REPORT DOCUMENTATION PAGE

AD-A160 184	**1b. RESTRICTIVE MARKINGS**
	3. DISTRIBUTION/AVAILABILITY OF REPORT
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	Approved for public release; distribution unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) N/A	**5. MONITORING ORGANIZATION REPORT NUMBER(S)** AFOSR-TR- 85-0856

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Center for Multivariate Analysis		AFOSR

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
515 Thackeray Hall University of Pittsburgh, PA 15260	Bldg. 410 Bolling AFB, D.C. 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR	NM	F49620-85-C-0008

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS.			
Bldg. 410 Bolling AFB, D.C. 20332-6448	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
	61102F	2304	A5	

11. TITLE *(Include Security Classification)*
An Extension of Spitzer's Integral Representation Theorem with an Application

12. PERSONAL AUTHOR(S)
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13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Interim	FROM ____ TO ____	April 1985	10

16. SUPPLEMENTARY NOTATION

17.	COSATI CODES		18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
FIELD	GROUP	SUB. GR.	Branching process, Cauchy equation, Damage Models, Potential theory of Markoff chains, Rao-Rubin condition
	XXXXXXXXXX		

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

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DTIC SELECTED OCT 15 1985

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☒ DTIC USERS ☐	Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Brian W. Woodruff, MAJ, USAF	(202)767-5027	NM

AN EXTENSION OF SPITZER'S INTEGRAL
REPRESENTATION THEOREM WITH AN APPLICATION*

by

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University of Pittsburgh, U.S.A. and
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April 1985
Technical Report No. 85-11

Center for Multivariate Analysis
Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

85 10 11 113

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Key Words: Branching process, Cauchy equation, Damage Models, Potential theory
of Markoff chains, Rao-Rubin condition.

AMS Subject Classification: 60E05, 62E10.

1. INTRODUCTION

Consider a modified discrete branching process $\{Z_n; n = 0, 1, \ldots\}$ with one step tansition probabilities given by

$$(1.1) \qquad p_{ij} = P(Z_{n+1} = j \mid Z_n = i) = \begin{cases} c\, p_j^{(i)}, & i = 0, 1, \ldots; j = 1, 2, \ldots \\ 1 - c + c\, p_0^{(i)}, & i = 0, 1, \ldots; j = 0 \end{cases}$$

where $0 < c \leq 1$ and $\{p_j^{(i)}\}$ is the i-fold convolution of some nondegenerate probability distribution $\{p_j\}$ with itself for $i > 0$ and the degenerate distribution at zero if $i = 0$. (It is seen that the process reduces to Bienaymé–Galton–Watson branching process when $c = 1$). Define

$$(1.2) \qquad m = \sum_{j=1}^{\infty} j\, p_j, \quad m^* = \sum_{j=1}^{\infty} (j \log j) p_j.$$

In this paper, we extend the Spitzer's integral representation for stationary measures of a Bienaymé–Galton–Watson process with $m < 1$ and $m^* < \infty$ to a branching process of the type defined in (1.1). Such an extension could be arrived at by using arguments similar to those of Spitzer (1967) based on the potential theory of Markoff chains. However, we give a new approach based on Bernstein's theorem on absolutely monotonic functions. We use the extended version of Spitzer's theorem to obtain a complete identification of the solutions to a certain functional equation in damage models considered earlier by Talwalker (1980) and Rao et al. (1980), and thus establish a link between branching processes and damage models.

2. AN EXTENDED VERSION OF SPITZER'S INTEGRAL REPRESENTATION

We establish the following theorems.

Theorem 1. The generating function $U(s) = \Sigma n_j s^j$ of any stationary measure $\{n_j\}$ of the modified branching process defined in (1.1) is analytic for $|s| < q$ where q is the smallest non-negative root of the equation $s = f(s)$, and (if normalized so that $U(p_0) = 1$) satisfies the equation

$$(2.1) \qquad cU(f(s)) = c + U(s)$$

where $f(s)$ is the g.f. (generating function) of $\{p_j\}$. Conversely, if $U(s) = \Sigma n_j s^j$, $n_j \geq 0$, $|s| < q$ satisfies (2.1), then $\{n_j\}$ is a stationary measure.

