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# THE MOTION OF ELLIPSOIDS IN A SECOND ORDER FLUID

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## ABSTRACT

The rigid body motion of an ellipsoid in a second order fluid (SOF) under the action of specified (time independent) external forces and torques have been obtained to first order in the Weissenberg number by inverting the resistance relations for the force and torque under specified rigid body motions. The reciprocal theorem of Lorentz was used to bypass the calculation of the O(W) velocity field. The results agree with known analytic solutions for a SOF with the secondary to primary normal stress ratio of -1/2.

The solution procedure was also tested by computing the torque on a translating prolate spheroid with aspect ratios ranging from slender bodies to near-spheres. One result is that for a SOF with zero secondary normal stress (Weissenberg fluid), previous asymptotic results for near-spheres were found to be accurate even at fairly large aspect ratios (e.g. 2).

New results for non-degenerate ellipsoids suggest that the orientation (as monitored by Euler angles) and trajectory of sedimenting, non-axisymmetric particles such as ellipsoids provide useful information on the rheology of the suspending fluid.

AMS (MOS) Subject Classifications: 76D05, 35Q10

Key Words: sedimentation, spheroids, ellipsoids, viscoelastic fluid, second-order fluid

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## SIGNIFICANCE AND EXPLANATION

Viscoelastic fluids are encountered in many manufacturing processes, especially in the form of solutions and melts of large macro-molecules (polymers). The characterization of the flow properties of these materials is an important step in the elucidation of the molecular structure and processibility of the material. The work presented here show how data from sedimentation experiments with non-axisymmetric particles e.g. ellipsoids, may be applied to determine such rheological properties (material flow properties) of the suspending viscoelastic fluid.

A-1

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

# THE MOTION OF ELLIPSOIDS IN A SECOND ORDER FLUID

Sangtae Kim

## 1. Introduction

It has been shown, both theoretically and experimentally, that particles suspended in a viscoelastic fluid behave differently than when suspended in a Newtonian fluid [1,2,3]. For example, an orthotropic particle translating in an unbounded Newtonian fluid does not experience a hydrodynamic torque. Consequently, such particles will settle in a Newtonian fluid without rotation. On the other hand, experiments have confirmed the rotation of needle-like particle settling in viscoelastic fluids, e.g. Separan solutions [1]. These observations are consistent with theoretical predictions of a torque on a slender particle translating in a second order fluid (SOF). This has lead earlier investigators to consider the feasibilty of these experiments as rheometric tools for dilute polymeric solutions.

Here, we examine the motion of an ellipsoidal particle in a viscoelastic fluid in the limit of small Weissenberg number, W. The problem is solved by a perturbation procedure in which the zero-th order solution is the Newtonian result and the first order solution includes the O(W) effect in a SOF. The method employed, classified as the "reciprocal theorem method" by Brunn [3], bypasses the solution of the O(W) velocity problem and requires knowledge of just the Newtonian velocity field.

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The solution of this problem is an extension of earlier works on shapes such as spheres and needles. However, the primary objective of this work is to determine whether motions of nonaxisymmetric particles can differ sufficiently from that of axisymmetric particles to the extent that such measurements could be usefully applied in rheological studies of the suspending fluid. To this end, the reader must keep in mind that the suspending fluid is modelled as an incompressible fluid with a pre-specified constitutive equation and that despite similarities in the techniques, the present work is not an effort on the modelling of the rheological behavior of suspensions.

The outline of the paper is as follows. The mathematical formulation of the problem is presented in Section 2. The Newtonian case is reviewed, including recently discovered [4] "singularity solutions" for the velocity field, in Section 3. Results which have been obtained for the non-Newtonian effects are described in Section 4, including the effect of non-axisymmetry.

## 2. Problem Formulation

In the limit of rheologically slow flows, the constitutive equation for the suspending fluid reduces to the n-th order Rivlin-Erickson fluid [5]. We shall confine our attention to the first deviations from Newtonian behavior by approximating the fluid as a second order fluid. The ultimate test of whether such a regime exists in real polymeric solutions can only be determined by comparison with experiments. The bulk stress is given by the SOF constitutive equation [5]

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \mu\underline{\underline{A}}_1 + (b_{11}-2b_2)\underline{\underline{A}}_1\cdot\underline{\underline{A}}_1 + b_2\underline{\underline{A}}_2,$$

where $\mu$ is the viscosity, $b_2$ and $b_{11}$ are material properties of the SOF and the $\underline{\underline{A}}_n$ are the Rivlin-Erickson tensors defined by

$$\underline{\underline{A}}_1 = \nabla\underline{v} + (\nabla\underline{v})^t, \qquad \underline{\underline{A}}_2 = D\underline{\underline{A}}_1/Dt + \underline{\underline{A}}_1\cdot(\nabla\underline{v})^t + (\nabla\underline{v})\cdot\underline{\underline{A}}_1.$$

The following dimensional analysis defines the Weissenberg number for this problem. If the stresses are divided by the viscous scale, $\mu U/\ell$, where U is the particle translational speed and $\ell$ is a characteristic particle dimension, then the constitutive equation becomes

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \underline{\underline{A}}_1 + W\underline{\underline{A}} = \underline{\underline{\tau}} + W\underline{\underline{A}},$$

where $\qquad \underline{\underline{A}} = (1+\alpha)\underline{\underline{A}}_1\cdot\underline{\underline{A}}_1 - \frac{1}{2}\underline{\underline{A}}_2,$

and $W = (-2b_2)U/(\mu\ell)$ is the Weissenberg number. The parameter $\alpha = b_{11}/(-2b_2)$. The definitions for W and $\alpha$ are motivated by the fact that in Couette flow, the SOF has the primary normal stress coefficient, $\Psi_1 = -2b_2$ and the secondary normal stress coefficient, $\Psi_2 = b_{11}$ [5]. Experimental results with polymeric systems and polymer kinetic theories suggest that $0 \leq -\alpha \leq 0.2$ [3,5].

