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CONSTRUCTION OF EXPONENTIAL MARTINGALES FOR COUNTING
PROCESSES(U) MASSACHUSETTS UNIV AMHERST DEPT OF
MATHEMATICS AND STATISTICS W A ROSENKRANTZ

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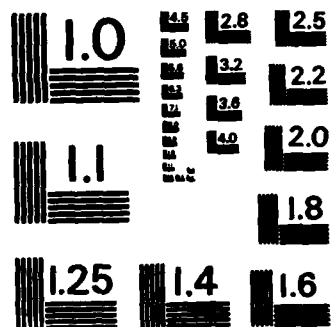
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Let $N(t)$ be a counting process with continuous $A(t)$ and $f(t)$ a bounded predictable process. If $E(\exp(2|f|N(t))) < \infty$ and $E(\exp(2(1 + \exp|f|)A(t))) < \infty$ then it is shown that $z(t) = \exp\{-\int_0^t f(u)dN(u) - \int_0^t [\exp(-f(u)) - 1]dA(u)\}$ is a martingale.

This is a partial extension of a theorem of Kabanov, Liptser, Shiryaev (1980) who assumed $A(t) \leq c$ but did not assume $A(t)$ is continuous.

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Construction of Exponential Martingales
for Counting Processes

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Abstract

also we have $\frac{d}{dt} \int_{\Omega} u^2 dx = 2 \int_{\Omega} u \frac{du}{dt} dx$
 $= 2 \int_{\Omega} u ((\exp f^2(t)) - \gamma u^2) dx$

→ this is negative for γ big enough
 for γ big enough

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Chief, Federal Bureau of Investigation

1. Introduction

If $p(t)$ is a standard Poisson process of unit intensity with $P(p(t) = j) = \exp(-t)t^j/j!$, $j = 0, 1, \dots$ then it is easy to see that

(1.1) $z(t) = \exp\{-\lambda p(t) - (e^{-\lambda} - 1)t\}$ is a martingale for every $\lambda \in \mathbb{R}$.

Formula (1) suggests that if f is bounded and predictable with respect to the filtration $F(t) = \sigma(p(s), 0 \leq s \leq t)$ then

$$(1.2) \quad z(t) = \exp\left\{-\int_0^t f(u)dp(u) - \int_0^t [\exp(-f(u)) - 1]du\right\}$$

is a martingale also. Note that by putting $f(u) \equiv \lambda$ in (1.2) we obtain (1.1). More generally Kabanov-Liptser-Shiryaev (1980) (henceforth abbreviated to K-L-S) have proved the following theorem.

THEOREM 1: Let $N(t)$ denote a counting process with continuous compensator $A(t)$ satisfying the condition $A(t, \omega) \leq c$ and let $f(t, \omega)$ denote a bounded predictable process with respect to the filtration $F(t) = \sigma(N(s), 0 \leq s \leq t)$. Then

$$(1.3) \quad z(t) = \exp\left\{-\int_0^t f(u)dN(u) - \int_0^t [\exp(-f(u)) - 1]dA(u)\right\}$$

is a martingale.

Remarks: (1) K-L-S use the martingale $z(t)$ to give a very nice proof of a Poisson limit theorem for point processes due to T. Brown (1978).

(ii) If we recall that the compensator of the Poisson process $A(t) = t$ then we see at once that the condition $A(t) \leq c$ is too restrictive since it excludes the Poisson process! These remarks suggest that a more natural condition to impose on $A(t)$ in order for the process $z(t)$ defined by (1.3) above to be a martingale is the following one:

$$(1.4) \quad E(\exp(cA(t))) < \infty, \quad E(\exp(dN(t))) < \infty$$

for non-negative constants c and d which may depend on $|f|$.

It is the purpose of this paper to give a statement and proof of just such an extension to Theorem 1.

THEOREM 2: Let $N(t)$ denote a counting process with continuous compensator $A(t)$ and let $f(t, \omega)$ denote a bounded predictable process.

(i) If $A(t)$ satisfies condition (1.4) with $c = 2(1 + \exp(|f|))$ and $d = 2|f|$ then the process $z(t)$ (defined at (1.3)) is a martingale.

(ii) If in addition $f(t, \omega) \geq 0$ and $A(t)$ satisfies condition (1.4) with $c = 1$ and $d = 0$ then $z(t)$ is a martingale.

When the hypothesis that $A(t)$ be continuous is dropped K-L-S (1980) have shown that

$$(1.8) \quad z(t) = \exp\left\{-\int_0^t f(u) dN(u) - \int_0^t [\exp(-f(u)) - 1] du - \sum_{s \leq t} \phi(\exp(f(s)) - 1) \Delta A(s)\right\}$$

is a martingale provided $A(t) \leq c$ where $\phi(x) = \ln(1 + x) - x$. We conjecture that (1.6) remains true under the less restrictive condition (1.4); the proof of this result however has so far escaped us.

