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MATHEMATICS AND STATISTICS..  W RISING ET AL. AUG 85
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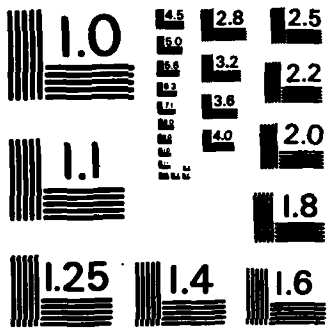
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AFOSR-TR- 85-1048

# A Birth and Death Process Approximation For the Slotted ALOHA Algorithm

## INTRODUCTION

Many authors have concerned themselves with the bistable behavior of the finite-user slotted ALOHA protocol under heaving loading. Recently Neslon used a catastrophe-theoretic approach to demonstrate that under a fluctuating load the protocol suffers hysteresis as well as bistability. He uses results from catastrophe theory to give a possible improved control algorithm.

Central to Nelson's approach is a diffusion approximation of the queue of backlogged users. This approximation has the advantage of yielding a continuing probability density for the process, thus allowing the use of (stochastic) catastrophe throry. Unfortunately, as will be seen later, the approximation requires difficult numerical integration and yields no closed form solution.

It is being proposed here that the process should remain discrete, and that it can be approximated reasonably well as a birth-death process. This allows rapid computation of the approximate stationary distribution.

## THE SLOTTED ALOHA MODEL

Assume there are N users, each of whom wishes to send packets across the communications channel at various times with equal probability. Time is divided into equal slots, and all packets are no bigger than one time slot. Note that if two or more users attempt to use the channel at once,



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their packets collide (i.e. become garbled), and no packet is transmitted successfully. These packets must still be transmitted at some future time; hence users of previously collided packets are referred to as backlogged. Those users who are not backlogged are called idle. Using these labels the ALOHA protocol can be defined. At the beginning of each slot each idle user receives a packet to transmit with probability $p_0$; these newly arrived packets (arrivals) are immediately transmitted; each backlogged user transmits its packet with probability $p_1$; In case of a successful transmission, the user reverts to being idle. Note that the users are independent of one another, and their transimission attempts are independent across time. It should also be noted that by definition a backlogged user has only one packet which it must transmit; buffering is not allowed.

This model can readily be modelled by a Markov chain $X_t$ on $0,1,\dots,N$ in which the states of the process represent the number of backlogged users. The transmission probabilities $P_{ij} \overset{\Delta}{=} \Pr\{X_{t+1} = j | X_t = i\}$ are given below:

| | | |
|---|---|---|
| $P_{00} = (1 - p_0)^N$ | : | empty queue, no arrivals |
| $P_{01} = 0$ | : | empty queue, if one packet arrives it is sent successfully |
| $P_{0k} = \binom{N}{k} p_0^k (1-p_0)^{N-k}$ | $2 \leq k \leq N$: | empty queue, collisions |
| $P_{n,n-k} = 0$ | $k \geq 2$: | no more than one packet can be sent at a time |
| $P_{n,n-1} = (1-p_0)^{N-n} n\, p_1 (1-p_1)^{n-1}$ | : | no arrivals, one line transmission |
| $P_{n,n} = (1-p_0)^{N-n}(1-np_1(1-p_1)^{n-1})$ | : | either no arrivals and no successful line transmissions or |
| $+(N-n)p_0(1-p_0)^{N-n-1}(1-(1-p_1)^n)$ | : | one arrival which is successfully transmitted |
| $P_{n,n+1} = (N-n)p_0(1-p_0)^{N-n-1}(1-(1-p_1)^n)$ | : | one arrival which is blocked by line transmissions |
| $P_{n,n+k} = \binom{N-n}{k} p_0^k (1-p_0)^{N-n-k}$ | : | more than one arrival. |

These transition probabilities make for a fairly complitated Markov chain. The only "nice" property is that all the entries below the subdiagonal are zero (by condition (4).