From hereon, the $\underline{\underline{A}}_n$ are dimensionless and defined by

$$\underline{\underline{A}}_1 = \nabla\underline{u} + (\nabla\underline{u})^t, \qquad \underline{\underline{A}}_2 = D\underline{\underline{A}}_1/Dt + \underline{\underline{A}}_1\cdot(\nabla\underline{u})^t + (\nabla\underline{u})\cdot\underline{\underline{A}}_1.$$

Throughout this paper, we will use $\underline{v}$ to denote the (dimensional) velocity field and reserve $\underline{u}$ for the velocity scaled by U.

We assume that inertial effects are small, and also assume that the fluid is weakly elastic. More specifically, we require $Re \ll W \ll 1$. The field variables can then be expanded in the small parameter W:

$$\underline{u} = \underline{u}^{(0)} + W\underline{u}^{(1)} + \ldots \qquad p = p^{(0)} + Wp^{(1)} + \ldots$$

and substituted into the equations of change (momentum and mass conservation), resulting in the following hierarchy of perturbation problems:

$$-\nabla p^{(0)} + \nabla^2\underline{u}^{(0)} = \underline{0}, \qquad \nabla\cdot\underline{u}^{(0)} = 0,$$

$$-\nabla p^{(1)} + \nabla^2\underline{u}^{(1)} = -\nabla\cdot\underline{\underline{A}}\big|_{u=u^{(0)}}, \qquad \nabla\cdot\underline{u}^{(1)} = 0.$$

Thus at zero-th order, we have the Newtonian problem, which we shall review in the following section, while the first-order problem reduces to the forced Stokes equation (which will be considered in Section 4).

## 3. The Newtonian Solutions

Consider an ellipsoid with semiaxes of lengths a, b and c, with $a \geq b \geq c$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{3.1}$$

We choose $\ell = a$. The governing equations for the velocity, $\underline{v}$, and pressure, p, are the Stokes equations and the equation of continuity for incompressible flow,

$$-\nabla p + \mu\nabla^2\underline{v} = \underline{0}, \tag{3.2}$$

$$\nabla\cdot\underline{v} = 0. \tag{3.3}$$

with $\mu$ denoting the fluid viscosity.

Since the governing equations are linear in this case, we may consider the translational and rotational motions separately. Furthermore, the translational and rotational problems are completely decoupled in the sense that the fluid exerts no force on a rotating ellipsoid and no torque on a translating ellipsoid. The boundary conditions for the two problems are, in the Cartesian coordinate system fixed with respect to the particle as specified by equation (3.1):

Translating Ellipsoid

$\underline{v} = \underline{0}$ on the particle surface (3.4a,b)

$\underline{v} \to -\underline{U}$ as $|\underline{x}| \to \infty$.

Rotating Ellipsoid

$\underline{v} = \underline{0}$ on the surface (3.5a,b)

$\underline{v} \to -\underline{\omega}\times\underline{x}$ as $|\underline{x}| \to \infty$.

The force-translation and torque-rotation relations may be expressed as [6,7,8],

$$\underset{\sim}{F} = -\mu\, {}^t\underset{\approx}{K}\cdot\underset{\sim}{U} \qquad \text{and} \qquad \underset{\sim}{T} = -\mu\, {}^r\underset{\approx}{R}\cdot\underset{\sim}{\omega},$$

where the components of the resistance tensors ${}^t\underset{\approx}{K}$ and ${}^r\underset{\approx}{R}$ with respect to the particle-fixed axes are given by ${}^tK_{ij} = {}^rR_{ij} = 0$ for $i \neq j$ and

$$ {}^tK_{11} = 16\pi abc(\chi_0 + \alpha_0 a^2)^{-1}, \tag{3.6}$$

$$ {}^rR_{11} = \frac{16}{3}\pi abc(b^2\beta_0 + c^2\gamma_0)^{-1}(b^2+c^2). \tag{3.7}$$

Here, $\chi_0$, $\alpha_0$, $\beta_0$ and $\gamma_0$ are constants which are obtained by evaluating the following ellipsoidal harmonics at $\lambda=0$.[1]

$$\chi(\lambda) = abc\int_\lambda^\infty [P(t)]^{-1}dt, \tag{3.8}$$

$$\alpha(\lambda) = abc\int_\lambda^\infty [(a^2+t)P(t)]^{-1}dt, \tag{3.9}$$

with $\qquad P(t) = [(a^2+t)(b^2+t)(c^2+t)]^{1/2}.$

The lower limit of the definite integral, $\lambda(x,y,z)$, is the ellipsoidal coordinate which is the positive root of

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

The functions $\beta(\lambda)$, $\gamma(\lambda)$ and the expressions for the other diagonal elements of ${}^t\underset{\approx}{K}$ and ${}^r\underset{\approx}{R}$ are obtained by successive cycling of the dependence on a, b and c and the subscripts 1, 2 and 3.

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1. $\alpha$, $\beta$ and $\gamma$ are as defined in Happel & Brenner [9] and differ from Jeffery's [8] definition by a factor of (abc).

