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STEADY STATE DISTRIBUTIONS FOR
MANPOWER MODELS UNDER
CONDITIONS OF GROWTH

by

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September 1985

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20. state distribution of personnel. The conditions of change include systems where recruitment is constant, increasing (decreasing) additively, or increasing (decreasing) multiplicatively and systems where the changes in total system size are additively or multiplicatively increasing (decreasing). Numerical examples utilizing these models are provided, along with a computer program of the formulas written in the language APL.

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Steady State Distributions for Manpower Models Under Conditions of Growth

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ABSTRACT

Markov Chain models have been used to forecast stocks in a wide range of manpower systems. Studies have been done in many areas such as education planning, hospital planning, manufacturing, private research and development, a women's military unit, the civilian work force supporting the U.S. Navy and a state police organization. This study looks at such systems under conditions of change and develops the equations that describe the steady state distribution of personnel. The conditions of change include systems where recruitment is constant, increasing (decreasing) additively, or increasing (decreasing) multiplicatively and systems where the changes in total system size are additively or multiplicatively increasing (decreasing). Numerical examples utilizing these models are provided, along with a computer program of the formulas written in the language APL.

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I. BACKGROUND

A. INTRODUCTION

According to Bartholomew and Forbes [Ref. 1: p. 1],

> Manpower planning is often defined as the attempt to match the supply of people with the jobs available for them.

This is a concept that is an integral part of military force planning, today. All branches of the Department of Defense are involved with studies concerning manpower planning. Since large numbers of personnel are involved, the use of manpower models to determine the results of changes to the existing system before they are actually implemented can prevent costly future problems. This is especially true in terms of having the right number of personnel in the right places with the right grade levels and skills, at the right time. To quote again from Bartholomew and Forbes [Ref. 1: p. 6],

> A manpower model is a mathematical description of how change takes place in the system.

Much work has been done in the area of manpower/ organizational modeling. The text by Bartholomew and Forbes (referenced above and hereafter referred to as B&F) compiles information on the subject from numerous disciplines and standardizes notation to describe some of the basic models and statistical techniques used in manpower planning. A central theme of this work is the idea of depicting an organization as a system of <u>stocks</u> and <u>flows</u> [Ref. 1: pp. 3,4]. The personnel within the organization under study are divided into classes, or categories, by specific characteristics (for example: age, time in service, grade, etc). The numbers of personnel within categories at a specific time are considered stocks. Flows are the movements of personnel from one category to another during a unit interval of time, such as a year.

Various mathematical models that predict future stocks from one year (or other time period) to the next are described in BaF. Of particular interest here is the transition model that is based on the theory of Markov Chains. In addition to describing the model [Ref. 1: pp. 85-98], BaF provides a BASIC computer program [Ref. 1: pp. 248-260], to calculate future stock vectors. That program was converted into an APL program at the Naval Postgraduate School (NPS) and is available on the NPS mainframe computer, an IBM system 370. The APL program, like its predecessor, computes stocks from one year to the next based on input such as, initial stocks, recruitment and wastage (or attrition) rates and the transition probabilities for the stocks. This can be done repetitively and theoretically could compute future stocks forever.

There are two versions of the basic Markov Chain model that are well developed in BaF. One is based on a fixed flow of recruits into the system and the other on a fixed total system size. Also of interest are other versions (or options) that deal with a system undergoing growth where either recruitment or total system size grows in an additive or multiplicative fashion. The APL program allows the computation of future stocks for the cases where:

(1) recruitment is fixed, grows additively or multiplicatively;

(2) the total system size grows additively or multiplicatively.

Steady state is the condition which a system reaches, after sufficient time has passed, so that the stocks no longer change. There have been steady state models developed for the cases where there is a fixed flow of recruits into the system and where the total system size is fixed. In both these cases the number of personnel in each category remains fixed after steady state is reached by the system. The vector whose components are these numbers is called the steady state stock vector (S.S.S.V.).

Under conditions of growth, the steady state of a system must be redefined. In those cases, steady state does not exist in the above sense, because the system is undergoing constant expansion and so stocks will never reach a fixed size. However, it turns out that the stock sizes do reach constant proportions and so it is possible to talk about a steady state distribution vector (S.S.D.V.). Components of this vector represent the <u>proportions</u> of personnel that remain in each category forever once steady state has been reached. In effect, in those cases where the distribution of personnel among the categories no longer changes with time the system has reached a steady state "in distribution."

Although, at the present time, the APL program at the Naval Postgraduate School predicts future stocks for a system that grows under conditions mentioned above, it is able to supply a steady state stock vector (S.S.S.V.) only for the case where there is a <u>fixed flow of recruits</u> into the system. The only way to get the steady state distribution vector under the various conditions of growth is to continue to run the program, i.e., to forecast stocks for as many years into the future as necessary until steady state is reached. This is not a satisfactory method for several reasons:

(1) it takes too long to run the model when the system reaches steady state only after many years have passed,

(2) it is a trial and error process of going back and forth, in time, to try and find out at what point in time steady state has been reached,

(3) occasionally, even if the program is run out as far as the present formatting of the program allows, there can still be instances where steady state has not yet been reached.

This thesis uses the 'basic prediction equation', as described in BaF [Ref. 1: pp. 86-88], to correct the aforementioned deficiencies by developing models to compute analytically the steady state distributions for systems under varying conditions of growth. Also developed was an APL program, to be added as a subroutine to the main program, that computes the S.S.D.V. for these options.

The next section in this chapter sets out the necessary notation and equations to enable the reader to follow the development of these steady state models. For more amplification on the basic Markov model see BaF [Ref. 1: pp 3-8, 86-132].

B. NOTATION AND BASIC EQUATIONS

Stocks, representing personnel at an organization, are divided into k categories. The stock vector

$$\underline{n}(t) = (n_1(t), n_2(t), \ldots, n_k(t))$$

is a row vector where each element $n_i(t)$, represents the number of personnel in category i at time t. Time is measured discretely with a time interval of unit length denoted (t-1,t). The most frequently used period of time is the year although it is sometimes convenient, or necessary, to use quarters, months, etc.

Flows describe the movements of personnel. There are two basic types of flows in a system. The first type of flow is internal to the system and can be thought of as <u>transfers</u> from one category to another within the system. Demotions and promotions would be examples of this type of flow. These are denoted by the k by k matrix, $N(t-1)$, where each element, $n_{ij}(t-1)$, represents the number of personnel moving from category i to category j during the time interval $(t-1,t)$.

The other type of flow involves the two-way transfer between the system (organization) and the outside world. First, there are <u>wastage</u> (or <u>attrition</u>) flows of the number of personnel leaving the system from category i, during the time interval $(t-1,t)$, denoted by $n_{i,k+1}(t-1)$, $(i = 1,2,\ldots,k$ and $t = 1,2,\ldots)$. Then there are <u>recruitment</u> (or <u>accession</u>) flows of personnel into category j from the outside world, during the period $(t-1,t)$, denoted by $n_{oj}(t)$, $(j = 1,2,\ldots,k$ and $t = 1,2,\ldots)$. Following BaF [Ref. 1: pp. 3,4], the notation for recruitment flows is slightly different from that for internal flows and wastage flows. This is to remind the reader that in most organizations attrition and internal flows (transfers) are accounted for first and then recruitment is determined at the end of the time period.

The Markov Chain model uses flow rates (or probabilities) rather than actual numbers. P, a k by k matrix of probabilities has elements

$$p_{ij} = \frac{n_{ij}(t-1)}{n_i(t-1)}, \qquad i,j = 1,2,\ldots,k .$$

There are certain basic assumptions that a Markov Chain model needs to meet to be valid:

(1) the transition probabilities, p_{ij}, do not change over time;

(2) the population in each category is homogeneous so that p_{ij} represents the probability of each individual in category i moving, independently of any other individual, to category j.

While these assumptions may not always be met, there have been enough studies done using the model to show they are sufficiently realistic to justify the use of the model. Even in cases where these assumptions don't hold very well, the model was fcund to be quite useful for predicting future stocks [Refs. 2,3].

Next, $\underline{w} = (w_1, w_2, \ldots, w_k)$ is defined as the wastage (attrition) rate vector where each element, w_i, is defined as

$$w_i = \frac{n_{i,k+1}(t-1)}{n_i(t-1)} .$$

Ncte that,

$$\sum_{j=1}^{k} p_{ij} + w_i = 1 , \quad i = 1,2,\ldots,k .$$

Finally, $\underline{r} = (r_1, r_2, \ldots, r_k)$ is defined as the recruitment proporticn vector where each element

$$r_j = \frac{n_{oj}(t)}{R(t)}$$

is the proportion of total recruitment that will be allocated to category j, with

$$R(t) = \sum_{j=1}^{k} n_{oj}(t)$$

being the total recruitment during the time interval (t-1,t). This implies $\sum_{j=1}^{k} r_j = 1$.

BaF [Ref. 1: pp. 87,88], show that future stocks may be calculated using the equation

$$n_j(t) = \sum_{i=1}^{K} n_i(t-1)\,p_{ij} + R(t)\,r_j \;, \qquad \text{(eqn 1.1)}$$

for $j = 1,2,\ldots,k$ and $t = 1,2,\ldots$. This equation states, that the stocks in category j at time t are equal to the number of personnel that moved into category j from anywhere within the system during time period $(t-1,t)$, plus the number of personnel recruited into category j during the same period. In matrix form the equation is

$$\underline{n}(t) = \underline{n}(t-1)P + R(t)\underline{r} \qquad t = 1,2,\ldots \;. \qquad \text{(eqn 1.2)}$$

BaF [Ref. 1,: p. 88] refers to this as "the basic prediction equation."

Another equation dealing with changes in total system size is also developed in BaF [Ref. 1: pp. 94-96]. To see this, note that

$$N(t) = \sum_{i=1}^{K} n_i(t)$$

is the total system size at time t and

$$M(t) = N(t) - N(t-1) \qquad \text{(eqn 1.3)}$$

represents the change in the total system size during the time interval $(t-1,t)$. This implies that total recruitment during $(t-1,t)$ can be expressed as:

$$R(t) = \sum_{i=1}^{K} n_i(t-1)\,w_i + M(t) \geq 0 \;. \qquad \text{(eqn 1.4)}$$

The first term in this expression is recruitment that is done to replace those who left (attrition from the various categories) while the second term consists of recruitment that is due to the change in total system size.

If $M(t) = 0$, then only those personnel who leave the system are replaced, which implies that the total system size remains fixed. If $M(t) > 0$, new personnel are supplied to cover an increase in total system size, i.e., to fill new jobs. If $M(t) < 0$, then some jobs must be eliminated as soon as they are vacated. Of course, $R(t) \geq 0$ always.