Outline of the Proof: We first prove Theorem 2 in the special case where $N(t) = p(t)$. The general case is then reduced to this one via a random time change. A similar method is used by Ikeda-Watanabe in their proof of a Theorem of Novikov's cf. IKEDA-WATANABE (1981) Theorem 5.3 pp 142-144.

NOTATION: Whenever convenient we will drop the ω and write $f(t)$ for $f(t, \omega)$, $A(t)$ for $A(t, \omega)$ etc.

2. Proof of Theorem 2:

Recall the setting of the introduction: $p(t)$ is a standard Poisson process of unit intensity and $F(t) = \sigma(p(s); 0 \leq s \leq t)$.

LEMMA 1: If $X(\omega)$ is $F(s)$ measurable and bounded then

$$E(\exp(-X(\omega)[p(t) - p(s)] | F(s)) = \exp([t - s](\exp(-X(\omega)) - 1)).$$

This is a consequence of the following lemma, the proof of which is left to the reader.

LEMMA 2: Let $h(x, y)$ be a bounded Borel measurable function and suppose $X(\omega)$ and $Y(\omega)$ are random variables such that $X(\omega)$ is G measurable. Then

$$\begin{aligned} E(h(X, Y) | G) &= g(X(\omega), \omega) \quad \text{where} \\ g(x, \omega) &= E(h(x, Y) | G). \end{aligned}$$

LEMMA 3: Let $f(t, \omega)$ be a bounded $F(t)$ adapted process with left continuous paths (and hence predictable). Then

$$\exp\left(-\int_0^t f(u) dp(u) - \int_0^t [\exp(-f(u)) - 1] du\right) \text{ is a martingale.}$$

Proof: Assume f is a simple function i.e.

$$(2.1) \quad f(u, \omega) = \sum_{i=0}^{n-1} f(t_i, \omega) I_{(t_i, t_{i+1}]}(u) \quad \text{where}$$

$0 = t_0 < t_1 < \dots < t_n$. It suffices to show that

$$(2.2) \quad E\left(\exp\left(-\int_s^t f(u) dp(u) - \int_s^t [\exp(-f(u)) - 1] du\right) \middle| F(s)\right) = 1 \quad 0 \leq s < t.$$

Assume $t_i \leq s < t \leq t_{i+1}$ so $\int_s^t f(u) dp(u) = f(t_i, \omega)(p(t) - p(s))$ and $\int_s^t [\exp(-f(u)) - 1] du = (t - s)(\exp(-f(t_i, \omega)) - 1)$ which is $F(s)$ measurable.

Consequently by Lemma 1

$$\begin{aligned} E\left(\exp\left(-\int_s^t f(u) dp(u)\right) \middle| F(s)\right) &= E(\exp(-f(t_i, \omega)(p(t) - p(s))) \middle| F(s)) \\ &= \exp\{(t - s)[\exp(-f(t_i, \omega)) - 1]\} \quad \text{which yields (2.2).} \end{aligned}$$

If $s < t_{i+1} < t$ then we can reduce it to the case just considered by successively conditioning on $F(t_{i+1})$ and then $F(s)$ etc.

For the next step we invoke Lemma 5.3 on p. 175 of Liptser-Shiryaev V.1 (1977) which asserts that sample functions of the form (2.1) are dense in the class of predictable functions satisfying the condition

$$(2.3) \quad E\left(\int_0^t (f(u, \omega))^2 dA(u)\right) < \infty.$$

Here density is of course understood to be with respect to the norm

$$E\left(\int_0^t (f(u,\omega) - g(u,\omega))^2 dA(u)\right)^{1/2}.$$

Bring in the square integrable martingale $M(t) = p(t) - t$ and recall that the compensator of $(M(t))^2$ is t . Let $f_n(t,\omega)$ denote a sequence of simple functions of the form (2.1) satisfying the conditions $|f_n| \leq |f|$ and

$$(2.4) \quad \lim_{n \rightarrow \infty} E\left(\int_0^t (f_n(u,\omega) - f(u,\omega))^2 du\right) = 0, \text{ i.e. set } A(u) = u \text{ in (2.3);}$$

It then follows that

$$(2.5) \quad \lim_{n \rightarrow \infty} E\left(\left|\int_0^t f_n(u,\omega) dM(u) - \int_0^t f(u,\omega) dM(u)\right|^2\right) = 0.$$