We shall see in the next section that the Newtonian velocity field is required in the calculations for the first non-Newtonian terms in the expansion solution. The classical solution of Oberbeck [6] for a translating ellipsoid and Edwardes [7] for a rotating ellipsoid are rewritten as velocity fields forced by a distribution of Stokes singularities (hence the name "singularity solutions") in order to highlight the common structure of the two solutions. The velocity field of the point force, i.e., the fundamental solution of the Stokes equation, is $\underline{\underline{I}}/(8\pi\mu)$, where $\underline{\underline{I}}$, the Oseen-Burgers tensor defined by

$$I_{ij} = \frac{1}{r}\delta_{ij} + \frac{1}{r^3}x_i x_j, \quad \text{with} \quad r = |\underline{x}|, \tag{3.10a}$$

and its pressure field, $$p_j = \frac{2\mu}{r^3}x_j, \tag{3.10b}$$

satisfy the Stokes equation with point forcing,

$$-\frac{\partial p_j}{\partial x_i} + \mu\nabla^2 I_{ij} = -8\pi\mu\delta_{ij}\delta(\underline{x}), \tag{3.11a}$$

and the continuity equation, $$\frac{\partial I_{ij}}{\partial x_i} = 0, \tag{3.11b}$$

as shown in reference [9], Chapter 2.

The Newtonian solutions of Oberbeck and Edwardes can then be rewritten as [4]:

$$\underline{v}(\underline{x}) = \underline{L}^{(n)}\cdot\iint_E f_{(n)}(x',y')\left\{1 + \frac{c^2q^2}{4n-2}\nabla^2\right\}\underline{\underline{I}}(\underline{x}-\underline{x}')/(8\pi\mu)\,dx'dy', \tag{3.12}$$

with n=1 for translation and n=2 for rotation. The two-dimensional integration is performed over the fundamental ellipse, E, which will be defined below. The terms in equation (3.12) are:

$$f_{(n)}(x,y) = \frac{(2n-1)}{2\pi a_E b_E} q^{2n-3}, \tag{3.13a-e}$$

$$q(x,y) = \left[1 - \frac{x^2}{a_E^2} - \frac{y^2}{b_E^2}\right]^{1/2},$$

$$a_E = (a^2-c^2)^{1/2}, \qquad b_E = (b^2-c^2)^{1/2},$$

and
$$\underset{\sim}{L}^{(n)} = \begin{cases} -\underset{\sim}{F} & \text{if } n=1, \\ \underset{=}{S}\cdot\nabla + \underset{=}{T}\cdot\nabla & \text{if } n=2. \end{cases}$$

For n=1, $\underset{\sim}{L}^{(1)}\cdot\underset{=}{I} = -\underset{\sim}{F}\cdot\underset{=}{I}$, so the translational solution is generated by a distribution of Stokes monopoles (or Stokeslets) over the fundamental ellipse with density $f_{(1)}$. For n=2, $\underset{\sim}{L}^{(2)}\cdot\underset{=}{I} = [(\underset{=}{S}+\underset{=}{T})\cdot\nabla]\cdot\underset{=}{I}$, so the rotational solution is generated by the Stokes dipole field, $\underset{=}{S}+\underset{=}{T}$ where $\underset{=}{S}$ and $\underset{=}{T}$ are respectively, the symmetric (stresslet) and anti-symmetric parts of the stress dipole,

$$\oint (\underset{=}{\sigma}\cdot\underset{\sim}{n})\underset{\sim}{x}\, dA.$$

The dyadic $\underset{=}{T}$ is related to $\underset{\sim}{T}$, the hydrodynamic torque exerted on the ellipsoid, by

$$T_{ij} = -\frac{1}{2}\epsilon_{ijk}T_k.$$

The domain of integration, E(x',y'), is the interior of the fundamental ellipse,

$$\frac{x^2}{a_E^2} + \frac{y^2}{b_E^2} = 1, \qquad z = 0.$$

The fundamental ellipse is the degenerate elliptical disk in the family of ellipsoids that are confocal to the particle ellipsoid. The major and minor semi-axes of the fundamental ellipse, $a_E$ and $b_E$, are given in equation (3.13c)[2]. The density function $f_{(n)}(x',y')$ in E(x',y') may also be

2. References [10,11] use k and $(k^2-h^2)1/2$ in place of our $a_E$ and $b_E$.

interpreted as the surface singularity distribution for an elliptical disk, as can be seen by looking at the limit $c = 0$ in equation (3.12).

The function $q(x,y)$ which appears in $f_{(n)}$ plays an analogous role in the potential theory for ellipsoidal particles (see [10]). In fact, in potential theory, $q^{-1}$ is the requisite charge distribution over the fundamental ellipse which generates ellipsoidal equipotential surfaces. Chwang & Wu [12] have noted that the distribution of Stokes multipoles in low-Reynolds-number problems is similar to the distribution of multipoles in analogous problems in potential theory, except for the presence of additional degenerate multipoles (the $\nabla^2 \underline{\underline{I}}$ term) in equation (3.12). The presence of such quadrupoles (or potential doublet) when n=1 and octupoles when n=2 in (3.12) are consistent with (and in fact extend) the rules stated in [12] for prolate spheroids.

