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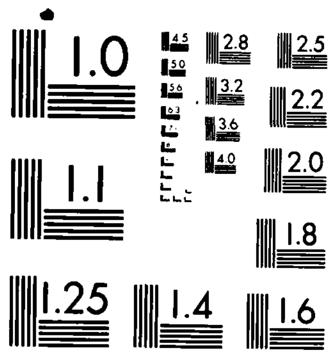

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NEARLY OPTIMAL STATE FEEDBACK CONTROLS
FOR STOCHASTIC SYSTEMS
WITH WIDEBAND NOISE DISTURBANCES

by

Harold J. Kushner and W. Runggaldier

July 1985 LCDS #85-23

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Nearly Optimal State Feedback Controls for Stochastic Systems with Wideband Noise Disturbances

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Abstract

Much of optimal stochastic control theory is concerned with diffusion models. Such models are often only idealizations (or limits in an appropriate sense) of the actual physical process, which might be driven by a wide bandwidth (not white) process or be a discrete parameter system with correlated driving noises. Optimal or nearly optimal controls, derived for the diffusion models, would not normally be useful or even of much interest, if they were not also 'nearly optimal' for the physical system which the diffusion approximates. It turns out that, under quite broad conditions, the 'nearly optimal' controls for the diffusions do have this desired robustness property and are 'nearly optimal' for the physical (say wide band noise driven) process, even when compared to controls which can depend on all the (past) driving noise. ~~We~~ The authors treat the problem over a finite time interval, as well as the average cost per unit time problem. Extensions to discrete parameter systems, and to systems stopped on first exit from a bounded domain are also discussed. Weak convergence methods provide the appropriate analytical tools.

1. Introduction

The paper is concerned with "approximately optimal" controls for a wide variety of systems driven by wide band-width noise, and their discrete parameter counterparts. Consider a system of the type

$$(1.1)\qquad \dot{x}^{\epsilon} = F_{\epsilon}(x^{\epsilon},\xi^{\epsilon},u), \quad x \in R^{r}, \text{ Euclidean r-space,}$$

where $\xi^{\epsilon}(\cdot)$ is a wide band-width noise process (the band-width $\to \infty$ as $\epsilon \to 0$), and the cost is

$$(1.2)\qquad R^{\epsilon}(u) = E\int_{0}^{T_1} k(x^{\epsilon}(s),u^{\epsilon}(s)),$$

for some $T_1 < \infty$. When we wish to emphasize the control, we write the solution to (1.1) as $x^{\epsilon}(u^{\epsilon},\cdot)$.

For the moment (and loosely speaking) suppose that (1.1) is 'close' to a controlled diffusion process, modelled by (1.3), in the sense that if $u^{\epsilon}(\cdot)$ is a sequence of 'nice' controls for (1.1), then there is a control $u(\cdot)$, and a corresponding controlled diffusion $x(u,\cdot)$ defined by (1.3), such that as $\epsilon \to 0$, $x^{\epsilon}(u^{\epsilon},\cdot) \Rightarrow x(u,\cdot)$, where $\Rightarrow$ denotes weak convergence (see the next section). Let $\bar{u}(\cdot)$ denote an optimal control for the limit diffusion (1.3), and $\bar{u}^{\delta}(\cdot)$ a 'smooth' δ-optimal control, where $\delta > 0$.

$$(1.3)\qquad dx = \bar{b}(x,u)dt + \sigma(x)dw$$

Now apply $\bar{u}^{\delta}(\cdot)$ to (1.1). Under fairly broad conditions, it is shown that

$$(1.4)\qquad \inf_{u \in RC^{\epsilon}} R^{\epsilon}(u) \geq R^{\epsilon}(\bar{u}^{\delta}) - \delta$$

for small $\epsilon > 0$, where RC^{ϵ} are the admissible (relaxed) controls for (1.1) (see Section 3). Since $\bar{u}^{\delta}(\cdot)$ is only a function of x and t, it would be considerably simpler than an optimal control for (1.1).

The methods also work well for the discrete parameter case

$$x_{n+1}^{\epsilon} = x_n^{\epsilon} + \epsilon F_{\epsilon}(x_n^{\epsilon},\xi_n^{\epsilon},u_n). \tag{1.5}$$

The $\{\xi_n^{\epsilon}\}$ and $\xi^{\epsilon}(\cdot)$ can be state dependent, and there are straightforward extensions to the discounted cost problem, to the problem where the process is stopped on first exit from a set, to the impulsive control problem, and to the average cost per unit time case.

The basic technique is that of weak convergence theory [1], [2], [3], which will be seen to provide a very natural and relatively simple basis for results of the type presented here. The relevant background results are listed in Section 2. In Section 3 the problem on a finite interval $[0,T_1]$ for a form of (1.1) is set up, and the assumptions stated. For convenience in dealing with the weak convergence, as well as to minimize detail and the number of hypotheses, we work with relaxed controls. The relevant estimates and approximations (the "chattering" lemma, etc.) are also stated in Section 3. In Section 4, the results for the finite interval are proved. Section 5 concerns the discrete parameter case. The average cost per unit time problem is in Section 6, and extensions are discussed in Section 8.

A related problem is discussed by Bensoussan and Blankenship in [4], [5]. They deal with the particular non-degenerate system

$$\begin{aligned} dx^{\epsilon} &= f(x^{\epsilon},y^{\epsilon},u)dt + \sqrt{2}\, dw \\ \epsilon dy^{\epsilon} &= g(x^{\epsilon},y^{\epsilon},u)dt + \sqrt{2\epsilon}\, dB, \end{aligned} \tag{1.6}$$

where $w(\cdot)$ and $B(\cdot)$ are mutually independent standard Wiener processes. The technique in [4,5] concerns an asymptotic expansion of the Bellman equation associated with the optimal control of (1.6). These expansions are hard to carry out, and rely heavily on various non-degeneracy properties associated with (1.6). In a "linear-quadratic" problem, they show that applying the optimal control for the limit problem to the pre-limit problem gives a cost increase of $O(\epsilon)$. There is negligible overlap in methodology with the ideas here. We can treat (1.6) if $g(\cdot)$ does not depend on $u(\cdot)$.

The results in [4,5] seem to require an analytical approach, rather than our purely probabilistic approach. The methods used here seem quite simple in comparison, and cover a broader collection of problems. Expansions of the value functions do not seem to be obtainable by ur methods. On the other hand, we can show, for many typical problem formulations, that the optimal or δ-optimal control for the limit system is a good (nearly optimal) control for the system which is driven by wide band-width noise. Such robustness is an important part of the statement of the control problem. In fact, the optimal or nearly optimal controls for diffusion models would not usually be of interest, were they also not good controls for the actual physical system which is 'idealized' by the diffusion model. The general ideas carry over to more general spaces (e.g., to measure valued processes).

2. Weak Convergence

Let $C^r[0,\infty)$ denote the space of R^r-valued continuous functions with the sup norm topology on bounded intervals, and let $D^r[0,\infty)$ denote the space of R^r-valued functions which are right continuous and have left hand limits. Endow $D^r[0,\infty)$ with the Skorohod topology [2]. Our processes (except for the discrete parameter case) have values in C^r, but it is easier to prove tightness in D^r, and then to show that all limits are continuous.

Let $\mathcal{F}_t^\epsilon$ denote the minimal σ-algebra over which $\{x^\epsilon(s),\xi^\epsilon(t),\ s \leq t\}$ is measurable, and let E_t^ϵ denote expectation conditioned on $\mathcal{F}_t^\epsilon$. Let $f(\cdot)$ be progressively measurable with respect to $\{\mathcal{F}_t^\epsilon\}$. We say that $f(\cdot)$ is in $\mathcal{D}(\hat{A}^\epsilon)$, the domain of the operator $\hat{A}^\epsilon$ and $\hat{A}^\epsilon f = g$ if for each $T < \infty$

$$\sup_{t\leq T} E|g(t)| < \infty, \quad E|g(t+\delta) - g(t)| \to 0 \text{ as } \delta \downarrow 0, \text{ each } t,$$

$$\sup_{\substack{T\geq t \\ \delta>0}} E\left|\frac{E_t^\epsilon f(t+\delta) - f(t)}{\delta} - g(t)\right| < \infty,$$

$$\lim_{\delta\downarrow 0} E\left|\frac{E_t^\epsilon f(t+\delta) - f(t)}{\delta} - g(t)\right| \to 0, \text{ each } t.$$

If $f(\cdot) \in \mathcal{D}(\hat{A}^\epsilon)$ then ([3], [6])

(2.1) $$f(t) - \int_0^t \hat{A}^\epsilon f(s)ds \quad \text{is a martingale}$$

and

(2.2) $$E_t^\epsilon f(t+s) - f(t) = \int_t^{t+s} E_t^\epsilon \hat{A}^\epsilon f(u)du.$$

The following condition for tightness in $D^r[0,\infty)$ (Theorem 3.4, [3]) is a sufficient condition for a criterion of Aldous and Kurtz [2]. Let $\hat{C}_0$ denote the continuous real values functions on R^r with compact support, and $\hat{C}_0^k$ the subset of functions all of whose mixed partial derivatives of order up to k are continuous.

Theorem 0.

Let $x^\epsilon(\cdot)$ have paths in $D^r[0,\infty)$ and let

$$\lim_{K\to\infty} \overline{\lim_{\epsilon}} \; P\left\{\sup_{t\le T}|x^\epsilon(t)| \ge K\right\} = 0, \quad \text{each} \quad T < \infty. \tag{2.3}$$

For each $f(\cdot) \in \hat{C}_0^\infty$ and $T < \infty$ let there be a sequence $f^\epsilon(\cdot) \in D(\hat{A}^\epsilon)$ such that either I or II below hold. Then $\{x^\epsilon(\cdot)\}$ is tight in $D^r[0,\infty)$.

