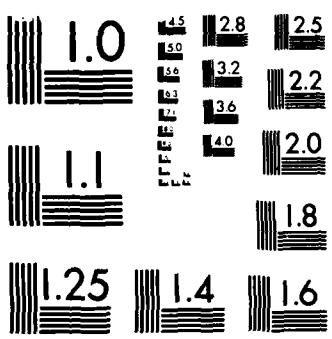
AD-A160 957 LIPSCHITZ CONTINUITY OF SOLUTIONS OF LINEAR INEQUALITIES PROGRAMS AND COM (U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER 1/1

UNCLASSIFIED O L MANGASARIAN ET AL JUL 85 MRC-TSR-2838 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(2)

AD-A160 957

MRC Technical Summary Report #2838

LIPSCHITZ CONTINUITY OF SOLUTIONS OF LINEAR INEQUALITIES, PROGRAMS AND COMPLEMENTARITY PROBLEMS

O. L. Mangasarian and T.-H. Shiau

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

July 1985

(Received June 25, 1985)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

LIPSCHITZ CONTINUITY OF SOLUTIONS OF LINEAR INEQUALITIES, PROGRAMS AND COMPLEMENTARITY PROBLEMS

O. L. Mangasarian & T.-H. Shiau

Technical Summary Report # 2838

July 1985

ABSTRACT

It is shown that solutions of linear inequalities, linear programs and certain linear complementarity problems (e.g. those with P-matrices or Z-matrices but not semidefinite matrices) are Lipschitz continuous with respect to changes in the right hand side data of the problem. Solutions of linear programs are not Lipschitz continuous with respect to the coefficients of the objective function. The Lipschitz constant given here is a generalization of the role played by the norm of the inverse of a nonsingular matrix in bounding the perturbation of the solution of a system of equations in terms of a right hand side perturbation.

Keyword: optimization,

AMS (MOS) Subject Classifications: 15A39, 90C05, 65F35

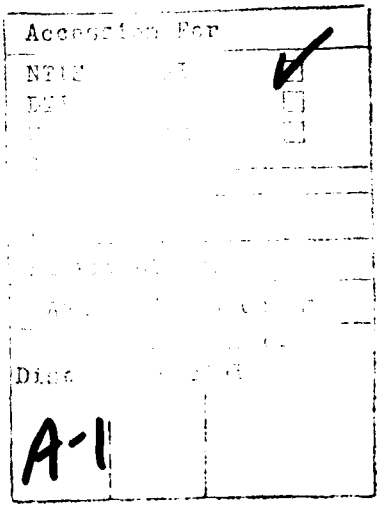
Key Words: Linear inequalities, linear programming, linear complementarity problems, Lipschitz continuity, perturbation analysis

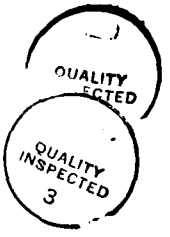
Work Unit Number 5: Optimization and Large Scale Systems

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under grants MCS-8200632, MCS-8420963 and DMS-8210950, Mod. 1.

SIGNIFICANCE AND EXPLANATION

In the solution of constrained optimization problems one often is faced with inaccurate or perturbed data. In this work we show how perturbations in the data of linear inequalities, linear programs and other optimization-related problems influence the solution of the problem.





LIPSCHITZ CONTINUITY OF SOLUTIONS OF LINEAR INEQUALITIES, PROGRAMS AND COMPLEMENTARITY PROBLEMS

O. L. Mangasarian & T.-H. Shiau

I. Introduction

The purpose of this work is to show that solutions of linear inequalities, linear programs and certain linear complementarity problems are Lipschitz continuous with respect to changes in the right hand side of the problem. Speaking in general and in somewhat loose terms, if we denote by r^1 and r^2, two distinct right hand sides, then there exist corresponding solutions x^1 and x^2 such that

$$\|x^1 - x^2\| \leq K\|r^1 - r^2\| \tag{1.1}$$

where the Lipschitz constant K depends only on the matrix defining the problem, but not on the right hand sides nor the objective function if there is one. A key role in determining the Lipschitz constant K is played by the condition number for linear inequalities introduced in [8] which is a generalization of the very useful concept of a condition number for a non-singular square matrix [2]. In [15] Robinson obtained local Lipschitz continuity results for generalized equations which include linear programs, convex quadratic programs and monotone linear complementarity problems. Robinson's Lipschitz constant [15, Theorem 2] involves a bound on the solution set which is assumed to be bounded. By constrast our Lipschitz constants are global, and our solution sets need not be bounded.

Sponsored by the United States Army under Contract No. DAAG29-80-C0041. This material is based upon work supported by the National Science Foundation under Grants MCS-8200632, MCS-8420963 and DMS-8210950, Mod. 1.

We give now a summary of our principal results. Theorem 2.2 deals with a system of linear inequalities and equalities (2.1) and shows that if the system is solvable for right hand sides r^1 and r^2, then for each solution x^1 for right hand side r^1 there exists a solution x^2 for right hand side r^2 such that (1.1) holds. The Lipschitz constant here plays the same role as the norm of the inverse of a nonsingular matrix does for a system of linear equations. Our Lipschitz constant for the system (2.1) defined by (2.5), is a minor variation of the constant (6) of [8]. Furthermore, the Lipschitz continuity Theorem 2.2 leads in a very elementary way to Theorem 2.2' which is essentially equivalent to Theorem 1 of [8] and which gives an estimate of the error in an approximate solution to the systems of linear inequalities and equalities (2.1) in terms of the residual of the approximate solution and the Lipschitz constant. Again the role played in Theorem 2.2' by the Lipschitz constant is an extension of the same role played by the norm of the inverse of a matrix for a system of linear equations. Computation of the Lipschitz constant (2.5) for the system of linear inequalities and equalities (2.1) is quite difficult, but an important fact is that such a constant exists and is finite. For some special cases such as when we have strongly stable linear inequalities only (that is linear inequalities solvable for all right hand sides) the Lipschitz constant can be computed by a single linear program as in (2.17) below. By using the Lipschitz constant for linear inequalities and equalities we show in Theorem 2.4 that solutions of linear programs are also Lipschitz continuous with respect to right hand side perturbations only. Proposition 2.6 shows that our Lipschitz constant (2.20) for the linear program (2.18) is sharper than that of Cook et al [3, Theorem 5]. By means of a simple example (2.26), we show that solutions

