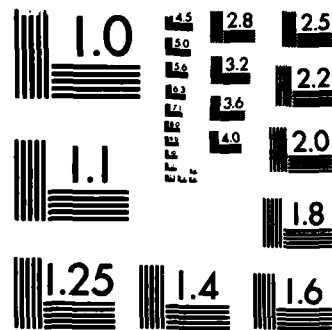END

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A NOTE ON LEAST TWO NORM SOLUTIONS OF
MONOTONE COMPLEMENTARITY PROBLEMS

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ABSTRACT

For the monotone nonlinear complementarity problem, ~~we~~ consider[s] Tihonov regularizations which reduce the solution of the problem to the solution of a sequence of strongly monotone complementarity problems. The sequence of solutions obtained are called approximate solutions and it is known that for a solvable monotone complementarity problem, the approximate solutions converge to the least two norm *solution of the given problem.* This paper provides new growth rates for these approximate solutions, sharpens some previously known results and gives a procedure for obtaining an approximate solution for any apriori prescribed tolerance.

AMS(MOS) Classification: 90C30, 90C25

Keywords: Monotone operators, complementarity, monotonicity, Tihonov regularization.

Work Unit Number 5: Optimization and Large Scale Systems

SIGNIFICANCE AND EXPLANATION

Tihonov regularization is a useful computational procedure for monotone complementarity problems which leads to approximate solutions when the given problem is solvable. Growth rates are given for these approximants which sharpen some known results. These results provide a unified framework for finding approximate solutions of important classes of constrained optimization problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NOTE ON LEAST TWO NORM SOLUTIONS

OF MONOTONE COMPLEMENTARITY PROBLEMS

P. K. Subramanian

1. Introduction

Given an operator $F\colon \Re^n \longrightarrow \Re^n$, the celebrated *complementarity problem* $NLCP(F)$ consists in finding $z \geq 0$ such that $F(z) \geq 0$ and $z^T F(z) = 0$. We say F is *monotone* if

$$\big(F(x) - F(y)\big)^T (x - y) \geq 0, \qquad \forall x, y \in \Re^n,$$

and *strongly monotone with modulus* λ if

$$\big(F(x) - F(y)\big)^T (x - y) \geq \lambda \|x - y\|^2$$

for some real number $\lambda > 0$. When F is an affine operator, that is, $F(x) = Mx + q$ for some $n \times n$ matrix M and a vector $q \in \Re^n$, $NLCP(F)$ is referred to as the *linear complementarity problem* and denoted by $LCP(M, q)$. It is well known that if M is positive semidefinite and $LCP(M, q)$ is feasible, that is there exists a $z \geq 0$ such that $F(z) \geq 0$,then it is solvable [Eaves, 1971]. However this is known to be false for $NLCP(F)$ in general as shown by the

counterexample of [Megiddo, 1977] and [Garcia, 1977] . On the other hand, if F is monotone and satisfies some growth conditions to be defined below or the *distributed Slater constraint qualification* [Mangasarian & McLinden, 1985], then $NLCP(F)$ is solvable.

In this note we shall be concerned with $NLCP(F)$ when F is monotone and the complementarity problem in this case will be referred to as the *monotone complementarity problem*. For such an operator F, given $\varepsilon > 0$, the *Tihonov regularization* of F is defined to be $F_\varepsilon := F + \varepsilon I$. It is well known that $NLCP(F_\varepsilon)$ has a unique solution $x(\varepsilon)$. The prinicpal theorem of this note provides new growth rates for $\|x(\varepsilon)\|$. As a corollary, we obtain the well known result $x(\varepsilon) \to x^*$ where x^* is the least two-norm solution of $NLCP(F)$, provided $NLCP(F)$ is solvable. These growth rates are also useful in obtaining *δ-approximate solutions* when $NLCP(F)$ needs only to be solved within a preassigned tolerance δ in some special cases.