I. For each $T < \infty$, $\{\hat{A}^\epsilon f^\epsilon(t), \epsilon > 0, t \le T\}$ is uniformly integrable and for each $\alpha > 0$

$$\lim_{\epsilon} P\left\{\sup_{t\le T}|f^\epsilon(t) - f(x^\epsilon(t))| \ge \alpha\right\} = 0. \tag{2.4}$$

II. (2.4) holds and for each $T < \infty$. There is a random variable $B_T^\epsilon(f)$ such that

$$\begin{aligned} &\sup_{t\le T} |\hat{A}^\epsilon f^\epsilon(t)| \le B_T^\epsilon(f) \\ &\lim_{K\to\infty} \overline{\lim_{\epsilon}} \; P\{B_T^\epsilon(f) \ge K\} = 0. \end{aligned} \tag{2.5}$$

Consider a discrete parameter case

$$x_{n+1}^\epsilon = x_n^\epsilon + F_\epsilon(x_n^\epsilon, \xi_n^\epsilon).$$

Let $\mathcal{F}_n^\epsilon$ denote the minimal σ-algebra over which $\{x_i^\epsilon, \xi_{i-1}^\epsilon, i \le n\}$ is measurable, with E_n^ϵ denoting the associated conditional expectation. We say that $f(\cdot) \in D(\hat{A}^\epsilon)$ if it is constant on each $[n\epsilon, n\epsilon+\epsilon)$ interval, $f(n\epsilon)$ is $\mathcal{F}_n^\epsilon$-measurable, and

$\sup_n E|f(n\epsilon)| < \infty$. Then we define

$$\hat{A}^\epsilon f(n\epsilon) = [E_n^\epsilon f(n\epsilon+\epsilon) - f(n\epsilon)]/\epsilon,$$

and the discrete parameter analogs of Theorem 0 and (2.1), (2.2) hold. In particular for $f \in \mathcal{D}(\hat{A}^\epsilon)$,

$$E_n^\epsilon f(n\epsilon + m\epsilon) - f(n\epsilon) = \epsilon \sum_{i=n}^{n+m-1} E_n^\epsilon \hat{A}^\epsilon f(i\epsilon).$$

Let $M(\infty)$ denote the collection of measures $\{m(\cdot)\}$ on the Borel subsets of $U \times [0,\infty)$, where U is compact and $m([0,t] \times U) = t$, for all $t \geq 0$. We will be working with weak convergence of a sequence of $M(\infty)$-valued random variables. Topologize $M(\infty)$ as follows. Let $\{f_{ni}(\cdot), i < \infty\}$ be a countable dense (sup norm) set of continuous functions on $[0,n] \times U$. Let $(m,f) = \int f(s,\alpha)m(ds \times d\alpha)$, and define

$$d(m',m'') = \sum_{n=1}^{\infty} 2^{-n} d_n(m',m''),$$

where

$$d_n(m',m'') = \sum_{i=1}^{\infty} \frac{2^{-i}|(m'-m'',f_{ni})|}{1+|(m'-m'',f_{ni})|} .$$

When we say that $m_n(\cdot) \Rightarrow m(\cdot)$ for a sequence of random measures, we always mean weak convergence in $M(\infty)$.

3. Assumptions and Rclaxcd Controls

We adopt a particular noise model which is a standard way of modelling wide band-width noise. The model can readily be generalized, since only a few properties of the processes are used. The model is convenient also because the relevant weak convergence results can be easily referred to. A control $u(\cdot)$ for (1.1) is said to be admissible if it takes values in U, a compact set, and it is progressively measurable with respect to the σ-algebras $\sigma\{\xi^\epsilon(s), s \leq t\}$.

A random measure $m(\cdot)$ with values in $M(\infty)$ is said to be an admissible relaxed control if $\int_0^t f(s,\alpha)m(ds \times d\alpha) \equiv (f,m)_t$ is progressively measurable with respect to $\{F_t^\epsilon\}$ for each bounded continuous $f(\cdot)$. If $m(\cdot)$ is admissible, then there is a measure valued function $m_t(\cdot)$ of (ω,t) such that for smooth $f(\cdot)$

$$\int f(s,\alpha)m(ds \times d\omega) = \int dt \int f(s,\alpha)m_s(d\alpha),$$

and $m_t(\cdot)$ is (weakly) progressively measurable in the sense that $\int_0^t ds \int f(s,\alpha)m_s(d\alpha)$ is progressively measurable. Let AC^ϵ and RC^ϵ denote the class of admissible and admissible relaxed controls, repsectively, for (1.1).

Assumptions

A1. $\xi^\epsilon(t) = \xi(t/\epsilon^2)$, where $\xi(\cdot)$ is a stationary zero mean process which is either (a) strongly mixing*, right continuous and bounded, with the mixing rate function $\phi(\cdot)$ satisfying $\int_0^\infty \phi^{1/2}(s)ds < \infty$ or (b) stationary Gauss-Markov with an integrable correlation function (which thus must go to zero exponentially).

---

* I.e., for A $\sigma(\xi(v), v \leq s)$, B $\sigma(\xi(v), v \geq s+t)$, $\sup_{A,B}|P(B|A) - P(B)| \leq \phi(s)$.

A2. $F_\epsilon(x,\xi,u) = b(x,u) + \tilde{b}(x,\xi) + g(x,\xi)/\epsilon$, where $E\tilde{b}(x,\xi) = Eg(x,\xi) = 0$ under (A1a), and $\tilde{g}(x,\xi) = g(x)\xi$, $\tilde{b}(x,\xi) = \tilde{b}(x)\xi$ under (A1b). $k(\cdot,\cdot)$ is bounded and continuous, and $b(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$, $g(\cdot,\cdot)$ are continuous. The derivative $g_x(\cdot,\xi)$ is continuous (in x,ξ). Also $b(\cdot,\alpha)$ satisfies a linear growth condition and a Lipschitz condition in x, uniformly in $\alpha \in U$. Under (A1a), $\tilde{b}(\cdot,\xi)$, $g(\cdot,\xi)$, $g_x(\cdot,\xi)$ satisfy the same uniform Lipschitz and growth condition, and under (A1b), $\tilde{b}(\cdot)$, $g(\cdot)$ and $g_x(\cdot)$ do.

Define

$$\{a_{ij}(x)\} = \int_{-\infty}^{\infty} Eg(x,\xi(t))g'(x,\xi(0))dt = a(x),$$

$$\bar{b}_i(x,u) = b_i(x,u) + \int_0^{\infty} E \sum_j g_{i,x_j}(x,\xi(t))g_j(x,\xi(0))dt, \quad i \leq r.$$

A3. Suppose that $\{a_{ij}(\cdot)\}$ has a Lipschitz continuous square root $\sigma(\cdot)$.

For the problem on $[0,T_1]$, the boundedness condition on $k(\cdot,\cdot)$ can be replaced by a polynomial growth condition. For the average cost per unit time problem, the stability methods and assumptions of Section 7 can be used for the same purpose.

The weak convergence and existence (of an optimal control) arguments are easier if one works with relaxed controls. It is convenient to work with relaxed controls on $[0,\infty)$. If the control problem is of interest on $[0,T_1]$ only, then define $u(\cdot)$ or $m(\cdot)$ in any admissible way on $[T_1,\infty)$.

Admissible controls for (1.3) or (3.1) below. An admissible control for (1.3) is any U-valued function $u(\cdot)$ which is non-anticipative with respect to $w(\cdot)$. An admissible relaxed control for (1.3) or (3.1) below is any $M(\infty)$ valued random variable $m(\cdot)$ such that for any collection $\{f_\beta(\cdot)\}$ of bounded continuous functions $f_\beta(\cdot)$, and each $t > 0$, $\{\int_0^t f_\beta(s,\alpha)m(ds \times d\alpha)\}$ is independent of $\{w(t+s) - w(t), s > 0\}$. If $m(\cdot)$ is an admissible relaxed control then there is a (ω,t-dependent) measure $m_t(\cdot)$ on the Borel sets of U such that

$$\int_0^t \int f(s,\alpha)m(ds \times d\alpha) = \int_0^t ds \int f(s,\alpha)m_s(d\alpha), \quad t < \infty,$$

for each bounded and continuous $f(\cdot)$ and almost all ω. When working with (1.3) or (3.1), we assume that $\bar{b}(\cdot)$ and $\sigma(\cdot)$ have the continuity, growth and Lipschitz conditions ascribed to $b(\cdot)$ and $\sigma(\cdot)$ in (A1)-(A3). Let AC and RC denote the class of admissible and admissible relaxed controls, respectively.

Theorem 1.

Let $m(\cdot)$ *be an admissible relaxed control* (*with respect to a Wiener proces* $w(\cdot)$). *Then there exists a non-anticipative solution to*

$$dx = dt \int \bar{b}(x,\alpha)m_t(d\alpha) + \sigma(x)dw, \quad x(0) = x, \tag{3.1}$$

and

$$E \sup_{t \le T} |x(t)|^2 \le K[1 + |x|^2], \tag{3.2}$$

where K *depends only on* T *and on the growth rates and Lipschitz constants on* $\bar{b}(\cdot)$ *and* $\sigma(\cdot)$. *The multivariate distributions of* $x(\cdot)$ *depend only on the multivariate distributions of the random variables* {m(B), Borel B}, *and on the fact that* $m(\cdot)$ *is 'non-anticipative'* (*thus if* $m(\cdot)$ *is replaced by another such process with the same multivariate distributions, then the multivariate distributions of* $x(\cdot)$ *will remain the same*).

Define $\{x_n^\Delta\}$ *by* $x_0^\Delta = x_1^\Delta = x$ *and for* $n \ge 1$,

$$x_{n+1}^\Delta = x_n^\Delta + \int_{n\Delta-\Delta}^{n\Delta} ds \int \bar{b}(x_n^\Delta,\alpha)m_s(d\alpha) + \sigma(x_n^\Delta)[w(n\Delta+\Delta) - w(n\Delta)]. \tag{3.3}$$

Define $x^{\Delta}(\cdot)$ *to be the piecewise constant interpolation* (*interval* Δ) *of* $\{x_n^{\Delta}\}$. *Then there is a* $K_{\Delta} \to 0$ *as* $\Delta \to 0$ (*and depending only on* T *and on the Lipschitz and growth constants*) *such that*

$$(3.4) \qquad E \sup_{t \leq T} |x^{\Delta}(t) - x(t)|^2 \leq K_{\Delta}(1 + |x|^2)$$

(K_{Δ} *does not depend on* $m(\cdot)$.)