## NELSON'S APPROXIMATION

It is common to assess the stability or instability of an infinite Markov chain by looking at the drift and variance of each state (Lamperti 1960). These are defined as the expected jump and the expected squared jump:

$$d_i \overset{\Delta}{=} E(X_{t+1} - X_t | X_t = i) = \sum_j (j - i) P_{ij}$$
$$v_i \overset{\Delta}{=} E((X_{t+1} - X_t)^2 | X_t = i) = \sum_j (j - i)^2 P_{ij}$$

for finite chains, such as the one under consideration, stability is of no consequence, but the qualitative behavior of the chain can be assessed by finding the places at which the drift changes sign.

As Nelson states, the drift and variance for the N-user slotted ALOHA are given by

$$d_i = (N-i)p_0 - (S_1(i) + S_0(i))$$
$$v_i = (N-i)p_0((N-i)p_0 + (1-p_0)) + S_1(i) - S_0(i)$$

where

$$S_0(i) = (N-i)p_0(1-p_0)^{(N-i-1)}(1-p_1)^i$$
$$S_1(i) = ip_1(1-p_1)^{(i-1)}(1-p_0)^{(N-i)} \quad .$$

He approximates the behavior of the queue by defining a diffusion process on (0,N) by taking the same functional form for the drift and variance, but allowing i to be real valued (instead of integer valued). To make the distinction clear, the real valued functions will be denoted as $\mu(x)$ and $\sigma^2(x)$. He then defines the c-drift function as $\frac{2\mu(x)}{\sigma^2(x)}$ and the c-potential function as $V(x) \overset{\Delta}{=} \int^x \frac{2\mu(s)}{\sigma^2(s)} ds$. (These are are denoted as "c-drift" and "c-potential" to make the connection of the drift and potential functions used in catastrophe theory while keeping the stochastic nature of the model distinct from its deterministic analog). It should be boted that $v(x) = -\ell n\ s(x)$ where $s(x)$ is the scale density of the diffusion process.

Using c-drift and c-potential, Nelson shows that the slotted ALOHA can be viewed as a stochastic cusp catastrophe. He also approximates the stationary distribution of the discrete valued Markov chain with the stationary density of the related diffusion:

$$\chi(x) = ce^{-(v(x) + \ell n \sigma^2(x))}$$

where $c$ is the normal constant.

This approach is very nice in its theoretical simplicity, since the catastrophe theoretic approach is interested only in the <u>shape</u> (i.e. location of the maxima, minima, and inflection points) of the stationary density $\chi(x)$. These can all be found by differentiating the c-drift function. Unfortunately finding error bounds for the final approximation of the stationary distribution by $\chi(x)$ appears to be difficult. This is the case because $v(x)$ has no closed form and can be found only via numerical integration. It is for this reason that the following material is being proposed.

## BIRTH-DEATH APPROXIMATION

In order to ease the computations while keeping the flavor of Nelson's work, it is proposed here that the approximation of the slotted ALOHA should be kept in a discrete state-space. To do this the chain is approximated with a birth-death chain which can be considered in continuous or discrete time. The transition probabilities are defined so that the drift and variance of each interior state equal those of the original chain. To do this, define:

$$\lambda_i = \frac{1}{2}(v_i + d_i)$$
$$\mu_i = \frac{1}{2}(v_i - d_i)$$
$$\lambda_0 = d_0 \qquad \mu_2 = 0$$
$$\lambda_N = 0 \qquad \mu_N = d_N$$

where the $\lambda$'s are the entries on the superdiagonal and the $\mu$'s are the entries on the subdiagonal of the transition (generator) matrix of the discrete (continuous) time birth-death chain. Note that with these definitions the $d_0$ indeed match the drift and variance:

$$E(X_{t+1} - X_t | X_t = i) = \lambda_i - \mu_i = d_i \qquad \text{(discrete time)}$$
$$E((X_{t+1} - X_t)^2 | X_t = i) = \lambda_i + \mu_i = v_i$$

(The continuous time calculations are the same)

Unfortunately, only the drift can be matched at the boundaries unless the original chain already has only next-neighbor transitions at 0 and N.

