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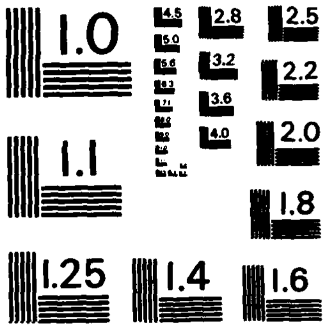

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A NOTE ON
AN INTEGRATED CAUCHY FUNCTIONAL EQUATION*

by

Ka-Sing Lau
University of Pittsburgh

and

Hua-Min Gu
South China Normal University

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Center for Multivariate Analysis
515 Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

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ABSTRACT

In characterizing the semistable law, Shimizu reduced the problem into solving a ~~the~~ equation $H(x) = \int_0^\infty H(x+y)\,d(\mu-\nu)(y), \quad x \geq 0$ where mu μ and upsilon ν infinity document are given positive measures on $[0,\infty)$. In this ~~note, we~~ obtains a simple proof and shows that some of his conditions can be weakened.

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§1. INTRODUCTION

Let μ be a positive regular Borel measure defined on $[0,\infty)$, we call the following equation

$$f(x) = \int_0^\infty f(x+y)d\mu(y), \text{ a.e. } x \geq 0 \tag{1.1}$$

an integrated Cauchy functional equation (ICFE(μ)). Lau and Rao (1982), Ramachandran (1982), gave two elementary methods to characterize all non-negative locally integrable solutions f of the ICFE(μ) as

$$f(x) = p(x)\, e^{\alpha x} \text{ a.e.} \tag{1.2}$$

where p is a periodic function of periods $\tau \in \text{supp}\,\mu$ and

$$\int_0^\infty e^{\alpha x} d\mu(x) = 1.$$

The theorem was used to characterize probability distributions arising from the strong lack of memory property, conditional expectation, record value problem, order statistics, Pareto Law (Lau and Rao (1982)). Generalizations of (1.1) were also investigated and used to study a damage model of Rao (Alzaid, Rao, Shanbhag (1983), Lau and Rao (1984)).

In characterizing the characteristic function ϕ of a random variable which satisfies the following generalized semistable law:

$$\phi(t) = \prod_{i=1}^{\infty} \phi^{\gamma_{2i}}(\beta_{2i}t) \prod_{i=1}^{\infty} \phi^{\gamma_{2i-1}}(\beta_{2i-1}t), \quad t \in R$$

$0 < \beta_i < 1$, $\gamma_i > 0$, one is confronted with solving the following equation

$$f(t) = \int_0^\infty f(x+y)\,d(\mu-\nu)(y), \quad x \geq 0 \tag{1.3}$$

where μ and ν are positive regular Borel measures on $[0,\infty)$ (Kagan, Linnik and Rao (1973), Shimizu (1978), Shimizu and Davies (1981)). Shimizu (1978) classified the solutions f of (1.2) under certain growth conditions on μ, ν and on f. In this note, we apply the result of the ICFE(μ) (1.1) and obtained a much simple proof of Shimizu's theorem. The hypotheses on μ and ν turn out to be redundant.

The authors wish to thank Professor C. R. Rao for bringing their attention to this problem.

§2. THE THEOREM

We assume that μ and ν are positive regular Borel measures on $[0,\infty)$, $\mu,\nu \not\equiv 0$.

PROPOSITION 2.1: Let f, g be nonnegative locally integrable solutions on $[0,\infty)$ satisfy the following equations

$$\begin{aligned} f(x) &= \int_0^\infty f(x+y)\,d\mu(y) + \int_0^\infty g(x+y)\,d\nu(y) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e.} \quad x \geq 0. \\ g(x) &= \int_0^\infty g(x+y)\,d\mu(y) + \int_0^\infty f(x+y)\,d\nu(y) \end{aligned} \tag{2.1}$$

Then $f(x) = p(x)\,e^{\alpha x}$, $g(x) = q(x)\,e^{\alpha x}$ a.e. where p, q are periodic functions with periods τ for $\tau \in \operatorname{supp}(\mu + \sum_{n=0}^{\infty} \mu^n * \nu^2)$ $(\mu^n = \mu * \ldots * \mu)$ and α satisfies

$$\int_0^\infty e^{\alpha x}\,d(\mu+\nu)(x) = 1.$$

PROOF: By adding the two equations in (2.1), we have

$$(f+g)(x) = \int_0^\infty (f+g)(x+y)\,d(\mu+\nu)(y), \quad \text{a.e.} \quad x \geq 0.$$