To complete the solution, we must relate $\underline{F}$, $\underline{T}$ and $\underline{\underline{S}}$ in terms of the knowns, $\underline{U}$ and $\underline{\omega}$. The expressions for $\underline{F}$ and $\underline{T}$ are obtained from equations (3.6) and (3.7). The stresslet is identically zero for a sphere undergoing rigid body motions in a quiescent fluid. However, for a nonspherical ellipsoid, a stresslet is generated by a rotational motion [13]. The linear relation between the stresslet and the angular velocity can be deduced from [8] as

$$\underline{\underline{S}} = -\mu\, {}^r\underline{\underline{Q}}\cdot\underline{\omega},$$

where ${}^rQ_{ijk} = 0$ unless $\{i,j,k\}$ is a permutation of $\{1,2,3\}$ and

$${}^rQ_{123} = {}^rQ_{213} = -\frac{8}{3}\pi\mu abc(a^2\alpha_0 + b^2\beta_0)^{-1}(a^2-b^2). \qquad (3.14)$$

The remaining elements of ${}^r\underline{\underline{Q}}$ are obtained by cycling 1,2,3 and a,b,c. This completes the description of the singularity solution. Further simplifications of these shape-dependent tensors are possible in the degenerate case of ellipsoids of revolution. These and other asymptotic expressions for ellipsoidal and spheroidal resistance functions are provided in [4].

In summary, the basic results for the Newtonian velocity field are:

1) The disturbance velocity field for a translating ellipsoid (or a fixed ellipsoid in a uniform stream) is generated by a distribution of stokeslets and potential doublets over the fundamental ellipse.

2) The disturbance fields for a rotating ellipsoid or a fixed ellipsoid in a constant vorticity field (and for a stationary ellipsoid in a rate-of-strain field) are generated by a distribution of rotlets, stresslets and Stokes-octupoles over the fundamental ellipse.

3) For prolate spheroids, the fundamental ellipse degenerates into a line segment from one focal point to the other and the singularity solutions of Chwang & Wu [12] are recovered. Furthermore, if the spheroid is slender, the Chwang & Wu distributions reduce into forms that are analogous to those used by Leal [1].

In actual computation, e.g. evaluation of the volume integrals over the fluid region in the following section, it is more efficient to first integrate these representations over the fundamental ellipse to obtain the classical solutions (i.e. use the classical forms). However, in the present form, these representations are more amenable to analyses such as asymptotic expansions in the limit of degenerate shapes. The present forms also provide a straightforward bridge to the earlier works on slender bodies.

## 4. The Non-Newtonian Contributions

### 4.1 The Torque on a Translating Ellipsoid

We first derive the expression for the torque on a translating ellipsoid. This relation is then "inverted" to determine the rotational motions of a torque-free, sedimenting ellipsoid.

The perturbation expansion for the torque is:

$$\underline{T} = \underline{T}^{(0)} + W\underline{T}^{(1)} + \ldots \tag{4.1}$$

As discussed in the previous section, the Newtonian term is identically zero for a translating ellipsoid. The expression for the O(W) term is:

$$\underline{T}^{(1)} = \mu U \ell^2 \{ \int_S \underline{x} \times (\underline{\underline{A}} \cdot \underline{n})\, dA + \int_S \underline{x} \times (\underline{\underline{\tau}}^{(1)} \cdot \underline{n})\, dA \}. \tag{4.2}$$

The first integral in (4.2) is simply the contribution from the non-Newtonian terms in the constitutive equation, evaluated with the Newtonian (zero-th order) velocity field. The second integral is the contribution from the stress field of $W\underline{u}^{(1)}$. Obviously, we need to retain only the Newtonian terms in this stress field. Thus, it would appear that, given the Newtonian solution, the first term can be calculated directly, whereas the second term would require the solution of the forced Stokes equation. However, as shown in [1], the Lorentz reciprocal theorem may be applied to convert the second integral into an expression which requires only the Newtonian velocity field, thereby bypassing the O(W) velocity problem. Explicitly, for particles of any shape, we have:

$$\int_S \varepsilon_{ijk} x_j (\underline{\underline{\tau}}^{(1)} \cdot \underline{n})_k \, dA = \int_{V_f} t_{ik} A_{k\ell,\ell} \, dV, \tag{4.3a}$$

where $t_{ij}$ is the i-th component of the Newtonian solution for the particle rotating with unit angular velocity in the j-th coordinate direction and the volume integration is over the unbounded fluid region exterior to the particle. Brunn [14] has derived a more compact formula by noting that if $t_{ji}$ is used in equation (4.3a) instead of $t_{ij}$ , an integration by parts cancels the surface integral in equation (4.2). Thus

$$T_i^{(1)} = -\mu U a^2 \left\{ \int_{V_f} \frac{1}{2}(t_{ki,\ell} + t_{\ell i,k})\, A_{k\ell}\, dV. \right. \tag{4.3b}$$

The results of the previous section imply that the required velocity fields may be expressed as:

$$u_i^{(0)} = -U_i + F_j \iint_E f_{(1)}(x',y')\left\{1 + \frac{1}{2}c^2q^2\nabla^2\right\} I_{ij}(\underline{x}-\underline{x}')/(8\pi)\, dx'dy', \tag{4.4}$$

$$t_{ik} = -({}^rQ_{\ell jk} + \frac{1}{2}{}^rR_{km}\epsilon_{mj\ell}) \tag{4.5}$$

$$\iint_E f_{(2)}(x',y')\left\{1+\frac{1}{6}c^2q^2\nabla^2\right\} I_{i\ell,j}(\underline{x}-\underline{x}')/(8\pi)dx'dy'.$$

The resulting torque-translation relation for the O(W) term may now be expressed as

$$T_i^{(1)} = N^1_{ijk}U_jU_k, \tag{4.6}$$

as can be seen from combining equations (4.2), (4.3b) and (4.4). Furthermore, $N^1_{ijj}$ (no sum) is identically zero for i=1,2,3 because of particle symmetry. Thus the O(W) contribution to the torque on a translating ellipsoid in a SOF is completely characterized by three shape-dependent parameters, $(N^1_{123}+N^1_{132})$, $(N^1_{231}+N^1_{213})$ and $(N^1_{312}+N^1_{321})$. Furthermore, the $N^1_{ijk}$ are linear with respect to $\alpha$, therefore, the complete solution may be determined from the solution at (only) two distinct values of $\alpha$. Before turning to the sedimentation problem, we examine the special case when $\alpha = -1/2$, for which an analytical solution is possible.