of linear programs are not Lipschitz continuous with respect to perturbations in the objective function coefficients. Finally in Section 3 by using the Lipschitz constant for linear inequalities and equalities we establish in Theorem 3.2 Lipschitz continuity of solutions of linear complementarity problems with respect to right hand side perturbations that generate unique solutions along the line segment joining perturbed and unperturbed right hand sides. A simple consequence of this result is Theorem 3.3 which shows that the solution of a linear complementarity problem with a P-matrix (that is a matrix with positive principal minors) is Lipschitz continuous with respect to right hand side perturbations. Example 3.4 shows that solutions of positive semidefinite linear complementarity problems are not Lipschitz continuous with respect to their right hand sides. Finally by exploiting the fact that for certain classes of matrices such as Z-matrices (real matrices with nonpositive off-diagonal elements) the linear complementarity problems can be solved as a linear program [7], Lipschitz continuity of solutions of such linear complementarity problems are obtained in Theorem 3.5.

A brief word about notation and some basic concepts employed. For a vector x in the n-dimensional real space R^n, $|x|$ and x_+ will denote the vectors in R^n with components $|x|_i := |x_i|$ and $(x_+)_i := \max\{x_i, 0\}$, $i=1,\ldots,n$ respectively. For a norm $\|x\|_\beta$ on R^n, $\|x\|_{\beta*}$ will denote the dual norm [6,13] on R^n, that is $\|x\|_{\beta*} := \max_{\|y\|_\beta=1} xy$, where xy denotes the scalar product $\sum_{i=1}^{n} x_i y_i$. The generalized Cauchy-Schwarz inequality $|xy| \leq \|x\|_\beta \cdot \|y\|_{\beta*}$, for x and y in R^n, follows immediately from this definition of the dual norm. For $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, the p-norm $\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ and the

q-norm are dual norms on R^n [13]. If $\|\cdot\|_\beta$ is a norm on R^n, we shall, with a slight abuse of notation, let $\|\cdot\|_\beta$ also denote the corresponding norm on R^m for $m \neq n$. For an $m \times n$ real matrix A, A_i denotes the ith row, $A_{\cdot j}$ denotes the jth column, $A_I := A_{i \in I}$, and $A_{\cdot J} := A_{\cdot j \in J}$, where $I \subset \{1,\ldots,m\}$ and $J \subset \{1,\ldots,n\}$. $\|A\|_\beta$ denotes the matrix norm [13,16] subordinate to the vector norm $\|\cdot\|_\beta$, that is $\|A\|_\beta = \max_{\|x\|_\beta = 1} \|Ax\|_\beta$. The consistency condition $\|Ax\|_\beta \leq \|A\|_\beta \|x\|_\beta$ follows immediately from this definition of a matrix norm. A monotonic norm on R^n is any norm $\|\cdot\|$ on R^n such that for a, b in R^n, $\|a\| \leq \|b\|$ whenever $|a| \leq |b|$ or equivalently if $\|a\| = \| |a| \|$ [6, p. 47]. The p-norm for $p \geq 1$ is monotonic [13]. A vector of ones in any real space will be denoted by e. The identity matrix of any order will be denoted by I. The nonnegative orthant in R^n will be denoted by R^n_+. The abbreviation rhs will denote "right hand side".

2. Linear Inequalities and Programs

We shall first be concerned with Lipschitz continuity of solutions of the following set of linear inequalities with respect to changes in the right hand side

$$Ax \leq b,\ Cx = d \tag{2.1}$$

where b and d are given points in R^m and R^k respectively, $A \in R^{m\times n}$, that is an $m \times n$ real matrix and $C \in R^{k\times n}$. We shall employ a slight variation of the condition constant introduced in [8, Equation (6)] for linear inequalities and programs as our Lipschitz constant for the linear inequalities (2.1) and subsequently for the linear program (2.18) and the linear complementarity problem (3.1).

We begin with a simple extension of the fundamental theorem on basic solutions [5, Theorem 2.11] to unrestricted as well as nonnegative variables.

2.1 Lemma (Basic solutions) Let $A \in R^{m\times n}$, $C \in R^{k\times n}$ and $p \in R^n$. The system

$$A^T u + C^T v = p,\ u \geq 0 \tag{2.2}$$

has a solution $(u,v) \in R^{m+k}$ if and only if it has a basic solution, that is a solution (u,v) such that the rows of $\begin{pmatrix} A \\ C \end{pmatrix}$ corresponding to nonzero components of (u,v) are linearly independent.

Proof The system (2.2) having a solution (u,v) implies that

$$A^T u + \tilde{C}^T v = p,\ (u,v) \geq 0 \tag{2.3}$$

has a solution where $\tilde{C}$ is obtained from C by multiplying by -1 those rows

of C corresponding to negative components of v. It follows from the fundamental theorem on basic solutions [5, Theorem 2.11] that (2.3) has a basic solution and consequently so does (2.2). □

We proceed now to establish Lipschitz continuity of solutions of (2.1) with respect to right hand side perturbations.

2.2 Theorem (Lipschitz continuity of feasible points of linear inequalities and equalities) Let the linear inequalities and equalities (2.1) have nonempty feasible sets S^1 and S^2 for the right hand sides (b^1, d^1) and (b^2, d^2) respectively. For each $x^1 \in S^1$ there exists an $x^2 \in S^2$ closest to x^1 in the ∞-norm such that

$$\|x^1 - x^2\|_\infty \leq \mu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \tag{2.4}$$

where $\|\cdot\|_\beta$ is some norm on R^{m+k},

$$\mu_\beta(A; C) := \sup_{u,v} \left\{ \left\| \begin{matrix} u \\ v \end{matrix} \right\|_{\beta^*} \middle| \begin{matrix} \|uA + vC\|_1 = 1, \ u \geq 0 \\ \text{Rows of } \begin{pmatrix} A \\ C \end{pmatrix} \text{ corresponding to nonzero} \\ \text{elements of } \begin{pmatrix} u \\ v \end{pmatrix} \text{ are lin. indep.} \end{matrix} \right\} \tag{2.5}$$

and $\|\cdot\|_{\beta^*}$ is the dual norm to $\|\cdot\|_\beta$.