Let $m^n(\cdot) \Rightarrow \bar{m}(\cdot)$, *where the* $m^n(\cdot)$ *are admissible with respect to some Wiener process, and let* $x^n(\cdot)$ *satisfy* (3.1) *with* $m(\cdot) = m^n$. *Then* $(x^n(\cdot),m^n(\cdot)) \Rightarrow (x(\cdot),\bar{m}(\cdot))$ *where* $x(\cdot),\bar{m}(\cdot)$ *satisfy* (3.1) *for some Wiener process* $w(\cdot)$ *and* $\bar{m}(\cdot)$ *is admissible with respect to* $w(\cdot)$.

Proof. The existence and uniqueness proof for the relaxed control case follows the same (standard) lines as when an admissible control $u(\omega,t)$ is used, and is discussed by Fleming [7] and Fleming and Nisio [8]. The proofs of the estimates (3.2), (3.4) also follow the classical lines. To get the weak convergence in the last paragraph, it is sufficient to work with the discrete parameter case (3.3), in view of the uniformity (in $m(\cdot)$) of K and K_{Δ}. But the result is obvious for the discrete parameter case, owing to the continuity of $\bar{b}(\cdot,\cdot)$ and the Lipschitz conditions and linear growth conditions. Q.E.D.

For (3.1), define

$$R(m) = \int_0^{T_1} \int k(x(s),\alpha)m_s(d\alpha)ds$$

where $x(\cdot)$ corresponds to $m(\cdot)$ via (3.1). We *sometimes* write the solution to (1.3) or (3.1) as $x(u,\cdot)$ or $x(m,\cdot)$.

Theorem 2. *In the class of admissible relaxed controls for* (3.1), *there is an optimal control*.

Proof. The theorem follows from Theorem 1. Simply choose a weakly convergent subsequence $m^{\delta}(\cdot)$, $\delta \to 0$, such that $R(m^{\delta}) \to \inf_{m \in RC} R(m) \equiv \bar{R}$. Denote the limit of $\{x(m^{\delta},\cdot),m^{\delta}(\cdot)\}$ by $(x(\bar{m},\cdot),\bar{m}(\cdot))$. Then by Theorem 1, $\bar{m}(\cdot)$ is admissible for some Wiener process $w(\cdot)$ and $(x(\bar{m},\cdot),\bar{m}(\cdot),w(\cdot))$ solve (3 .1). By the weak convergence,

$$E \int_0^{T_1}\int k(x^{\delta}(s),\alpha)m^{\delta}(ds\times d\alpha) \to E \int_0^{T_1}\int k(x(s),\alpha)\bar{m}(ds\times d\alpha) = \bar{R} = R(\bar{m})$$

Q.E.D.

Since we wish to show (in the following sections) that any smooth and nearly optimal feedback control for (1.3) is a nearly optimal control of (1.1) for small $\epsilon > 0$, it is important to know that there is a smooth nearly optimal control for (1.1). This is shown in the next two theorems.

The chattering lemma.

Theorem 3. For each $\delta > 0$, there is a piecewise constant admissible control $u^{\delta}(\cdot)$ for (1.3) such that

$$R(u^{\delta}) \leq \inf_{m \in RC} R(m) + \delta.$$

Remark. A proof is in [7], [8]. We only give a rough outline of the construction. Let $\bar{m}(\cdot)$ be an optimal admissible relaxed control. Let $u_1^{\rho}, \ldots, u_k^{\rho}$ be a ρ-grid in U. Define A_1^{ρ} by $A_1^{\rho} = \{\alpha \in U: |\alpha-u_1^{\rho}| \leq \rho\}$. For $k \geq n > 1$, define

$$A_n^{\rho} = \{\alpha \in U: |\alpha-u_n^{\rho}| \leq \rho\} - \bigcup_1^{n-1} A_i^{\rho}.$$

For $\Delta > 0$ and $i \geq 0$, define

$$\tau_{in}^{\Delta\rho} = \int_{i\Delta}^{i\Delta+\Delta} \bar{m}_s(A_n^\rho)ds,$$

the total integrated time that the optimal relaxed control 'takes values' in the set A_n^ρ in the time interval $[i\Delta, i\Delta+\Delta)$. Define the piecewise constant admissible control $\tilde{u}^\delta(\cdot)$ by $\tilde{u}^\delta(t) = u_0^\rho$ for $t \leq \Delta$, where u_0^ρ is any value in U; in general, set $\tilde{u}^\delta(t) = u_n^\rho$ on

$$\left[(i+1)\Delta + \sum_{\ell=1}^{n-1} \tau_{i\ell}^{\Delta}{}^{\rho}, (i+1)\Delta + \sum_{\ell=1}^{n} \tau_{i\ell}^{\Delta}{}^{\rho}\right], \quad i \geq 0, \quad n \leq k.$$

Then, for small ρ and Δ, $\tilde{u}^\delta(\cdot)$ satisfies our needs, even though the intervals of constancy are random.

We can also get a control whose intervals of constancy are non-random. Let $\Delta_1 > 0$ be such that $\Delta/\Delta_1 \equiv \bar{k}$ is a large integer, and write $k_{i\ell}^{\Delta\rho} = [\tau_{i\ell}^{\Delta\rho}/\Delta_1]$. Then define $\hat{u}^\delta(\cdot)$ as $\tilde{u}^\delta(\cdot)$ was defined but with $k_{i\ell}^{\Delta\rho}\, \Delta_1$ replacing $\tau_{i\ell}^{\Delta\rho}$, and on the non-assigned set, simply set $\hat{u}^\delta(t) = u_0^\rho$, where u_0^ρ is any value in U. For small Δ_1, Δ and ρ, and large $\bar{k}$, $\hat{u}^\delta(\cdot)$ also satisfies our needs.

Theorem 4. For each $\delta > 0$, there is a piecewise constant (in t) and locally Lipschitz continuous in x (uniformly in t) control $\bar{u}^\delta(\cdot)$ such that

$$R(\bar{u}^\delta) \leq \inf_{m \in RC} R(m) + \delta.$$

Proof. Fix $\delta > 0$. By the previous theorem, we can find a $\Delta > 0$ and an admissible control $u^\delta(\cdot)$, constant on each interval $[i\Delta, i\Delta+\Delta)$, and such that

$$R(u^\delta) \leq \inf_{m \in RC} R(m) + \delta/4.$$

By examining the imbedded Markov chain $\{x(i\Delta), i\Delta \leq T_1\}$, we see that there is an admissible control $\hat{u}^\delta(t)$ which is piecewise constant and has the form $\hat{u}^\delta(t)$

$= \hat{u}^\delta(x(i\Delta), i\Delta)$ for $t \in [i\Delta, i\Delta+\Delta)$ for some function $\hat{u}^\delta(x,t)$, and is such that

$$R(\hat{u}^\delta) \leq R(u^\delta).$$

In fact we can suppose that the $\hat{u}^\delta(t)$ take only a finite number of values $u_1, \ldots, u_k$, where k might depend on δ but not otherwise on Δ. Let $x(\cdot)$ denote the process corresponding to the control $\hat{u}^\delta(\cdot)$. Define $B^i_\ell = \{x: \hat{u}^\delta(x,i\Delta) = u_\ell\}$. There are open sets $\tilde{B}^i_\ell$ with smooth boundaries (say, unions of a finite number of spheres) and whose closures are disjoint and such that (∂B denotes the boundary of the set B)

$$\text{(3.5)} \quad \begin{aligned} &P\{x(i\Delta) \in \partial\tilde{B}^i_\ell\} = 0, \quad \text{all } i,\ell, \quad i\Delta \leq T_1, \\ &\sum_{i=0}^{T_1/\Delta-1} P\{x(i\Delta) \in \bigcup_\ell (B^i_\ell \Delta \tilde{B}^i_\ell)\} \leq \frac{\delta}{4T} /[1 + \sup_{x,\alpha}|k(x,\alpha)|] . \end{aligned}$$

For each i, define $\tilde{u}^\delta(x,i\Delta)$ to equal u_ℓ on $\tilde{B}^i_\ell$, and use any locally Lipschitz continuous interpolation for $x \notin \bigcup_\ell \tilde{B}^i_\ell$. Thus the costs with use of $\hat{u}^\delta(\cdot)$ (on one hand) and use (on the other hand) of $\tilde{u}^\delta(x(i\Delta),i\Delta)$ for $t \in [i\Delta,i\Delta+\Delta)$ and each i differ by at most $\delta/2$. In fact the latter control and $\hat{u}^\delta(\cdot)$ differ on a set whose probability is less than $\delta/2$ plus the right side of (3.5). Define $\bar{u}^\delta(x,t) = \tilde{u}^\delta(x,i\Delta)$ for t $(i\Delta,i\Delta+\Delta)$.

For small Δ, the $\bar{u}^\delta(\cdot)$ satsify our needs. Q.E.D.

4. Weak Convergence of and Approximation of the Optimal Controls for $x^\epsilon(\cdot)$

In this section we work with the control problem on $[0,T_1]$ and prove (Theorem 5) that the weak limit of any (weakly convergent) sequence of admissible relaxed controls for (4.1) is an admissible relaxed control for (3.1) and that the corresponding costs converge. Then, in Theorem 6, we show that any smooth 'nearly optimal ' feedback control for (3.1) also is 'nearly optimal' for (4.1) for small ϵ.

Let $\delta_\epsilon \to 0$, and let $\hat{m}^\epsilon(\cdot)$ be a δ_ϵ-optimal admissible relaxed control for the process defined by

$$\dot{x}^\epsilon = \int b(x^\epsilon,\alpha)m_t(d\alpha) + \tilde{b}(x^\epsilon,\xi^\epsilon) + g(x^\epsilon,\xi^\epsilon)/\epsilon, \tag{4.1}$$

with cost function (1.2). For convenience, we define all $m(\cdot)$ on $[0,\infty)$. In the analysis below it is convenient (but not necessary) to have $\int b(x^\epsilon(s),\alpha)m_t(d\alpha)$ right continuous (in order to be able to readily evaluate $\hat{A}^\epsilon$). Owing to the Lipschitz, continuity and growth conditions, for each ϵ we can suppose (w.l.o.g.) that $\hat{m}_t^\epsilon(\cdot)$, is, in fact, constant on intervals $[i\Delta_\epsilon, i\Delta_\epsilon+\Delta_\epsilon)$ for small enough Δ_ϵ.