Giesekus [15] has noted that for $\alpha = -1/2$, $\nabla \cdot \underline{\underline{A}}$ is irrotational and may be written as the gradient of a scalar potential. The O(W) equation may be solved directly by incorporating the right-hand-side into the pressure. Brunn [16] has applied this simplification to the general motion of a rigid particle in a SOF. For a particle undergoing steady translation, his results reduce to the extremely simple formula,

$$\underline{T}^{(1)} = \frac{1}{2}(\ell/U)\ \underline{F}^{(0)} \times \underline{U}. \tag{4.7}$$

For ellipsoids, this reduces to

$$T_3^{(1)} = \frac{1}{2}(\mu a/U)({}^tK_{22} - {}^tK_{11})U_1U_2, \tag{4.8}$$

and two analogous expressions obtained by cycling the indices. Necessarily, the general solution described below must reproduce these results when $\alpha = -1/2$.

Results for the torque on a translating ellipsoid were obtained by numerical integration of the right-hand-side of equation (4.3b) using an adaptive quadrature package provided by the Numerical Algorithms Group. These results were also reproduced using three nested one-dimensional integrations in the ellipsoidal coordinates, using the intuitively obvious procedure of finer grid sizes in the region near the surface. The linearity with respect to $\alpha$ was exploited, i.e., the integrand was split into terms independent of and first power in $\alpha$.

The (dimensionless) torque on prolate spheroids with shapes ranging from slender bodies to near-spheres is presented in Figure 1. The torque for arbitrary $\alpha$ may be obtained from the figure by using the plots for $\alpha = 0$ and the $\alpha$-coefficient. For $\alpha = 0$, the asymptotic solution for near-spheres remains a good approximation even at $b/a = 0.5$. In contrast, there is a significant deviation of the $\alpha$-coefficient from the asymptotic solution (the x-axis) at this value of $b/a$. When $\alpha = -1/2$, the results agree with equation (4.8) over the entire range of $\alpha$. Finally, the $-\alpha$ coefficient vanishes when $b = 0$. There is a sharp "boundary layer" in the curve which does not appear in the scale of Figure 1. The maximum in the curve lies at a value of $b/a$ between 0.01 and 0.001. (In comparison, when $\alpha = -1/2$ the maximum is near $b/a = 0.15$).

Thus the results presented here agree with those in reference [14] for near-spheres and satisfy equation (4.8) over the entire range of aspect ratios. In the slender-body limit, the results do not approach those presented in [1]. At the present time, I suspect that this disagreement is due to an error in [1], in part because that result does not agree with the analytic solution for $\alpha = -1/2$. However, it should be noted that there are differences between slender spheroids and slender-bodies generated by uniform distribution of stokeslets and rotlets[3].

4.2 Sedimentation

We shall first consider the general formulation for the translational and rotational velocities of a particle whose motion is driven by specified (time-independent) forces and torques. The sedimentation problem is a special case. The relations between the force and torque and the translational and rotational velocities are, accurate to $O(W)$,

$$F_i = -\mu^t K_{ij} U_j + (M^1_{ijk} U_j U_k + M^2_{ijk} U_j \omega_k + M^3_{ijk} \omega_j \omega_k) W + O(W^2), \tag{4.9a}$$

3. Chwang & Wu [17] note that there are fundamental differences between slender spheroids and those generated with a uniform distribution of rotlets.

$$T_i = -\mu^r R_{ij}\omega_j + (N^1_{ijk}U_jU_k + N^2_{ijk}U_j\omega_k + N^3_{ijk}\omega_j\omega_k)W + O(W^2). \qquad (4.9b)$$

For ellipsoids and other orthotropically-symmetric particles, $M^1=0$ as shown by [1]. The consequences for sedimentation are now considered.

The force-translation relation implies an O(1) translational velocity, which when inserted into equation (4.9b) implies that $\boldsymbol{\omega}$ is O(W) smaller than $\mathbf{U}$. Thus (4.9a) and (4.9b) may be inverted to arrive at the following expressions for the sedimenation and rotational velocities:

$$U_i = (\mu^t K_{ij})^{-1} F_j^{ext} + O(W^2), \qquad (4.10a)$$

$$\omega_i = (\mu^r R_{ij})^{-1}\ (N^1_{jk\ell}U_kU_\ell) + O(W^2). \qquad (4.10b)$$

Since the sedimentation and rotational motions are small, the sedimentation trajectory and the evolution of the particle orientatation may be determined by using small time steps and (4.10a) and (4.10b). The trajectory of the ellipsoid center and the three Euler angles (defined by Figure 2) are traced for different ratios of the normal-stress coefficients (Figures 3a-3d) for the representative case with b/a = 0.25 and c/a = 0.05. Several points pertaining o the use of non-axisymmetric particles are raised by these figures.