Referring back to Equation 1.1, replace $R(t)$ using Equation 1.4 and let

$$q_{ij} = p_{ij} + w_i r_j \ .$$

Then, as BaF [Ref. 1: pp. 94,95], shows

$$n_j(t) = \sum_{i=1}^{k} n_i(t-1) q_{ij} + M(t) r_j \ , \qquad \text{(eqn 1.5)}$$

for $j = 1,2,\ldots,k$ and $t = 1,2,\ldots$. To put this into matrix notation let

$$Q = \{q_{ij}\} \ , \qquad i,j = 1,2,\ldots,k \ ,$$

or

$$Q = P + \underline{w}'\underline{r} \ .$$

Then Equation 1.5 may be expressed as

$$\underline{n}(t) = \underline{n}(t-1)Q + M(t)\underline{r} \ , \qquad t = 1,2,\ldots \qquad \text{(eqn 1.6)}$$

Equations 1.2 and 1.6 form the basis from which five steady state submodels (or options) will be derived. The next section introduces these options and the direction of development.

C. OBJECTIVES

The objective of this thesis is to develop steady state equations modeling five different conditions of growth. Each of these conditions will be addressed separately as a "growth option." The first three options deal with different types of recruitment. They use Equation 1.2 where $R(t)$ is the total recruitment during the time period $(t-1,t)$ and in these three options, $R(t)$ is the only term changing. The last two cptions deal with changes in total system size. They utilize Equation 1.6 where $M(t)$ represents the change in total system size during the period $(t-1,t)$ and in these two options, $M(t)$ is the only term that changes. A complete list of these five options is:

(1) Fixed recruitment: $R(t) = R$, $t \geq 1$, $R>0$ fixed, i.e., recruitment is always constant;

(2) Additive increase (decrease) of recruitment:
$R(t) = R + (t-1)M$, $t \geq 1$
where M is the constant amount of yearly increase (decrease);

(3) Multiplicative increase (decrease) of recruitment:
$R(t) = R\theta^{t-1}$, $t \geq 1$,
where $100(\theta-1)$ is the percent of yearly increase (decrease);

(4) Additive increase (decrease) in system size:
$M(t) = M$, $t \geq 1$
where M is the constant amount of yearly increase (decrease) in total system size;

(5) Multiplicative increase (decrease) in system size:
$M(t) = (\theta-1)\theta^{t-1}N(0)$, $t \geq 1$
where $100(\theta-1)$ is the percent of yearly increase (decrease) in total system size.

If $M = 0$ in option 2 or $\theta = 1$ in option 3, then those options revert to option 1, of fixed recruitment. If $M = 0$ in option 4 or $\theta = 1$ in option 5, then $M(t) = 0$ and the result is a system with a fixed total system size. That case will be dealt with as subcases of options 4 and 5. The fixed total system size case and option one, where R is fixed, are the only cases where it is possible to achieve a steady state stock vector, i.e., where the stocks don't change with time. In all other cases steady state will be considered to have been reached when a steady state distribution of personnel is achieved.

Chapter II addresses, in depth, the analytical development of the steady state equation for each of the options. It starts with an account of the work done on option 1 in BaF. Then it proceeds with the derivation of the steady state equations for options 2 through 5 from either Equation 1.2 or Equation 1.6 . It also presents examples, one a four-grade hierarchical system, the other a three grade non-hierarchical system. In addition to formulating analytical models for options 2 through 5, an APL program has also been developed that will compute the steady state distribution for those options. This program is contained in Appendix F. Appendices A through E are computer printouts of terminal sessions of the two examples for each of options 1 through 5.

II. OPTION DEVELOPMENT

The options mentioned in Chapter I will be discussed here under the usual restrictions of manpower systems. These restrictions imply that $w_i > 0$ for all $i=1,2,...,k$, since in manpower systems attrition must be allowed from any category. This means that the P matrix, as defined in Chapter I, is composed of non-negative elements and has row sums all strictly less than one. The Q matrix, also defined in Chapter I, is a stochastic matrix (i.e., it is composed of non-negative elements and has row sums all equal to one). These facts along with the following two mathematical theorems are needed in the derivation of the steady state results for the five options.

Theorem 1

> If P is a matrix composed of non-negative elements and row sums strictly less than one and θ is a scalar then the matrix $\theta I-P$ has a unique inverse for all values of $\theta \geq 1$ [Ref. 4: p. 42].

Theorem 2

> If Q is a stochastic matrix and θ is a scalar then the matrix $\theta I-Q$ has a unique inverse for all values of $\theta > 1$ [Ref. 4: pp. 60,61].

A. RECRUITMENT OPTIONS

The following three options are all variations of Equation 1.2, where R(t), representing total recruitment, is the term that takes on different forms. Consequently, the steady state equations for these options are also obtained in somewhat different forms. These steady state equations are derived in the next three subsections.

1. Option 1: Fixed Recruitment

The steady state equation for this first option is developed in BaF [Ref. 1: p.90]. The analysis presented here is a brief summary of that in BaF, using the same assumptions. Recalling the basic prediction equation, (Equation 1.2), and assuming that

$$\lim_{t \to \infty} R(t) = R \quad \text{is fixed}$$

it follows that the steady state stock vector,

$$\underline{n} = \lim_{t \to \infty} \underline{n}(t)$$

also exists [Ref. 4: pp.40-43,48-50]. Therefore, letting $t \to \infty$ in Equation 1.2 results in

$$\underline{n} = \underline{n}P + R\underline{r} \quad ,$$

or

$$\underline{n}(I - P) = R\underline{r} \quad .$$

Finally, knowing the matrix I-P has a unique inverse (see Theorem 1), the result

$$\underline{n} = R\underline{r}(I - P)^{-1} \; , \qquad \text{(eqn 2.1)}$$

is obtained for the steady state stock vector (S.S.S.V.). After steady state has been reached, it follows that

$$\underline{n}(t) = \underline{n} \quad ,$$

for "large" t. The following two examples will be used to illustrate the use of this steady state equation.

Example (a)

This first example uses data from Example 4.6 in BaF [Ref. 1: p. 97], which is taken from the women officer's system of one of the British services. It is a hierarchical, four category system.

The inital stock vector is:

$$\underline{n}(0) = (129,\ 74,\ 28,\ 11) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .728 & .102 & 0 & 0 \\ 0 & .83 & .046 & 0 \\ 0 & 0 & .867 & .033 \\ 0 & 0 & 0 & .902 \end{pmatrix} .$$

The total recruitment is:

$$R(t) = 35 \ , \quad \text{fixed for all } t \geq 1 \ .$$

The recruitment proportion vector is:

$$\underline{r} = (1,\ 0,\ 0,\ 0) \quad .$$

Then

$$I - P = \begin{pmatrix} .272 & -.102 & 0 & 0 \\ 0 & .17 & -.046 & 0 \\ 0 & 0 & .133 & -.033 \\ 0 & 0 & 0 & .098 \end{pmatrix}$$

and the inverse is

$$(I - P)^{-1} = \begin{pmatrix} 3.676 & 2.206 & .763 & .257 \\ 0 & 5.882 & 2.034 & .685 \\ 0 & 0 & 7.519 & 2.532 \\ 0 & 0 & 0 & 10.204 \end{pmatrix} .$$

Using Equation 2.1

$$\underline{n} = (35)(1, 0, 0, 0)\begin{pmatrix} 3.676 & 2.206 & .763 & .257 \\ 0 & 5.882 & 2.034 & .685 \\ 0 & 0 & 7.519 & 2.532 \\ 0 & 0 & 0 & 10.204 \end{pmatrix}$$

$$= (128.66, 77.21, 26.705, 8.995) .$$

After rounding to the nearest integer the S.S.S.V. is:

$\underline{n} = (129, 77, 27, 9)$.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50 and 100, can be found in Appendix A, confirming the steady state stock vector above. As the printout shows, steady state is achieved by t = 10.

Example (b)

This is a non-hierarchical, three-grade system.

The inital stock vector is:

$$\underline{n}(0) = (300, 200, 100) .$$

The transition matrix is:

$$P = \begin{pmatrix} .60 & .15 & .05 \\ .75 & .20 & 0 \\ .05 & 0 & .90 \end{pmatrix} .$$

The total recruitment is:

$$R(t) = 100 \text{, fixed for all } t \geq 1 .$$

The recruitment proportion vector is:

$$\underline{r} = (.70, .07, .23) .$$

Then

$$I - P = \begin{pmatrix} .4 & -.15 & -.05 \\ -.75 & .8 & 0 \\ -.05 & 0 & .10 \end{pmatrix}$$

and the inverse is

$$(I - P)^{-1} = \begin{pmatrix} 4.267 & .8 & 2.133 \\ 4 & 2 & 2 \\ 2.133 & .4 & 11.066 \end{pmatrix} .$$

Using Equation 2.1

$$\underline{n} = (100)(.70, .07, .23)\begin{pmatrix} 4.267 & .8 & 2.133 \\ 4 & 2 & 2 \\ 2.133 & .4 & 11.066 \end{pmatrix}$$

$$= (375.7, 79.2, 417.8) .$$

After rounding to the nearest integer the S.S.S.V. is:

$\underline{n} = (376, 79, 418)$.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50 and 100, can be found in Appendix A, confirming the steady state stock vector above. As the printout shows, steady state is achieved by t = 50.

2. Option 2: Additive Recruitment

This option addresses the fixed additive increase (decrease) in recruitment for each time period. Here, $R + (t-1)M$, replaces $R(t)$ in Equation 1.2 resulting in

$$\underline{n}(t) = \underline{n}(t-1)P + R\underline{r} + (t-1)M\underline{r} . \qquad \text{(eqn 2.2)}$$

In this option, $M \geq 0$ must be true, otherwise there would be negative recruitment $(R(t) < 0)$ which does not make sense in manpower models. This means there are two possible cases.

Case 1: $M = 0$

Then

$$\lim_{t \to \infty} R(t) = R \quad ,$$

which is identical to option 1.

Case 2: $M > 0$

Assume that,

$$\lim_{t \to \infty} \frac{\underline{n}(t)}{tM} = \hat{\underline{n}}$$

exists. This is the same as assuming that for "large" t, $\underline{n}(t)$ behaves like $tM\hat{\underline{n}}$. Using the above assumption, Equation 2.2 can be rewritten as

$$\frac{\underline{n}(t)}{tM} = \frac{\underline{n}(t-1)}{tM} P + \frac{R}{tM}\underline{r} + \frac{(t-1)M}{tM}\underline{r}$$

or

$$\frac{\underline{n}(t)}{tM} = \frac{\underline{n}(t-1)}{(t-1)M} \cdot \frac{(t-1)}{t} \cdot P + \frac{R}{tM}\underline{r} + \frac{(t-1)M}{tM}\underline{r} \quad .$$

Taking the limit as $t \to \infty$ results in

$$\hat{\underline{n}} = \hat{\underline{n}} P + \underline{r}$$

or

$$\hat{\underline{n}} (I-P) = \underline{r} \quad .$$

Since the I-P matrix has a unique inverse (see Theorem 1), the result

$$\hat{\underline{n}} = \underline{r}(I-P)^{-1}$$

is obtained. The steady state distribution vector is achieved by normalizing $\hat{\underline{n}}$, which results in

$$\tilde{\underline{n}} = \frac{\hat{\underline{n}}}{\hat{\underline{n}}\underline{1}'} = \frac{\underline{r}\,[I-P]^{-1}}{\hat{\underline{n}}\underline{1}'} \quad . \qquad \text{(eqn 2.3)}$$

After steady state has been reached it follows that the formula

$$\underline{n}(t) = tM\hat{\underline{n}} \quad \text{for "large" } t,$$

can be used to approximate the stocks at time t. The two examples from section A1 are worked out below to illustrate Equation 2.3 .