Define L^m, the infinitesimal operator of $x(m,\cdot)$ defined by (3.1), by

$$L^m f(x) = f_x'(x) \int \bar{b}(x,\alpha)m_t(d\alpha) + \frac{1}{2}\sum_{i,j} f_{x_i x_j}(x)a_{ij}(x).$$

<u>Theorem 5</u>. <u>Assume</u> (A1)-(A3). <u>Then</u> $\{x^\epsilon(\hat{m}^\epsilon,\cdot),\hat{m}^\epsilon(\cdot)\}$ <u>is tight in</u> $D^r[0,\infty) \times M(\infty)$. <u>Let</u> $\hat{m}^\epsilon(\cdot) \Rightarrow \hat{m}(\cdot)$. <u>There is a</u> $w(\cdot)$ <u>such that</u> $\hat{m}(\cdot)$ <u>is admissible with respect to</u> $w(\cdot)$ <u>and</u> $(x^\epsilon(\hat{m}^\epsilon,\cdot), \hat{m}^\epsilon(\cdot)) \Rightarrow (x(\hat{m},\cdot),\hat{m}(\cdot))$, <u>where</u>

$$dx = dt \int \bar{b}(x,\alpha)\hat{m}_t(d\alpha) + \sigma(x)dw. \tag{4.2}$$

<u>Also</u>

$$R^{\epsilon}(\hat{m}^{\epsilon}) = E\int_0^{T_1}\int k(x^{\epsilon}(s),\alpha)\hat{m}^{\epsilon}(ds\times d\alpha)$$
$$\to E\int_0^{T_1}\int k(x(s),\alpha)\hat{m}(ds\times d\alpha) = R(\hat{m})$$

Proof. We first work with a truncated system, since tightness is easier to prove if the $x^{\epsilon}(\cdot)$ paths are all bounded (see, e.g., [3], Chapter 3.3 or 4.6.4 or [9]). Let $q_N(\cdot)$ be a twice continuously differentiable function satisfying $q_N(x) = 1$ for $|x| \leq N$, $q_N(x) = 0$ for $|x| \geq N+1$ and $q_N(x)$ [0,1] for all x. Define $b_N(x,\alpha) = b(x,\alpha)q_N(x)$, $g_N(x,\xi) = g(x,\xi)q_N(x)$, etc., and let $x^{\epsilon,N}(\cdot)$ denote the solution to (4.1) corresponding to the use of b_N, $\tilde{b}_N$, g_N, and $\hat{m}^{\epsilon}(\cdot)$.

Part I. Tightness of $\{x^{\epsilon,N}(\cdot)\}$

Since $U \times [0,t_1]$ is compact for each $t_1 < \infty$, $\{\hat{m}^{\epsilon}(\cdot)\}$ is tight in $M(\infty)$. To prove the tightness of $\{x^{\epsilon,N}(\cdot)\}$, we use the first order perturbed test function method of [3, Chapter 3] (see also [9]). Let $f(\cdot) \in \hat{C}_0^2$. Then (write x for $x^{\epsilon,N}(t)$ for convenience)

$$\hat{A}^{\epsilon}f(x) = f_x'(x)\left[\int b_N(x,\alpha)\hat{m}_t^{\epsilon}(d\alpha) + \tilde{b}_N(x,\xi^{\epsilon}(t)) + g_N(x,\xi^{\epsilon}(t))/\epsilon\right]$$

For arbitrary $T < \infty$ and for $t \leq T$, define $f_1^{\epsilon}(t) = f_1^{\epsilon}(x^{\epsilon,N}(t),t)$, where

$$f_1^{\epsilon}(x,t) = \int_t^T f_x'(x)E_t^{\epsilon}g_N(x,\xi^{\epsilon}(s))ds/\epsilon$$
$$= \epsilon\int_{t/\epsilon^2}^{T/\epsilon^2} f_x'(x)E_t^{\epsilon}g_N(x,\xi(s))ds.$$

Under (A1a), $f_1^{\epsilon}(t) = O(\epsilon)$. Under (A1b), $f_1^{\epsilon}(t) = O(\epsilon)|\xi^{\epsilon}(t)|$. In either case

$$\sup_{t\leq T}|f_1^{\epsilon}(t)| \xrightarrow{P} 0 \quad \text{as} \quad \epsilon\to 0.$$

We have

$$\hat{A}^{\epsilon} f_1^{\epsilon}(t) = -f_x'(x^{\epsilon,N}(t))\, g_N(x^{\epsilon,N}(t),\xi^{\epsilon}(t))/\epsilon$$

$$+\frac{1}{\epsilon}\int_t^T ds[f_x'(x^{\epsilon,N}(t))E_t^{\epsilon} g_N(x^{\epsilon,N}(t),\xi^{\epsilon}(s))]_x'\, \dot{x}^{\epsilon,N}(t).$$

Define $f^{\epsilon}(t) = f(x^{\epsilon,N}(t)) + f_1^{\epsilon}(t)$. Then, writing x for $x^{\epsilon,N}(t)$, using the above results and a scale change $s/\epsilon^2 \to s$,

$$\hat{A}^{\epsilon} f^{\epsilon}(t) = f_x'(x)\int b_N(x,\alpha)\hat{m}_t^{\epsilon}(d\alpha) + f_x'(x)\tilde{b}_N(x,\xi^{\epsilon}(t)) \tag{4.3}$$

$$+\int_{t/\epsilon^2}^{T/\epsilon^2} ds\; E_t^{\epsilon}[f_x'(x)g_N(x,\xi(s))]_x'\; g_N(x,\xi^{\epsilon}(t))$$

$$+\epsilon\int_{t/\epsilon^2}^{T/\epsilon^2} ds\; E_t^{\epsilon}[f_x'(x)g_N(x,\xi(s))]_x'\left[\int b(x,\alpha)\hat{m}_t^{\epsilon}(d\alpha) + \tilde{b}(x,\xi^{\epsilon}(t))\right].$$

Under (A1a), the second and third terms in (4.3) are O(1). Under (A1b), they are $O(1)[1 + |\xi^{\epsilon}(t)|^2]$. Under (A1a), the last term is $O(\epsilon)$, and under (A1b) it is $O(\epsilon)[1 + |\xi^{\epsilon}(t)|^2]$. In either case the conditions of Theorem 0 hold. Hence $\{x^{\epsilon,N}(\cdot)\}$ is tight in $D^r[0,\infty)$.

Part 2. The martingale problem satisfied by the limit

Let ϵ index a weakly convergent subsequence with limit denoted by $x^N(\cdot)$, $\hat{m}(\cdot)$, i.e., $\{x^{\epsilon,N}(\cdot), \hat{m}^{\epsilon}(\cdot)\} \Rightarrow (x^N(\cdot), \hat{m}(\cdot))$. There is an (ω,t)-measurable $\hat{m}_t(\cdot)$ such that $\hat{m}_t(U) = 1$ and

$$\int_0^t\int f(s,\alpha)\hat{m}_s(d\alpha)ds = \int_0^t\int f(s,\alpha)\hat{m}(ds\times d\alpha)$$

for each continuous $f(\cdot)$. This is a consequence of the fact that $\hat{m}\{A \times [0,t]\}$ is absolutely continuous for each Borel A, uniformly in ω,A, which implies that the (measurable) limit

$$\lim_{\Delta} [\hat{m}\{A \times [0,t]\} - \hat{m}\{A \times [0,t-\Delta]\}]/\Delta \equiv \hat{m}_t(A)$$

exists for a.a. (ω,t) for each Borel A.

Define $\mathcal{L}^m_N$ as $\mathcal{L}^m$ was defined, but with the use of $\bar{b}_N$ and g_N instead of $\bar{b}$ and $\bar{g}$. Let $f(\cdot) \in \hat{C}^2_0$ and define $M^N_f(\cdot)$ by

$$M^N_f(t) = f(x^N(t)) - f(x(0)) - \int_0^t \mathcal{L}^{\hat{m}}_N f(x^N(s))ds.$$

We next show that $M^N_f(\cdot)$ is a martingale with respect to $\mathcal{B}^N(t) \equiv \sigma\{x^N(s), \hat{m}(A \times [0,s]), \text{Borel } A, s \leq t\}$.

We know that $x^N(\cdot)$ has paths in $D^r[0,\infty)$, but we haven't yet proved that the paths are in $C^r[0,\infty)$. There are at most a countable set of t-points such that $P\{x^N(\cdot)$ is discontinuous at $t\} > 0$. Denote this set by $\mathcal{T} = \{\tau_i\}$. In what follows, until continuity is established, the t_i, t, t+s do not take values in $\mathcal{T}$. Let $h(\cdot)$ be bounded and continuous and let $t_i < t < t+s$. Let q_1 and q_2 be arbitrary integers and $k_j(\cdot)$ arbitrary bounded and continuous functions. By (2.1), (2.2), and a change of scale $(s/\epsilon^2 \to s)$ for one of the terms, we have

$$\begin{aligned}
(4.4) \qquad & Eh(x^{\epsilon,N}(t_i),(k_j,\hat{m}^\epsilon)_{t_i}, i \leq q_1, j \leq q_2)\{f(x^{\epsilon,N}(t+s)) \\
& \qquad - f(x^{\epsilon,N}(t)) + f^\epsilon_1(t+s) - f^\epsilon_1(t) \\
& - \int_t^{t+s}\int f'_x(x^{\epsilon,N}(\tau))b_N(x^{\epsilon,N}(\tau),\alpha)\hat{m}^\epsilon(d\tau \times d\alpha) \\
& - E^\epsilon_t \int_t^{t+s} f'_x(x^{\epsilon,N}(\tau))\tilde{b}_N(x^{\epsilon,N}(\tau),\xi^{\epsilon,N}(\tau))d\tau \\
& - \int_t^{t+s} d\tau\, E^\epsilon_\tau \int_{t/\epsilon^2}^{T/\epsilon^2} [f'_x(x^{\epsilon,N}(\tau))g_N(x^{\epsilon,N}(\tau), \\
& \qquad \xi(v))]'_x g_N(x^{\epsilon,N}(\tau),\xi^\epsilon(\tau))dv \\
& + \text{terms which go to 0 in mean as } \epsilon \to 0\} = 0.
\end{aligned}$$

Owing to (2.1) and (2.2), (4.4) holds with or without the E_t^ϵ term on the right hand side. Recall that $(f,m)_t \equiv \int_0^t \int f(s,\alpha)m(ds \times d\alpha)$.