(1.2) implies that

$$(f+g)(x) = r(x)\,e^{\alpha x} \quad \text{a.e.}$$

where r is a periodic function with periods $\tau \in \mathrm{supp}(\mu+\nu)$ and

$$\int_0^\infty e^{\alpha x}\, d\mu(x) = 1.$$

We will assume, without loss of generality, that $\mu_1 + \mu_2$ is a probability measure so that $\alpha = 0$. Consider the following identities:

$$\begin{aligned} f(x) &= \int_0^\infty f(x+y)\,d\mu(y) + \int_0^\infty g(x+y)\,d\nu(y) \\ &= \int_0^\infty f(x+y)\,d\mu(y) + \int_0^\infty f(x+y)\,d\nu^2(y) + \int_0^\infty g(x+y)\,d(\mu * \nu)(y) \\ &= \int_0^\infty f(x+y)\,d\mu(y) + \sum_{n=0}^{k-1} \int_0^\infty f(x+y)\,d(\mu^n * \nu^2)(y) + \int_0^\infty g(x+y)\,d(\mu^k * \nu)(y). \end{aligned}$$

Since $\mu + \nu$ is a probability measure, the total variation $||\mu||$ of μ is strictly less than 1. Also since

$$0 \le g(x) \le r(x) \quad \text{a.e.}$$

and r is bounded, we can conclude that

$$\lim_{k\to\infty} \int_0^\infty g(x+y)\,d\mu^k * \nu = 0, \text{ a.e. } x \ge 0$$

and

$$f(x) = \int_0^\infty f(x+y)\,d\omega(y), \text{ a.e. } x \ge 0 \tag{2.2}$$

where $\omega = \mu + \sum_{n=0}^{\infty} \mu^n * \nu^2$. Note that

$$||\tau|| = ||\mu|| + \sum_{n=0}^{\infty} ||\mu||^n \cdot ||\nu||^2$$

$$= ||\mu|| + \frac{1}{1 - ||\mu||} ||\nu||^2$$

$$= ||\mu|| + ||\nu||$$

$$= 1,$$

the solution f of (1.2) hence equals p a.e. where p is a periodic function with periods $\tau \in \text{supp}(\mu)$. Similarly, we can show that $g(x) = q(x)$ a.e. where q has the same property as p.

<u>COROLLARY 2.2</u>: Let μ, ν, $f(x) = p(x)\, e^{\alpha x}$ and $g(x) = q(x)\, e^{\alpha x}$ be as in Theorem 2.1. Then either

(i) If there exists a $\rho > 0$ such that

$$\text{supp } \mu \subseteq \{2\rho, 4\rho, 6\rho, \ldots\} \quad \text{and} \quad \text{supp } \nu \subseteq \{\rho, 3\rho, 5\rho, \ldots\},$$

then

$$p(x+\rho) = q(x), \quad q(x+\rho) = p(x) \quad \text{a.e.,} \quad x \geq 0, \text{ or}$$

(ii) $p = q$ for the other cases.

PROOF: Without loss of generality, we assume that $\mu+\nu$ is a probability measure. Let

$$A(\rho) = \{\rho, 2\rho, 3\rho, \ldots\}, \quad B(\rho) = \{\rho, 3\rho, 5\rho, \ldots\}.$$

(i) By assumption, $\operatorname{supp}\mu \subseteq A(2\rho)$ implies that

$$\operatorname{supp}\left(\mu + \sum_{n=0}^{\infty} \mu^n * \nu^2\right) \subseteq A(2\rho).$$

It follows that p,q have periods in $A(2\rho)$. By substituting p,q into (2.2), we have

$$\left.\begin{aligned} p(x) &= p(x)\,\mu[0,\infty) + \int_0^\infty q(x+y)\,d\nu(y) \\ q(x) &= q(x)\,\mu[0,\infty) + \int_0^\infty p(x+y)\,d\nu(y). \end{aligned}\right\} \quad \text{a.e. } x \geq 0. \tag{2.3}$$

From the first equation, and make use of $(\mu+\nu)[0,\infty) = 1$, we have for $x \geq 0$

$$\begin{aligned} 0 &= \int_0^\infty (p(x) - q(x+y))\,d\nu(y) \\ &= \sum_{n=0}^{\infty} (p(x) - q(x+(2n+1)\rho)\nu((2n+1)\rho) \quad (\text{since } \operatorname{supp}\nu \subseteq B(\rho)) \\ &= \sum_{n=0}^{\infty} (p(x) - q(x+\rho))\nu((2n+1)\rho) \quad (\text{since } q \text{ has periods in } A(2\rho) \\ &= (p(x) - q(x+p)) \sum_{n=0}^{\infty} \nu((2n+1)\rho). \end{aligned} \tag{2.4}$$

This implies that

$$q(x+\rho) = p(x) \quad \text{a.e.} \quad x \geq 0.$$

By applying the same argument to the second equation of (2.3), we have

$$p(x+\rho) = q(x) \quad \text{a.e.}$$

(ii) We divide it into two cases: (a) $\operatorname{supp}\mu \cup \operatorname{supp}\nu$ generates a lattice but does not satisfy (i), (b) $\operatorname{supp}\mu \cup \operatorname{supp}\nu$ does not generate a lattice.