<u>Example</u> (<u>a</u>)
This is a hierarchical, four category system.

The inital stock vector is:

$$\underline{n}(0) = (129,\ 74,\ 28,\ 11) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .728 & .102 & 0 & 0 \\ 0 & .83 & .046 & 0 \\ 0 & 0 & .867 & .033 \\ 0 & 0 & 0 & .902 \end{pmatrix} .$$

The total recruitment is:

$$R(t) = R + (t-1)M \quad \text{where } R = 35 \text{ and } M = 5 .$$

The recruitment proportion vector is:

$$\underline{r} = (1,\ 0,\ 0,\ 0) \quad .$$

Then

$$I - P = \begin{pmatrix} .272 & -.102 & 0 & 0 \\ 0 & .17 & -.046 & 0 \\ 0 & 0 & .133 & -.033 \\ 0 & 0 & 0 & .098 \end{pmatrix}$$

and the inverse is

$$(I - P)^{-1} = \begin{pmatrix} 3.676 & 2.206 & .763 & .257 \\ 0 & 5.882 & 2.034 & .685 \\ 0 & 0 & 7.519 & 2.532 \\ 0 & 0 & 0 & 10.204 \end{pmatrix} .$$

Using Equation 2.3

$$\underset{\sim}{\tilde{n}} = \frac{\hat{\underline{n}}}{\hat{\underline{n}}\underline{1}'} = \frac{(1,\ 0,\ 0,\ 0)}{6.902}\begin{pmatrix} 3.676 & 2.206 & .763 & .257 \\ 0 & 5.882 & 2.034 & .685 \\ 0 & 0 & 7.519 & 2.532 \\ 0 & 0 & 0 & 10.204 \end{pmatrix}$$

$$= (.53,\ .32,\ .11,\ .04)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50, 100, 900, and 975 can be found in Appendix B, confirming that in steady state there will be 53%, 32%, 11%, and 4% respectively, in categories 1,2,3, and 4. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by t = 975 .

<u>Example</u> (<u>b</u>)
This is a non-hierarchical, three-grade system.

The inital stock vector is:

$$\underline{n}(0) = (300,\ 200,\ 100) \ .$$

The transition matrix is:

$$P = \begin{pmatrix} .60 & .15 & .05 \\ .75 & .20 & 0 \\ .05 & 0 & .90 \end{pmatrix} .$$

The total recruitment is:

$$R(t) = R + (t-1)M \quad \text{where } R = 100 \text{ and } M = 25 .$$

The recruitment proportion vector is:

$$\underline{r} = (.70, .07, .23) .$$

Then

$$I - P = \begin{pmatrix} .4 & -.15 & -.05 \\ -.75 & .8 & 0 \\ -.05 & 0 & .10 \end{pmatrix}$$

and the inverse is

$$(I - P)^{-1} = \begin{pmatrix} 4.267 & .8 & 2.133 \\ 4 & 2 & 2 \\ 2.133 & .4 & 11.066 \end{pmatrix} .$$

Using Equation 2.3

$$\underline{\tilde{n}} = \frac{\underline{\hat{n}}}{\underline{\hat{n}}\underline{1}'} = \frac{(.7, .07, .23)}{8.728} \begin{pmatrix} 4.267 & .8 & 2.133 \\ 4.0 & 2.0 & 2.0 \\ 2.133 & .4 & 11.066 \end{pmatrix}$$

$$= (.43, .09, .48)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50, 100, 350, and 400 can be found in Appendix B, confirming that in steady state there will be 43%, 9%, and 48% respectively, in categories 1, 2, and 3. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by $t = 400$.

3. Option 3: Multiplicative Recruitment

This cption addresses the multiplicative increase (decrease) in recruitment for each time period. Here, $R\theta^{t-1}$, replaces $R(t)$ in Equation 1.2 resulting in

$$\underline{n}(t) = \underline{n}(t-1)P + R\theta^{t-1}\underline{r} \quad . \qquad \text{(eqn 2.4)}$$

The values that θ can assume play an important part in the derivation of the steady state equation. First, $\theta \geq 0$ must be true, otherwise in even numbered years there would be negative recruitment ($R(t)<0$) which does not make sense in manpower models. There are three cases to be examined.

Case 1 : $0 \leq \theta < 1$

Then

$$\lim_{t \to \infty} R(t) = 0 \quad ,$$

because $\theta^{t-1} \longrightarrow 0$ when $0 \leq \theta < 1$. Therefore, letting $t \to \infty$ in Equation 1.2 results in

$$\underline{n} = \underline{n}P \ ,$$

or

$$\underline{n}(I - P) = \underline{0} \ .$$

Since I-P has a unique inverse matrix (see Theorem 1), the only solution to the above equation is $\underline{n} = \underline{0}$. This means that such a system becomes empty in steady state.

Case 2: $\theta = 1$

Then

$$\lim_{t \to \infty} R(t) = R \quad ,$$

which is identical to option 1.

Case 3: $\theta > 1$

Assume that,

$$\lim_{t \to \infty} \frac{\underline{n}(t)}{\theta^t} = \hat{\underline{n}}$$

exists. This amounts to assuming that for "large" t, $\underline{n}(t)$ behaves like $\theta^t \hat{\underline{r}}$. Using the above assumption, Equation 2.4 can be rewritten as

$$\frac{\underline{n}(t)}{\theta^t} = \frac{\underline{n}(t-1)}{\theta^{t-1}\,\theta} \bullet P + \frac{R\theta^{t-1}}{\theta^t}\,\underline{r} \quad .$$

Taking the limit as $t \to \infty$ results in

$$\hat{\underline{n}} = \hat{\underline{n}} \bullet \frac{1}{\theta} \bullet P + \frac{R}{\theta}\,\underline{r} ,$$

or

$$\hat{\underline{n}}\,[\theta I - P] = R\underline{r} \quad .$$

Finally, knowing that the matrix $\theta I-P$ has a unique inverse, since $\theta > 1$ (see Theorem 1), the result

$$\hat{\underline{n}} = R\underline{r}\,[\theta I - P]^{-1}$$

is obtained. When $\hat{\underline{n}}$ is normalized, the resulting vector is $\tilde{\underline{n}}$, the steady state distribution vector where,

$$\tilde{\underline{n}} = \frac{\hat{\underline{n}}}{\hat{\underline{n}}\underline{1}'} = \frac{R\underline{r}[\theta I - P]^{-1}}{\hat{\underline{n}}\underline{1}'} \quad . \qquad \text{(eqn 2.5)}$$

After steady state has been reached it follows that the formula

$$\underline{n}(t) = \theta^{t}\hat{\underline{n}} \quad \text{for "large" } t,$$

can be used to approximate the stocks at time t. Two examples are worked out below to illustrate Equation 2.5

Example (a)
This is a hierarchical, four category system.

The inital stock vector is:

$$\underline{n}(0) = (129,\ 74,\ 28,\ 11) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .728 & .102 & 0 & 0 \\ 0 & .83 & .046 & 0 \\ 0 & 0 & .867 & .033 \\ 0 & 0 & 0 & .902 \end{pmatrix} .$$

The total recruitment is:

$$R(t) = R\theta^{t-1} \quad \text{where } R = 35\ , \quad \theta = 1.01 \ .$$

The recruitment proportion vector is:

$$\underline{r} = (1,\ 0,\ 0,\ 0) \quad .$$

Then

$$\theta I - P = \begin{pmatrix} .282 & -.102 & 0 & 0 \\ 0 & .18 & -.046 & 0 \\ 0 & 0 & .143 & -.033 \\ 0 & 0 & 0 & .108 \end{pmatrix}$$

and the inverse is

$$[\,\theta I - P\,]^{-1} = \begin{pmatrix} 3.546 & 2.009 & .646 & .198 \\ 0 & 5.556 & 1.787 & .546 \\ 0 & 0 & 6.993 & 2.137 \\ 0 & 0 & 0 & 9.259 \end{pmatrix} .$$

Using Equation 2.5

$$\tilde{\underline{n}} = \frac{\hat{\underline{n}}}{\hat{\underline{n}}\underline{1}'} = \frac{(35)(1,\ 0,\ 0,\ 0)}{223.965} \begin{pmatrix} 3.546 & 2.009 & .646 & .198 \\ 0 & 5.556 & 1.787 & .546 \\ 0 & 0 & 6.993 & 2.137 \\ 0 & 0 & 0 & 9.259 \end{pmatrix}$$

$$= (.554,\ .314,\ .101,\ .031)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50 and 100, can be found in Appendix C, confirming that in steady state there will be 55%, 31%, 10%, and 3% respectively, in categories 1,2,3, and 4. As tne printout shows, steady state distribution (to two decimal accuracy) is achieved by $t = 50$.

Example (b)

This is a non-hierarchical, three-grade system.

The inital stock vector is:

$$\underline{n}(0) = (300,\ 200,\ 100) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .60 & .15 & .05 \\ .75 & .20 & 0 \\ .05 & 0 & .90 \end{pmatrix} .$$

The total recruitment is:

$$R(t) = R\theta^{t-1} \text{ where } R = 100 \text{ , } \theta = 1.01 \text{ .}$$

The recruitment proportion vector is:

$$\underline{r} = (.70, .07, .23) \text{ .}$$

Then

$$\theta I - P = \begin{pmatrix} .41 & -.15 & -.05 \\ -.75 & .81 & 0 \\ -.05 & 0 & .11 \end{pmatrix}$$

and the inverse is

$$[\theta I - P]^{-1} = \begin{pmatrix} 4.03 & .75 & 1.83 \\ 3.73 & 1.92 & 1.69 \\ 1.83 & .34 & 9.92 \end{pmatrix} \text{ .}$$

Using Equation 2.5

$$\tilde{\underline{n}} = \frac{\hat{\underline{n}}}{\hat{\underline{n}}\underline{1}'} = \frac{(100)(.70, .07, .23)}{792.15} \begin{pmatrix} 4.03 & .75 & 1.83 \\ 3.73 & 1.92 & 1.69 \\ 1.83 & .34 & 9.92 \end{pmatrix}$$

$$= \quad (.44, .09, .47)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50 and 100, can be found in Appendix C, confirming that in steady state there will be 44%, 9%, and 47% respectively, in categories 1,2, and 3. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by t = 100 .

B. TOTAL SYSTEM SIZE OPTIONS

The options in this section are variations of Equation 1.6 where $M(t)$, representing the change in total system size, is the term that will take on different forms. As a result, the steady state equations for these options are also obtained in somewhat different forms. In the next two subsections the steady state equations will be derived for the system size options.

1. Option 4: Additive System Size

This option addresses the additive increase (decrease) in total system size for each time period. In this case, $M(t) = M$, a fixed amount of increase (decrease) per time period. Rewriting Equation 1.6 results in

$$\underline{n}(t) = \underline{n}(t-1)Q + M\underline{r} \quad . \qquad \text{(eqn 2.6)}$$

There are 3 cases to be considered.