Now take limits ($\epsilon \to 0$) in (4.4) and use Skorohod imbedding ([10], Theorem 3.1.1). The imbedding allows us to define the probability space so that the weak convergence becomes w.p.l. in the topology of the space $D^r[0,\infty) \times M(\infty)$. We use the imbedding without changing the notation, where convenient. The f_1^ϵ terms in (4.4) disappear as $\epsilon \to 0$. Also by the weak convergence and Skorohod imbedding,

$$\int_t^{t+s} \int b_N(x^{\epsilon,N}(\tau),\alpha)\hat{m}^\epsilon(d\tau \times d\alpha) \to \int_t^{t+s} \int b_N(x^N(\tau),\alpha)\hat{m}(d\tau \times d\alpha),$$

$$(k_j,\hat{m}^\epsilon)_t \to (k_j,\hat{m})_t,$$

w.p.l., uniformly on each finite interval. Next consider the second integral term in (4.4). We will show that

$$\lim_\epsilon E \left|\int_t^{t+s} E_t^\epsilon \tilde{b}_N(x^{\epsilon,N}(\tau), \xi^\epsilon(\tau))d\tau\right| = 0. \tag{4.5}$$

Since $\{x^{\epsilon,N}(\cdot)\}$ is tight in $D^r[0,\infty)$ it is essentially a right equicontinuous set in the following sense. Given $\rho > 0$ and $T < \infty$, there is a compact set $\Omega_\rho \subset D^r[0,T]$ such that

$$P\{x^{\epsilon,N}(\cdot) \in \Omega_\rho\} \geq 1 - \rho.$$

For $y(\cdot) \in D^r[0,T]$, define $w_y[a,b) = \sup\{|y(s) - y(t)|: s,t \quad [a,b)\}$ and define

$$w_y'(\delta) = \inf_{\substack{\{t_i\} \\ q}} \max_{i \leq q} w_y[t_i,t_{i+1}),$$

where $0 = t_0 < \dots < t_q = T$ and $t_{i+1} - t_i \geq \delta$.

Then [1, p. 116]

$$\text{(4.6)} \qquad \lim_{\delta} \sup_{y(\cdot) \in \Omega_p} w_y'(\delta) = 0$$

Because of this 'equi rightcontinuity' characterization, to get the limit (4.5) it is sufficient to evaluate

$$\lim_{\Delta \downarrow 0} \overline{\lim_{\epsilon}} \, E\left|\int_t^{t+s} E_t^\epsilon \tilde{b}_N(x^{\epsilon,N}(\tau-\Delta), \xi^\epsilon(\tau))d\tau\right|$$

$$\leq \lim_{\Delta \downarrow 0} \overline{\lim_{\epsilon}} \, E\left|\int_t^{t+s} E_{\tau-\Delta}^\epsilon \tilde{b}(x^{\epsilon,N}(\tau-\Delta),\xi^\epsilon(\tau))d\tau\right| \equiv \lim_{\Delta \downarrow 0} \overline{\lim_{\epsilon}} \, K_\Delta^\epsilon.$$

There are constants C_N and C_N' depending only on N such that, under (A1a)

$$|E_{\tau-\Delta}^\epsilon \tilde{b}(x^{\epsilon,N}(\tau-\Delta), \xi^\epsilon(\tau))| \leq C_N \phi(\Delta/\epsilon^2),$$

and under (A1b)

$$|E_{\tau-\Delta}^\epsilon \tilde{b}(x^{\epsilon,N}(\tau-\Delta),\xi^\epsilon(\tau))| \leq C_N'[\exp - \lambda\Delta/\epsilon^2]|\xi^\epsilon(\tau-\Delta)|,$$

where $\phi(\cdot)$ is the mixing rate (A1a) for $\xi(\cdot)$, and $\exp - \lambda t$ is a bound on the norm of the correlation matrix (under (A1b)). Thus, under (A1a), $\overline{\lim}_\epsilon K_\Delta^\epsilon = 0$ for each $\Delta > 0$. Under (A1b) $K_\Delta^\epsilon \leq O(\exp - \lambda\Delta/\epsilon^2) \int_t^{t+s}|\xi^\epsilon(\tau)|d\tau$. Thus (4.5) holds.

By a very similar technique we can show that, as $\epsilon \to 0$, the double integral term in the brackets in (4.4) converges (in mean) to

$$\text{(4.7)} \qquad \int_t^{t+s} d\tau \int_0^\infty E[f_x'(x^N(\tau))g_N(x^N(\tau),\xi(s))]_x' g_N(x^N(\tau),\xi(0))ds.$$

The expectation in (4.7) is over the $\xi(\cdot)$ only. The $x^N(\tau)$ is considered to be a fixed parameter when taking the expectation. This last limit result is, in fact, a special case of [3, Theorem 5.11]. Thus

$$(4.8)\qquad Eh(x^N(t_i),(k_j,\hat{m})_{t_i},\ i \le q_1,\ j \le q_2)\cdot[f(x^N(t+s)) - f(x^N(t)) - \int_t^{t+s} \mathcal{L}_N^{\hat{m}} f(x^N(\tau))d\tau] = 0$$

Since q_1, q_2, $h(\cdot)$ and the $k_j(\cdot)$, t_i, t, s are arbitrary (with t_i, t, $t+s \notin \mathcal{T} = \{\tau_i\}$), the assertion that the $M_f^N(\cdot)$ are $\{\mathcal{B}^N(t)\}$ martingales is proved.

It follows from the fact that $x^N(\cdot)$ solves the martingale problem in $D^r[0,\infty)$ associated with the <u>local operator</u> $\mathcal{L}_N^{\hat{m}}$ that $x^N(\cdot)$ has continuous paths w.p.1.

Part 3. Representation of the limit

Define $\sigma_N(x) = \sigma(x)q_N(x)$. Since the $M_f^N(\cdot)$ are martingales with respect to $\mathcal{B}^N(t)$, there is a standard Wiener process $w^N(\cdot)$ (augmenting the probability space if necessary, via the addition of an independent Wiener process if $a(\cdot)$ is degenerate) such that $w^N(t)$ is $\mathcal{B}^N(t)$ adapted, $x^N(\cdot)$ is nonanticipative with respect to $w^N(\cdot)$ and

$$(4.9)\qquad dx^N = dt \int \bar{b}_N(x^N,\alpha)\hat{m}_t(d\alpha) + \sigma_N(x^N)dw^N.$$

Also, since $w^N(\cdot)$ is $\mathcal{B}^N(t)$ adapted, the $\hat{m}\,(A \times [0,t])$ and $\hat{m}_t(A)$ are non-anticipative with respect to $w^N(\cdot)$. Hence $\hat{m}\,(\cdot)$ is an admissible relaxed control for the problem with coefficients $\bar{b}_N$, σ_N.

Define $\tau_N = \min\{t: |x^N(t)| \ge N\}$. Let $w(\cdot)$ be any Wiener process such that $\hat{m}(\cdot)$ is non-anticipative with respect to $w(\cdot)$. For this pair (4.2) has a unique solution whose distributions do not depend on the particular $w(\cdot)$ (and with no explosion w.p.1. on any bounded time interval). So does the system (4.9) with $w^N(\cdot)$ replaced by $w(\cdot)$. Replace $w^N(\cdot)$ in (4.9) by $w(\cdot)$. Then the sets $\{x^N(t \cap \tau_N), \hat{m}(A \times [0,t]),$ Borel A, $t < \infty\}$ and $\{x(t \cap \tau_N), \hat{m}(A \times [0,t]),$ Borel A, $t < \infty\}$

have the same distributions. Since $P\{\tau_N \leq T\} \to 0$ as $N \to \infty$ for each $T < \infty$, we then have that $\{x^\epsilon(.),\hat{m}^\epsilon(\cdot)\}$ is tight and converges weakly to a solution of (4.2).

The last assertion of the theorem follows from the weak convergence $(x^\epsilon(\cdot),\hat{m}^\epsilon(\cdot)) \Rightarrow (x(\cdot),\hat{m}(\cdot))$, and the continuity of the process $x(\cdot)$. Q.E.D.

Remark. With a simpler proof (not requiring working with $\{\hat{m}^\epsilon(\cdot)\}$) we have the following. Let $u(\cdot)$ be a (time-dependent) feedback control which is continuous in x, uniformly in t on each bounded (x,t) set, and for which the martingale problem associated with (1.3) has a unique solution. Then $x^\epsilon(u,\cdot) \Rightarrow x(u,\cdot)$. Also $R^\epsilon(u) \to R(u)$.

Theorem 6. *Assume* (A1)-(A3). *Then for each* $\delta > 0$, *there is a Lipschitz continuous (uniformly in* t) *control* $\bar{u}^\delta(\cdot)$ *such that*

$$\text{(4.10)} \qquad \overline{\lim_\epsilon}[R^\epsilon(\bar{u}^\delta) - \inf_{m \in RC^\epsilon} R^\epsilon(m)] \leq \delta.$$

Proof. Use the $\bar{u}^\delta(\cdot)$ of Theorem 4. By the weak convergence argument of Theorem 5, $x^\epsilon(\bar{u}^\delta,\cdot) \Rightarrow x(\bar{u}^\delta,\cdot)$ and $R^\epsilon(\bar{u}^\delta) \to R(\bar{u}^\delta)$. The theorem follows from this since

$$R^\epsilon(\hat{m}^\epsilon) \to R(\hat{m}) \geq \inf_{m \in RC} R(m) \geq R(\bar{u}^\delta) - \delta$$

Q.E.D.

5. The Discrete Parameter Case

An advantage of the weak convergence point of view is that the discrete parameter case can be treated in almost the same way as the continuous parameter case.