Proof We note that $\mu_\beta(A; C)$ is finite. For if not, there would exist fixed subsets I and J of $\{1,\ldots,m\}$ and $\{1,\ldots,k\}$ respectively and a sequence $\{u_I^i, v_J^i\}$ such that $\{\|u_I^i, v_J^i\|\} \to \infty$ and the rows of $\begin{pmatrix} A_I \\ C_J \end{pmatrix}$ are linearly independent. Hence a subsequence $\{(u_I^{i_j}, v_J^{i_j})/\|u_I^{i_j}, v_J^{i_j}\|\}$ converges to $(\bar{u}_I, \bar{v}_J)$ satisfying $\bar{u}_I A_I + \bar{v}_J C_J = 0$, $\|\bar{u}_I, \bar{v}_J\| = 1$, which

contradicts the linear independence of the rows of $\begin{pmatrix} A_I \\ C_J \end{pmatrix}$.

Now let $x^1 \in S^1$. Choose $x^2 \in S^2$ which is closest to x^1 in the ∞-norm. Thus x^2 must solve

$$(2.6) \qquad \min_x \|x - x^1\|_\infty \quad \text{s.t.} \quad Ax \leq b^2,\ Cx = d^2$$

which is equivalent to the linear program

$$(2.7) \qquad \min_{x,\delta} \delta \quad \text{s.t.} \quad Ax \leq b^2,\ Cx = d^2,\ x + e\delta \geq x^1,\ -x + e\delta \geq -x^1$$

Hence (x^2, δ^2) and some $(u^2, v^2, r^2, s^2) \in R^{m+k+2n}$ satisfy the following Karush-Kuhn-Tucker conditions for (2.7)

$$(2.8) \qquad \begin{cases} Ax^2 \leq b^2,\ Cx^2 = d^2,\ \|x^1 - x^2\|_\infty = \delta^2 \\ u^2(-Ax^2 + b^2) = 0,\ r^2(x^2 + e\delta^2 - x^1) = 0,\ s^2(-x^2 + e\delta^2 + x^1) = 0 \\ -u^2A + v^2C + r^2 - s^2 = 0,\ e(r^2 + s^2) = 1,\ (u^2, r^2, s^2) \geq 0 \end{cases}$$

Note that if $0 = \delta^2 = \|x^1 - x^2\|_\infty$, then (2.4) is trivially true. So assume that $\delta^2 > 0$. It follows from $\delta^2 > 0$ and $r_j^2(x^2 + e\delta^2 - x^1)_j = 0$ and $s_j^2(-x^2 + e\delta^2 + x^1)_j = 0$ that $r_j^2 s_j^2 = 0$, for $j=1,\ldots,n$. Hence

$$(2.9) \quad -u^2A + v^2C + r^2 - s^2 = 0,\ e(r^2 + s^2) = 1,\ r^2s^2 = 0,\ (u^2, r^2, s^2) \geq 0$$

By Lemma 2.1 it follows that we may take $u^2 = \begin{pmatrix} u_I^2 \\ 0 \end{pmatrix} \geq 0$ and $v^2 = \begin{pmatrix} v_J^2 \\ 0 \end{pmatrix}$ such that the rows of $\begin{pmatrix} A_I \\ C_J \end{pmatrix}$ are linearly independent. Hence (2.9) becomes

$$\left\langle \begin{array}{l} \|-u_I^2 A_I + v_J^2 C_J\|_1 = \|r^2 - s^2\|_1 = e(r^2 + s^2) = 1,\ u_I^2 \geq 0, \\ \text{Rows of } \begin{pmatrix} A_I \\ C_J \end{pmatrix} \text{ lin. indep.} \end{array} \right.$$

Hence by (2.5) we have that

$$\left\| \begin{matrix} u^2 \\ v^2 \end{matrix} \right\|_{\beta^*} \leq \mu_\beta(A; C) \tag{2.10}$$

We now have

$$\begin{aligned} \|x^1 - x^2\|_\infty = \delta^2 &= -b^2 u^2 + d^2 v^2 + x^1(r^2 - s^2) \\ &= -b^2 u^2 + d^2 v^2 + x^1(A^T u^2 - C^T v^2) \\ &= u^2(Ax^1 - b^2 + b^1 - b^1) + v^2(-Cx^1 + d^2 + d^1 - d^1) \\ &\leq u^2(b^1 - b^2) + v^2(d^2 - d^1) \\ &\leq \left\| \begin{matrix} u^2 \\ v^2 \end{matrix} \right\|_{\beta^*} \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \\ &\leq \mu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \qquad \text{(By (2.10))} \qquad \square \end{aligned} \tag{2.11}$$

Note that the Lipschitz constant $\mu_\beta(A; C)$ of (2.4) plays the same role as that of the norm of the inverse of a nonsingular matrix of a system of linear equations. This fact can be seen more clearly from the following corollary to Theorem 2.2. Note also that we can get a sharper result by replacing $(b^1 - b^2)$, in (2.4) and (2.11) onward, by $(b^1 - b^2)_+$.