In case (a) let $\rho > 0$ be the largest real number so that

$$\operatorname{supp}\mu \cup \operatorname{supp}\nu \subseteq A(\rho).$$

Since (i) cannot be satisfied, then either (α) $\operatorname{supp}\mu \not\subseteq A(2\rho)$ or (β) $\operatorname{supp}\mu \subseteq A(2\rho)$, $\operatorname{supp}\nu \not\subseteq B(\rho)$. In ($\alpha$), the greatest common divisor of members in $\operatorname{supp}(\mu + \sum_{n=0}^{\infty} \mu^n * \nu^2)$ is ρ. Hence p,q are periodic functions with period ρ. By substituting into (2.1), we have

$$p(x) = p(x)\mu[0,\infty) + q(x)\nu[0,\infty). \quad \text{a.e.}$$

This implies that $p(x) = q(x)$ a.e. In (β), by using the same argument as in (i), we obtain

$$p(x) = q(x)a + q(x+\rho)b \quad \text{a.e.}$$

where $a = \sum_{n=0}^{\infty} \nu((2n)\rho)$, $b = \sum_{n=0}^{\infty} \nu((2n+1)\rho)$ (instead of (2.4)). Similarly,

$$q(x) = p(x)a + p(x+\rho)b \quad \text{a.e.}$$

By a simple calculation, the two equations imply

$$p(x) = q(x) = 0 \quad \text{a.e.}$$

For case (b), since $\operatorname{supp}\mu \cup \operatorname{supp}\nu$ does not generate a lattice, it is easy to show that p,q are constants, and (2.1) implies that $p=q$ a.e.

<u>THEOREM 2.3</u>: Let $\mu,\nu \neq 0$ be positive measures on $[0,\infty)$ such that $\mu+\nu$ is a probability measure, and H is a bounded measurable function on $[0,\infty)$ satisfies

$$H(x) = \int_0^\infty H(x+y)d(\mu(y) - \nu(y)) \tag{2.5}$$

Then either one of the following hold:

(i) if there exists a $\rho > 0$ such that

$$\operatorname{supp}\mu \subseteq \{2\rho, 4\rho, \ldots\}, \quad \operatorname{supp}\nu \subseteq \{\rho, 3\rho, 5\rho, \ldots\}$$

then $H(x+\rho) = -H(\rho)$ a.e.

(ii) $H(x) = 0$ a.e. otherwise.

<u>PROOF</u>: Let G_1 be a nonnegative locally integrable solution of the ICFE$(\mu+\nu)$. Let K be the bound of H, and let

$$G(x) = G_1(x) + K, \quad x \geq 0.$$

Then G also satisfies the ICFE $(\mu+\nu)$, i.e.

$$G(x) = \int_0^\infty G(x+y)d(\mu+\nu)(y) \quad \text{a.e.} \quad x \geq 0. \tag{2.6}$$

By adding and subtracting (2.5) and (2.6), we have

$$(G+H)(x) = \int_0^\infty (G+H)(x+y)d\mu(y) + \int_0^\infty (G-H)(x+y)d\nu(y)$$

$$(G+H)(x) = \int_0^\infty (G-H)(x+y)d\mu(y) + \int_0^\infty (G+H)(x+y)d\nu(y)$$

and $G+H,\ G-H \geq 0$. Theorem 2.1 implies that

$$(G+H)(x) = p(x), \quad (G-H)(x) = q(x) \quad \text{a.e.}$$

where p,q are periodic functions with periods in $\mathrm{supp}(\mu + \sum_{n=0}^{\infty} \mu^n * \nu^n)$. Therefore

$$H(x) = \frac{1}{2}(p(x)-q(x)) \quad \text{a.e.}$$

The conclusion about H follows from Corollary 2.2 .

COROLLARY 2.4: The conclusions of Theorem 2.3 also hold if we replace "H is bounded" by "for each $y \geq 0$, $H(x+y)-H(x)$ is a bounded function on x".

PROOF: For each fixed y, apply Theorem 2.3 to $H(x+y)-H(x)$. In case (i), we have

$$H(x+y+\rho) - H(x+\rho) = -(H(x+y)-H(x)) \quad \text{a.e.}$$

This implies that

$$H(x+y+\rho) + H(x+y) = -(H(x+\rho) + H(x)) \quad \text{a.e.}$$

As y is arbitrary, we can conclude that

$$H(x+\rho) + H(x) = 0, \quad \text{a.e.}$$

For case (ii), we have for each y,

$$H(x+y) - H(x) = 0, \text{ a.e.}$$

This implies that H is a constant, and (2.5) shows that it must be zero.

REMARK: It is clear that if α satisfies

$$\int_0^{\infty} e^{\alpha x} d(\mu-\nu)(x) = 1$$

then $H(x) = e^{\alpha x}$ is a positive unbounded solution of (2.5), however we are still unable to characterize all those solutions.

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In characterizing the semistable law, Shimizu reduced the problem into solving the quotation $H(x) = \int_0^\infty H(x+y)d(\mu-\nu)(y)$, $x \geq 0$ where μ and ν are given positive measures on $[0,\infty)$. In this note, we obtain a simple proof and show that some of his conditions can be weakened.

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