Let the system be given by

$$x_{n+1}^{\epsilon} = x_n^{\epsilon} + \epsilon \int b(x_n^{\epsilon},\alpha)m_n(d\alpha) + \epsilon \tilde{b}(x_n^{\epsilon},\xi_n) + \sqrt{\epsilon}\, g(x_n^{\epsilon},\xi_n), \tag{5.1}$$

where $\{\xi_n\}$ satisfies the discrete parameter form of (A1a) or (A1b) and the conditions on $g(\cdot)$, $b(\cdot)$, $\tilde{b}(\cdot)$ and $k(\cdot)$ in (A2)-(A3) hold. Also, assume that the discrete parameter relaxed control $m_n(\cdot)$ depends on $\{\xi_{j-1},x_j,\ j \le n\}$ only. For any admissible relaxed control $m(\cdot)$ for (3.1), define the infinitesimal operator L^m by (which implicitly defines $\bar{b}(\cdot)$ and $\sigma(\cdot)$)

$$\begin{aligned} L^m f(x) &= f_x'(x) \int b(x,\alpha)m_t(d\alpha) \\ &+ \frac{1}{2}\sum_{-\infty}^{\infty} E[f_x'(x)g(x,\xi_n)]_x'\, g(x,\xi_0) \\ &\equiv f_x'(x) \int \bar{b}(x,\alpha)m_t(d\alpha) + \frac{1}{2}\sum_{i,j} f_{x_i x_j}(x)a_{ij}(x). \end{aligned} \tag{5.2}$$

The discrete parameter case can easily be put into the framework of the last section. The optimal policy for the discrete parameter case would not usually be 'relaxed', but it is convenient to represent it as a relaxed control, since the limit controls might be relaxed. Define $x^{\epsilon}(\cdot)$ by $x^{\epsilon}(t) = x_n^{\epsilon}$ on $[n\epsilon,n\epsilon+\epsilon)$, and define $m(\cdot)$ by

$$m(A \times [0,t]) = \epsilon \sum_{n=0}^{[t/\epsilon]-1} m_n(A) + \epsilon(t - \epsilon[t/\epsilon])m_{[t/\epsilon]}(A). \tag{5.3}$$

Let $\delta_\epsilon \to 0$ and let $\hat{m}^\epsilon(\cdot)$ be a δ_ϵ-optimal control for (5.1).

Theorem 7. Under the conditions of this section, Theorems 5 and 6 hold for the discrete parameter case.

Remark. The proof is nearly identical to that of Theorems 5 and 6. One uses the discrete parameter versions (in [3]) of the theorems which were cited to that reference and the definition of $\hat{A}^\epsilon f(n\epsilon)$ and E_n^ϵ given in Section 2.

6. Average Cost Per Unit Time

In this section, $(x^\epsilon(\cdot),\xi^\epsilon(\cdot))$ will be a Markov-Feller process with a stationary transition function when the control is of the feedback form $u(x,\xi)$, and $\xi(\cdot)$ is a Markov-Feller process. Let PM denote the class of U-valued functions of x for which (1.3) has a unique (weak sense) solution for each initial condition, and let PM^ϵ denote the class of U-valued continuous functions of (x,ξ) for which the corresponding $(x^\epsilon(\cdot),\xi^\epsilon(\cdot))$ is a Markov-Feller process (e.g., PM^ϵ includes all U-valued locally Lipschitz continuous functions). We work with (6.1), the same system dealt with in the previous section.

$$\dot{x}^\epsilon = b(x^\epsilon,u) + \tilde{b}(x^\epsilon,\xi^\epsilon) + g(x^\epsilon,\xi^\epsilon)/\epsilon. \tag{6.1}$$

Let SR denote the class of stationary admissible relaxed controls for (3.1) such that for each $m(\cdot) \in SR$, there is a process $x(m,\cdot)$ where the pair $(x(m,\cdot),m(\cdot))$ is *stationary, and define* SR^ϵ *analogously for* (6.1). When writing $\inf_{m \in SR} F(x(\cdot))$ for some function $F(\cdot)$, we infimize the functional values over these <u>stationary pairs</u> $(x(m,\cdot),m(\cdot))$.

The cost function (for a relaxed admissible control) is

$$\overline{\lim_T}\ \frac{1}{T}\int_0^T \int Ek(x^\epsilon(t),\alpha)m_t(d\alpha)dt \equiv \gamma^\epsilon(m)$$

and, for a feedback control,

$$\overline{\lim_T}\ \frac{1}{T}\int_0^T Ek(x^\epsilon(t),u(x^\epsilon(t),\xi^\epsilon(t))dt \equiv \gamma^\epsilon(u).$$

We define the costs $\gamma(u)$ and $\gamma(m)$ for the controlled diffusion $x(\cdot)$ in the analogous way.

It is convenient to start our analysis with some additional assumptions. They will be discussed and sufficient conditions given for them in the next section.

(C1)-(C4) hold in very many cases of interest. (C1) and (C3) are basically uniform (in the control) recurrence conditions. They certainly hold if the $x^\epsilon(t)$ are confined to a compact set. But, more generally, if the system has a stability property for large $|x|$, then it can often be exploited to get (C1) and (C3). See Section 7.3. Also, a nearly optimal stabilizing control for (1.3) is often a stabilizing control for (6.1).

C1. There is $\epsilon_0 > 0$ such that for each $\delta > 0$, there are δ-optimal controls $u^{\epsilon,\delta}(\cdot,\cdot) \in PM^\epsilon$ such that $\{x^\epsilon(u^{\epsilon,\delta},t),\ t < \infty,\ \epsilon \le \epsilon_0\}$ is tight in R^r.

C2. For each $\delta > 0$, there is a continuous δ-optimal control $\bar{u}^\delta(\cdot)$ in PM for (1.3) for which (1.3) has a unique invariant measure $\mu^\delta(\cdot)$, and such that $\bar{u}^\delta(\cdot) \in PM^\epsilon$ for small ϵ.

C3. For the $\bar{u}^\delta(\cdot)$ in (C2), $\{x^\epsilon(\bar{u}^\delta,t),\ t < \infty,\ \epsilon > 0\}$ is tight in R^r.

C4.

$$\inf_{u \in PM} \gamma(u) = \inf_{m \in SR} \gamma(m).$$

Theorem 8 says that if $\bar{u}^\delta(\cdot)$ is a δ-optimal control for the diffusion, then its use with the $x^\epsilon(\cdot)$ gives a nearly (3δ-optimal) result for small ϵ.

Theorem 8. Assume (A1)-(A3) and (C1)-(C4). Then for each $\delta > 0$, and small ϵ,

$$\gamma^\epsilon(\bar{u}^\delta) \le \inf_{u \in PM^\epsilon} \gamma^\epsilon(u) + 3\delta. \tag{6.2}$$

Proof. Fix $\delta > 0$. $\bar{u}^\delta(\cdot)$ will be the function defined in (C2), and $u^{\epsilon,\delta}(\cdot)$ will be the function defined in (C1). Let $P^{\epsilon,\delta}(x,\xi,t,\cdot)$ denote the transition function for the Markov-Feller process $(x^\epsilon(\cdot), \xi^\epsilon(\cdot))$, under the control $u^{\epsilon,\delta}(\cdot)$. Define the measures

$$P_T^{\epsilon,\delta}(\cdot) = \frac{1}{T} E \int_0^T P^{\epsilon,\delta}(x^\epsilon(0),\xi^\epsilon(0),t,\cdot)dt,$$

where the average E is over the possibly random initial condition $(x^\epsilon(0), \xi^\epsilon(0))$. Then

$$\gamma^\epsilon(u^{\epsilon,\delta}) = \overline{\lim_T} \int P_T^{\epsilon,\delta}(dx \times d\xi)k(x,u^{\epsilon,\delta}(x,\xi)). \tag{6.3}$$

Let $\xi^\epsilon(t)$ take values in R^k, and let M(0) denote the set of probability measures on R^{r+k} with the weak topology. By (C1), the set of M(0)-valued measures $\{P_T^{\epsilon,\delta}(\cdot), T < \infty\}$ is in a compact set in M(0). It follows from Beneš [11] that the limit of any weakly convergent (in the topology of M(0)) subsequence is an invariant measure for $(x^\epsilon(\cdot), \xi^\epsilon(\cdot))$, with the control $u^{\epsilon,\delta}(\cdot)$ used.

Let $T_n \to \infty$ be a sequence such that it yields the $\overline{\lim}_T$ in (6.3) and also $P_{T_n}^{\epsilon,\delta}(\cdot)$ converges weakly to an invariant measure $\mu^{\epsilon,\delta}(\cdot)$ for $(x^\epsilon(\cdot), \xi^\epsilon(\cdot))$. Thus

$$\gamma^\epsilon(u^{\epsilon,\delta}) = \int k(x,u^{\epsilon,\delta}(x,\xi))\mu^{\epsilon,\delta}(dx \times d\xi).$$

Let $(\hat{x}^\epsilon(\cdot), \hat{\xi}^\epsilon(\cdot))$ denote a <u>stationary process</u> corresponding to the invariant measure $\mu^{\epsilon,\delta}(\cdot)$.