We briefly indicate the notation used in this paper. We denote by $\Re^n$ the space of real ordered n-tuples. We use the Euclidean two-norm throughout. All vectors are column vectors. Given a vector x, we indicate its ith component by x_i. We say $x \geq 0$ if one has $x_i \geq 0 \, \forall i$ and the set of all such vectors in $\Re^n$ is denoted by $\Re_+^n$. Given x, y in $\Re^n$, we shall indicate their inner product $x^T y$ by $< x, y >$. Given $NLCP(F)$, we define

its *feasible set* $S(F)$ and *solution set* $\overline{S}(F)$ by

$$S(F) = \{x \in \Re^n_+ : F(x) \in, \Re^n_+\}$$

$$\overline{S}(F) = \{x \in S(F) :< x, F(x) >= 0\}.$$

The end of a proof is indicated by ∎.

2. Variational inequalities and NLCP(F)

The following notions are essential for this paper and the reader is referred

to [Auslender, 1976] for proofs.

2.1 Definition. *Let $D \subseteq \Re^n$, $F: D \to \Re^n$. The variational inequality prob-*

lem consists in finding $z_o \in D$, if it exists, such that

$$< F(z_o), \ x - z_o > \geq 0 \quad \forall x \in D.$$

In this case we say that z_o solves the variational inequality

$$(\text{VI}): \quad < F(z), \ x - z > \geq 0 \quad \forall x \in D.$$

Although many problems can be cast as variational inequality problems,

our interest in them stems from the following well known proposition (see

e.g., [Karamardian, 1972]).

2.2 Proposition. *Let $F: \Re^n_+ \to \Re^n$. Then z_o solves $NLCP(F)$ if and only*

if z_o solves (VI).

2.3 Definition. *Let C be a closed convex set in $\Re^n$, and let $F: \Re^n \to \Re^n$.*

We say F is hemicontinuous on C if for all x, $y \in C$, the map

$$\lambda \longmapsto < F(\lambda x + (1 - \lambda)y), \ x - y >$$

is continuous on the interval $[0, 1]$.

2.4 Proposition. *Let C be a closed convex set contained in D and let $F: D \longrightarrow \Re^n$ be monotone and hemicontinuous on C. Then*

$$< F(z^\star), \ (z - z^\star) > \ \geq \ 0 \qquad \forall z \in C$$

if and only if

$$< F(z), \ (z - z^\star) > \ \geq \ 0 \qquad \forall z \in C. \qquad (2.5)$$

Further, $Z_o = \{z^\star : z^\star \text{ solves } (4.5)\}$ is closed and convex.

See Auslender [1976, page 121] for a proof.

2.6 Definition. *Let $C \subseteq D$ be a nonempty closed convex set and assume $F: D \to \Re^n$. We say F is coercive (strongly coercive) if there exist $v_o \in C$, $\lambda \in \Re$ positive such that*

$$v \in C, \ \|v\| \geq \lambda \ \Longrightarrow \ F(v)(v - v_o) \ > \ 0$$

(respectively,

$$v \in C, \ \|v\| \to \infty \ \Longrightarrow \ \frac{F(v)(v - v_o)}{\|v - v_o\|} \to +\infty).$$

The proof of the following Theorem may be found in [Auslender, 1976].

2.7 Theorem. *Let $F : \Re^n \to \Re^n$ be a monotone operator, coercive and hemiconintuous on $\Re^n_+$. Then $NLCP(F)$ is solvable. If in addition F is strongly coercive, then $NLCP(F)$ has a unique solution.*

We now define the *Tihonov regularization* of an operator.

2.8 Definition. *Let $F : \Re^n \longrightarrow \Re^n$ and let $\varepsilon > 0$. The Tihonov regularization F_ε of F is defined by $F_\varepsilon(x) = F(x) + \varepsilon\, x$.*

If $F : \Re^n \to \Re^n$ is monotone and hemicontinuous, then F_ε is also hemicontinuous and strongly monotone with modulus of monotonicity at least ε. It is immediate that F_ε is strongly coercive. Thus we get the following useful corollary to Theorem 2.7.

2.9 Corollary. *Let $F : \Re^n \to \Re^n$ be monotone and hemicontinuous. Then $\forall \varepsilon > 0$, there exists a unique $x(\varepsilon)$ (called ε-approximant or simply approximant), which solves $NLCP(F)$.*

3. Properties of approximants

In this section we shall prove the principal theorems of this paper on the growth rate of ε-approximants.