2.3 Corollary (Lipschitz continuity of feasible points of strongly stable linear inequalities) Let $A \in R^{m \times n}$ and $C \in R^{k \times n}$ be such that

(2.12) $$\begin{cases} \text{Rows of } C \text{ are linearly independent and} \\ Ax < 0,\ Cx = 0 \text{ has a solution } x \end{cases}$$

Then the linear inequalities (2.1) are solvable for all right hand sides $(b, d) \in R^{m+k}$. For each x^1 in the solution set of (2.1) with rhs (b^1, d^1), there exists an x^2 in the solution set of (2.1) with rhs (b^2, d^2) such that

(2.13) $$\|x^1 - x^2\|_\infty \leq \bar{\mu}_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta$$

where $\|\cdot\|_\beta$ is some norm on R^{m+k} and

(2.14) $$\bar{\mu}_\beta(A; C) := \max_{(u,v)\in R^{m+k}} \left\{ \left\| \begin{matrix} u \\ v \end{matrix} \right\|_{\beta^*} \middle| \begin{matrix} \|uA + vC\|_1 = 1 \\ u \geq 0 \end{matrix} \right\}$$

Proof That (2.1) is solvable for any right hand side (b, d) follows from solving $Cx = d$ for x^d for any given d and then taking as the desired solution $x^d + \lambda\bar{x}$ for sufficiently large positive λ, where $\bar{x}$ solves $Ax < 0$, $Cx = 0$. The rest of the proof of the corollary is similar to the proof of Theorem 2.2, except that u^2 and v^2 are not decomposed into $\begin{pmatrix} u_I^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} v_J^2 \\ 0 \end{pmatrix}$. The finiteness of $\bar{\mu}_\beta(A; C)$ of (2.14) follows from the boundedness of the feasible region of (2.14). For it were unbounded, there would exist $\{u^i, v^i\}$ such that $\{\|u^i, v^i\| \to \infty$ and consequently an accumulation point $(\bar{u}, \bar{v})$ would exist such that

(2.15) $$\bar{u}A + \bar{v}C = 0,\ \bar{u} \geq 0,\ (\bar{u}, \bar{v}) \neq 0$$

This however would contradict the linear independence of the rows of C if

$\bar{u} = 0$, and if $\bar{u} \neq 0$ would contradict the solvability of $Ax < 0$, $Cx = 0$, because then $0 = \bar{u}Ax + \bar{v}Cx = \bar{u}Ax < 0$. □

Note that if A is vacuous and C is a nonsingular square matrix, then

$$(2.16) \qquad \bar{u}_\infty(\phi; C) = \max_{v \in R^k} \{\|v\|_1 \mid \|vC\|_1 = 1\} = \|(C^T)^{-1}\|_1 = \|C^{-1}\|_\infty$$

This was already pointed out in [8, Remark 2]. Note also that (2.14) can be written in the equivalent form

$$(2.14') \qquad \bar{\mu}_\beta(A; C) = \max_{(u,v,z) \in R^{m+k+n}} \left\{ \left\| \begin{matrix} u \\ v \end{matrix} \right\|_{\beta^*} \middle| \begin{matrix} -z \leq uA + vC \leq z \\ u \geq 0, \ ez = 1 \end{matrix} \right\}$$

This is a difficult convex-function <u>maximization</u> problem on a polyhedral set which is closely related to the NP-complete problem of a norm-maximization problem on a polyhedral set for positive integer β^* [9]. However for $\beta^* = \infty$, that is $\beta = 1$, it can be shown, as in [9], that (2.14') is in P. In addition a good bound for $\bar{\mu}_\beta(A; C)$ for any β can be obtained by solving a single linear program [9]. When C is empty and $\beta = \infty$, (2.14') degenerates to the following linear programs

$$(2.17) \qquad \bar{\mu}_\infty(A; \phi) = \max_{(u,z) \in R^{m+n}} \{eu \mid -z \leq uA \leq z, \ u \geq 0, \ ez = 1\}$$

We note that the Lipschitz constants $\mu_\beta(A; C)$ and $\bar{\mu}_\beta(A; C)$ which play the role of the norm of the inverse of a nonsingular matrix of a system of linear equations, can also be used, just as the norm of the inverse can, to obtain a bound on the error in an approximate solution in terms of the residual. Thus if we assume for the moment that A is vacuous and that C

is $n \times n$ and nonsingular, then $\bar{\mu}_\beta(\phi; C) = \|C^{-1}\|_\infty$ by (2.16). Thus (2.5) and (2.13) are the extensions to a system of linear inequalities and equalities of the following simple Lipschitz continuity property of $Cx = d$

$$\|x^1 - x^2\|_\infty \leq \|C^{-1}\|_\infty \|d^1 - d^2\|_\infty$$

where $x^1 = C^{-1}d^1$ and $x^2 = C^{-1}d^2$. Since $\|C^{-1}\|_\infty$ can also be used to estimate the error in an approximate solution x to $Cx^1 = d^1$ in terms of its residual $\|Cx - d^1\|_\infty$ as follows

$$\|x - x^1\|_\infty = \|C^{-1}(Cx - d^1)\|_\infty \leq \|C^{-1}\|_\infty \|Cx - d^1\|_\infty$$

it follows that the Lipschitz constants $\mu_\beta(A; C)$ and $\bar{\mu}_\beta(A; C)$ can be similarly used to give an estimate on the error in an approximate solution to (2.1) in terms of its residual. In fact this estimate was given in [8, Theorem 1], but it also follows very easily from Theorem 2.2 above as follows.

2.2' Theorem (Error bound for approximate solution of linear inequalities and equalities) Let the linear inequalities and equalities (2.1) have a nonempty feasible set S^1 for the right hand side (b^1, d^1). For each x in R^n there exists an $x^1 \in S^1$ such that

$$\|x - x^1\|_\infty \leq \mu_\beta(A; C) \left\| \begin{matrix} (Ax - b^1)_+ \\ Cx - d^1 \end{matrix} \right\|_\beta$$

where $\mu_\beta(A; C)$ is defined by (2.5).

Proof Since for each $x \in R^n$

$$Ax \leq b^1 + (Ax - b^1)_+,\ Cx = d^1 + (Cx - d^1)$$

it follows by Theorem 2.2 that there exists an $x^1 \in S^1$ such that the conclusion of the theorem holds. □

A similar error bound holds for strongly stable linear inequalities which is based on (2.13).