Despite this drawback this approximation should yield good results for "slowly moving" processes, i.e. processes where large jumps are rare. The heuristics reason for this is that the first two moments of the approximating process are the same as those in the original chain at each interior state. If large jumps are rare, then the higher order moments on the two chains will also be close.

By far the biggest advantage of the birth-death approximation is the ease with which the approximate stationary distribution can be calculated. For a b - d chain of birth parameters $\lambda_i$ and death parameters $\mu_i$, the stationary distribution $\pi$ is given by

$$\pi_i = \frac{p_i}{\Sigma p_i}$$

where

$$p_0 \triangleq 1 \quad \text{and} \quad p_i \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i} .$$

Note that the simple form of $\pi$ also allows rapid calculation of the approximate expected queue length $\Sigma i\pi_i$ and the approximate expected length of a busy cycle $\frac{1}{\pi_0}$. This type of approximation also appears more promising in the area of error-bounds than a diffusion approximation.

## APPLICATION TO SLOTTED-ALOHA

As was mentioned earlier, the birth-death approximation works well for "slowly moving" processes. The slotted-ALOHA is such a process because most of the probability in the transition matrix is concentrated on the main-, super- and sub-diagonals, i.e. is almost a birth-death process.

It should be noted that the slotted-ALOHA allows transitions from state 0 to all states except state 1, which is a rather unfortunate situation since the approximation allows transitions only to state 1. Although this does not pose a major problem (because $P_{00}$'s large), it does cause the approximate stationary probability of state 1 to be too large.

It is not too difficult to calculate the stationary distribution of an "altered" approximation which allows the transition probabilities from one state, in this case state 0, to be equal to the actual transition probabilities, while approximating the rest of the chain with a b - d process. This, however, makes the form of the stationary distribution much more complicated and thus destroys the simplicity of the approximation.

Tables 1, 2, and 3 present examples of computations, where

APPX1 - simple birth-death approximation

APPX2 - altered birth-death approximation

EXACT - exact stationary distribution (calculated via $P^n$ $n \to \infty$)

SHORT - exact stationary distribution (calculated via $\pi R$ $P^{P^n}$ $n \to \infty$ see below)

As these computations show, the approximation is good although the relative error is fairly large for those states whose exact stationary distribution is less than $10^{-5}$. The absolute errors are small. It is important to note that the approximate distributions have their relative maxima and minima at the same states.

## SOME SIDE NOTES IN THE COMPUTATIONS

It is, of course, possible to calculate the exact stationary distribution of a Markov chain by repeatedly squaring the transition matrix. This is tantamount to grinding out $P^n$ $n \to \infty$. This type of calculational bullying has several drawbacks. First of all, it doesn't lend any insight into the possible changes in performance which would result from small changes in the parameters. It also is subject to roundoff errors since the number of necessary computations is large, as is the range of values. Lastly, the "brute-force" method is wasteful of computer time when applied to large chains. The computations of the exact disk given here required 27 CPU seconds, whereas the approximation required less than three.

As is noted, another use of the b-d approximation is in calculating the exact distribution. This can be done by calculating successive values of $\pi P^n$. Although this is also subject to roundoff error, it requires far less time thatn the brute-force method - the calculations here (SHORT) required 20 CPU sec.

## CONCLUSION

Taking Nelson's lead, it appears that approximations of Markov chains are possible and useful. It also appers that a birth-death approximation though less elegant, is easier to use and just as accurate as a diffusion approximation. The direction of the research at this time is two fold, namely error bounds for the birth-death approximation are being computed with the end of perturbation theory, and the shape preserving properties of the approximation are being investigated.