The projection of the trajectory of the particle center onto a horizontal plane deserves special mention. Since the ellipsoid eventually achieves the "vertical" orientation and falls without horizontal drift, the plane curve of Figure 2 approaches a limit point at large t. Moreover, the fluid rheology affects the shape of this plane curve only through the ratio, $\alpha$. In contrast, for axisymmetric particles, the projected curve does not provide as much useful information since it is simply a straight line in the X-Y plane. A change in $\alpha$ results in a change in the length, but not the direction of the line.

Another behavior which may be exploited is the different time scales in the decay of the Euler angles $\theta$ and $\psi$. The particle of Figure 3 represents the case where $\psi$ decays faster than $\theta$. In general, the two time scales may be adjusted relative to each other by varying b/a and c/b.

In conclusion, the theoretical analyses presented here suggest that the sedimentation configurations and trajectories of non-axisymmetric particles may ion

is being initiated to follow this lead.

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## REFERENCES

1. L.G. Leal, J. Fluid Mech. **69** (1975) 305.

2. P.C. Chan and L.G. Leal, J. Fluid Mech. **82** (1977) 549.

3. P. Brunn, J. Non-Newtonian Fluid Mech. **7** (1980) 271.

4. S. Kim, Int. J. Multiphase Flow (in press).

5. R.B. Bird, R.C. Armstrong and O. Hassager, Dynamics of Polymeric Fluids, Wiley New York, 1977.

6. A. Oberbeck, J. reine. angew. Math. **81** (1876) 62.

7. D. Edwardes, Quart. J. Math. **26** (1892) 70.

8. G.B. Jeffery, Proc. Roy. Soc. London **A102** (1922) 161.

9. J. Happel and H. Brenner, Low Reynolds Number Hydrodynamics, Sijthoff & Noordhoff, Leyden, The Netherlands, 1973.

10. T. Miloh, SIAM J. Appl. Math. **26** (1974) 334.

11. E.W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Chelsea, New York, 1955.

12. A.T. Chwang and T. Wu, J. Fluid Mech. **67** (1975) 787.

13. G.K. Batchelor, J. Fluid Mech. **41** (1970) 545.

14. P. Brunn, Rheol. Acta **18** (1979) 482.

14. H. Giesekus, Rheol. Acta **3** (1963) 59.

14. P. Brunn, J. Fluid Mech. **82** (1977) 529.

17. A.T. Chwang and T. Wu, J. Fluid Mech. **63** (1974) 607.

Captions for Figures

1. The torque, $T_z = W\mu Ua^2(f - \alpha g)U_x U_y/U^2$, on a prolate spheroid translating in a SOF, with $W = \Psi_1 U/(\mu a)$ and $\alpha = \Psi_2/\Psi_1$, f (---) and g (- - -). The solid curve is Brunn's [14] asymptotic result for f(b/a) for near-spheres. The asymptotic curve for g(b/a) is coincident with the horizontal axis.

2. The definition for the Euler angles of the subsequent figures. The body-fixed axes, x, y, and z are obtained from the laboratory-fixed axes, X, Y and Z as follows: rotation about the Z-axis by an angle $\phi$ until the X-axis coincides with the line-of-nodes, followed by rotation about the line-of-nodes by an angle $\theta$ until the Z axis coincides with the x-axis. The third Euler angle, $\psi$, is the angle between the y-axis and the line-of-nodes.

3. The initial orientation is $\theta=45°$, $\phi=0°$ and $\psi=45°$ and the initial location of the center is (0,0,0). The curves are for $\alpha=0$ (——), $\alpha=-0.1$ (---) and $\alpha=-0.2$ (- - -). All time axes have been scaled with the characteristic time, $(a/U)W^{-1}$.

   a. The trajectory for the ellipsoid center projected onto the X-Y plane.

   b. The evolution of $\theta$.

   c. The evolution of $\phi$. Without loss of generality, the X and Y axes are defined so that $\phi=0$ initially.

   d. The evolution of $\psi$.

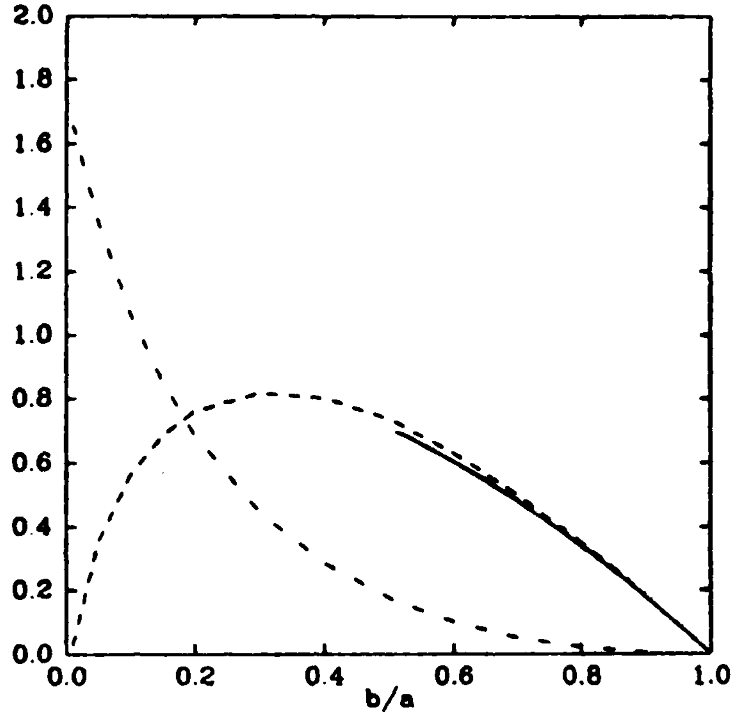


Figure 1.