Case 1: $M < 0$

In this case jobs are being eliminated by a fixed amount, M, each year which implies that eventually $N(t) = 0$ is reached in a finite amount of time. This implies that $\underline{n} = \underline{0}$ is the steady state stock vector. This means that such a system becomes empty in steady state.

Case 2: $M = 0$

When $M(t) = 0$, $N(t)$ is fixed and can be denoted N. It follows that Equation 1.6 can be written as

$$\underline{n}(t) = \underline{n}(t-1)Q \quad . \qquad \text{(eqn 2.7)}$$

Since the

$$\lim_{t \to \infty} \underline{n}(t) = \underline{n}$$

exists [Ref. 4: pp. 40-43, 48-50], Equation 2.7 can be written as

$$\underline{n} = \underline{n}\, Q \quad ,$$

or

$$\underline{n}\ (I - Q) = \underline{0} \quad .$$

Because the first column of I-Q is a linear combination of all the other columns the inverse of I-Q does not exist. So, to solve for $\underline{n}$, it is necessary to introduce the additional constraint of $\underline{n}\,\underline{1}' = N$ [Ref. 5: p. 80]. Replacing the first column of the I-Q matrix with a column of ones, denoted by (I-Q)*, and replacing the first element of $\underline{0}$ with N, denoted $\underline{0}$*, results in the following equation

$$\underline{n}\ (I-Q)^{*} = \underline{0}^{*} \quad .$$

Taking the inverse results in

$$\underline{n} = \underline{0}^{*}[\,(I-Q)^{*}\,]^{-1} \quad , \qquad \text{(eqn 2.8)}$$

the steady state stock vector. After steady state is reached, then $\underline{n}(t) = \underline{n}$ for "large" t. Two examples are worked out below to illustrate Equation 2.8 .

Example (a)

This is a hierarchical, four category system.

The inital stock vector is:

$$\underline{n}(0) = (129,\ 74,\ 28,\ 11) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .728 & .102 & 0 & 0 \\ 0 & .83 & .046 & 0 \\ 0 & 0 & .867 & .033 \\ 0 & 0 & 0 & .902 \end{pmatrix} ,$$

and

$$Q = \begin{pmatrix} .898 & .102 & 0 & 0 \\ .124 & .83 & .046 & 0 \\ .1 & 0 & .867 & .033 \\ .098 & 0 & 0 & .902 \end{pmatrix} .$$

The $\underline{0}^*$ vector is:

$$\underline{0}^* = (242, 0, 0, 0) .$$

The recruitment proportion vector is:

$$\underline{r} = (1, 0, 0, 0) .$$

Then

$$I - Q = \begin{pmatrix} .102 & -.102 & 0 & 0 \\ -.124 & .17 & -.046 & 0 \\ -.1 & 0 & .133 & -.033 \\ -.098 & 0 & 0 & .098 \end{pmatrix} .$$

Replacing the first column with ones gives

$$(I - Q)^* = \begin{pmatrix} 1 & -.102 & 0 & 0 \\ 1 & .17 & -.046 & 0 \\ 1 & 0 & .133 & -.033 \\ 1 & 0 & 0 & .098 \end{pmatrix}$$

and the inverse is

$$((I - Q)^*)^{-1} = \begin{pmatrix} .533 & .320 & .111 & .037 \\ -4.582 & 3.133 & 1.084 & .365 \\ -5.354 & -3.212 & 6.408 & 2.158 \\ -5.435 & -3.261 & -1.128 & 9.824 \end{pmatrix} .$$

Using Equation 2.8

$$\underline{n} = (242,\ 0,\ 0,\ 0)\begin{pmatrix} .533 & .320 & .111 & .037 \\ -4.582 & 3.133 & 1.084 & .365 \\ -5.354 & -3.212 & 6.408 & 2.158 \\ -5.435 & -3.261 & -1.128 & 9.824 \end{pmatrix}$$

$$= (128.99,\ 77.44,\ 26.86,\ 8.95)\ .$$

After rounding to the nearest integer the S.S.S.V. is:

$\underline{n} = (129,\ 77,\ 27,\ 9)$.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50 and 100, can be found in Appendix D, confirming the steady state stock vector above. As the printout shows, steady state is achieved by t = 50 .

Example (b)

This is a non-hierarchical, three-grade system.

The inital stcck vector is:

$$\underline{n}(0) = (300,\ 200,\ 100)\ .$$

The transition matrix is:

$$P = \begin{pmatrix} .60 & .15 & .05 \\ .75 & .20 & 0 \\ .05 & 0 & .90 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} .74 & .164 & .096 \\ .785 & .2035 & .0115 \\ .085 & .0035 & .9115 \end{pmatrix}.$$

The $\underline{0}^*$ vector is:

$$\underline{0}^* = (\ 600,\ 0,\ 0\ )\ .$$

The recruitment proportion vector is:

$$\underline{r} = (.70,\ .07,\ .23)\ .$$

Then

$$I - Q = \begin{pmatrix} .26 & -.164 & -.096 \\ -.785 & .797 & -.012 \\ -.085 & -.004 & .089 \end{pmatrix} .$$

Replacing the first column with ones gives

$$(I - Q)^* = \begin{pmatrix} 1 & -.164 & -.096 \\ 1 & .797 & -.012 \\ 1 & -.004 & .089 \end{pmatrix}$$

and the inverse is

$$((I - Q)^*)^{-1} = \begin{pmatrix} .430 & .090 & .479 \\ -.611 & 1.127 & -.516 \\ -4.888 & -.981 & 5.869 \end{pmatrix} .$$

Using Equation 2.8

$$\underline{n} = (600,\ 0,\ 0) \begin{pmatrix} .430 & .090 & .479 \\ -.611 & 1.127 & -.516 \\ -4.888 & -.981 & 5.869 \end{pmatrix}$$

$$= (258.0,\ 54.4,\ 287.4)\ .$$

After rounding to the nearest integer the S.S.S.V. is:

$$\underline{n} = (258,\ 54,\ 287)\ .$$

The computer printout of the terminal session for this example, showing the stocks for years 10, 50 and 100, can be found in Appendix D, confirming the steady state stock vector above. As the printout shows, steady state is achieved by $t = 50$.

Case 3: $M > 0$

Equation 1.6 can then be rewritten as

$$\frac{\underline{n}(t)}{N(t)} = \frac{\underline{n}(t-1)}{N(t)} Q + \frac{M}{N(t)} \underline{r} .$$

Iterating the equation, $N(t) = N(t-1) + M$, for $t = 1,2,...$ will show that, $N(t) = N(0) + tM$ for all $t = 1,2,...$ and therefore,

$$\lim_{t \to \infty} \frac{M}{N(t)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{N(t-1)}{N(t-1) + M} = 1 .$$

Then writing

$$\frac{\underline{n}(t)}{N(t)} = \frac{\underline{n}(t-1)}{N(t-1)} \bullet \frac{N(t-1)}{[N(t-1) + M]} \bullet Q + \frac{M}{N(t)} \underline{r} ,$$

and taking the limit as $t \to \infty$ yields

$$\underline{\tilde{n}} = \underline{\tilde{n}} Q , \text{ or } \underline{\tilde{n}} (I-Q) = \underline{0}$$

where

$$\underline{\tilde{n}} = \lim_{t \to \infty} \frac{\underline{n}(t)}{N(t)}$$

is the steady state distribution vector.

This is identical to the $M \doteq 0$ case, except here the constraint $\underline{n}\,\underline{1}' = 1$ must hold. Therefore, by replacing the first column of the I-Q matrix with a column of ones, denoted by $(I-Q)^*$, and replacing the first element of $\underline{0}$ with 1, denoted $\underline{0}^{**}$, the solution is

$$\underline{\tilde{n}} = \underline{0}^{**}[(I-Q)^*]^{-1} \quad , \qquad \text{(eqn 2.9)}$$

the steady state distribution vector.

After steady state has been achieved, $\underline{n}(t)$ can be found using

$$\underline{n}(t) = Q[N(0) + tM]\underline{\tilde{n}}$$

for "large" t. Two examples are worked out below to illustrate Equation 2.9 .

Example (a)

This is a hierarchical, four category system.

The inital stock vector is:

$$\underline{n}(0) = (129,\ 74,\ 28;\ 11) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .728 & .102 & 0 & 0 \\ 0 & .83 & .046 & 0 \\ 0 & 0 & .867 & .033 \\ 0 & 0. & 0 & .902 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} .898 & .102 & 0 & 0 \\ .124 & .83 & .046 & 0 \\ .1 & 0 & .867 & .033 \\ .098 & 0 & 0 & .902 \end{pmatrix}.$$

The $\underline{0}^{**}$ vector is:

$$\underline{0}^{**} = (\ 1,\ 0,\ 0,\ 0\ ) \ .$$

The recruitment proportion vector is:

$$\underline{r} = (1, 0, 0, 0) \quad .$$

Then

$$I - Q = \begin{pmatrix} .102 & -.102 & 0 & 0 \\ -.124 & .17 & -.046 & 0 \\ -.1 & 0 & .133 & -.033 \\ -.098 & 0 & 0 & .098 \end{pmatrix} .$$

Replacing the first column with ones, gives

$$(I - Q)^* = \begin{pmatrix} 1 & -.102 & 0 & 0 \\ 1 & .17 & -.046 & 0 \\ 1 & 0 & .133 & -.033 \\ 1 & 0 & 0 & .098 \end{pmatrix}$$

and the inverse is

$$((I - Q)^*)^{-1} = \begin{pmatrix} .533 & .320 & .111 & .037 \\ -4.582 & 3.133 & 1.084 & .365 \\ -5.354 & -3.212 & 6.408 & 2.158 \\ -5.435 & -3.261 & -1.128 & 9.824 \end{pmatrix} .$$

Using Equation 2.9

$$\underline{\tilde{n}} = (1, 0, 0, 0) \begin{pmatrix} .533 & .320 & .111 & .037 \\ -4.582 & 3.133 & 1.084 & .365 \\ -5.354 & -3.212 & 6.408 & 2.158 \\ -5.435 & -3.261 & -1.128 & 9.824 \end{pmatrix}$$

$$= (.53, .32, .11, .04)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50, 100, 900, and 950 can be found in Appendix D, confirming that in steady state there will be 53%, 32%, 11%, and 4% respectively, in categories 1,2,3, and 4. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by t = 950 .

Example (b)

This is a non-hierarchical, three-grade system.