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TABLE 1

CALC'F0=.003 F1=.06'

| STATE | AFFX1 | AFFX2 | LOG AFFX1 | LOG AFFX2 |
|---|---|---|---|---|
| 0 | 0.000029 | 0.000055 | ¯4.543581 | ¯4.259922 |
| 1 | 0.000039 | 0.000023 | ¯4.403736 | ¯4.640807 |
| 2 | 0.000041 | 0.00004 | ¯4.384059 | ¯4.399743 |
| 3 | 0.000037 | 0.000037 | ¯4.435848 | ¯4.437015 |
| 4 | 0.000029 | 0.000029 | ¯4.534418 | ¯4.5345 |
| 5 | 0.000022 | 0.000022 | ¯4.665206 | ¯4.665214 |
| 6 | 0.000015 | 0.000015 | ¯4.818784 | ¯4.818788 |
| 7 | 0.00001 | 0.00001 | ¯4.988653 | ¯4.988657 |
| 8 | 0.000007 | 0.000007 | ¯5.170116 | ¯5.17012 |
| 9 | 0.000004 | 0.000004 | ¯5.359656 | ¯5.35966 |
| 10 | 0.000003 | 0.000003 | ¯5.554563 | ¯5.554566 |
| 11 | 0.000002 | 0.000002 | ¯5.752698 | ¯5.752701 |
| 12 | 0.000001 | 0.000001 | ¯5.952341 | ¯5.952344 |
| 13 | 0.000001 | 0.000001 | ¯6.152083 | ¯6.152087 |
| 14 | 0. | 0. | ¯6.350758 | ¯6.350761 |
| 15 | 0. | 0. | ¯6.547382 | ¯6.547386 |
| 16 | 0. | 0. | ¯6.741122 | ¯6.741125 |
| 17 | 0. | 0. | ¯6.931261 | ¯6.931265 |
| 18 | 0. | 0. | ¯7.117181 | ¯7.117184 |
| 19 | 0. | 0. | ¯7.298341 | ¯7.298345 |
| 20 | 0. | 0. | ¯7.474269 | ¯7.474273 |
| 21 | 0. | 0. | ¯7.644547 | ¯7.64455 |
| 22 | 0. | 0. | ¯7.808803 | ¯7.808806 |
| 23 | 0. | 0. | ¯7.966707 | ¯7.966711 |
| 24 | 0. | 0. | ¯8.117964 | ¯8.117968 |
| 25 | 0. | 0. | ¯8.262309 | ¯8.262313 |
| 26 | 0. | 0. | ¯8.399504 | ¯8.399508 |
| 27 | 0. | 0. | ¯8.529333 | ¯8.529336 |
| 28 | 0. | 0. | ¯8.651602 | ¯8.651606 |
| 29 | 0. | 0. | ¯8.766136 | ¯8.76614 |
| 30 | 0. | 0. | ¯8.872776 | ¯8.87278 |
| 31 | 0. | 0. | ¯8.97138 | ¯8.971383 |
| 32 | 0. | 0. | ¯9.061817 | ¯9.06182 |
| 33 | 0. | 0. | ¯9.143971 | ¯9.143975 |
| 34 | 0. | 0. | ¯9.217739 | ¯9.217742 |
| 35 | 0. | 0. | ¯9.283026 | ¯9.283029 |
| 36 | 0. | 0. | ¯9.339749 | ¯9.339753 |
| 37 | 0. | 0. | ¯9.387836 | ¯9.38784 |
| 38 | 0. | 0. | ¯9.427223 | ¯9.427226 |
| 39 | 0. | 0. | ¯9.457854 | ¯9.457857 |
| 40 | 0. | 0. | ¯9.479682 | ¯9.479686 |
| 41 | 0. | 0. | ¯9.492671 | ¯9.492675 |
| 42 | 0. | 0. | ¯9.496789 | ¯9.496793 |
| 43 | 0. | 0. | ¯9.492014 | ¯9.492018 |
| 44 | 0. | 0. | ¯9.478332 | ¯9.478336 |
| 45 | 0. | 0. | ¯9.455735 | ¯9.455739 |
| 46 | 0. | 0. | ¯9.424225 | ¯9.424229 |