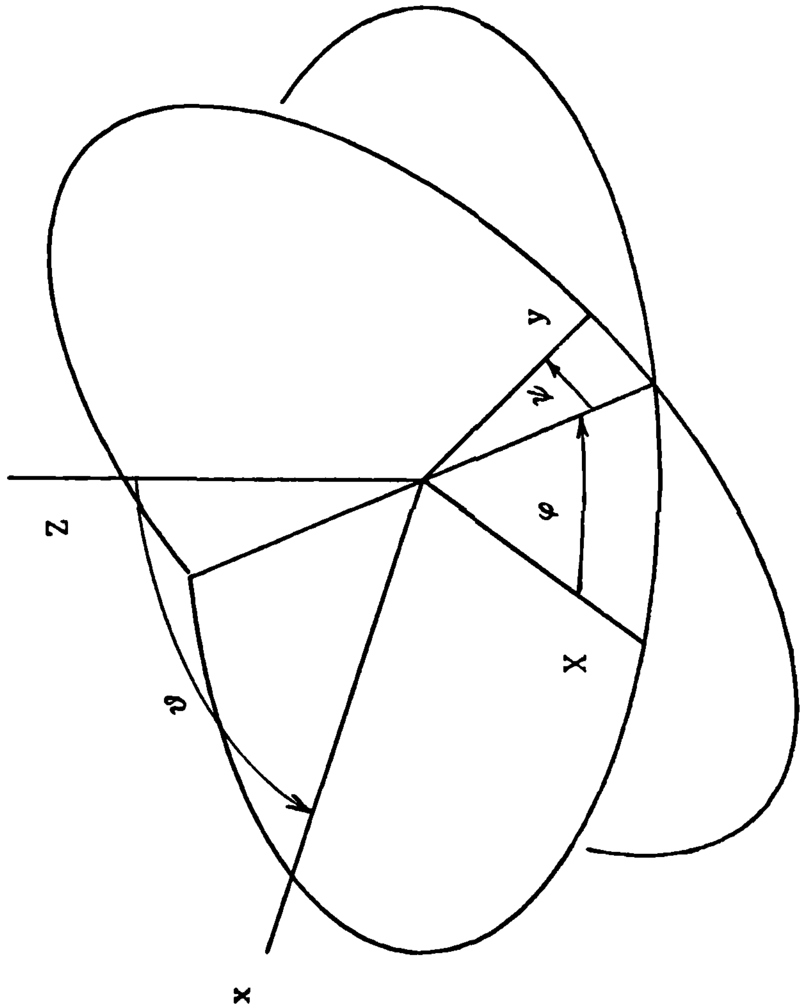


Figure 2.

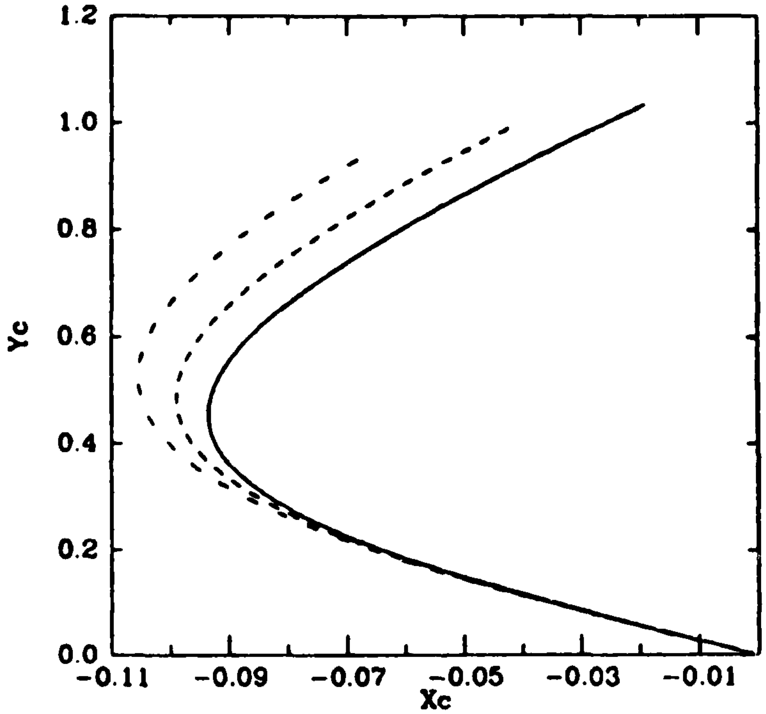


Figure 3a.

