20)(or Hill and Davis formula) we obtain with u in $\bar{G}_f(u) = \alpha$

$$x_i = u - \frac{1}{n_0} b_{i0} + O(n_0^{-2}) \tag{6.24}$$

where $b_{i0} = -uC_i$ with C_i in (6.23). Hence using (6.20) and (6.24), we obtain the following theorem.

Theorem 6.1. Let $\pi(\phi_i, \Phi)$ be the power function of the test ϕ_i of size α with critical region $T_i > x_i (i=1,2,3)$. Then for u satisfying $\bar{G}_f(u) = \alpha$ and for δ small, it is evaluated as

$$\pi(\phi_i, \Phi) = \bar{G}_f(u:\delta) + \frac{1}{n_0} H_i(\delta) g_f(u) + O(n_0^{-2})(i=1,2,3) \tag{6.25}$$

where

$$H_i(\delta) = c_{i1}\delta + c_{i2}\delta^2 + c_{i3}\mathrm{tr}\Phi^2 + O(n_0^{-2})(i=1,2,3) \tag{6.26}$$

$$c_{11} = c_{21} = c_{31} = \frac{2u}{f}\mathrm{tr}KK' - \frac{su^2}{f(f+2)}$$

$$c_{12} = -\frac{2u}{f}\mathrm{tr}KK' + (s+2\mathrm{tr}KK')\frac{u^2}{f(f+2)} - \frac{su^3}{f(f+2)(f+4)}$$

$$c_{22} = c_{32} = c_{12} + \frac{2su^4}{f(f+2)^2(f+4)(f+6)}$$

$$c_{13} = \frac{2u}{f} + \frac{2u^2}{f(f+2)} - \frac{2u^3}{f(f+2)(f+4)} \quad \text{and}$$

$$c_{23} = c_{33} = c_{13} - \frac{2u^4}{f(f+2)(f+4)(f+6)}$$

Proof. Using (6.20) and (6.24), the power functions in (6.21) are directly shown to be equal to those in (6.25) after some algebra, where $g_{f+3j}(u:\delta) = e^{-\delta}\Sigma_{k=0}^{\infty}(\delta^k/k!)g_{f+2j+2k}(u)$ and $g_{f+2}(u) = ug_f(u)/f$ were used.

In the expression (6.25), $\delta = \sum_{i=1}^{p_2} \omega_i$ has been chosen to be small. That is, the power functions were first asymptotically expanded in the orders of $n_0^{-k}(k=0,1,2...)$, and then for δ small the terms of order n_0^{-1} were asymptotically expanded in the orders of δ^k $(k=0,1,...)$ since the terms of order n_0^0 are common to all the tests, i.e., $\bar{G}_f(u:\delta)$. From this expression, the local behaviors of the power functions are compared as follows.

Theorem 6.2. For the tests ϕ_i, it follows that with $\delta = \mathrm{tr}\Phi$

(1) $\lim_{\delta\to 0} \lim_{n_0\to\infty} n_0[\pi(\phi_j,\Phi) - \pi(\phi_1,\Phi)]/\delta = 0$ $(j=2,3)$

(2) $\lim_{\delta\to 0} \lim_{n_0\to\infty} n_0[\pi(\phi_j,\Phi) - \pi(\phi_1,\Phi]/\delta^2 = \Delta(\phi)(j=2,3)$

and

(3) $\lim_{\delta\to 0} \lim_{n_0\to\infty} n_0[\pi(\phi_3,\Phi) - \pi(\phi_2,\Phi)]/\delta^2 = 0$

where $\Delta(\Phi) = [2u^4 g_f(u)/f(f+2)^2(f+4)(f+6)]\gamma(\Phi)$ with

$$\gamma(\Phi) = (p_1+p_2+1) - (p_1p_2+2) \lim_{\delta\to 0} [\mathrm{tr}\Phi^2/(\mathrm{tr}\Phi)^2] \qquad (6.27)$$

Proof. Immediate from Theorem 6.1.

Now (1) implies that in terms of power all the three tests are asymptotically equivalent up to $O(n_0^{-1})$ and $O(\delta)$. The asymptotic difference between the LBI test ϕ_3 (or the trace test ϕ_2) and the LRT-like test ϕ_1 appears in the term of $O(n_0^{-1})$ and $O(\delta^2)$ as is shown in (2). Setting $\tau \equiv \lim_{\delta\to 0}[\mathrm{tr}\Phi^2/(\mathrm{tr}\Phi)^2]$, from (2), if $\gamma(\Phi) > 0$, or equivalently $(p_1+p_2+1)/(p_1p_2+2) > \tau$, the LBI test is asymptotically better up to $O(n_0^{-1})$ and $O(\delta^2)$ than the LRT-like test, while if $\gamma(\Phi) < 0$, the LRT-like test is asymptotically better. Since $\mathrm{tr}\Phi^2 = \Sigma\omega_i^2$ and $\mathrm{tr}\Phi = \Sigma\omega_i = \delta$, the inequality

$$\Sigma\omega_i^2 \le [\Sigma\omega_i]^2 \le p_2[\Sigma\omega_i^2]$$

follows from $\omega_i \geq 0$ and Schwartz's inequality. This implies

$$\frac{1}{p_2} \leq \tau \doteq \lim_{\delta \to 0} \mathrm{tr}\Phi^2/[\mathrm{tr}\Phi]^2 \leq 1 \tag{6.28}$$