Proof If $U(s)$ is analytic for $|s| < s_0$ for some $s_0 > 0$, then

$$U(s) = \sum_{j=1}^{\infty} n_j s^j = \sum_{i=1}^{\infty} n_i \sum_{j=1}^{\infty} p_{ij} s^j$$

$$= c \sum_{i=1}^{\infty} n_i ([f(s)]^i - [f(0)]^i)$$

$$= c(U(f(s)) - U(p_0)), \quad |s| < s_0$$

which implies that (2.1) is valid at least for $|s| < s_0$. Then using the arguments exactly as in A.N. (Theorem 2, p. 68), [where we use the abbreviation A.N. for the book by Athreya and Ney (1972)], we find that $U(s)$ is analytic for $|s| < q$. The converse assertion easily follows by equating the coefficients of s^j in (2.1).

Theorem 2. (An extended version of Spitzer's theorem). If $m < 1$ and $m^* < \infty$ with f as in Theorem 1, then for every probability measure ν on $[0,1)$

$$(2.2) \qquad\qquad U(s) = K \int_{[0,1)} U(s,t)d\nu(t)$$

is the g.f. of a stationary measure, where

$$U(s,t) = \sum_{n=-\infty}^{\infty} [\exp\{(\mathcal{B}(s)-1)m^{n-t}\} - \exp\{-m^{n-t}\}]c^{n-t}$$

with $\mathcal{B}(s)^+$ as the unique p.g.f. among those vanishing at $s = 0$ and satisfying the equation

$$(2.3) \qquad\qquad \mathcal{B}(f(s)) = m\mathcal{B}(s) + 1-m.$$

Conversely, every stationary measure has the representation (2.2) for some probability measure ν on $[0,1)$.

   <u>Proof</u> The first part of the Theorem is easily verified. To prove the converse, it is sufficient to establish that the representation is valid for every $s \in [0,1)$. Define then for every $s \in [0,1)$

$$U^*(s) = U(\mathcal{B}^{-1}(s))$$

where $\mathcal{B}$ is as mentioned in the statement of Theorem 2. In view of Theorem 1 and the equation (2.3) we have

$$(2.4) \qquad c^n U^*(m^n s+1- m^n) = U^*(s) + \xi_n, \quad n = 1,2,\ldots,; \; 0 \le s < 1,$$

where

$$\xi_n = \begin{cases} c(\dfrac{1-c^n}{1-c}) & \text{if } c \ne 1, \\[2mm] n & \text{if } c = 1. \end{cases}$$

We can write (2.4) also as

$^+$For an interpretation of $\mathcal{B}$ in branching processes, see A.N. (p. 17).

$$U^*(s) = c^n U(1 + m^n Q_n(\beta^{-1}(s))) - \xi_n, \quad n = 1,2,\ldots; \quad 0 \le s < 1,$$

where $Q_n(s) = [f_n(s)-1]/m^n$ with $f_n(s)$ as the n-th iterate of $f(s)$. Consequently

$$(2.5) \qquad U^*(s) = \lim_{n \to \infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)\beta_n(\beta^{-1}(s))) - \xi_n\}$$

for $0 \le s < 1$, where $\beta_n(s) = [Q_n(0) - Q_n(s)]/Q_n(0)$. If $0 \le s_1 < s < s_2 < 1$, then noting in particular that $\beta_n \to \beta$ pointwise (see A.N., p. 47), and β and β^{-1} are strictly increasing on $[0,1)$, we find that

$$(2.6) \qquad -m^n Q_n(0)\beta_n(\beta^{-1}(s_1)) < -m^n Q_n(0)s = -m^n Q_n(0)\beta(\beta^{-1}(s))$$

$$< -m^n Q_n(0)\beta_n(\beta^{-1}(s_2))$$

for large enough n. Since U is increasing on $[0,1)$ we obtain from (2.5) using (2.6)

$$(2.7) \qquad U^*(s_1) \le \varliminf_{n \to \infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)s) - \xi_n\}$$

$$\le \varlimsup_{n \to \infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)s) - \xi_n\} \le U^*(s_2).$$