TABLE 1. (con't)

| | | | | |
|---|---|---|---|---|
| 47 | 0. | 0. | ¯9.383809 | ¯9.383813 |
| 48 | 0. | 0. | ¯9.334503 | ¯9.334507 |
| 49 | 0. | 0. | ¯9.276331 | ¯9.276335 |
| 50 | 0. | 0. | ¯9.209324 | ¯9.209328 |
| 51 | 0. | 0. | ¯9.133522 | ¯9.133526 |
| 52 | 0. | 0. | ¯9.048972 | ¯9.048976 |
| 53 | 0. | 0. | ¯8.955731 | ¯8.955734 |
| 54 | 0. | 0. | ¯8.853863 | ¯8.853866 |
| 55 | 0. | 0. | ¯8.743442 | ¯8.743445 |
| 56 | 0. | 0. | ¯8.624552 | ¯8.624555 |
| 57 | 0. | 0. | ¯8.497286 | ¯8.49729 |
| 58 | 0. | 0. | ¯8.361749 | ¯8.361752 |
| 59 | 0. | 0. | ¯8.218055 | ¯8.218058 |
| 60 | 0. | 0. | ¯8.066331 | ¯8.066335 |
| 61 | 0. | 0. | ¯7.906717 | ¯7.906721 |
| 62 | 0. | 0. | ¯7.739365 | ¯7.739369 |
| 63 | 0. | 0. | ¯7.564443 | ¯7.564447 |
| 64 | 0. | 0. | ¯7.382133 | ¯7.382137 |
| 65 | 0. | 0. | ¯7.192635 | ¯7.192639 |
| 66 | 0. | 0. | ¯6.996165 | ¯6.996169 |
| 67 | 0. | 0. | ¯6.792961 | ¯6.792964 |
| 68 | 0. | 0. | ¯6.58328 | ¯6.583284 |
| 69 | 0. | 0. | ¯6.367405 | ¯6.367408 |
| 70 | 0.000001 | 0.000001 | ¯6.145642 | ¯6.145645 |
| 71 | 0.000001 | 0.000001 | ¯5.918327 | ¯5.91833 |
| 72 | 0.000002 | 0.000002 | ¯5.685826 | ¯5.68583 |
| 73 | 0.000004 | 0.000004 | ¯5.448541 | ¯5.448544 |
| 74 | 0.000006 | 0.000006 | ¯5.20691 | ¯5.206914 |
| 75 | 0.000011 | 0.000011 | ¯4.961417 | ¯4.961421 |
| 76 | 0.000019 | 0.000019 | ¯4.712592 | ¯4.712595 |
| 77 | 0.000035 | 0.000035 | ¯4.461018 | ¯4.461021 |
| 78 | 0.000062 | 0.000062 | ¯4.207342 | ¯4.207345 |
| 79 | 0.000112 | 0.000112 | ¯3.952278 | ¯3.952282 |
| 80 | 0.000201 | 0.000201 | ¯3.696622 | ¯3.696626 |
| 81 | 0.000362 | 0.000362 | ¯3.441258 | ¯3.441262 |
| 82 | 0.00065 | 0.00065 | ¯3.187177 | ¯3.187181 |
| 83 | 0.00116 | 0.00116 | ¯2.935493 | ¯2.935496 |
| 84 | 0.002054 | 0.002054 | ¯2.68746 | ¯2.687464 |
| 85 | 0.003593 | 0.003593 | ¯2.444506 | ¯2.444509 |
| 86 | 0.006191 | 0.006191 | ¯2.208256 | ¯2.20826 |
| 87 | 0.010457 | 0.010457 | ¯1.980583 | ¯1.980587 |
| 88 | 0.017232 | 0.017232 | ¯1.763654 | ¯1.763658 |
| 89 | 0.027542 | 0.027542 | ¯1.560003 | ¯1.560007 |
| 90 | 0.042401 | 0.0424 | ¯1.372629 | ¯1.372632 |
| 91 | 0.062356 | 0.062356 | ¯1.205119 | ¯1.205123 |
| 92 | 0.086728 | 0.086727 | ¯1.061841 | ¯1.061845 |
| 93 | 0.112665 | 0.112664 | ¯0.948211 | ¯0.948215 |
| 94 | 0.134552 | 0.134551 | ¯0.87111 | ¯0.871114 |
| 95 | 0.144691 | 0.14469 | ¯0.839559 | ¯0.839563 |
| 96 | 0.136181 | 0.13618 | ¯0.865884 | ¯0.865888 |
| 97 | 0.107662 | 0.107661 | ¯0.967936 | ¯0.96794 |
| 98 | 0.067004 | 0.067004 | ¯1.173899 | ¯1.173902 |
| 99 | 0.029167 | 0.029166 | ¯1.535114 | ¯1.535118 |
| 100 | 0.006657 | 0.006656 | ¯2.176752 | ¯2.176756 |