The inital stock vector is:

$$\underline{n}(0) = (300,\ 200,\ 100) \ .$$

The transition matrix is:

$$P = \begin{pmatrix} .60 & .15 & .05 \\ .75 & .20 & 0 \\ .05 & 0 & .90 \end{pmatrix} ,$$

and

$$Q = \begin{pmatrix} .74 & .164 & .096 \\ .785 & .2035 & .0115 \\ .085 & .0035 & .9115 \end{pmatrix} .$$

The $\underline{0}$** vector is:

$$\underline{0}^{**} = (\ 1,\ 0,\ 0,\ ) \ .$$

The recruitment proportion vector is:

$$\underline{r} = (.70,\ .07,\ .23) \ .$$

Then

$$I - Q = \begin{pmatrix} .26 & -.164 & -.096 \\ -.785 & .797 & -.012 \\ -.085 & -.004 & .089 \end{pmatrix} .$$

Replacing the first column with ones, gives

$$(I - Q)^* = \begin{pmatrix} 1 & -.164 & -.096 \\ 1 & .797 & -.012 \\ 1 & -.004 & .089 \end{pmatrix}$$

and the inverse is

$$((I - Q)^*)^{-1} = \begin{pmatrix} .430 & .091 & .479 \\ -.611 & 1.127 & -.516 \\ -4.888 & -.981 & 5.869 \end{pmatrix} .$$

Using Equation 2.9

$$\underset{\sim}{n} = (1, 0, 0) \begin{pmatrix} .430 & .091 & .479 \\ -.611 & 1.127 & -.516 \\ -4.888 & -.981 & 5.869 \end{pmatrix}$$

$$= (.43. \ .09, \ .48)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50, 100, 350, and 400 can be found in Appendix D, confirming that in steady state there will be 43%, 9%, and 48% respectively, in categories 1,2, and 3. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by $t = 400$.

2. Option 5: Multiplicative System Size

This section addresses the option of multiplicative increase (decrease) in total system size for each time period. Starting with

$$N(t) = \theta N(t-1), \quad \text{for } t = 1,2,...$$

it follows that

$$N(t) = \theta^t N(0), \quad \text{for } t = 1,2,...$$

from which $M(t)$ can be determined as

$$M(t) = N(t) - N(t-1) = (\theta - 1)\theta^{t-1} N(0) .$$

Then, letting $(\theta-1)\theta^{t-1}N(0)$ replace $M(t)$ in Equation 1.6, results in

$$\underline{n}(t) = \underline{n}(t-1)Q + [(\theta-1)\theta^{t-1}N(0)]\,\underline{r} \quad . \qquad \text{(eqn 2.10)}$$

First, $\theta > 0$ must be true otherwise negative or zero system size may result. This leaves two cases to be examined.

Case 1: $0 < \theta \leq 1$

In this case

$$\lim_{t \to \infty} N(t) = 0$$

because as $t \to \infty$, $\theta^{t-1} \to 0$ for $0 < \theta < 1$. Finally, if $\theta = 1$ then $\theta - 1 = 0$ and $N(t) = 0$ for all t. Therefore,

$$\underline{n} = \lim_{t \to \infty} \underline{n}(t) = \underline{0}$$

must also hold. This means that such a system becomes empty in steady state.

Case 2: $\theta > 1$

Equation 2.10 can be rewritten as

$$\frac{\underline{n}(t)}{N(t)} = \frac{\underline{n}(t-1)}{N(t)}Q + \frac{(\theta-1)\theta^{t-1}N(0)}{N(t)}\underline{r} \quad .$$

Since $N(t)$ can be written as $\theta N(t-1)$ or as $\theta^{t}N(0)$, the above equation gives

$$\frac{\underline{n}(t)}{N(t)} = \frac{\underline{n}(t-1)}{N(t-1)} \bullet \frac{1}{\theta} \bullet Q + \frac{(\theta-1)\theta^{t-1}N(0)}{\theta^{t}N(0)}\underline{r}$$

or

$$\frac{\underline{n}(t)}{N(t)} = \frac{\underline{n}(t-1)}{N(t-1)} \bullet \frac{1}{\theta} \bullet Q + \frac{\theta-1}{\theta}\underline{r} \quad .$$

Taking the limit as $t \to \infty$ results in

$$\tilde{\underline{n}} = \tilde{\underline{n}} \frac{1}{\theta} Q + \frac{(\theta-1)}{\theta} \underline{r}$$

or

$$\tilde{\underline{n}} [\theta I - Q] = (\theta - 1) \underline{r}$$

where

$$\tilde{\underline{n}} = \lim_{t \to \infty} \frac{\underline{n}(t)}{N(t)}$$

is the steady state distributicn vector.

Since $\theta > 1$ in this case, the matrix $\theta I-Q$ has a unique inverse according to Theorem 2, so the result

$$\tilde{\underline{n}} = (\theta-1) \underline{r} [\theta I-Q]^{-1} \qquad \text{(eqn 2.11)}$$

is obtained for the steady state distribution vector. After steady state has been reached it follows that the formula

$$\underline{n}(t) = \theta^{t} N(0) \tilde{\underline{n}}$$

can be used tc find the stocks at time t. Two examples are worked out below to illustrate Equation 2.11 .

Example (a)

This is a hierarchical, four category system.

The inital stcck vector is:

$$\underline{n}(0) = (129, 74, 28, 11) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .728 & .102 & 0 & 0 \\ 0 & .83 & .046 & 0 \\ 0 & 0 & .867 & .033 \\ 0 & 0 & 0 & .902 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} .898 & .102 & 0 & 0 \\ .124 & .83 & .046 & 0 \\ .1 & 0 & .867 & .033 \\ .098 & 0 & 0 & .902 \end{pmatrix}.$$

The multiplicative factor is:

$$\theta = 1.01 \ .$$

The recruitment proportion vector is:

$$\underline{r} = (1, 0, 0, 0) \ .$$

Then

$$\theta I - Q = \begin{pmatrix} .112 & -.102 & 0 & 0 \\ -.124 & .18 & -.046 & 0 \\ -.1 & 0 & .143 & -.033 \\ -.098 & 0 & 0 & .108 \end{pmatrix}$$

and the inverse is

$$(\theta I - Q)^{-1} = \begin{pmatrix} 55.41 & 31.40 & 10.10 & 3.09 \\ 51.04 & 34.48 & 11.09 & 3.39 \\ 50.35 & 28.53 & 16.17 & 4.94 \\ 50.28 & 28.49 & 9.17 & 12.06 \end{pmatrix}.$$

Using Equation 2.11

$$\underline{\tilde{n}} = (.01)\ (1,0,0,0) \begin{pmatrix} 55.41 & 31.40 & 10.10 & 3.09 \\ 51.04 & 34.48 & 11.09 & 3.39 \\ 50.35 & 28.53 & 16.17 & 4.94 \\ 50.28 & 28.49 & 9.17 & 12.06 \end{pmatrix}$$

$$= (.554, .314, .101, .031)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50, and 100, can be found in Appendix E, confirming that in steady state there will be 55%, 31%, 10%, and 3% respectively, in categories 1, 2, 3, and 4. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by t = 50 .

Example (b)

This is a non-hierarchical, three-grade system.

The inital stock vector is:

$$\underline{n}(0) = (300,\ 200,\ 100) \quad .$$

The transition matrix is:

$$P = \begin{pmatrix} .60 & .15 & .05 \\ .75 & .20 & 0 \\ .05 & 0 & .90 \end{pmatrix} ,$$

and

$$Q = \begin{pmatrix} .74 & .164 & .096 \\ .785 & .2035 & .0115 \\ .085 & .0035 & .9115 \end{pmatrix} .$$

The multiplicative factor is:

$$\theta = 1.01 \quad .$$

The recruitment proportion vector is:

$$\underline{r} = (.70,\ .07,\ .23) \ .$$

Then

$$\theta I - Q = \begin{pmatrix} .27 & -.164 & -.096 \\ -.785 & .8065 & -.0115 \\ -.085 & -.0035 & .9985 \end{pmatrix}$$

and the inverse is

$$(\theta I - Q)^{-1} = \begin{pmatrix} 45.32 & 9.41 & 45.27 \\ 44.69 & 10.52 & 44.79 \\ 40.70 & 8.50 & 50.81 \end{pmatrix} .$$

Using Equation 2.11

$$\underline{\tilde{n}} = (.01)\ (.7,.07,.23) \begin{pmatrix} 45.32 & 9.41 & 45.27 \\ 44.69 & 10.52 & 44.79 \\ 40.70 & 8.50 & 50.81 \end{pmatrix} .$$

$$= (.44,\ .09,\ .47)$$

is the steady state distribution vector.

The computer printout of the terminal session for this example, showing the stocks for years 10, 50, and 100, can be found in Appendix E, confirming that in steady state there will be 44%, 9%, and 47% respectively, in categories 1,2, and 3. As the printout shows, steady state distribution (to two decimal accuracy) is achieved by t = 100 .

III. SUMMARY

This chapter is a synopsis of the options that describe the five different conditions of growth. In all options, the restriction of $w_i > 0$ for all $i = 1,2,\ldots,k$ was assumed, since attrition must be allowed from any category in a manpower system. Options 1 through 3 employed the basic prediction equation (Equation 1.2) and used constant, additively and multiplicatively increasing (decreasing) recruitment, $R(t)$, to derive the steady state behavior of a manpower system. For options 4 and 5, Equation 1.6 used additively and multiplicatively increasing (decreasing) total system size, $M(t)$, to derive the steady state behavior of a manpower system.

Derivation of the steady state stock vector or distribution vector required some standard results of matrix algebra summarized in Theorems 1 and 2 of Chapter II. In addition to the analytical derivation of the steady state equations, examples have been worked out in each case (excepting some trivial cases) to illustrate the use of the analytical results. The formulas have been programmed in the APL language and the program listing is given in Appendix F. This program (called a function in APL) has been integrated into the main APL program at NPS which forecasts future stocks of a manpower system using the Markov Chain theory summarized in Chapter I. Appendices A through E show the printouts of the terminal sessions where the examples referred to above are worked out to verify the steady state results obtained in each analytical example.