Write the control $u^{\epsilon,\delta}(\cdot)$ for $(\hat{x}^\epsilon(\cdot), \hat{\xi}^\epsilon(\cdot))$ in the form of a relaxed control, which we call $m^{\epsilon,\delta}(\cdot)$, with derivative $m_t^{\epsilon,\delta}(\cdot)$. Let $\dot{m}^{\epsilon,\delta}$ denote the measure valued process which is the time derivative of $m^{\epsilon,\delta}(\cdot\ x[0,t])$. Then the pair (state, relaxed control derivative) of processes $(\hat{x}^\epsilon(\cdot),\dot{m}^{\epsilon,\delta})$ is stationary. Alternatively, for any sequence $\{t_i\}$ and set of increasing numbers $\{s_i\}$, the distributions of $\{\hat{x}^\epsilon(t+t_i), m^{\epsilon,\delta}(\cdot x[s_j+t, s_{j+1}+t]),i,j\}$ do not depend on t. By the stationarity, we can write

$$(6.4) \qquad \gamma^{\epsilon}(u^{\epsilon,\delta}) = E\int_0^1 dt \int k(x^{\epsilon}(t),\alpha)m_t^{\epsilon,\delta}(d\alpha).$$

By (C1), the collection of invariant measures $\{\mu^{\epsilon,\delta}(\cdot), \epsilon > 0\}$ lies in a compact set in M(0). Thus, by Theorem 5, $\{\hat{x}^{\epsilon}(\cdot),m^{\epsilon,\delta}(\cdot)\}$ is tight in $D^r[0,\infty) \times M(\infty)$. Let ϵ index a weakly convergent subsequence with limit $(\hat{x}(\cdot),m^{\delta}(\cdot))$. The limit is of the form (4.2), with the admissible $m^{\delta}(\cdot)$ replacing the $\hat{m}(\cdot)$ there. Let $m_{\cdot}^{\delta}$ denote the measure-valued process which is the time derivative of $m^{\delta}(\cdot \times [0,t])$. By the stationarity of $(x^{\epsilon}(\cdot), m_{\cdot}^{\epsilon,\delta})$, the limit pair (state, relaxed control derivative), $(\hat{x}(\cdot), m_{\cdot}^{\delta})$ is also stationary, and by the weak convergence

$$(6.5) \qquad \gamma^{\epsilon}(u^{\epsilon,\delta}) \to E\int_0^1 dt \int k(\hat{x}(t),\alpha)m_t^{\delta}(d\alpha).$$

Owing to the stationarity of $(\hat{x}(\cdot),m_{\cdot}^{\delta})$, the right side of (6.5) equals

$$(6.6) \qquad \gamma(m^{\delta}) = \lim \frac{1}{T}E\int_0^T dt \int k(\hat{x}(t),\alpha)m_t^{\delta}(d\alpha).$$

We now apply $\bar{u}^{\delta}(\cdot)$ to $(x^{\epsilon}(\cdot), \xi^{\epsilon}(\cdot))$. Define $\tilde{P}_T^{\epsilon,\delta}(\cdot)$ as $P_T^{\epsilon,\delta}(\cdot)$ was defined, but with $(x^{\epsilon}(\bar{u}^{\delta},\cdot), \xi^{\epsilon}(\cdot))$ used. Choose $T_n \to \infty$ such that $\tilde{P}_{T_n}^{\epsilon,\delta}(\cdot) \Rightarrow \tilde{\mu}^{\epsilon,\delta}(\cdot)$, an invariant measure for $(x^{\epsilon}(\bar{u}^{\delta},\cdot), \xi^{\epsilon}(\cdot))$, and such that

$$\gamma^{\epsilon}(\bar{u}^{\delta}) = \lim_n \int \tilde{P}_{T_n}^{\epsilon,\delta}(dx \times d\xi)k(x,\bar{u}^{\delta}(x)).$$

Let $(\tilde{x}^{\epsilon}(\cdot),\tilde{\xi}^{\epsilon}(\cdot))$ denote the stationary process corresponding to the invariant measure $\tilde{\mu}^{\epsilon,\delta}(\cdot)$ and control $\bar{u}^{\delta}(\cdot)$.

By (C3), $\{\tilde{\mu}^{\epsilon,\delta}(\cdot), \epsilon > 0\}$ lies in a compact set in M(0). Then, by Theorem 5, $\{\tilde{x}^{\epsilon}(\cdot)\}$ is tight in $D^r[0,\infty)$. Let ϵ index a weakly convergent subsequence with limit $\tilde{x}(\cdot)$, and control $\bar{u}^{\delta}(\cdot)$. Then $\tilde{x}(\cdot)$ is stationary and is, in fact, the unique stationary process of the form (1.3) corresponding to the control $\bar{u}^{\delta}(\cdot)$. We have, by Theorem 5,

$$(6.7) \qquad \gamma^{\epsilon}(\overline{u}^{\delta}) = E\int_0^1 k(\tilde{x}^{\epsilon}(t),\overline{u}^{\delta}(\tilde{x}^{\epsilon}(t)))dt \to E\int_0^1 k(\tilde{x}(t),\overline{u}^{\delta}(\tilde{x}(t)))dt = \gamma(\overline{u}^{\delta}).$$

Also by the definition of $\overline{u}^{\delta}(\cdot)$ and (C4),

$$\gamma(\overline{u}^{\delta}) \leq \inf_{u \in PM} \gamma(u) + \delta,$$

(6.8)

$$\inf_{u \in PM} \gamma(u) = \inf_{m \in SR} \gamma(m) \leq \gamma(m^{\delta}).$$

The Theorem follows from inequalities (6.8) and the convergence in (6.5) to (6.7). Q.E.D.

7. On (C1) - (C4)

7.1. Consider (C4) first. Let there be an optimal (average cost per unit time) policy $\bar{u}(\cdot)$ in PM for (1.3) and such that the associated diffusion $\bar{x}(\cdot)$ has a unique invariant measure which we denote by $\mu^{\bar{u}}(\cdot)$. Let the potential

$$C(x) = \int_0^\infty E_x[k(\bar{x}(s),\bar{u}(\bar{x}(s))) - \bar{\gamma}]ds$$

and constant $\bar{\gamma}$ satisfy the Bellman equation

$$(7.1) \qquad \bar{\gamma} = \min_{u \in U} [\mathcal{L}^u C(x) + k(x,u)].$$

See [12] for one set of conditions guaranteeing this. Let $m(\cdot) \in SR$, with the associated stationary process $x(m,\cdot) \equiv x^m(\cdot)$ and stationary measure $\mu^m(\cdot)$, where $x^m(\cdot)$ satisfies (4.2) for $\hat{m}(\cdot) = m(\cdot)$. Suppose that for any such $m(\cdot)$ with finite $\gamma(m)$,

$$(7.2) \qquad \int |C(x)|\mu^m(dx) < \infty.$$

Then (7.1) implies that for any $T < \infty$

$$\bar{\gamma}\, T \leq EC(x^m(T)) - EC(x^m(0)) + E\int_0^T \int k(x^m(t),\alpha)m_t(d\alpha).$$

Then, by the stationarity of $x^m(\cdot)$, $\bar{\gamma} \leq \gamma(m)$, and (C4) holds. A sufficient condition for (7.2) will be given in Subsection 7.3 below.

7.2. On Condition (C2).

We use results from [13], where the system $\dot{x} = \bar{b}(x,u)$ was assumed to have a stability property, uniformly in $u(\cdot)$ PM. Write $\bar{b}(x,u) = B(x) + \hat{B}(x,u)$, where $B(\cdot)$ and $\hat{B}(\cdot)$ satisfy the conditions on $b(\cdot)$ in (A2), and $\hat{B}(\cdot)$ and $\sigma(\cdot)$ are bounded, $k(\cdot,\cdot)$ is bounded and continuous, and $\{a_{ij}(x)\}$ is uniformly positive definite and satisfies (A3). The model is such that the stabilizing effects of $B(\cdot)$ overpower the effects of $\hat{B}(x,u)$ for large $|x|$. This, together with the positive definitiveness, will esentially guarantee (C4). To quantify the stability property for large $|x|$, let there be a twice continuously differentiable function $V(\cdot)$ such that $0 \leq V(x) \to \infty$ as $|x| \to \infty$ and, for some compact set K and $\beta > 0$, $\mathcal{L}^u V(x) \leq -\beta$, for $x \notin K$ and all $u(\cdot) \in$ PM. (L^u is the differential generator of (1.3).) Let there be $c > 0$, $\alpha > 0$, $q(x) \geq 0$ such that $\mathcal{L}^u V^2(x) \leq c-q(x)$, where $\inf_x q(x)/V(x) \geq \alpha$. Typically $V(\cdot)$ would be a Liapunov function for the system $\dot{x} = B(x)$; e.g., if $B(x) = Ax$ where A is stable and for $Q > 0$, P can be defined by $A'P + PA = -Q$, and we use the Liapunov function $x'Px = V(x)$. Note that our c and $V(x)$ are called c_2 and $W_1(x)$ in [13].

Under the above conditions, Theorems 3.1, 4.2, 4.3 and the proof of Theorem 4.4 of [13] imply the following facts: To any $u(\cdot) \in$ PM, there is a unique invariant measure $\mu^u(\cdot)$ for (1.3) and $\{\mu^u(\cdot), u(\cdot) \in \text{PM}\}$ is in a compact set in M(0); let $u^\delta(\cdot)$ be a $\delta/2$-optimal control in PM, smooth or not, and let

$$(7.3) \qquad u^n(x) \to u^\delta(x) \text{ in } L_1(R^r), \quad u^n(\cdot) \quad \text{PM}.$$

Then for each Borel set A, $\mu^{u^n}(A) \to \mu^{u^\delta}(A)$ and

$$(7.4) \qquad \int k(x,u^n(x))\mu^{u^n}(dx) \to \int k(x,u^\delta(x))\mu^{u^\delta}(dx).$$

These facts imply that for any given $\delta/2$-optimal $u^\delta(\cdot)$, there is a locally Lipschitz continuous $\bar{u}^\delta(\cdot)$ such that

$$\gamma(\bar{u}^{\delta}) - \gamma(u^{\delta}) \leq \delta/2.$$

Reference [13] uses a convexity condition ((A3) there) on the set $\{\bar{b}(x,U), k(x,U)\}$ and on U. But, this convexity condition was used only to prove the existence of an optimal control. The $\delta/2$-optiml control always exists.

7.3. On the Assumption (7.2)

Again, we use results of [13]. Let $C(\cdot)$ satisfy (7.1) and assume the conditions of Subsection 7.2. Then, [13, proof of Lemma 5.1],

$$|C(x)| \leq K(1 + V(x)),$$

for some $K < \infty$ (our $C(x)$ is called $V^{\bar{u}}(x)$ in [13]). Adapting the proof of [13, Lemma 5.1] to our 'relaxed' control case and using the c and α of Subsection 7.2, we get for <u>any</u> $M < \infty$ and relaxed control $m(\cdot)$.

$$c \geq \lim_{t\to\infty} \int_0^t \alpha E \min[M, V(x^m(s))]ds$$

By the stationarity, the integral equals $\alpha E \min[M, V(x^m(0))]$. Since M is arbitrary and c does not depend on $m(\cdot)$, (7.2) holds.