Applying Schwarz's inequality and (2.4) we see that

$$(2.6) \quad \lim_{n \rightarrow \infty} E\left(\int_0^t |f_n(u) - f(u)| du\right) \leq \sqrt{t} \lim_{n \rightarrow \infty} E\left(\int_0^t |f_n(u) - f(u)|^2 du\right) = 0.$$

In addition the condition $|f_n| \leq |f|$ implies that

$$\left|\int_0^t \exp(-f_n(u)) du - \int_0^t \exp(-f(u)) du\right| \leq K \int_0^t |f_n(u) - f(u)| du;$$

thus

$$(2.7) \quad \lim_{n \rightarrow \infty} E\left(\left|\int_0^t \exp(-f_n(u)) du - \int_0^t \exp(-f(u)) du\right|\right) = 0.$$

Next we observe that $\int_0^t f_n(u) dp(u) + \int_0^t [\exp(-f_n(u)) - 1] du =$
 $\int_0^t f_n(u) dM(u) + \int_0^t \exp(-f_n(u)) du$ and that

$$(2.8) \quad \left| \int_0^t f_n(u) dp(u) + \int_0^t [\exp(-f_n(u)) - 1] du \right| \leq |f|p(t) + t(1 + \exp(|f|)) .$$

Consequently

$$(2.9) \quad \left| \exp\left\{-\int_0^t f_n(u) dp(u) + \int_0^t [\exp(-f_n(u)) - 1] du\right\} \right| \leq \exp(|f|p(t) + t(1 + \exp|f|))$$

which is obviously an integrable function. It is clear we can now extract a subsequence $f_{n_i}(u)$ such that

$$(2.10) \quad \begin{cases} (a) & \lim_{n_i \rightarrow \infty} \int_0^t f_{n_i}(u) dM(u) = \int_0^t f(u) dM(u) \quad \text{a.s.} \\ (b) & \lim_{n \rightarrow \infty} \int_0^t \exp(-f_{n_i}(u)) du = \int_0^t \exp(-f(u)) du \quad \text{a.s.} \end{cases}$$

On the other hand we've already shown for simple functions f_n , that

$$(2.11) \quad E\left(\exp\left\{-\int_s^t f_{n_i}(u) dp(u) - \int_s^t [\exp(-f_{n_i}(u)) - 1] du\right\} \middle| \mathcal{F}(s)\right) = 1.$$

The bound (2.9) and the existence of the limits in (2.10) now permit us to pass to the limit in (2.11) and deduce that (2.2) remains valid for bounded predictable f . Q.E.D.

LEMMA 4: Let $N(t)$ be a counting process with a continuous strictly increasing compensator $A(t)$ satisfying condition (1.4) with $c = 2(1 + \exp|f|)$ and $d = 2|f|$ (or $c = 1, d = 0$ if $f(t) \geq 0$). Then

$$(2.12) \quad z(t) = \exp\left\{-\int_0^t f(u) dN(u) - \int_0^t [\exp(-f(u)) - 1] dA(u)\right\}$$

is a martingale.

Proof: Bring in the random time change $A^{-1}(t) = \inf\{u: A(u) > t\}$ and note that $A^{-1}(t)$ is also continuous and strictly increasing. It is easy to see that $N(A^{-1}(t))$ is again a counting process with compensator $A(A^{-1}(t)) = t$ and therefore $N(A^{-1}(t)) = p(t)$ is a Poisson process relative to the filtration $F'(t) = F(A^{-1}(t))$. Assume f is left continuous which implies that $f(A^{-1}(t))$ is predictable and therefore by Lemma 3

$$(2.13) \quad \exp\left\{-\int_0^t f(A^{-1}(u))dN(A^{-1}(u)) - \int_0^t [\exp(-f(A^{-1}(u))) - 1]du\right\} = v(t)$$

is a martingale. Now $A(t)$ is a stopping time relative to the filtration $F'(t) = F(A^{-1}(t))$ and so Doob's optimal stopping theorem implies $v(t \wedge A(s))$ is also a martingale. Let us assume that $f(t) \geq 0$ which, combined with the fact that $N(A^{-1}(u))$ is monotone increasing, implies the inequality

$$(2.14) \quad -\int_0^{t \wedge A(s)} f(A^{-1}(u))dN(A^{-1}(u)) - \int_0^{t \wedge A(s)} [\exp(-f(A^{-1}(u))) - 1]du \leq t \wedge A(s).$$