It is interesting to note that Theorem 2.2 is stronger than Theorem 2.2' in the sense that the latter follows directly from the former as was demonstrated above, whereas the converse holds with the additional assumption that the norm $\|\cdot\|_\beta$ is a monotonic norm [6, 13]. Thus to obtain Theorem 2.2 from Theorem 2.2', we have from Theorem 2.2' that for each $x^1 \in S^1$ there exists an $x^2 \in S^2$ such that

$$\|x^2 - x^1\|_\infty \leq \mu_\beta(A; C) \left\| \begin{matrix} (Ax^1 - b^2)_+ \\ Cx^1 - d^2 \end{matrix} \right\|_\beta \leq \mu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta$$

where the last inequality follows from

$$(Ax^1 - b^2)_+ = (Ax^1 - b^2 + b^1 - b^1)_+ \leq (b^1 - b^2)_+ \leq |b^1 - b^2|$$
$$|Cx^1 - d^2| = |Cx^1 - d^2 + d^1 - d^1| = |d^1 - d^2|$$

and the monotonicity of the norm $\|\cdot\|_\beta$.

Next we establish the Lipschitz continuity with respect to right hand side perturbation of solutions of the linear program

$$\max_x\ px \quad \text{s.t.}\ Ax \leq b,\ Cx = d \tag{2.18}$$

where $p \in R^n$ and A, b, C, d are as in (2.1). For the Lipschitz continuity results for linear programs we have to restrict the norms employed to monotonic norms [6, 13] and have to drop $u \geq 0$ from (2.5).

2.4 Theorem (Lipschitz continuity of solutions of linear programs with respect to rhs perturbation) Let the linear program (2.18) have nonempty solution sets S^1 and S^2 for right hand sides (b^1, d^1) and (b^2, d^2) respectively. For each $x^1 \in S^1$ there exists an $x^2 \in S^2$ such that

$$\|x^1 - x^2\|_\infty \leq \nu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta \tag{2.19}$$

where $\|\cdot\|_\beta$ is some monotonic norm on R^{m+k} and

$$\nu_\beta(A; C) := \sup_{u,v} \left\{ \left\| \begin{matrix} u \\ v \end{matrix} \right\|_{\beta^*} \middle| \begin{matrix} \|uA + vC\|_1 = 1 \\ \text{Rows of } \begin{pmatrix} A \\ C \end{pmatrix} \text{ corresponding to nonzero} \\ \text{elements of } \begin{pmatrix} u \\ v \end{pmatrix} \text{ are lin. indep.} \end{matrix} \right\} \tag{2.20}$$

Proof Given $x^1 \in S^1$, let

$$A_I x^1 = b_I^1, \; A_J x^1 < b_J^1$$

where $I \cup J = \{1,2,\ldots,m\}$. Fix any $\bar{x}^2 \in S^2$ and let $I = I_1 \cup I_2$ where

$$I_1 := \{i \in I \,|\, A_i \bar{x}^2 = b_i^2\}, \; I_2 := \{i \in I \,|\, A_i \bar{x}^2 < b_i^2\}$$

Since $x = \bar{x}^2$ satisfies the system of constraints

$$(2.21)\qquad \begin{cases} \text{(i)} & A_{I_1}x = b^2_{I_1}, \\ \text{(ii)} & A_{I_2}\bar{x}^2 \leq A_{I_2}x,\ A_{I_2}x \leq b^2_{I_2} \\ \text{(iii)} & A_J x \leq b^2_J,\ Cx = d^2 \end{cases}$$

it follows that (2.21) is nonvacuous. Let x^2 be a solution of

$$(2.22)\qquad \min \|x - x^1\|_\infty \quad \text{s.t.} \quad (2.21).$$

Since (2.22) is a convex program, x^2 remains optimal after we remove any number of inactive constraints. For each $i \in I_2$, at least one of the two constraints of (2.21) (ii) is inactive because $A_i\bar{x}^2 < b^2_i$. So we can remove one inactive constraint for each $i \in I_2$ thus obtaining

$$(2.23)\qquad \|x^2 - x^1\|_\infty = \min \|x - x^1\|_\infty \ \text{s.t. (2.24)} = \min \|x - x^1\|_\infty \ \text{s.t. (2.21)}$$

where

$$(2.24)\qquad \begin{cases} \text{(i)} & A_{I_1}x = b^2_{I_1} \\ \text{(iia)} & A_K\bar{x}^2 \leq A_K x \\ \text{(iib)} & A_L x \leq b^2_L \\ \text{(iic)} & A_J x \leq b^2_J,\ Cx = d^2 \end{cases}$$

where $K \cup L = I_2$, $K \cap L = \phi$. So $I_1 \cup K \cup L \cup J = \{1,2,\ldots,m\}$ and I_1, K, L and J are all disjoint. On the other hand, since

$$A_K x^1 = b^1_K - b^2_K + b^2_K \geq b^1_K - b^2_K + A_k\bar{x}^2$$

it follows that $x = x^1$ satisfies the following system

$$(2.24') \qquad \begin{cases} \text{(i)} & A_{I_1} x = b^1_{I_1} \\ \text{(iia)} & b^1_K - b^2_K + A_K \bar{x}^2 \leq A_K x \\ \text{(iib)} & A_L x \leq b^1_L \\ \text{(iic)} & A_J x \leq b^1_J, \; Cx = d^1 \end{cases}$$

It follows by (2.23), (2.24'), Theorem 2.2 and the norm monotonicity that

$$\|x^1 - x^2\|_\infty \leq \mu_\beta \left(\begin{pmatrix} -A_K \\ A_L \\ A_J \end{pmatrix}; \begin{pmatrix} A_{I_1} \\ C \end{pmatrix} \right) \left\| \begin{matrix} -b^1_K + b^2_K \\ b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta$$

$$\leq \nu_\beta(A; C) \left\| \begin{matrix} b^1 - b^2 \\ d^1 - d^2 \end{matrix} \right\|_\beta$$

where $H = I_1 \cup L \cup J$ is the complement of K.

It remains to show that $x^2 \in S^2$. Since $x^1 \in S^1$, we have by the Karush-Kuhn-Tucker optimality conditions that

$$(2.25) \qquad A_I^T u_I^1 + C^T v^1 = p \quad \text{for some} \quad u_I^1 \geq 0 \quad \text{and some} \quad v^1.$$

Since both $\bar{x}^2$ and x^2 satisfy (2.21) it follows that

$$px^2 = u_I^1 A_I x^2 + v^1 C x^2 \geq u_I^1 A_I \bar{x}^2 + v^1 C \bar{x}^2 = p\bar{x}^2$$

and the proof is complete. □

2.5 Remark We note that Cook-Gerards-Schrijver and Tardos [3, Theorem 5] have a similar result to Theorem 2.4 for integer entries for A but without

the equality constraints $Cx = d$. However their Lipschitz constant is bigger than or equal to our Lipschitz constant. In fact their Lipschitz constant $n\Delta(A)$ is only for $\beta = \infty$, where ΔA is the maximum of the absolute values of the determinants of the square submatrices of A. We formalize the relation between the two Lipschitz constants as follows.