On (C1), (C3)

Under a suitable stability condition on the limit system $x(\cdot)$, both (C1) and (C3) can be shown via a perturbed Liapunov function method. In particular, we use some of the results of [3, Chapter 6.6] and [14]. We use the form $b(x,u) = B(x) + \hat{B}(x,u)$ and

$$\dot{x}^{\epsilon} = B(x) + \hat{B}(x,u) + \tilde{b}(x,\xi^{\epsilon}) + g(x,\xi^{\epsilon})/\epsilon \tag{7.5}$$

and (A2), (A3), (A1a). Assume that $B(\cdot)$ and $\hat{B}(\cdot)$ satisfy the conditions on $b(\cdot)$ in (A2). Analogous results can be obtained under (A1b), via the method in [3, Chapter 6.8]. We require the existence of a Liapunov function $V(\cdot)$ satisfying certain inequalities. In applications, the assumptions are essentially equivalent to $B(\cdot)$ strongly dominating the effects of the other terms for large $|x|$.

We begin with an adaptation of a perturbed Liapunov function method of [14], but with a simpler perturbation. Let $V(\cdot)$ be a twice continuously differentiable non-negative function such that $V(x) \to \infty$ as $|x| \to \infty$ and (D1)-(D4) hold. The K below are constants.

<u>D1</u>. <u>There are</u> $\alpha > 0$, $c < \infty$, <u>such that</u>

$$V_x'(x)B(x) \leq -\alpha V(x) + c \text{ and } |V_x'(x)\hat{B}(x,u)|/V(x) \to 0 \text{ as } |x| \to \infty.$$

<u>D2.</u> $|V_x'(x)g(x,\xi)| + |V_x'(x)\tilde{b}(x,\xi)| \leq K(1 + V(x))$

<u>D3.</u> $|(V_x'(x)q(x))_x'p(x)| \leq K(1 + V(x))$, <u>for the pairs</u>

$$q(\cdot) = \tilde{b}(\cdot),\ \ p(\cdot) = B(\cdot), \hat{B}(\cdot), \tilde{b}(\cdot) \text{ and } g(\cdot), \text{ and}$$

$$q(\cdot) = g(\cdot),\ p(\cdot) = B(\cdot),\ \hat{B}(\cdot),\ \tilde{b}(\cdot).$$

<u>D4</u>. $|[V_x'(x)g(x,\xi)]_x'g(x,\xi)|/V(x) \to 0$ <u>as</u> $|x| \to \infty$.

Define $V_1^\epsilon(t) = V_1^\epsilon(x^\epsilon(t),t)$, where

$$(7.6) \qquad V_1^\epsilon(x,t) = \int_t^\infty V_x'(x)E_t^\epsilon \tilde{b}(x,\xi^\epsilon(s))ds + \frac{1}{\epsilon}\int_t^\infty V_x'(x)E_t^\epsilon g(x,\xi^\epsilon(s))ds.$$

By a change of scale $s/\epsilon^2 \to s$ and (A1a), (D2), we get that the first term is $O(\epsilon^2)[1 + V(x)]$ and the second is $O(\epsilon)[1 + V(x)]$. Define the perturbed Liapunov function $V^\epsilon(t) = V(x^\epsilon(t)) + V_1^\epsilon(t)$. Then (write x for $x^\epsilon(t)$ and

$\dot{x}^\epsilon$ for $\dot{x}^\epsilon(t)$, where convenient)

$$\hat{A}^\epsilon V(x) = V_x'(x)[B(x) + \hat{B}(x,u) + \tilde{b}(x,\xi^\epsilon(t)) + g(x,\xi^\epsilon(t))/\epsilon],$$

$$\hat{A}^\epsilon V_1^\epsilon(x,t) = -V_x'(x)\tilde{b}(x,\xi^\epsilon(t)) - \frac{1}{\epsilon}V_x'(x)\,g(x,\xi^\epsilon(t))$$
$$+ \int_t^\infty ds[V_x'(x)E_t^\epsilon\,\tilde{b}(x,\xi^\epsilon(s))]_x'\dot{x}^\epsilon$$
$$+ \frac{1}{\epsilon}\int_t^\infty ds[V_x'(x)E_t^\epsilon g(x,\xi^\epsilon(s))]_x'\,\dot{x}^\epsilon\ .$$

By using the scale change $s/\epsilon^2 \to s$, (A1a) and (D1) to (D4), we get that there is a function $h(x) \geq 0$ such that $h(x)/V(x) \to 0$ as $|x| \to \infty$ and such that

$$\text{(7.7)}\qquad \hat{A}^\epsilon V^\epsilon(t) \leq -\alpha V(x^\epsilon(t)) + h(x^\epsilon(t)).$$

By the bound on $V_1^\epsilon(x,t)$ below (7.6), we can write (for small $\epsilon > 0$)

$$\text{(7.8)}\qquad \hat{A}^\epsilon V^\epsilon(t) \leq -\frac{\alpha}{2}V^\epsilon(x^\epsilon(t)) + c_1,$$

for some $c_1 < \infty$. Inequality (7.8) yields, for some $c_2 < \infty$,

$$\text{(7.9)}\qquad EV^\epsilon(t) \leq e^{-\alpha t/2}EV^\epsilon(0) + c_2.$$

Now use the bound on $V^\epsilon(x,0)$ obtained from the estimates below (7.6) to get that (for some $\epsilon_0 > 0$)

$$\sup_{\epsilon_0 \geq \epsilon, t} EV(x^\epsilon(t)) < \infty,$$

which yields (C1) and (C3).

By using the method and conditions in [3, Chapter 6.8], the conditions

(D1)-(D4) can be weakened. In particular, $V_x'(x)B(x) \leq -\alpha V(x) + c$ can be replaced by the condition that $V_x'(x)B(x) \leq -\alpha < 0$ for large $|x|$, and some $\alpha > 0$.

8. Extensions

Extensions of the results in Sections 4 to 6 to all the standard control problem formulations are quite possible. Here, we mention only a few possibilities.

8.1. Stopping Times

Let G be a bounded open set with a piecewise differentiable boundary, and define

$$R^\epsilon(m) = E\int_0^{\tau^\epsilon(m)} ds \int k(x^\epsilon(s),\alpha)m_s(d\alpha),$$

$$\tau^\epsilon(m) = \inf\{t: x^\epsilon(t) \notin G\},$$

where $x^\epsilon(\cdot)$ is the solution to (4.1) which corresponds to m. Define R(m), the cost for (3.1), in a similar way, with $\tau(m) = \inf\{t: x(t) \notin G\}$.

In extending Theorem 5 to this case, only two problems arise. First, is $\sup_\epsilon E_x\tau^\epsilon(m^\epsilon) < \infty$ for the various sequences $\{m^\epsilon(\cdot)\}$ which are used? Second, if $(x^\epsilon(\cdot), m^\epsilon(\cdot)) \Rightarrow (x(\cdot),m(\cdot))$, do the exit times also converge? The answers are affirmative under broad conditions, certainly if $\{a_{ij}(x)\}$ is uniformly positive definite in G. We discuss the questions in the simple case where $\xi^\epsilon(\cdot)$ is Markov and bounded.

Suppose that there are $\delta > 0$ and $\rho > 0$ such that

$$\inf_{\substack{x \in G \\ m \in RC}} P_x\{x(m,t) \notin N_\delta(G), \text{ some } t \leq T\} \geq \rho, \tag{8.1}$$

where $N_\delta(G)$ is a δ-neighborhood of G and P_x denotes the probability given the initial condition x. Then it follows that there is a $\rho_1 > 0$ such that for any sequence of $m^\epsilon(\cdot) \in RC^\epsilon$,

$$\underline{\lim}_{\epsilon} \; P_{x,\xi}\{x^\epsilon(m^\epsilon,t) \notin G, \text{ some } t \leq 2T\} \geq \rho_1. \tag{8.2}$$

where $P_{x,\xi}$ denotes the probability given the initial conditions x,ξ.

Suppose that (8.2) is false. Then there are $\epsilon \to 0$, and (bounded) initial conditions $x_\epsilon \in G$ and ξ_ϵ, such that

$$\lim_{\epsilon} \; P_{x_\epsilon,\xi_\epsilon}\{x^\epsilon(m^\epsilon,t) \notin G, \text{ some } t \leq 2T\} = 0. \tag{8.3}$$

There is a subsequence (indexed by ϵ) and $m(\cdot) \in RC$ such that $\{x^\epsilon(m^\epsilon,\cdot),m^\epsilon(\cdot)\} \Rightarrow \{x(m,\cdot),m(\cdot)\}$. Then (8.3) is contradicted by (8.1). It follows from (8.2) that there is an $\epsilon_0 > 0$ such that

$$\sup_{\substack{\epsilon_0>0 \\ x\in G,\xi}} E_{x,\xi} \, \tau^\epsilon(m) < \infty.$$

In the non-degenerate case, if $\{x^\epsilon(m^\epsilon,\cdot),m^\epsilon(\cdot)\} \Rightarrow (x(m,\cdot),m(\cdot))$, then the exit times also converge. This follows from the weak convergence and the fact that $x(m,\cdot)$ crosses the boundary of G infinitely often in $[\tau(m), \tau(m)+\Delta]$, for any $\Delta > 0$.

8.2. State Dependent Noise

The results of Sections 4 to 6 can be extended to the case where the evolution of $\xi^\epsilon(\cdot)$ depends on $x^\epsilon(\cdot)$ or $\{\xi_n^\epsilon\}$ depends on $\{x_n^\epsilon\}$. The technique is a combination of the control 'representation' results of this paper, and the weak convergence methods of the (state dependent noise or singular perturbations sections of [3]). The main problems concern, as before, tightness and the representation of the limit as a particular control problem.

One particular case in [3] concerns Markov $(x_n^\epsilon, \xi_{n-1}^\epsilon)$, where if x_n^ϵ is fixed at x, the $\{\xi_n^\epsilon\}$ is a Markov process with a unique invariant measure (see, e.g., Chapter 5.8.3 of [3]). Systems such as (1.6), or the wide band-noise driven forms can also be treated if the $g(\cdot)$ there does not depend on u.

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