Consequently $0 \leq v(t \wedge A(s)) \leq \exp(t \wedge A(s)) \leq \exp(A(s))$. We may now apply the dominated convergence to conclude $\lim_{t \rightarrow \infty} v(t \wedge A(s)) = v(A(s))$ in L_1 and hence $v(A(s))$ itself is a martingale. Now

$$\begin{aligned} (2.15) \quad v(A(s)) &= \exp\left\{-\int_0^{A(s)} f(A^{-1}(u))dN(A^{-1}(u)) - \int_0^{A(s)} [\exp(-f(A^{-1}(u))) - 1]du\right\} \\ &= \exp\left\{-\int_0^s f(u)dN(u) - \int_0^s [\exp(-f(u)) - 1]dA(u)\right\} \\ &= z(s) \text{ is a martingale.} \end{aligned}$$

We have thus proved (ii) of Theorem 2, at least in the case where f is continuous and $A(t)$ is strictly increasing. It is easy to extend this

result to simple functions of the form (2.1) by means of the following device: for each i construct a sequence of non-negative continuous functions $\phi_{k,i}(t)$, with compact support, such that $\lim_{k \rightarrow \infty} \phi_{k,i}(t) = I_{(t_i, t_{i+1}]}(t)$. Set $f_k(t) = \sum_{i=0}^{n-1} f(t_i, \omega) \phi_{k,i}(t)$ and note that we can arrange matters so that $f_k(t)$ is $F(t)$ adapted as well. Clearly $\lim_{k \rightarrow \infty} f_k(t) = f(t)$ in the sense of bounded pointwise convergence and from this it is easy to see that (ii) of Theorem 2 remains valid for non-negative simple functions of the form (2.1). The extension to arbitrary non-negative bounded predictable processes via the methods used in deriving (2.4)-(2.11) is left to the reader.

If we assume that f is bounded then inequality (2.14) is replaced by

$$(2.16) \quad \left| \int_0^{t \wedge A(s)} f(A^{-1}(u)) dN(A^{-1}(u)) + \int_0^{t \wedge A(s)} [\exp(-f(A^{-1}(u))) - 1] du \right| \leq$$

$$|f|p(A(s)) + (1 + \exp(|f|))A(s) = |f|N(s) + KA(s).$$

By Schwarz's inequality a sufficient condition for the integrability of $\exp(|f|N(s) + KA(s))$ is given by condition (1.4) with $c = 2K = 2(1 + \exp(|f|))$ and $d = 2|f|$. The proof of Theorem 2 is now complete, at least in the case where $A(t)$ is strictly increasing.

To complete the proof of Theorem 2 we drop the assumption that $A(t)$ be strictly increasing. It is still true however that $p(t) = N(A^{-1}(t))$ is a standard Poisson process with the property that $p(A(t)) = N(t)$ except possibly for an evanescent set and moreover matters can be arranged so that $A(t)$ is independent of $p(t)$ - see T. Brown (1981) Theorem 2 on

p. 308. Bring in the natural (strictly) increasing process $A_\epsilon(t) = A(t) + \epsilon t$ and note that $A_\epsilon(t)$ decreases to $A(t)$ as ϵ decreases to 0 and therefore $\lim_{\epsilon \rightarrow 0} p(A_\epsilon(t)) = p(A(t))$ since p is right continuous - in particular $p(A_\epsilon(t))$ converges weakly to $p(A(t))$. We observe that $N_\epsilon(t) = p(A_\epsilon(t))$ is again a counting process with strictly increasing compensator $A_\epsilon(t)$. By Lemma 4 then

$$z_\epsilon(t) = \exp\left(-\int_0^t f(u)dp(A_\epsilon(u)) - \int_0^t [\exp(-f(u)) - 1]dA_\epsilon(u)\right)$$

is a martingale for every $\epsilon > 0$. In order to pass to the limit as $\epsilon \downarrow 0$ we first assume f is continuous and then use the weak convergence of $p(A_\epsilon(u))$ to $p(A(u))$ to conclude

$$\begin{aligned} (2.17) \quad \lim_{\epsilon \rightarrow 0} \int_0^t f(u)dp(A_\epsilon(u)) &= \int_0^t f(u)dp(A(u)) \\ &= \int_0^t f(u)dN(u) \quad \text{a.s.} \end{aligned}$$

Similarly it is easy to check that

$$(2.18) \quad \lim_{\epsilon \rightarrow 0} \int_0^t [\exp(-f(u)) - 1]dA_\epsilon(u) = \int_0^t [\exp(-f(u)) - 1]dA(u) \quad \text{a.s.}$$

Clearly this implies that $\lim_{\epsilon \rightarrow 0} z_\epsilon(t) = z(t)$ is a martingale at least when $f(t)$ is continuous. Proceeding as we did just after (2.15) it can be shown that $z(t)$ is a martingale for step functions of the form (2.1) and finally the proof for arbitrary bounded predictable f is carried out by means of the standard approximation procedure used in (2.4)-(2.11). The proof of Theorem 2 is complete.

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