2.6 Proposition For integer A, $\nu_\infty(A; \phi) \leq n\Delta(A)$.

Proof For any u_I for which $\|u_I A_I\|_1 = 1$ and the rows of A_I are linearly independent, we can assume that

$$A_I = [B \quad N]$$

where B is a nonsingular square submatrix.

Let $q := u_I B$, then $\|q\|_1 \leq \|u_I A_I\|_1 = 1$ since $u_I B$ is a subvector of $u_I A_I$. It follows that

$$\|u_I\|_1 = \|(B^T)^{-1} q\|_1 \leq \|(B^T)^{-1}\|_1 \, \|q\|_1 \leq \|(B^T)^{-1}\|_1 = \max_i \sum_j |h_{ij}|$$

where h_{ij} is the (i,j) entry of B^{-1} [13, p. 22]. Hence

$$h_{ij} = \frac{1}{\det B}(-1)^{i+j} B_{ji},$$

where B_{ij} is the (i,j) cofactor of B which is the determinant of a square submatrix of A. Hence

$$|B_{ji}| \leq \Delta(A).$$

If A is integral $|\det B| \geq 1$ is an integer, hence

$$|h_{ij}| \leq \frac{1}{|\det B|} |B_{ji}| \leq \Delta(A).$$

Consequently

$$\|u_I\|_1 \leq \|(B^T)^{-1}\|_1 = \max_i \sum_j |h_{ij}| \leq n\Delta(A).$$

Since u_I is arbitrary, we have

$$\nu_\infty(A;\phi) = \sup \{\|u_I\|_1 \mid \|u_I A_I\|_1 = 1, \text{ rows of } A_I \text{ lin. indep.}\} \leq n\Delta(A). \qquad \square$$

2.7 Remark Note that it is not true that solutions of linear programs are Lipschitzian with respect to perturbations in the objective function coefficients as evidenced by the following simple example:

$$(2.26) \qquad \max\ (1+\delta)x_1 + x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 1,\ (x_1, x_2) \geq 0$$

The solution to this problem is:

$$x(\delta) = \begin{cases} (1,0) & \text{for } \delta > 0 \\ (0,1) & \text{for } -1 < \delta < 0 \end{cases}$$

Hence

$$\lim_{\delta \to 0+} \frac{\|x(\delta) - x(-\delta)\|}{2\delta} = \infty$$

and hence $x(\delta)$ is not Lipschitzian with respect to δ.

3. Linear Complementarity Problems

In this section we shall employ the Lipschitz constant $\mu_\beta(A; C)$ developed in Theorem 2.2 for linear inequalities and equalities to obtain a Lipschitz constant for linear complementarity problems with matrices that have positive principal minors [4] or which are hidden Z-matrices [14]. We will show by means of Example 3.4 that solutions of linear complementarity problems with a positive semidefinite matrix are not Lipschitz continuous with respect to right hand side perturbations.

We consider the linear complementarity problem (M, q) of finding an x in R^n such that

$$(3.1) \qquad Mx + q \geq 0,\ x \geq 0,\ x(Mx+q) = 0$$

where $M \in R^{n\times n}$ and $q \in R^n$. Note that given $J \subset \{1,\ldots,n\}$, any solution of the following system of $2n$ linear inequalities and equalities

$$(3.2) \qquad \begin{cases} M_j x + q_j \geq 0,\ x_j = 0, & j \in J \\ M_j x + q_j = 0,\ x_j \geq 0, & j \notin J \end{cases}$$

is a solution of (M, q). For $J \subset \{1,\ldots,n\}$ let $Q(J)$ denote the set of all q vectors for which (3.2) has a solution. It is easy to verify that $Q(J)$ is a closed convex cone. In fact it is called a complementary cone of (M, q) [11, p. 482]. It is also obvious that $\bigcup_{J\subset\{1,\ldots,n\}} Q(J)$ is the set of all q for which (M, q) is solvable. Define

$$(3.3) \qquad \sigma_\beta(M) := \max_{J\subset\{1,\ldots,n\}} \mu_\beta\left(\begin{bmatrix} M_J \\ I_{\bar{J}} \end{bmatrix}; \begin{bmatrix} I_J \\ M_{\bar{J}} \end{bmatrix}\right)$$

where μ_β is defined by (2.5) and $\bar{J}$ is the complement of J in $\{1,\ldots,n\}$. We shall prove (Theorem 3.3) that $\sigma_\beta(M)$ will serve as a Lipschitz constant for solutions of (M,q) when M is a P-matrix, that is a matrix with positive principal minors [4, 1], or more generally (Theorem 3.2) for perturbations of q such that the linear complementarity problem is uniquely solvable along the line joining the original q and the perturbed q. We will also establish Lipschitz continuity for solutions of (M,q) when M is a hidden Z-matrix (Theorem 3.5). We begin with a lemma. A related result to this lemma appears in [12].