3.1 Theorem. *Let $F : \Re^n \to \Re^n$ be a monotone operator which is hemicontinuous on $\Re^n_+$. Let $\{\varepsilon_n\}$ be a sequence of positive reals such that $\varepsilon_n \downarrow 0$. Let $F_n = F + \varepsilon_n I$ be the Tihonov regularization of F and let x_n be the unique solution of $NLCP(F_n)$. Let $m > n$ and assume that $F(0) \ngeq 0$. Then*

$$\|x_m - x_n\|^2 \leq \frac{(\varepsilon_n - \varepsilon_m)}{(\varepsilon_n + \varepsilon_m)} \cdot \left\{\|x_m\|^2 - \|x_n\|^2\right\}.$$

Proof

From Proposition 2.2, it follows that

$$< F_m(x_m),\ x - x_m > \ \geq\ 0 \qquad \forall\, x \in \Re^n_+.$$

By taking $x = x_n$,

$$< F_m(x_m),\ x_n - x_m > \ \geq\ 0. \tag{3.2}$$

Likewise,

$$< F_n(x_n),\ x_m - x_n > \ \geq\ 0,$$

which we rewrite as

$$< -F_n(x_n),\ x_n - x_m > \ \geq\ 0. \tag{3.3}$$

Adding (3.2) and (3.3) we get

$$< F_m(x_m) - F_n(x_n),\ x_n - x_m > \ \geq\ 0.$$

Hence remembering that $F_m = F + \varepsilon_m I$,

$$< F(x_m) + \varepsilon_m x_m - F(x_n) - \varepsilon_n x_n,\ x_n - x_m > \ \geq\ 0.$$

From the monotonicity of F this yields

$$< \varepsilon_m x_m - \varepsilon_n x_n,\ x_n - x_m > \ \geq\ < F(x_n) - F(x_m),\ x_n - x_m > \ \geq\ 0,$$

that is,

$$\varepsilon_m < x_m - x_n,\ x_n - x_m > \ + (\varepsilon_m - \varepsilon_n) < x_n,\ x_n - x_m > \ \geq\ 0.$$

By assumption $m > n$ so that $\varepsilon_m < \varepsilon_n$. We now have

$$(\varepsilon_n - \varepsilon_m) < x_n,\ x_m - x_n > \ \geq\ \varepsilon_m \| x_m - x_n \|^2. \tag{3.4}$$

Obviously,

$$\|x_m\|^2 = \|x_m - x_n\|^2 + \|x_n\|^2 + 2 < x_m - x_n, \ x_n >$$

so that from (3.4) we now get

$$\|x_m\|^2 \ \geq \ \|x_m - x_n\|^2 + \|x_n\|^2 + \left\{ \frac{2\varepsilon_m}{\varepsilon_n - \varepsilon_m} \right\} \|x_m - x_n\|^2.$$

Hence,

$$\|x_m\|^2 - \|x_n\|^2 \ \geq \ \left\{ \frac{\varepsilon_n + \varepsilon_m}{\varepsilon_n - \varepsilon_m} \right\} \left(\|x_m - x_n\|^2 \right).$$

This completes our proof. ∎

Theorem 3.1 has some interesting consequences. We present them in the following corollary.

3.5 Corollary. *Assume that the hypotheses of Theorem 3.1 are satisfied. Then*

a) $\|x_m\| \ > \ \|x_n\|$

b) $\varepsilon_m \|x_m\| \ \leq \varepsilon_n \|x_n\|$

Let $\overline{S} \ = \ \{x : x \ solves \ NLCP(F)\}$. Then

c) $\sup\{\|x_n\|\} \ < \ \infty \Longleftrightarrow x_n \ \longrightarrow \ \overline{x} \ = \ P_{\overline{S}}(0) \Longleftrightarrow \overline{S} \neq \emptyset.$

where $P_{\overline{S}}(0)$ denotes the projection of the origin on $\overline{S}(F)$, that is the closest point to 0 in $\overline{S}(F)$ in the two-norm.