It is seen that $U^*(s)$ is continuous and $U^*(0) = 0$, and hence from (2.7) we have for $s \in [0,1)$

$$(2.8) \qquad U^*(s) = \lim_{n \to \infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)s) - \xi_n\}$$

$$= \lim_{n \to \infty} \{c^n [U(1 + m^n Q_n(0) - m^n Q_n(0)s) - U(1 + m^n Q_n(0))]\}$$

for $s \in [0,1)$, with the limits well defined. Since for each $n \ge 1$, the expression within the second limit in (2.8) can be expressed as the g.f. of some non-negative sequence for $s \in [0,1)$, the extended continuity theorem given in Feller (1971,

p. 4 33) implies that U^* is the g.f. of a non-negative sequence. Define now a function $\hat{U}$ on $(-\infty,1)$ such that its restriction to $[0,1)$ is indeed U^* and for any $s \in (-\infty,0)$

$$\hat{U}(s) = c^n U^*(m^n s + 1 - m^n) - \xi_n$$

where n is the smallest integer for which $m^n s + 1 - m^n \in [0,1)$. It is easily seen that

$$(2.9) \qquad c^n \hat{U}(m^n s + 1 - m^n) = \hat{U}(s) + \xi_n, \quad n = 1,2,\ldots; \quad s \in (-\infty,1)$$

which, since $\hat{U} = U^*$ on $[0,1)$, implies that $\hat{U}'$ (i.e., $d\hat{U}(s)/ds$, $s \in (-\infty,1)$) exists and is absolutely monotonic on $(-\infty,1)$ with $\lim \hat{U}'(s) = 0$ as $s \to -\infty$. Since $\hat{U}(0) = 0$, we conclude from Bernstein's theorem that for some measure μ on $(0,\infty)$

$$(2.10) \qquad \hat{U}(s) = \int_{(0,\infty)} (e^{sx} - 1) d\mu(x), \qquad s \in (-\infty,1).$$

From the equation (2.9), we find that the measure μ is such that

$$(2.11) \qquad c^n \int_{(0,\infty)} (e^{sxm^n} - 1) e^{(1-m^n)x} \, d\mu(x) = \int_{(0,\infty)} (e^{sx} - 1) d\mu(x)$$

$$n = 1,2,\ldots; \quad s \in (-\infty,1).$$

If $s_0 \in (-\infty,1)$ and $s \in (-\infty,0]$, then subtracting the identity (2.11) from the corresponding identity with s replaced by $s + s_0$ we obtain

$$(2.12) \quad c^n \int_{(0,\infty)} e^{sxm^n} (e^{s_0 xm^n} - 1) e^{(1-m^n)x} \, d\mu(x) = \int_{(0,\infty)} e^{sx} (e^{s_0 x} - 1) d\mu(x)$$

which is valid for all $s \in (-\infty,0]$. Then, in view of the uniqueness theorem for Laplace-Stieltjes transforms, (2.12) implies

$$(2.13) \quad \int_{[m^n, m^{n-1})} (e^{s_0 x} - 1) d\mu(x) = c^n \int_{[1, m^{-1})} (e^{s_0 x m^n} - 1) e^{(1-m^n)x} d\mu(x)$$

for $n = 0, \pm 1, \pm 2, \ldots$. Since s_0 in (2.13) is arbitrary, we conclude using (2.10) that

$$(2.14) \quad U^*(s) = \hat{U}(s) = \sum_{n=-\infty}^{\infty} \int_{[m^n, m^{n-1})} (e^{sx} - 1) d\mu(x)$$

$$= \int_{[1, m^{-1})} \sum_{n=-\infty}^{\infty} c^n (e^{-(1-s)xm^n} - e^{-xm^n}) e^x d\mu(x)$$

$$= K \int_{[0,1)} \left\{ \sum_{n=-\infty}^{\infty} c^{n-t} (e^{-m^{n-t}(1-s)} - e^{-m^{n-t}}) \right\} d\nu(t)$$

for $s \in [0,1)$, where K is a positive constant and ν is a probability measure on $[0,1)$ such that for every Borel subset A of $[0,1)$ we have

$$(2.15) \quad \nu(A) = K^{-1} \int_{S_A} e^x c^{-\log x / \log m} d\mu(x),$$

where $S_A = \{x: (-\log x / \log m) \in A\}$. [The operation of interchanging the order of summation and integration in (2.14) is justified by either Fubini's theorem or monotone convergence theorem.] The required result now follows on observing that $U(s) = U^*(\beta(s))$, $s \in [0,1)$.