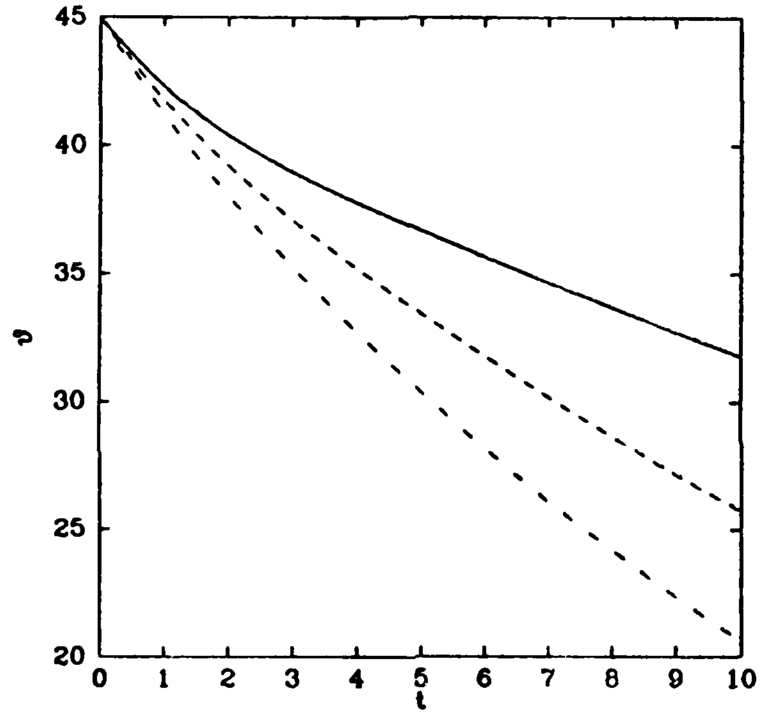


Figure 3b.

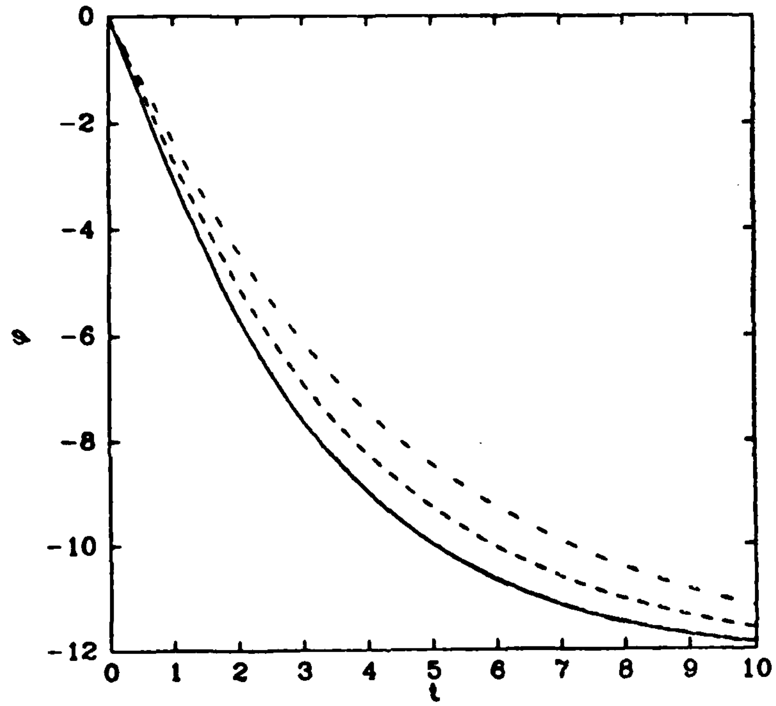


Figure 3c.

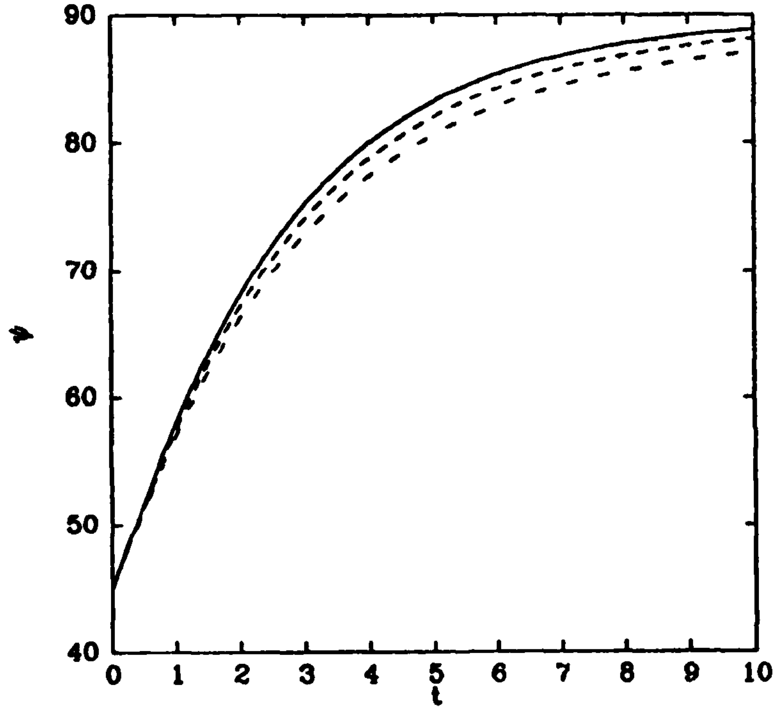


Figure 3d.

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The rigid body motion of an ellipsoid in a second order fluid (SOF) under the action of specified (time independent) external forces and torques have been obtained to first order in the Weissenberg number by inverting the resistance relations for the force and torque under specified rigid body motions. The reciprocal theorem of Lorentz was used to bypass the calculation of the O(W) velocity field. The results agree with known analytic solutions for a SOF with the secondary to primary normal stress ratio of -1/2.

The solution procedure was also tested by computing the torque on a

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ABSTRACT (cont.)

translating prolate spheroid with aspect ratios ranging from slender bodies to near-spheres. One result is that for a SOF with zero secondary normal stress (Weissenberg fluid), previous asymptotic results for near-spheres were found to be accurate even at fairly large aspect ratios (e.g. 2).

New results for non-degenerate ellipsoids suggest that the orientation (as monitored by Euler angles) and trajectory of sedimenting, non-axisymmetric particles such as ellipsoids provide useful information on the rheology of the suspending fluid.