Proof

Observe that $m \ > \ n$ implies that $x_m \ \neq \ x_n$. To see this, suppose the contrary and write $x_m \ = \ x_n \ = \ x$. Then x solves $NLCP(F_i)$ for $i \ = \ m, \ n$

so that

$$< F(x) + \varepsilon_m x,\ x > =\ 0$$

and

$$< F(x) + \varepsilon_n x,\ x > =\ 0$$

which imply $< \varepsilon_m x - \varepsilon_n x,\ x > =\ 0$. Since $\varepsilon_m\ <\ \varepsilon_n$ we must have $x\ =\ 0$. But $F_m(x)\ \geq\ 0$, so that we must have $F(0)\ \geq\ 0$ contradicting our hypothesis. Hence $\|x_m - x_n\|\ >\ 0$ and (a) follows from Theorem 3.1.

Next we prove (b). By Cauchy-Schwarz inequality from (3.4),

$$\frac{\varepsilon_n - \varepsilon_m}{\varepsilon_m} \|x_n\| \|x_m - x_n\| \geq \|x_m - x_n\|^2.$$

Hence

$$\frac{\varepsilon_n - \varepsilon_m}{\varepsilon_m} \|x_n\|\ \geq\ \|x_m - x_n\|\ \geq \|x_m\| - \|x_n\|$$

and

$$\varepsilon_n \|x_n\|\ \geq\ \varepsilon_m \|x_m\|.$$

This proves (b).

Finally we prove (c).

We start by showing that $\{\|x_n\|\}$ bounded $\Rightarrow$ x_n converges to an element of $\bar{S}$. From (a), since $\{\|x_n\|\}$ is strictly increasing, $\sup \|x_n\| = \lim \|x_n\|$. Taking $m\ >\ n$ and letting $m,\ n \to \infty$, it follows from Theorem 3.1 that $\{x_n\}$ is Cauchy. Hence x_n converges. Let $x_n \to \xi$. Since x_n solves $NLCP(F_n)$,

$$x_n\ \geq\ 0,\ \ F_n(x_n)\ \geq\ 0,\ \ < x_n,\ F_n(x_n) > =\ 0$$

which implies

$$\xi \geq 0, \;\; F(\xi) \geq 0, \;\; , < \xi, \, F(\xi) > \, = \, 0$$

so that $\xi \in \bar{S}$.

On the other hand, if $\bar{S} \neq \emptyset$, let $\bar{z}$ be any arbitrary element of $\bar{S}$. Assume that n is arbitrary but fixed. By Proposition (2.2),

$$< F_n(x_n), \; x - x_n > \, \geq \, 0 \qquad \forall \, x \in \Re_+^n.$$

Take $x = \bar{z}$ to get

$$< F(x_n) + \varepsilon_n x_n, \; \bar{z} - x_n > \, \geq \, 0. \qquad\qquad (3.6a)$$

Since $\bar{z}$ solves $NLCP(F)$, by Proposition (2.4),

$$< F(x), \; x - \bar{z} > \, \geq \, 0 \qquad \forall \, x \in \Re_+^n.$$

Taking $x = x_n$,

$$< F(x_n), \; x_n - \bar{z} > \, \geq \, 0. \qquad\qquad (3.6b)$$

From (3.6a) and (3.6b) we get

$$\varepsilon_n \; < x_n, \; \bar{z} - x_n > \, \geq \, 0 \qquad\qquad (3.7)$$

so that $< x_n, \; \bar{z} > \, \geq \, \|x_n\|^2$. Hence $\|x_n\| \leq \|\bar{z}\|$, that is, $\sup_n \|x_n\|$ is bounded proving the converse.

It remains only to show that if $x_n \to \xi$ then $\xi = P_{\bar{S}}(0)$. But from (3.7) we have $< \xi, \bar{z} - \xi > \geq 0$ and since $\bar{z}$ was an arbitrary element of $\bar{S}$ it follows that $\xi = P_{\bar{S}}(0)$. This completes the proof. ∎

Remark

Parts (a) and (c) of Corollary 3.5 are known when F is a multifunction on a Hilbert space H. For a proof using the theory of Yosida approximations, see [Brézis, 1974], who also proves a weaker form of Theorem 3.1.