Remark 1. In the definition 2 on page 432 of Feller (1971), $\overline{0,\infty}$ should be changed to $\overleftarrow{0,\infty}$ to make the extended continuity theorem given on page 433 non-ambiguous. [We used this theorem in the proof of our Theorem 2.]

Remark 2. If we define $G(i,j)$ to be the Green function corresponding to the substochastic matrix (p_{ij}), $i, j = 1, 2, \ldots$, then an argument essentially as in A.N. (p. 70) implies that for every subsequence of positive integers $\{k_i\}$ such

that the fractional part of $(-\log k_1)/\log m \to t$ with $t \in [0,1)$ we have

$$\frac{\sum_{j=1}^{\infty} G(k_1,j)s^j}{G(k_1,1)} \to U^*(s,t) = \frac{\tilde{U}(s,t)}{\tilde{U}'(0,t)}$$

where $\tilde{U}(s,t)$ is the $U(s,t)$ of Theorem 2 with m^{n-t} replaced by $(-Q(0)m^{n-t})$, $Q(0)$ being as defined in A.N. (p. 40), and

$$\tilde{U}'(0,t) = \frac{\partial \tilde{U}(s,t)}{\partial s}\Bigg|_{s=0}.$$

We note that the proof in AN (p. 69) remains valid in the present case as well with $U(s,t)$ changed to $U^*(s,t)$. Consequently, the Poisson-Martin integral re-presentation for a stationary measure analogous to the one given in Seneta (1973, p. 151) for a super regular vector yields the validity of our Theorem 2 with $U(s,t)$ replaced by $\tilde{U}(s,t)$. Now, it is easy to verify that Theorem 2 remains valid even when $Q(0)$ is not necessarily equal to -1 and hence we have an alternative proof of Theorem 2.

Remark 3. The version of Theorem 2 with $U(s,t)$ replaced by $\tilde{U}(s,t)$ of Remark 2 yields Spitzer's theorem when $c = 1$.

Corollary. If $f(s) = 1-m+ms$, then every sequence $\{n_j\}$ is a stationary measure iff it is of the following form:

$$(2.16) \qquad n_j = K \sum_{n=-\infty}^{\infty} \int_{[0,1)} c^{n-t} e^{-m^{n-t}} \frac{m^{(n-t)j}}{j!} d\nu(t), \quad j = 1,2,\ldots$$

where ν is a probability measure on $[0,1)$, and K is a positive constant as in Theorem 2.

The result (2.16) is obvious from Theorem 2 in view of the fact that $B(s) = s$.

3. AN APPLICATION TO DAMAGE MODELS

Let (X,Y) and (X',Y') be 2-vector random variables with non-negative integer valued components such that X and X' have the same marginal distribution $\{g_n\}$ with $g_0 < 1$, and Y and Y' are such that for each n with $g_n > 0$

$$P(Y = r | X = n) = \binom{n}{r}\pi^r(1-\pi)^{n-r}, \quad r = 0,1,\ldots,n,$$

$$P(Y' = r' | X' = n) = \binom{n}{r}\pi'^r(1-\pi')^{n-r}, \quad r = 0,1,\ldots,n,$$

where $0 < \pi',\pi < 1$ are fixed. Talwalker (1980) and Rao et al. (1980) considered the problem of characterizing the distributions of X and X' by the equation

$$(3.1) \qquad P(Y = r) = P(Y' = r | X' = Y'), \quad r = 0,1,\ldots$$

which is an extended version of the Rao-Rubin condition (Rao and Rubin, 1964). It is seen that (3.1) is equivalent to

$$(3.2) \qquad G(1-\pi + \pi s) = G(s\pi')/G(\pi'), |s| \leq 1$$

where G is the g.f. of $\{g_n\}$.