The equality in the first of (6.28) holds if and only if $\omega_1 = \omega_2 = \ldots = \omega_{p_2}$, while the equality in the second inequality holds if and only if $\omega_2 = \ldots = \omega_{p_2} = 0$ since $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_{p_2} \geq 0$. On the other hand,

$$\frac{1}{p_2} < \tau_0 \equiv (p_1+p_2+1)/(p_1p_2+1) < 1$$

since $p_1 \geq p_2 > 0$. Hence both the cases $1 \geq \tau \geq \tau_0$ and $\tau_0 > \tau \geq 1/p_2$ can occur. The above observation will show that the closer τ is to 1, the more concentrated ω_i's are around $\omega_1 = \ldots = \omega_{p_2}$, while the closer τ is to $1/p_2$, the more spread ω_i's are.

Next, we consider the small-small asymptotic comparison between the LRT and the Pillai test of independence in the MANOVA model. As has been pointed out, the problem of testing independence in the MANOVA model is included as a special case of our problem with $X_1 = X_2$. In case $X_1 = X_2$, $\bar{Z}_i = 0$ in (6.4) so that $S = G$ in (6.7), $K = 0$ in (6.8), $n_1 = n_2 = n_0$ in (6.12), and $T_2 = T_3$ in (6.9) and (6.11). However this does not cause any changes in the results of Theorem 6.1 and 6.2 except for the slight changes of the coefficients c_{ij}'s in Theorem 6.1. That is, by setting $K = 0$ in c_{11} and c_{12}, the results in Theorem 6.1 holds as it is, while Theorem 6.1 is effective whether or not $X_1 = X_2$.

<u>Corollary 6.1</u>. For testing independence in the MANOVA model with $X_1 = X_2$, all the results in Theorem 6.1 hold. If $\tau_0 > \tau(\geq 1/p_2)$, Pillai's test, which is LBI for fixed n_0, is asymptotically better up to $O(n_0^{-1})$ and $O(\delta^2)$ than the LRT while if $1 > \tau > \tau_0$, the LRT is asymptotically better.

7. REMARKS

In this section, we first consider Problem [IV]:$\theta_{12} = 0$ and $\theta_{12} = 0$. For this problem, it is easy to see that a canonical form of the model is also given by the model in (4.2) where the joint hypothesis $\theta_{12} = 0$, $\theta_{21} = 0$ and $\Sigma_{12} = 0$ in Problem [II] was tested, and the hypothesis here becomes $\eta_{12} = 0$ and $\eta_{21} = 0$. Further, the same group $\mathcal{G} = G\ell(p_1)\times G\ell(p_2)\times R^{r_1p_1}\times R^{r_2p_2}$ acting on (W,S) as in (4.5) leaves this problem. Hence the class of invariant tests in Problem [II] is exactly the same as the class of invariant test in Problem [IV] we are presently considering. This implies that the power function of an invariant test in Problem [IV] is also a function of Ω, ξ_{12} and ξ_{21} in (4.6). However, under the null hypothesis $\xi_{12} = 0$ and $\xi_{21} = 0$, it still depends on Ω because Ω may not be zero in Problem [IV]. Therefore in general an invariant test in the present problem is not similar. In fact, this follows from the fact that the group does not act transitively on the parameter space of the null hypothesis. This is true even in the case $X_1'X_2 = 0$ where the group is enlarged to $\bar{G} = \mathcal{O}(r_1)\times\mathcal{O}(r_2)\times G$ as in (   ). For example, suppose we construct such statistics as

$$L_2 = |S_{22}|/|S_{22} + W_{12}'W_{12}| \text{ and } L_3 = |S_{11}|/|S_{11} + W_{21}'W_{21}| \quad (7.1)$$

for $\eta_{12} = 0$ and $\eta_{21} = 0$ respectively analogously to (4.8), or

$$L_2' = \mathrm{tr}W_{12}S_{22}^{-1}W_{12}' \quad \text{and} \quad L_3' = \mathrm{tr}W_{21}S_{11}^{-1}W_{21}'.$$

But here L_2 and L_3 (or L_2' and L_3') are correlated under the mull hypothesis so that any test combining L_2 and L_3 (or L_2' and L_3') is not similar unless one of the two statistics is completely ignored. Because of the non-similarity feature of the problem we leave it here. One might use a non-similar test by combining L_2 and L_3 in (7.1) in such a way as L_2L_3 or might test the two hypothesis separately. It is noted that an explicit form of the LRT for the present problem is difficult to derive.

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In this paper, the authors consider various procedures for testing the hypotheses of independence of two sets of variables and certain regression coefficients are zero under the classical multivariate regression model. Various properties of these procedures and the asymptotic distributions associated with these procedures are also considered.

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