APPENDIX A

COMPUTER PRINTOUTS OF EXAMPLES FOR OPTION 1

1. Example (a)

```
      START
DO YOU WISH TC ENTER DATA?
  0   NC
  1   YES
 :
      1

ENTER THE NUMBER OF THE MODEL TYPE
  1   MARKOV HIERARCHICAL
  2   MARKOV LENGTH OF SERVICE
  3   MARKOV GENERAL
  4   VACANCY
 :
      1

ENTER N(INITIAL STOCK VECTOR)
 :
      129 74 28 11
ENTER THE PROMOTION RATE VECTOR.
THIS VECTOR SHOULD INCLUDE THE PROMOTION RATES FOR THE FIRST
 3 CLASSES.
 :
      .102 .046 .033
ENTER THE WASTAGE RATE VECTOR.
THE 4TH VALUE SHOULD INCLUDE THE TOTAL RATE FOR  THAT
STATE DUE TO EITHER PROMOTION OR WASTAGE.
 :
      .17 .124 .1 .098
ENTER THE NUMBER OF THE RECRUIT TYPE
  1   FIXED RECRUIT VECTOR
  2   ADDITIVE (RECRUIT SIZE)
  3   MULTIPLICATIVE (RECRUIT SIZE)
  4   ADDITIVE (SYSTEM SIZE)
  5   MULTIPLICATIVE (SYSTEM SIZE)
 :
      1
ENTER R (RECRUITMENT VECTOR)
 :
      35 0 0 0

ENTER THE PERCENT CODE
  0  NO GRADE PERCENTAGES
  1  GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2  GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7  QUIT PROGRAM
 :
      1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0    NO
  1    YES
 :
      1
```

```
   P MATRIX
 0.728 0.102 0     0
 0     0.83  0.046 0
 0     0     0.867 0.033
 0     0     0     0.902
   N VECTOR
129 74 28 11
OPTION =1
   R VECTOR
35 0 0 0

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
    T                 N          PERCENT      R
=============================================
     0      1           129 (        53)
            2            74 (        31)
            3            28 (        12)
            4            11 (         5)
       TOTAL            242 (       100)
-------------------------------
    10      1           129 (        53)
            2            77 (        32)
            3            27 (        11)
            4            10 (         4)
       TOTAL            242 (       100)          35
-------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
    50      1           129 (        53)
            2            77 (        32)
            3            27 (        11)
            4             9 (         4)
       TOTAL            242 (       100)          35
-------------------------------
```

```
DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
   100      1          129  (        53)
            2           77  (        32)
            3           27  (        11)
            4            9  (         4)
        TOTAL          242  (       100)         35
--------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       0

DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1
   999      1          129  (        53)
            2           77  (        32)
            3           27  (        11)
            4            9  (         4)
        TOTAL          242  (       100)         35
--------------------------------

ARE YOU THROUGH?
  0    NC
  1    YES
 :
       1
```

2. Example (b)

```
       START

DC YOU WISH TC ENTER DATA?
  0    NC
  1    YES
 :
       1

ENTER THE NUMBER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       3
```

```
ENTER N(INITIAL STOCK VECTOR)
 :
       300 200 100
ENTER P (TRANSITION MATRIX) BY ROWS
ENTER 1TH ROW
 :
       .6 .15 .05
ENTER 2TH ROW
 :
       .75 .2 0
ENTER 3TH ROW
 :
       .05 0 .9
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       1
ENTER R (RECRUITMENT VECTOR)
 :
       70 7 23

ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
       1

   P MATRIX
 0.6  0.15 0.05
 0.75 0.2  0
 0.05 0    0.9
   N VECTOR
300 200 100
OPTION =1
   R VECTOR
70 7 23

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
```

```
     T                N        PERCENT       R
==============================================
     0      1        300  (          50)
            2        200  (          33)
            3        100  (          17)
         TOTAL       600  (         100)
----------------------------------------
    10      1        351  (          48)
            2         75  (          10)
            3        302  (          41)
         TOTAL       728  (         121)          100
----------------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
    50      1        375  (          43)
            2         79  (           9)
            3        415  (          48)
         TOTAL       868  (         145)          100
----------------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
   100      1        376  (          43)
            2         79  (           9)
            3        418  (          48)
         TOTAL       873  (         146)          100
----------------------------------------

DC YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       0

DO YOU WISH TC SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1
```

999	1	376 (	43)	
	2	79 (	9)	
	3	418 (	48)	
	TOTAL	873 (	100)	100

```
ARE YOU THROUGH?
  0    NO
  1    YES
 :
       1
```

APPENDIX B

COMPUTER PRINTOUTS OF EXAMPLES FOR OPTION 2

1. Example (a)

```
START

DO YOU WISH TO ENTER DATA?
  0    NC
  1    YES
 :
       1

ENTER THE NUMBER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       1

ENTER N(INITIAL STOCK VECTOR)
 :
       129 74 28 11
ENTER THE PROMOTION RATE VECTOR.
THIS VECTOR SHOULD INCLUDE THE PROMOTICN RATES FOR THE FIRST
 3 CLASSES.
 :
       .102 .046 .033
ENTER THE WASTAGE RATE VECTOR.
THE 4TH VALUE SHCULD INCLUDE THE TOTAL RATE FOR  THAT
STATE DUE TO EITHER PROMOTION OR WASTAGE.
 :
       .17 .124 .1 .098
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       2
ENTER R (RECRUITMENT VECTOR)
 :
       35 0 0 0
ENTER ADDITIVE INCREASE
 :
       5

ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1
```

```
WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
      1

   P MATRIX
 0.728 0.102 0     0
 0     0.83  0.046 0
 0     0     0.867 0.033
 0     0     0     0.902
   N VECTOR
129 74  28  11
OPTION =2
INC    =5
   TOTAL RECRUITMENT
35
   RECRUITMENT PROPORTION VECTOR
1 0 0 0

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
      0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0

     T              N        PERCENT      R
========================================
      0      1       129 (        53)
             2        74 (        31)
             3        28 (        12)
             4        11 (         5)
         TOTAL       242 (       100)
-------------------------------
     10      1       248 (        63)
             2       106 (        27)
             3        29 (         7)
             4        10 (         3)
         TOTAL       392 (       162)         80
-------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
      1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
```

```
    50     1          980  (        58)
           2          523  (        31)
           3          152  (         9)
           4           39  (         2)
         TOTAL       1695  (       700)       280
-----------------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
   100     1         1899  (        56)
           2         1075  (        31)
           3          343  (        10)
           4          102  (         3)
         TOTAL       3419  (      1413)       530
-----------------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMEER OF THE YEAR YOU WISH TO SEE
 :
       900

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
   900     1        16605  (        54)
           2         9898  (        32)
           3         3395  (        11)
           4         1130  (         4)
         TOTAL      31028  (     12821)      4530
-----------------------------------------

DC YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       975

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
```

```
   975     1         17984 (         53)
           2         10725 (         32)
           3          3681 (         11)
           4          1226 (          4)
        TOTAL        33617 (       13891)       4905
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       0

DO YOU WISH TC SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1

PERCENTAGES ARE IN STEADY STATE
   999     1         17984 (         53)
           2         10725 (         32)
           3          3681 (         11)
           4          1226 (          4)
        TOTAL        33617 (        100)       4905
--------------------------------

ARE YOU THROUGH?
  0    NO
  1    YES
 :
       1
```

2. Example (b)

```
       START

DO YOU WISH TC ENTER DATA?
  0    NO
  1    YES
 :
       1

ENTER THE NUMBEP OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       3

ENTEP N(INITIAL STOCK VECTOR)
 :
       300 200 100
ENTER P (TRANSITION MATRIX) BY ROWS
ENTER 1TH ROW
 :
       .6 .15 .05
ENTER 2TH ROW
 :
       .75 .2 0

ENTER 3TH ROW
 :
       .05 0 .9
```

```
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       2

ENTER R (RECRUITMENT VECTOR)
 :
       70 7 23

ENTER ADDITIVE INCREASE
 :
       25

ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
       1

   P MATRIX

 0.6  0.15 0.05
 0.75 0.2  0
 0.05 0    0.9

   N VECTOR

300 200 100

OPTION =2

INC    =25

   TOTAL RECRUITMENT

100

   RECRUITMENT PROPORTION VECTOR

0.7 0.07 0.23

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       0

ENTER THE NUMEER OF THE YEAR YOU WISH TO SEE
 :
       10
```

```
DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
   T              N       PERCENT      R
========================================
   0     1       300  (       50)
         2       200  (       33)
         3       100  (       17)
      TOTAL      600  (      100)
--------------------------------
  10     1       801  (       53)
         2       162  (       11)
         3       561  (       37)
      TOTAL     1523  (      254)        325
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
  50     1      4374  (       46)
         2       914  (       10)
         3      4260  (       45)
      TOTAL     9548  (     1591)       1325
--------------------------------

DC YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
 100     1      9066  (       44)
         2      1903  (        9)
         3      9469  (       46)
      TOTAL    20438  (     3406)       2575
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       1
```

```
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      350

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
   350     1          32550 (         43)
           2           6853 (          9)
           3          35585 (         47)
       TOTAL          74987 (      12498)       8825
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
      1

ENTER THE NUMBER OF THE YEAR YOU FISH TO SEE
 :
      400

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
   400     1          37246 (         43)
           2           7843 (          9)
           3          40808 (         48)
       TOTAL          85897 (      14316)      10075
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
      0

DO YOU WISH TC SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
      1

PERCENTAGES ARE IN STEADY STATE
   999     1          37246 (         43)
           2           7843 (          9)
           3          40808 (         48)
       TOTAL          85897 (        100)      10075
--------------------------------

ARE YOU THROUGH?
  0    NC
  1    YES
 :
      1
```

APPENDIX C

COMPUTER PRINTOUTS OF EXAMPLES FOR OPTION 3

1. Example (a)

```
        START

DO YOU WISH TO ENTER DATA?
  0     NC
  1     YES
 :
        1

ENTER THE NUMBER OF THE MODEL TYPE
  1     MARKOV HIERARCHICAL
  2     MARKOV LENGTH OF SERVICE
  3     MARKOV GENERAL
  4     VACANCY
 :
        1

ENTER N(INITIAL STOCK VECTOR)
 :
        129 74 28 11
ENTER THE PROMOTION RATE VECTOR.
THIS VECTOR SHOULD INCLUDE THE PROMOTICN RATES FOR THE FIRST
 3 CLASSES.
 :
        .102 .046 .033
ENTER THE WASTAGE RATE VECTOR.
THE 4TH VALUE SHOULD INCLUDE THE TOTAL RATE FOR  THAT
STATE DUE TO EITHER PROMOTION OR WASTAGE.
 :
        .17 .124 .1 .098
ENTER THE NUMBER OF THE RECRUIT TYPE
  1     FIXED RECRUIT VECTOR
  2     ADDITIVE (RECRUIT SIZE)
  3     MULTIPLICATIVE (RECRUIT SIZE)
  4     ADDITIVE (SYSTEM SIZE)
  5     MULTIPLICATIVE (SYSTEM SIZE)
 :
        3
ENTER R (RECRUITMENT VECTOR)
 :
        35 0 0 0
ENTER MULTIPLICATIVE FACTOR
 :
        1.01

ENTER THE PERCENT CODE
  0    NO GRADE PERCENTAGES
  1    GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2    GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7    QUIT PROGRAM
 :
        1
```

```
WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
      1

   P MATRIX
 0.728 0.102 0     0
 0     0.83  0.046 0
 0     0     0.867 0.033
 0     0     0     0.902
   N VECTOR
129 74  28 11
OPTION =3
FACT   =1.01
   TOTAL RECRUITMENT
35
   RECRUITMENT PROPORTION VECTOR
1 0 0 0

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
      0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
    T              N        PERCENT      R
=============================================
     0    1           129  (       53)
          2            74  (       31)
          3            28  (       12)
          4            11  (        5)
       TOTAL          242  (      100)
------------------------------------
    10    1           137  (       54)
          2            79  (       31)
          3            27  (       11)
          4            10  (        4)
       TOTAL          253  (      105)         38
------------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
      1

ENTER THE NUMEER OF THE YEAR YOU WISH TO SEE
 :
      50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
```

```
     50      1          204  (       55)
             2          116  (       31)
             3           37  (       10)
             4           11  (        3)
          TOTAL         368  (      152)        57
--------------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0     NO
  1     YES
  7     QUIT PROGRAM
 :
        1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
        100

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
        0
    100      1          336  (       55)
             2          190  (       31)
             3           61  (       10)
             4           19  (        3)
          TOTAL         606  (      250)        94
--------------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0     NO
  1     YES
  7     QUIT PROGRAM
 :
        0

DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0     NO
  1     YES
 :
        1

PERCENTAGES ARE IN STEADY STATE
    999      1          336  (       55)
             2          190  (       31)
             3           61  (       10)
             4           19  (        3)
          TOTAL         606  (      100)        94
--------------------------------------

ARE YOU THROUGH?
  0     NO
  1     YES
 :
        1
```