TABLE 2.

USING THE APPROXIMATION TO START.......
AFTER LOOPING 290 TIMES.......

| STATE | SHORT | LOG SHORT |
|---|---|---|
| 0 | 0.000044 | ¯4.360016 |
| 1 | 0.000036 | ¯4.444273 |
| 2 | 0.00004 | ¯4.396149 |
| 3 | 0.000033 | ¯4.484199 |
| 4 | 0.000026 | ¯4.582656 |
| 5 | 0.000019 | ¯4.712498 |
| 6 | 0.000014 | ¯4.857873 |
| 7 | 0.00001 | ¯5.016323 |
| 8 | 0.000007 | ¯5.183849 |
| 9 | 0.000004 | ¯5.357937 |
| 10 | 0.000003 | ¯5.536503 |
| 11 | 0.000002 | ¯5.717892 |
| 12 | 0.000001 | ¯5.90074 |
| 13 | 0.000001 | ¯6.083908 |
| 14 | 0.000001 | ¯6.266431 |
| 15 | 0. | ¯6.447484 |
| 16 | 0. | ¯6.626352 |
| 17 | 0. | ¯6.802414 |
| 18 | 0. | ¯6.975129 |
| 19 | 0. | ¯7.144026 |
| 20 | 0. | ¯7.308692 |
| 21 | 0. | ¯7.468773 |
| 22 | 0. | ¯7.623961 |
| 23 | 0. | ¯7.773998 |
| 24 | 0. | ¯7.918664 |
| 25 | 0. | ¯8.05778 |
| 26 | 0. | ¯8.191196 |
| 27 | 0. | ¯8.318791 |
| 28 | 0. | ¯8.440462 |
| 29 | 0. | ¯8.556116 |
| 30 | 0. | ¯8.665664 |
| 31 | 0. | ¯8.769004 |
| 32 | 0. | ¯8.866019 |
| 33 | 0. | ¯8.956558 |
| 34 | 0. | ¯9.040433 |
| 35 | 0. | ¯9.117409 |
| 36 | 0. | ¯9.187199 |
| 37 | 0. | ¯9.249468 |
| 38 | 0. | ¯9.303837 |
| 39 | 0. | ¯9.349894 |
| 40 | 0. | ¯9.387217 |
| 41 | 0. | ¯9.415392 |
| 42 | 0. | ¯9.434043 |
| 43 | 0. | ¯9.442855 |
| 44 | 0. | ¯9.441594 |
| 45 | 0. | ¯9.430119 |
| 46 | 0. | ¯9.408383 |