4. Application to LCP(M, q)

We now consider an application of Corollary 3.5 to $LCP(M, q)$ in the case when M is positive semidefinite. From (c) of Corollary 3.5, the solution of $LCP(M, q)$ is reduced to the solution of the sequence $LCP(M + \varepsilon_n I, q)$. We shall not be concerned here with an algorithm for the solution of the positive definite $LCP(M + \varepsilon_n I, q)$. However, we would like to show that if the solution set $\overline{S}(M, q)$ is *bounded* then for any preassigned tolerance δ, it suffices to solve $LCP(M + \varepsilon I, q)$ for a single value of the parameter ε to obtain a δ-*approximate* solution. We make this precise in the following Theorem.

4.1 Theorem. *Let $\delta > 0$ be a preassigned tolerance. Assume that M is positive semidefinite and that $\overline{S}(M, q)$ is nonempty and bounded. Then there exists $\bar{\varepsilon} > 0$ such that $\forall \varepsilon, \quad 0 < \varepsilon < \bar{\varepsilon}$, the unique solution $x(\varepsilon)$ of*

$LCP(M + \varepsilon I, \ q)$ *satisfies*

$$x(\varepsilon) \geq 0, \qquad \|w(\varepsilon) - w(\varepsilon)_+\| \leq \delta \quad and \quad | < x(\varepsilon), \ w(\varepsilon) > | \ < \ \delta$$

where $w(\varepsilon) = M x(\varepsilon) + q.$

Proof

Assume that $x(\varepsilon)$ solves $LCP(M + \varepsilon I, q)$. Let $v(\varepsilon) = w(\varepsilon) + \varepsilon x(\varepsilon)$. By assumption, $\exists K \geq 1$ such that $\|\overline{S}(M, \ q)\| \leq K$. Now choose $\overline{\varepsilon} = \delta / K^2$. Since $x(\varepsilon)$ solves $LCP(M + \varepsilon I, q)$, we have

$$x(\varepsilon) v(\varepsilon) = 0, \quad v(\varepsilon) - w(\varepsilon) = \varepsilon x(\varepsilon).$$

If $x^\star$ is the least two-norm solution of $LCP(M, q)$, then

$$|x(\varepsilon) w(\varepsilon)| = \varepsilon \|x(\varepsilon)\|^2 \ \leq \ K \, \varepsilon \, \|x(\varepsilon)\| \quad \text{(by 3.5(a) and (c))}$$

$$\leq K \, \overline{\varepsilon} \|x^\star\| \quad \text{(by 3.5(a) and (c))}$$

$$< K^2 \cdot \delta / K^2$$

$$= \delta.$$

Also since $w(\varepsilon)_+$ is the closest point to $w(\varepsilon)$ in $\Re^n_+$, we have

$$\|w(\varepsilon) - w(\varepsilon)_+\| \leq \|w(\varepsilon) - v(\varepsilon)\| = \varepsilon \cdot \|x(\varepsilon)\|$$

$$\leq \overline{\varepsilon} \cdot \|x(\overline{\varepsilon})\| \leq \frac{\delta}{K^2} \cdot K$$

$$< \delta.$$

This completes our proof. ∎

Remark

We remark that a sufficient condition for the boundedness of the solution set is that the interior of the feasible set $S(M, q)$ be nonempty (see for instance [Mangasarian, 1982] where this is proved for the more general case when M is *copositive plus*, that is, (i) $x \geq 0 \Rightarrow xMx \geq 0$ and (ii) $xMx = 0 \Rightarrow Mx = 0$.) To find the constant K one can use the bounds obtained by Mangasarian [1985] by solving a single linear program if necessary.

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For the monotone nonlinear complementarity problem, we consider Tihonov regularizations which reduce the solution of the problem to the solution of a sequence of strongly monotone complementarity problems. The sequence of solutions obtained are called approximate solutions and it is known that for a solvable monotone complementarity problem, the approximate solutions converge to the least two norm solution of the given problem. This paper provides new growth rates for these approximate solutions, sharpens some previously known results and gives a procedure for obtaining an approximate solution for any apriori prescribed tolerance.

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