2. <u>Example</u> (<u>b</u>)

```
        START

DO YOU WISH TO ENTER DATA?
  0     NO
  1     YES
 :
        1
```

```
ENTER THE NUMEER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       3

ENTER N(INITIAL STOCK VECTOR)
 :
       300 200 100
ENTER P (TRANSITION MATRIX) BY ROWS
ENTER 1TH ROW
 :
       .6 .15 .05
ENTER 2TH ROW
 :
       .75 .2 0
ENTER 3TH ROW
 :
       .05 0 .9
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       3

ENTER R (RECRUITMENT VECTOR)
 :
       70 7 23
ENTER MULTIPLICATIVE FACTOR
 :
       1.01

ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
       1

   P MATRIX
 0.6  0.15 0.05
 0.75 0.2  0
 0.05 0    0.9

   N VECTOR
300 200 100

OPTION =3

FACT   =1.01

   TOTAL RECRUITMENT
100

   RECRUITMENT PROPORTION VECTOR
0.7 0.07 0.23
```

```
WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
     T                N         PERCENT      R
=======================================
      0      1          300  (        50)
             2          200  (        33)
             3          100  (        17)
          TOTAL         600  (       100)
-----------------------------
     10      1          370  (        49)
             2           78  (        10)
             3          312  (        41)
          TOTAL         761  (       127)        109
-----------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMEER CF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
     50      1          575  (        44)
             2          121  (         9)
             3          603  (        46)
          TOTAL        1298  (       216)        163
-----------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
```

```
   100     1            947 (          44)
           2            199 (           9)
           3            996 (          47)
        TOTAL          2141 (         357)        268
--------------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       0

DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0    NO
  1    YES
 :
       1

PERCENTAGES ARE IN STEADY STATE
   999     1            947 (          44)
           2            199 (           9)
           3            996 (          47)
        TOTAL          2141 (         100)        268
--------------------------------------

ARE YOU THROUGH?
  0    NC
  1    YES
 :
       1
```

APPENDIX D

COMPUTER PRINTOUTS OF EXAMPLES FOR OPTION 4

1. Case 1 Example (a)

```
START

DO YOU WISH TO ENTER DATA?
  0   NO
  1   YES
 :
      1

ENTER THE NUMBER OF THE MODEL TYPE
  1   MARKOV HIERARCHICAL
  2   MARKOV LENGTH OF SERVICE
  3   MARKOV GENERAL
  4   VACANCY
 :
      1

ENTER N(INITIAL STOCK VECTOR)
 :
      129 74 28 11
ENTER THE PROMOTION RATE VECTOR.
THIS VECTOR SHOULD INCLUDE THE PROMOTION RATES FOR THE FIRST
 3 CLASSES.
 :
      .102 .046 .033
ENTER THE WASTAGE RATE VECTOR.
THE 4TH VALUE SHOULD INCLUDE THE TOTAL RATE FOR  THAT
STATE DUE TO EITHER PROMOTION OR WASTAGE.
 :
      .17 .124 .1 .098
ENTER THE NUMBER OF THE RECRUIT TYPE
  1   FIXED RECRUIT VECTOR
  2   ADDITIVE (RECRUIT SIZE)
  3   MULTIPLICATIVE (RECRUIT SIZE)
  4   ADDITIVE (SYSTEM SIZE)
  5   MULTIPLICATIVE (SYSTEM SIZE)
 :
      4
ENTER RPROP (RECRUITMENT PROPORTION VECTOR)
 :
      1 0 0 0
ENTER ADDITIVE INCREASE
 :
      0

ENTER THE PERCENT CODE
  0  NO GRADE PERCENTAGES
  1  GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2  GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7  QUIT PROGRAM
 :
      1
WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0    NO
  1    YES
 :
      1
```

```
   P MATRIX
 0.728 0.102  0      0
 0      0.83   0.046  0
 0      0      0.867  0.033
 0      0      0      0.902
   N VECTOR
129  74  28  11
OPTION  =4
INC     =0
   RECRUITMENT PROPORTION VECTOR
1 0 0 0

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0

    T              N        PERCENT      R
========================================
     0    1           129  (        53)
          2            74  (        31)
          3            28  (        12)
          4            11  (         5)
       TOTAL          242  (       100)
--------------------------------
    10    1           129  (        53)
          2            77  (        32)
          3            27  (        11)
          4            10  (         4)
       TOTAL          242  (       100)        35
--------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0

    50    1           129  (        53)
          2            77  (        32)
          3            27  (        11)
          4             9  (         4)
       TOTAL          242  (       100)        35
--------------------------------
```

```
DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
   100      1           129  (         53)
            2            77  (         32)
            3            27  (         11)
            4             9  (          4)
        TOTAL           242  (        100)          35
--------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       0
DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1

PERCENTAGES ARE IN STEADY STATE
   999      1           129  (         53)
            2            77  (         32)
            3            27  (         11)
            4             9  (          4)
        TOTAL           242  (        100)          35
--------------------------------

ARE YOU THROUGH?
  0    NO
  1    YES
 :
       1
```

2. Case 1   Example (b)

```
START

DO YOU WISH TC ENTER DATA?
  0    NO
  1    YES
 :
       1
ENTER THE NUMEER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       3
```

```
ENTER N(INITIAL STOCK VECTOR)
 :
      300 200 100
ENTER P (TRANSITION MATRIX) BY ROWS
ENTER 1TH ROW
 :
      .6 .15 .05
ENTER 2TH ROW
 :
      .75 .2 0
ENTER 3TH ROW
 :
      .05 0 .9
ENTER THE NUMBER OF THE RECRUIT TYPE
  1   FIXED RECRUIT VECTOR
  2   ADDITIVE (RECRUIT SIZE)
  3   MULTIPLICATIVE (RECRUIT SIZE)
  4   ADDITIVE (SYSTEM SIZE)
  5   MULTIPLICATIVE (SYSTEM SIZE)
 :
      4

ENTER RPROP (RECRUITMENT PROPORTION VECTOR)
 :
      .7 .07 .23
ENTER ADDITIVE INCREASE
 :
      0

ENTER THE PERCENT CODE
  0  NO GRADE PERCENTAGES
  1  GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2  GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7  QUIT PROGRAM
 :
      1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
      1

   P MATRIX
 0.6  0.15 0.05
 0.75 0.2  0
 0.05 0    0.9
   N VECTOR
300 200 100
OPTION =4
INC    =0

   RECRUITMENT PROPORTION VECTOR
0.7 0.07 0.23


WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0   NO
  1   YES
  7   QUIT PROGRAM
 :
      0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      10
```

```
DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
        0
    T              N        PERCENT      R
========================================
    0      1        300 (          50)
           2        200 (          33)
           3        100 (          17)
       TOTAL        600 (         100)
--------------------------------
   10      1        284 (          47)
           2         61 (          10)
           3        255 (          43)
       TOTAL        600 (         100)          73
--------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
   50      1        258 (          43)
           2         54 (           9)
           3        287 (          48)
       TOTAL        600 (         100)          69
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0

  100      1        258 (          43)
           2         54 (           9)
           3        287 (          48)
       TOTAL        600 (         100)          69
--------------------------------
```

```
DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       0

DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1

PERCENTAGES ARE IN STEADY STATE
   999     1         258 (        43)
           2          54 (         9)
           3         287 (        48)
       TOTAL         600 (       100)        69
--------------------------------

ARE YOU THROUGH?
  0    NC
  1    YES
 :
       1
```

3. <u>Case 2</u> <u>Example</u> (a)

```
       START

DO YOU WISH TC ENTER DATA?
  0    NC
  1    YES
 :
       1
ENTER THE NUMBER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       1
ENTER N(INITIAL STOCK VECTOR)
 :
       129 74 28 11

ENTER THE PROMOTION RATE VECTOR.
THIS VECTOR SHOULD INCLUDE THE PROMOTION RATES FOR THE FIRST
 3 CLASSES.
 :
       .102 .046 .033

ENTER THE WASTAGE RATE VECTOR.
THE 4TH VALUE SHOULD INCLUDE THE TOTAL RATE FOR  THAT
STATE DUE TO EITHER PROMOTION OR WASTAGE.
 :
       .17 .124 .1 .098

ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       4
```

```
ENTER RPROP (RECRUITMENT PROPORTION VECTOR)
 :
      1 0 0 0
ENTER ADDITIVE INCREASE
 :
      5

ENTER THE PERCENT CODE
 0   NO GRADE PERCENTAGES
 1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
 2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
 7   QUIT PROGRAM
 :
      1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
 0     NO
 1     YES
 :
      1

   P MATRIX
 0.728  0.102  0      0
 0      0.83   0.046  0
 0      0      0.867  0.033
 0      0      0      0.902

   N VECTOR
129 74 28 11

OPTION =4

INC    =5

   RECRUITMENT PROPORTION VECTOR
1 0 0 0

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
 0     NO
 1     YES
 7     QUIT PROGRAM
 :
      0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      10

DO YOU WISH TO SEE THE INTERVENING YEARS?
 0   NO
 1   YES
 :
      0

    T             N       PERCENT      R
=====================================
    0     1      129  (        53)
          2       74  (        31)
          3       28  (        12)
          4       11  (         5)
       TOTAL     242  (       100)
---------------------------------
   10     1      165  (        57)
          2       89  (        30)
          3       28  (        10)
          4       10  (         3)
       TOTAL     292  (       121)        47
---------------------------------
```

```
DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50
DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
    50      1          273  (        56)
            2          155  (        31)
            3           49  (        10)
            4           15  (         3)
        TOTAL          492  (       203)        76
-----------------------------------------
DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       1
ENTER THE NUMEER OF THE YEAR YOU WISH TO SEE
 :
       100
DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
   100      1          407  (        55)
            2          235  (        32)
            3           77  (        10)
            4           24  (         3)
        TOTAL          742  (       307)       113
-----------------------------------------
DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PRCGRAM
 :
       1
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       900
DO YOU WISH TC SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
   900      1         2537  (        54)
            2         1513  (        32)
            3          519  (        11)
            4          173  (         4)
        TOTAL         4742  (      1960)       692
-----------------------------------------
```

```
DO YOU WISH TC SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMEER OF THE YEAR YOU WISH TO SEE
 :
       950

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0

   950      1          2670  (          53)
            2          1593  (          32)
            3           547  (          11)
            4           182  (           4)
        TOTAL          4992  (        2063)         728
--------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PRCGRAM
 :
       0

DO YOU WISH TC SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1

PERCENTAGES ARE IN STEADY STATE
   999      1          2670  (          53)
            2          1593  (          32)
            3           547  (          11)
            4           182  (           4)
        TOTAL          4992  (         100)         728
--------------------------------

ARE YOU THROUGH?
  0    NO
  1    YES
 :
       1
```