TABLE 2 (con't)

| | | |
|---|---|---|
| 47 | 0. | ¯9.376424 |
| 48 | 0. | ¯9.33435 |
| 49 | 0. | ¯9.282317 |
| 50 | 0. | ¯9.22051 |
| 51 | 0. | ¯9.149122 |
| 52 | 0. | ¯9.068343 |
| 53 | 0. | ¯8.97835 |
| 54 | 0. | ¯8.879304 |
| 55 | 0. | ¯8.771348 |
| 56 | 0. | ¯8.654615 |
| 57 | 0. | ¯8.529226 |
| 58 | 0. | ¯8.3953 |
| 59 | 0. | ¯8.252957 |
| 60 | 0. | ¯8.102321 |
| 61 | 0. | ¯7.943527 |
| 62 | 0. | ¯7.776723 |
| 63 | 0. | ¯7.602072 |
| 64 | 0. | ¯7.419753 |
| 65 | 0. | ¯7.229969 |
| 66 | 0. | ¯7.032939 |
| 67 | 0. | ¯6.82891 |
| 68 | 0. | ¯6.61815 |
| 69 | 0. | ¯6.400957 |
| 70 | 0.000001 | ¯6.177656 |
| 71 | 0.000001 | ¯5.948604 |
| 72 | 0.000002 | ¯5.714193 |
| 73 | 0.000003 | ¯5.474849 |
| 74 | 0.000006 | ¯5.231043 |
| 75 | 0.00001 | ¯4.983287 |
| 76 | 0.000019 | ¯4.732144 |
| 77 | 0.000033 | ¯4.478234 |
| 78 | 0.00006 | ¯4.222236 |
| 79 | 0.000108 | ¯3.9649 |
| 80 | 0.000196 | ¯3.707055 |
| 81 | 0.000355 | ¯3.449621 |
| 82 | 0.00064 | ¯3.19362 |
| 83 | 0.001148 | ¯2.940194 |
| 84 | 0.002039 | ¯2.690626 |
| 85 | 0.003578 | ¯2.446364 |
| 86 | 0.006179 | ¯2.209051 |
| 87 | 0.010458 | ¯1.98057 |
| 88 | 0.017255 | ¯1.763089 |
| 89 | 0.027597 | ¯1.559138 |
| 90 | 0.042492 | ¯1.371695 |
| 91 | 0.062471 | ¯1.20432 |
| 92 | 0.086829 | ¯1.061334 |
| 93 | 0.112696 | ¯0.948091 |
| 94 | 0.134465 | ¯0.871391 |
| 95 | 0.144493 | ¯0.840153 |
| 96 | 0.135964 | ¯0.866575 |
| 97 | 0.107559 | ¯0.968352 |
| 98 | 0.067068 | ¯1.173484 |
| 99 | 0.029303 | ¯1.533092 |
| 100 | 0.006728 | ¯2.172089 |

TABLE 3

AFTER STRAIGHTFORWARD BUT TEDIOUS CALCULATIONS,
(RAISING P TO THE 1.0737E9TH POWER).....

| STATE | EXACT | LOG EXACT |
|---|---|---|
| 0 | 0.000018 | ¯4.746165 |
| 1 | 0.000015 | ¯4.830447 |
| 2 | 0.000017 | ¯4.782327 |
| 3 | 0.000013 | ¯4.870391 |
| 4 | 0.000011 | ¯4.968858 |
| 5 | 0.000008 | ¯5.09871 |
| 6 | 0.000006 | ¯5.244095 |
| 7 | 0.000004 | ¯5.402554 |
| 8 | 0.000003 | ¯5.570088 |
| 9 | 0.000002 | ¯5.74418 |
| 10 | 0.000001 | ¯5.922746 |
| 11 | 0.000001 | ¯6.104126 |
| 12 | 0.000001 | ¯6.286952 |
| 13 | 0. | ¯6.47008 |
| 14 | 0. | ¯6.652537 |
| 15 | 0. | ¯6.833483 |
| 16 | 0. | ¯7.012188 |
| 17 | 0. | ¯7.18801 |
| 18 | 0. | ¯7.360379 |
| 19 | 0. | ¯7.528786 |
| 20 | 0. | ¯7.692772 |
| 21 | 0. | ¯7.851922 |
| 22 | 0. | ¯8.00586 |
| 23 | 0. | ¯8.154241 |
| 24 | 0. | ¯8.29675 |
| 25 | 0. | ¯8.433095 |
| 26 | 0. | ¯8.563009 |
| 27 | 0. | ¯8.686246 |
| 28 | 0. | ¯8.802574 |
| 29 | 0. | ¯8.911784 |
| 30 | 0. | ¯9.013677 |
| 31 | 0. | ¯9.108071 |
| 32 | 0. | ¯9.194797 |
| 33 | 0. | ¯9.273698 |
| 34 | 0. | ¯9.34463 |
| 35 | 0. | ¯9.407459 |
| 36 | 0. | ¯9.462062 |
| 37 | 0. | ¯9.508326 |
| 38 | 0. | ¯9.54615 |
| 39 | 0. | ¯9.57544 |
| 40 | 0. | ¯9.596114 |
| 41 | 0. | ¯9.608097 |
| 42 | 0. | ¯9.611324 |
| 43 | 0. | ¯9.605742 |
| 44 | 0. | ¯9.591302 |
| 45 | 0. | ¯9.567969 |
| 46 | 0. | ¯9.535715 |