3.1 Lemma Let q^1 and q^2 be fixed distinct vectors in R^n and let $q(t) := (1-t)q^1 + tq^2$ for $t \in [0,1]$. Assume that $(M,q(t))$ is solvable for $t \in [0,1]$. Then there exists a partition $0 = t_0 < t_1 < \ldots\ldots < t_N = 1$ such that for $1 \leq i \leq N$

$$(3.4) \qquad q(t_{i-1}) \in Q(J_i),\ q(t_i) \in Q(J_i) \text{ for some } J_i \subset \{1,\ldots,n\}$$

Proof Let

$$T(J) := \{t \mid t \in [0,1],\ q(t) \in Q(J)\}$$

for $J \subset \{1,\ldots,n\}$. It is easy to see that $T(J)$ is closed and convex and hence it is a closed interval which may degenerate to a single point or to the empty set. Since $(M,q(t))$ is solvable for $t \in [0,1]$ it follows that

$$[0,1] \subset \bigcup_{J \subset \{1,\ldots,n\}} T(J)$$

Let

$$L := \{[\ell_1, u_1], \ldots\ldots, [\ell_K, u_K]\}$$

be the set of maximal intervals in $\{T(J) \mid J \subset \{1,\ldots,n\}\}$, that is there is no other interval $T(J)$, $J \subset \{1,\ldots,n\}$ that properly contains $[\ell_i, u_i]$. By removing duplicates from L if needed, we can assume that $[\ell_1, u_1],\ldots,[\ell_K, u_K]$ are distinct and that $\ell_1 < \ell_2 < \ldots\ldots < \ell_K$. Since each $t \in [0,1]$ belongs $T(J)$ (for some $J \subset \{1,\ldots,n\}$) which is either in L or contained in some interval of L, we have that

$$[0,1] \subset \bigcup_{i=1}^{K} [\ell_i, u_i]$$

Thus $\ell_i \leq u_{i-1}$, otherwise (u_{i-1}, ℓ_i) would be an uncovered gap of $[0,1]$. Also $u_{i-1} < u_i$, otherwise $[\ell_i, u_i]$ would not be maximal because it would be contained in $[\ell_{i-1}, u_{i-1}]$.

Hence $\ell_1 = 0$, $\ell_{i-1} < \ell_i \leq u_{i-1} < u_i$ and $u_K = 1$. Let $0 = t_0 < t_1 < \ldots\ldots < t_N = 1$ be the sorted numbers of $\{\ell_1, u_1, \ell_2, u_2,\ldots,\ell_K, u_K\}$ with duplicates removed. Then each interval $[t_{i-1}, t_i]$ is contained in some interval $T(J_i)$ in L and so

$$q(t_{i-1}) \in Q(J_i) \quad \text{and} \quad q(t_i) \in Q(J_i) \qquad \square$$

We establish now the Lipschitz continuity of linear complementarity problems with unique solutions along the line segment $q(t) := (1 - t)q^1 + tq^2$, $t \in [0,1]$.

3.2 Theorem (Lipschitz continuity of uniquely solvable linear complementarity problems) Let q^1 and q^2 be points in R^n such that the linear complementarity problem $(M, q(t))$ with $q(t) := (1 - t)q^1 + tq^2$ has a unique solution for each $t \in [0,1]$. Then the unique solutions x^1 of (M, q^1) and x^2 of (M, q^2) satisfy

$$\|x^1 - x^2\|_\infty \leq \sigma_\beta(M)\|q^1 - q^2\|_\beta$$

where $\sigma_\beta(M)$ is defined by (3.3)

Proof There exist $0 = t_0 < t_1 < \ldots\ldots < t_N = 1$ with properties stated in Lemma 3.1. Let $x(t_i)$ be the unique solution of $(M, q(t_i))$. Since for $1 \leq i \leq N$, $q(t_{i-1})$ and $q(t_i)$ belong to $Q(J_i)$ for some $J_i \subset \{1,\ldots,n\}$, there exists a solution $y(t_{i-1})$ of $(M, q(t_{i-1}))$ such that by (2.4) and (3.3) it follows that

$$\|x(t_i) - y(t_{i-1})\|_\infty \leq \mu_\beta \begin{pmatrix} -M_{J_i} ; & I_{J_i} \\ I_{\bar{J}_i} ; & M_{\bar{J}_i} \end{pmatrix} \|q(t_i) - q(t_{i-1})\|_\beta$$

(3.5)

$$\leq \sigma_\beta(M)\ (t_i - t_{i-1})\|q^1 - q^2\|_\beta$$

where $\bar{J}_i$ is the complement of J_i in $\{1,\ldots,n\}$. Summing up for $i=1,\ldots,N$ gives

$$\sum_{i=1}^{N} \|x(t_i) - y(t_{i-1})\|_\infty \leq \sigma_\beta(M)\ \|q^1 - q^2\|_\beta$$

Since $(M, q(t_{i-1}))$ has a unique solution, $y(t_{i-1}) = x(t_{i-1})$. Hence

$$\|x_1 - x_2\|_\infty \leq \sum_{i=1}^{N} \|x(t_i) - x(t_{i-1})\|_\infty \leq \sigma_\beta(M)\ \|q^1 - q^2\|_\beta \qquad \square$$

Since for P-matrix M, the linear complementarity problem (M, q) has a unique solution for each $q \in R^n$ [10], the following theorem is an immediate corollary to Theorem 3.2.

3.3 Theorem (Lipschitz continuity of solutions of linear complementarity problems with P-matrices) Let M be a P-matrix. For each q^1 and q^2 in R^n the corresponding unique solutions x^1 and x^2 of (M, q^1) and (M, q^2) respectively, satisfy

$$\|x^1 - x^2\|_\infty \leq \sigma_\beta(M) \|q^1 - q^2\|_\beta$$

where $\sigma_\beta(M)$ is defined by (3.3).

The following example shows that solutions of positive semidefinite linear complementarity problems may not be Lipschitzian.

3.4 Example $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $q^1 = \begin{pmatrix} -\varepsilon \\ 1 \end{pmatrix}$, $q^2 = \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix}$, $\varepsilon > 0$

$$q(t) = \begin{pmatrix} -\varepsilon + 2\varepsilon t \\ 1 \end{pmatrix}, \quad t_0 = 0, \; t_1 = \tfrac{1}{2}, \; t_2 = 1$$

$$J_1 = \phi, \; J_2 = \{1,2\}$$

$q(t_0)$ and $q(t_1)$ are in $Q(J_1) = \{q \in R^2 \,|\, q_1 \leq 0, \; q_2 \geq 0\}$

$q(t_1)$ and $q(t_2)$ are in $Q(J_2) = R^2_+$

$$y(t_0) = x(t_0) = \begin{pmatrix} 1 \\ \varepsilon - 2\varepsilon t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

$$x(t_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In order to satisfy (3.5), $y(t_1)$ must be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. However (3.5) also requires that $x(t_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $x(t_1) \neq y(t_1)$ and the proof of Theorem 3.2 fails.