4. Case 2   Example (b)

```
       START

DO YOU WISH TC ENTER DATA?
  0    NC
  1    YES
 :
       1
```

```
ENTER THE NUMBER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       3

ENTER N(INITIAL STOCK VECTOR)
 :
       300 200 100
ENTER P (TRANSITION MATRIX) BY ROWS
ENTER 1TH ROW
 :
       .6 .15 .05
ENTER 2TH ROW
 :
       .75 .2 0
ENTER 3TH ROW
 :
       .05 0 .9
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       4

ENTER RPROP (RECRUITMENT PROPORTION VECTOR)
 :
       .7 .07 .23

ENTER ADDITIVE INCREASE
 :
       25


ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
       1

   P MATRIX
 0.6  0.15 0.05
 0.75 0.2  0
 0.05 0    0.9

   N VECTOR
300 200 100

OPTION =4
INC    =25

   RECRUITMENT PROPORTION VECTOR
0.7 0.07 0.23
```

```
WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       10

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
    T              N        PERCENT      R
============================================
     0     1          300 (          50)
           2          200 (          33)
           3          100 (          17)
       TOTAL          600 (         100)
-----------------------------------
    10     1          417 (          49)
           2           88 (          10)
           3          345 (          41)
       TOTAL          850 (         142)        128
-----------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
    50     1          827 (          45)
           2          173 (           9)
           3          849 (          46)
       TOTAL         1850 (         308)        239
-----------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       100

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
       0
```

```
 100      1          1365 (         44)
          2           287 (          9)
          3          1448 (         47)
        TOTAL        3100 (        517)      382
--------------------------------
DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1 .  YES
  7    QUIT PROGRAM
 :
       1
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       350
DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
 350      1          4056 (         43)
          2           854 (          9)
          3          4440 (         47)
        TOTAL        9350 (       1558)      1098
--------------------------------
DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       1
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
       400

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
       0
 400      1          4594 (         43)
          2           967 (          9)
          3          5039 (         48)
        TOTAL       10600 (       1767)      1241
--------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       0
DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0    NO
  1    YES
 :
       1
```

```
PERCENTAGES ARE IN STEADY STATE
   999      1         4594 (        43)
            2          967 (         9)
            3         5039 (        48)
       TOTAL         10600 (       100)        1241
--------------------------------

ARE YOU THROUGH?
  0    NO
  1    YES
 :
       1
```

APPENDIX E

COMPUTER PRINTOUTS OF EXAMPLES FOR OPTION 5

1. Example (a)

```
       START

DO YOU WISH TO ENTER DATA?
  0    NC
  1    YES
 :
       1

ENTER THE NUMBER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       1

ENTER N(INITIAL STOCK VECTOR)
 :
       129 74 28 11
ENTER THE PROMOTION RATE VECTOR.
THIS VECTOR SHOULD INCLUDE THE PROMOTION RATES FOR THE FIRST
 3 CLASSES.
 :
       .102 .046 .033
ENTER THE WASTAGE RATE VECTOR.
THE 4TH VALUE SHOULD INCLUDE THE TOTAL RATE FOR  THAT
STATE DUE TO EITHER PROMOTION OR WASTAGE.
 :
       .17 .124 .1 .098
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       5
ENTER RPROP (RECRUITMENT PROPORTION VECTOR)
 :
       1 0 0 0
ENTER MULTIPLICATIVE FACTOR
 :
       1.01

ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1
```

```
WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
        1

    P MATRIX
 0.728  0.102  0      0
 0      0.83   0.046  0
 0      0      0.867  0.033
 0      0      0      0.902
    N VECTOR
129  74  28  11
OPTION =5
FACT   =1.01
    RECRUITMENT PROPORTION VECTOR
1 0 0 0

WOULD YOU LIKE TC CHANGE ANY OF THE DATA?
  0     NC
  1     YES
  7     QUIT PROGRAM
 :
        0

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
        10

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
        0
     T                 N          PERCENT       R
=========================================
      0      1           129  (        53)
             2            74  (        31)
             3            28  (        12)
             4            11  (         5)
         TOTAL           242  (       100)
--------------------------------
     10      1           148  (        55)
             2            82  (        31)
             3            27  (        10)
             4            10  (         4)
         TOTAL           268  (       111)          42
--------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0     NC
  1     YES
  7     QUIT PRCGRAM
 :
        1

ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
        50

DO YOU WISH TO SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
        0
```

```
     50      1           222  (          55)
             2           125  (          31)
             3            40  (          10)
             4            12  (           3)
         TOTAL           400  (         165)        62
------------------------------------------------

DO YOU WISH TC SEE ANY OTHER YEARS?
  0     NC
  1     YES
  7     QUIT PRCGRAM
 :
        1

ENTER THE NUMEER OF THE YEAR YOU WISH TO SEE
 :
        100

DO YOU WISH TC SEE THE INTERVENING YEARS?
  0   NO
  1   YES
 :
        0
    100      1           362  (          55)
             2           206  (          31)
             3            66  (          10)
             4            20  (           3)
         TOTAL           654  (         270)        101
------------------------------------------------

DO YOU WISH TO SEE ANY OTHER YEARS?
  0     NC
  1     YES
  7     QUIT PRCGRAM
 :
        0

DO YOU WISH TO SEE THE STEADY STATE VECTOR?
  0     NC
  1     YES
 :
        1

PERCENTAGES ARE IN STEADY STATE
    999      1           362  (          55)
             2           206  (          31)
             3            66  (          10)
             4            20  (           3)
         TOTAL           654  (         100)        101
------------------------------------------------

ARE YOU THROUGH?
  0     NO
  1     YES
 :
        1
```

2. <u>Example</u> (<u>b</u>)

```
        START

DO YOU WISH TC ENTER DATA?
  0     NC
  1     YES
 :
        1
```

```
ENTER THE NUMBER OF THE MODEL TYPE
  1    MARKOV HIERARCHICAL
  2    MARKOV LENGTH OF SERVICE
  3    MARKOV GENERAL
  4    VACANCY
 :
       3

ENTER N(INITIAL STOCK VECTOR)
 :
       300 200 100
ENTER P (TRANSITION MATRIX) BY ROWS
ENTER 1TH ROW
 :
       .6 .15 .05
ENTER 2TH ROW
 :
       .75 .2 0
ENTER 3TH ROW
 :
       .05 0 .9
ENTER THE NUMBER OF THE RECRUIT TYPE
  1    FIXED RECRUIT VECTOR
  2    ADDITIVE (RECRUIT SIZE)
  3    MULTIPLICATIVE (RECRUIT SIZE)
  4    ADDITIVE (SYSTEM SIZE)
  5    MULTIPLICATIVE (SYSTEM SIZE)
 :
       5
ENTER RPROP (RECRUITMENT PROPORTION VECTOR)
 :
       .7 .07 .23
ENTER MULTIPLICATIVE FACTOR
 :
       1.01

ENTER THE PERCENT CODE
  0   NO GRADE PERCENTAGES
  1   GRADE SIZE AS PERCENT OF TOTAL GRADE SIZE
  2   GRADE SIZE AS PERCENT OF ORIGINAL GRADE SIZE
  7   QUIT PROGRAM
 :
       1

WOULD YOU LIKE TO SEE THE ENTERED DATA?
  0     NO
  1     YES
 :
       1

   P MATRIX
 0.6  0.15 0.05
 0.75 0.2  0
 0.05 0    0.9
   N VECTOR
300 200 100
OPTION =5
FACT   =1.01
   RECRUITMENT PROPORTION VECTOR
0.7 0.07 0.23

WOULD YOU LIKE TO CHANGE ANY OF THE DATA?
  0    NO
  1    YES
  7    QUIT PROGRAM
 :
       0
```

```
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      10
DO YOU WISH TC SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
    T                N          PERCENT      R
=========================================
      0      1            300 (          50)
             2            200 (          33)
             3            100 (          17)
         TOTAL            600 (         100)
-------------------------------------
     10      1            316 (          48)
             2             67 (          10)
             3            277 (          42)
         TOTAL            661 (         110)           87
-------------------------------------
DO YOU WISH TO SEE ANY OTHER YEARS?
  0   NC
  1   YES
  7   QUIT PRCGRAM
 :
      1
ENTER THE NUMBER OF THE YEAR YOU WISH TO SEE
 :
      50
DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
     50      1            435 (          44)
             2             91 (           9)
             3            458 (          46)
         TOTAL            985 (         164)          123
-------------------------------------
DO YOU WISH TC SEE ANY OTHER YEARS?
  0   NC
  1   YES
  7   QUIT PROGRAM
 :
      1
ENTER THE NUMEER CF THE YEAR YOU WISH TO SEE
 :
      100
DO YOU WISH TO SEE THE INTERVENING YEARS?
  0  NO
  1  YES
 :
      0
    100      1            717 (          44)
             2            150 (           9)
             3            754 (          47)
         TOTAL           1621 (         270)          203
-------------------------------------
```

```
DO YOU WISH TO SEE ANY OTHER YEARS?
  0    NC
  1    YES
  7    QUIT PROGRAM
 :
       0

DO YOU WISH TC SEE THE STEADY STATE VECTOR?
  0    NC
  1    YES
 :
       1

PERCENTAGES ARE IN STEADY STATE
   999     1         717 (        44)
           2         150 (         9)
           3         754 (        47)
       TOTAL        1621 (       100)       203
--------------------------------------

ARE YOU THROUGH?
  0    NC
  1    YES
 :
       1
```

APPENDIX F

APL COMPUTER PROGRAM FOR STEADY STATE DISTRIBUTION FUNCTION

This is the APL function that is used to find the steady state distribution vector for all options.

```
    ∇ STDST;Q;IDENT;RV;W;NSS;NHAT1;NHAT2;FACTI;IQ;ZERO
    N←N
    RV←(1,K)ρRPROP
    IDENT←(K,K)ρ(1,Kρ0)
    W←(1,K)ρ1-(+/[2] P)
    Q←P+((⍉W)+.×RV)
    →(TYPE≡ 1 2 3 4 5)/FIX,ADDREC,MULTREC,ADDSYS,MULTSYS
   FIX:N←R+.×⌹(IDENT-P)
    NSS←N÷(+/N)
    N←0 ROUND N
    →END
   ADDREC:NHAT1←RPROP+.×⌹(IDENT-P)
    NSS←NHAT1÷(+/NHAT1)
    →END
   MULTREC:FACTI←FACT×IDENT
    NHAT2←R+.×⌹(FACTI-P)
    NSS←NHAT2÷(+/NHAT2)
    →END
   ADDSYS:IQ←IDENT-Q
    IQ[;1]←Kρ1
    ZERO←1,(K-1)ρ0
    NSS←ZERO+.×⌹IQ
    →END
   MULTSYS:FACTI←FACT×IDENT
    NSS←(FACT-1)×RPROP+.×⌹(FACTI-Q)
   END:TCOUNT←999
    PERCENT←⌊(0.5+100×NSS)
    TOTPERCENT←100
    OUTPUT
    →0
```

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