TABLE 3 (con't)

| | | |
|---|---|---|
| 47 | 0. | ¯9.49452 |
| 48 | 0. | ¯9.444377 |
| 49 | 0. | ¯9.385285 |
| 50 | 0. | ¯9.317255 |
| 51 | 0. | ¯9.240307 |
| 52 | 0. | ¯9.154472 |
| 53 | 0. | ¯9.059791 |
| 54 | 0. | ¯8.956317 |
| 55 | 0. | ¯8.844112 |
| 56 | 0. | ¯8.723252 |
| 57 | 0. | ¯8.593824 |
| 58 | 0. | ¯8.455929 |
| 59 | 0. | ¯8.30968 |
| 60 | 0. | ¯8.155206 |
| 61 | 0. | ¯7.992648 |
| 62 | 0. | ¯7.822167 |
| 63 | 0. | ¯7.643936 |
| 64 | 0. | ¯7.458148 |
| 65 | 0. | ¯7.265017 |
| 66 | 0. | ¯7.064773 |
| 67 | 0. | ¯6.857671 |
| 68 | 0. | ¯6.643989 |
| 69 | 0. | ¯6.424029 |
| 70 | 0.000001 | ¯6.198121 |
| 71 | 0.000001 | ¯5.966627 |
| 72 | 0.000002 | ¯5.729939 |
| 73 | 0.000003 | ¯5.488486 |
| 74 | 0.000006 | ¯5.242736 |
| 75 | 0.00001 | ¯4.993203 |
| 76 | 0.000018 | ¯4.740447 |
| 77 | 0.000033 | ¯4.485083 |
| 78 | 0.000059 | ¯4.227789 |
| 79 | 0.000107 | ¯3.969309 |
| 80 | 0.000195 | ¯3.710466 |
| 81 | 0.000353 | ¯3.452174 |
| 82 | 0.000638 | ¯3.195448 |
| 83 | 0.001144 | ¯2.941423 |
| 84 | 0.002035 | ¯2.691373 |
| 85 | 0.003575 | ¯2.446737 |
| 86 | 0.006178 | ¯2.209148 |
| 87 | 0.01046 | ¯1.980478 |
| 88 | 0.017263 | ¯1.762886 |
| 89 | 0.027613 | ¯1.558887 |
| 90 | 0.042516 | ¯1.371449 |
| 91 | 0.062501 | ¯1.204116 |
| 92 | 0.086857 | ¯1.061197 |
| 93 | 0.112712 | ¯0.94803 |
| 94 | 0.134462 | ¯0.8714 |
| 95 | 0.144475 | ¯0.840208 |
| 96 | 0.135946 | ¯0.866635 |
| 97 | 0.107557 | ¯0.968359 |
| 98 | 0.067087 | ¯1.173359 |
| 99 | 0.029327 | ¯1.532737 |
| 100 | 0.006739 | ¯2.171384 |

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