When $\pi = \pi'$, (3.1) is the Rao-Rubin condition, and it is shown by Rao and Rubin (1964) using Bernstein's theorem and in a simpler way by Shanbhag (1974), that in this case G is the p.g.f. of a Poisson distribution. When $\pi > \pi'$, it is easily seen that G is the p.g.f. of a binominal distribution with an arbitrary index and success probability $(\pi-\pi')/\pi(1-\pi')$.

When $\pi < \pi'$, the picture is totally different, and the family of distributions $\{g_n\}$ for which (3.1) holds is somewhat curious and fairly large. Talwalkar (1980) and Rao et al (1980) identified the family as a mixture of Poisson distributions with the mixing measure itself satisfying a further functional equation. The

following theorem gives a complete identification of the solution and provides a more satisfactory answer and a rigorous proof to the characterization problem than that given earlier.

Theorem 3. Let (X,Y) and (X',Y') be 2-vector random variables as considered above with $0 < \pi < \pi' < 1$. Then (3.1) is valid if and only if

$$g_j = K \sum_{-\infty}^{\infty} \int_{[0,1)} c^{n-t} e^{-(\pi/\pi')^{n-t}} \frac{(\pi/\pi')^{(n-t)j}}{j!} [\frac{\pi'-\pi}{\pi'(1-\pi)}]^j d\nu(t), \quad j = 0,1,\ldots$$

where ν is a probability measure on $[0,1)$ and c is a real number lying in $(0,1)$ and K is a normalizing constant.

The result follows from the Corollary to Theorem 2 (see equation (2.16)) since (3.1) is equivalent to (3.2) which can be written as

$$cU(1 - m + ms) = U(s) + cU(1-m)$$

with

$$g_0 = \frac{c}{1-c} U(1-m), \quad m = \frac{\pi}{\pi'}$$

$$U(s) = G(\frac{(1-\pi)\pi'}{\pi'-\pi} s) - g_0.$$

Observe that (3.2) implies that the p.g.f. $G(s)$ is defined for all s such that $|s| < \pi'(1-\pi)/(\pi'-\pi)$ and it satisfies (3.2) for every

$$s \in (- \frac{\pi'(1-\pi)}{(\pi'-\pi)}, \frac{\pi'(1-\pi)}{\pi'-\pi}).$$

Remark 4. If the measure ν in Theorem 3 is taken as the Lebesgue measure on $[0,1)$, then the distribution $\{g_n\}$ in question reduces to a negative binomial distribution.

Remark 5. It is interesting to note that if a p.g.f. G satisfies (3.2) simultaneously for two pairs (π_i, π_i'), $i = 1,2$, where $0 < \pi_i < \pi_i' < 1$ and $(\log \pi_1 - \log \pi_1')/(\log \pi_2 - \log \pi_2')$ is irrational, then G is the p.g.f. of a negative binomial distribution of the form

$$G(s) = \frac{\{[\pi_1'(1-\pi_1)/(\pi_1'-\pi_1)]-1\}^{\alpha}}{\{[\pi_1'(1-\pi_1)/(\pi_1'-\pi_1)]-s\}^{\alpha}}$$

for some $\alpha > 0$. Since the condition implies G to be well defined also on $(1, s_0)$ where $s_0 = \pi_1'(1-\pi_1)/(\pi_1'-\pi_1)$, the result in question follows as a corollary to the result of Marsaglia and Tubilla (1975) by noting in particular that $f(x) = G(s_0 - s_0 e^{-x})/G(0)$, $x \geq 0$ is well defined and satisfies the equation $f(t_i + x) = f(t_i)f(x)$, $x \geq 0$, $i = 1,2$ with $t_i = \log(\pi_i'/\pi_i)$, $i = 1,2$. The same result was established in Rao et al (1980) by a different and slightly more involved method.

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