In fact since $\lim_{\varepsilon \to 0} \frac{\|x(t_2) - x(t_0)\|_\infty}{\|q^2 - q^1\|_\infty} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} = \infty$, the solutions of the problem cannot be Lipschitzian.

We conclude by showing that other linear complementarity problems which can be formulated as linear programs [7] have solutions which are Lipschitzian with respect to their right hand sides as a consequence of Theorem 2.4. In particular if M satisfies the condition of Theorem 2 of [7] with $c = 0$, that is

$$MZ_1 = Z_2,\ rZ_1 + sZ_2 > 0,\ (r,s) \geq 0 \tag{3.6}$$

for some $n \times n$ Z-matrices Z_1 and Z_2, and some n-vectors r and s, then a solution to such a linear complementarity problem is obtained by solving the single linear program

$$\min px \quad \text{s.t. } Mx + q \geq 0,\ x \geq 0$$

where $p = r + M^T s$, and hence p is independent of q. In the terminology of [14], such a matrix M is called a hidden Z-matrix and is a generalization of Z-matrix which includes such matrices as those with a strictly dominant diagonal, and all matrices of Table 1 of [7] except cases 12 to 14.

3.5 Theorem (Lipschitz continuity of solutions of linear complementarity problems with hidden Z-matrices) Let M be a hidden Z-matrix, that is M satisfies (3.6). For each q^1 and q^2 in R^n for which (M, q^1) and (M, q^2) are solvable, there exist solutions x^1 of (M, q^1) and x^2 of (M, q^2) such as

$$\|x^1 - x^2\|_\infty \leq \nu_\beta \begin{pmatrix} M \\ I \end{pmatrix}; \phi \|q^1 - q^2\|_\beta$$

where $\|\cdot\|_\beta$ is some norm on R^n and ν_β is defined by (2.20).

Proof By [7], there exist solutions of (M, q^1) and (M, q^2) which are obtained by solving the linear programs

$$\min \{px \mid Mx + q^1 \geq 0,\ x \geq 0\}$$

$$\min \{px \mid Mx + q^2 \geq 0,\ x \geq 0\}$$

where p is a fixed vector independent of q^1 and q^2. The conclusion of the theorem follows immediately from Theorem 2.4. □

We note that for the case of a strictly diagonally dominant and hence positive definite matrix M, (M, q) is uniquely solvable for each q in R^n, and the Lipschitz continuity of the solution follows also from either Theorem 3.5 or Theorem 3.3.

References

1. A. Berman & R. J. Plemmons: "Nonnegative matrices in the mathematical sciences", Academic Press, New York 1979.

2. S. D. Conte & C. de Boor: "Elementary numerical analysis", 3rd Edition, McGraw-Hill, New York 1980.

3. W. Cook, A.M.H. Gerards, A. Schrijver & É. Tardos: "Sensitivity results in integer linear programming", manuscript, presented at Mathematische Optimierung, Oberwolfach January 6-12, 1985.

4. R. W. Cottle & G. B. Dantzig: "Complementary pivot theory in mathematical programming", Linear Algebra and Its Applications 1, 1968, 103-125.

5. D. Gale: "The theory of linear economic models", McGraw-Hill, New York 1960.

6. A. S. Householder: "The theory of matrices in numerical analysis", Blaisdell Publishing, New York 1964.

7. O. L. Mangasarian: "Characterization of linear complementarity problems as linear programs", Mathematical Programming Study 7, 1978, 74-87.

8. O. L. Mangasarian: "A condition number for linear inequalities and linear programs", in "Methods of Operations Research 43", Proceedings of 6. Symposium über Operations Research Universität Augsburg, September 7-9, 1981, G. Bamberg & O. Opitz (editors), Verlagsgruppe Athenäum/Hain/Scriptor/Hanstein, Konigstein 1981, 3-15.

9. O. L. Mangasarian & T.-H. Shiau: "A variable-complexity norm maximization problem", University of Wisconsin, Madison, Computer Sciences Technical Report #568, December 1984.

10. K. G. Murty: "On the number of solutions of the complementarity problem and spanning properties of complementarity cones", Linear Algebra and Its Applications 5, 1972, 65-108.

11. K. G. Murty: "Linear and combinatorial programming", Wiley, New York 1976.

12. K. G. Murty: "Linear complementarity, linear and nonlinear programming", Heldermann Verlag, West Berlin 1985.

13. J. M. Ortega: "Numerical analysis a second course", Academic Press, New York 1972.

14. J.-S. Pang: "Hidden Z-matrices with positive principal minors", Linear Algebra and Its Applications 23, 1979, 201-215.

15. S. M. Robinson: "Generalized equations and their solutions, Part I: Basic theory", Mathematical Programming Study 10, 1979, 128-141.

16. G. W. Stewart: "Introduction to matrix computations", Academic Press, New York 1973.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2838	2. GOVT ACCESSION NO. AD-A160952	3. RECIPIENT'S CATALOG NUMBER ?
4. TITLE (and Subtitle) Lipschitz Continuity of Solutions of Linear Inequalities, Programs and Complementarity Problems		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) O. L. Mangasarian and T.-H. Shiau		8. CONTRACT OR GRANT NUMBER(s) MCS-8200632 & MCS-8420963 DAAG29-80-C-0041 DMS-8210950, Mod. 1
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 5 - Optimization and Large Scale Systems
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE July 1985 13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linear inequalities, linear programming, linear complementarity problems, Lipschitz continuity, perturbation analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that solutions of linear inequalities, linear programs and certain linear complementarity problems (e.g. those with P-matrices or Z-matrices but not semidefinite matrices) are Lipschitz continuous with respect to changes in the right hand side data of the problem. Solutions of linear programs are not Lipschitz continuous with respect to the coefficients of the objective function. The Lipschitz constant given here is a generalization of the role played by the norm of the inverse of a nonsingular matrix in bounding the perturbation of the solution of a system of equations in terms of a right hand side perturbation.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

FILMED

12-85

DTIC