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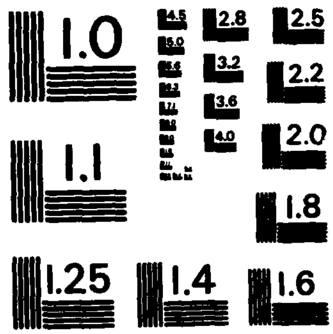
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by

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Key Words: bivariate exponential and geometric distributions, associated r.v.s., bivariate moving averages and bivariate point processes.

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Abstract

Two classes of finite and infinite moving average sequences of bivariate random vectors are considered. The first class has bivariate exponential marginals while the second class has bivariate geometric marginals. The theory of positive dependence is used to show that in various cases the two classes consist of associated random variables. Association is then applied to establish moment inequalities and to obtain approximations to some joint probabilities of the related bivariate point processes. Additional keywords: mathematical models.

1. Introduction and Summary.

A reigning stationary model in time series analysis is the p×1 moving average (M.A.) model given by:

$$X(n) = \sum_{j=-\infty}^{\infty} A(j)\,\epsilon(n-j), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{1.1}$$

where $A(j)$, $j = 0, \pm 1, \pm 2, \ldots$, is a sequence of p×p parameter matrices s.t. $\sum_{j=-\infty}^{\infty} \|A(j)\| < \infty$, and $\epsilon(n)$, $n = 0, \pm 1, \pm 2, \ldots$ is a sequence of uncorrelated p×1 random vectors (r.ve.s) with mean zero and common covariance matrix. It is well known that this model emerges from many physically relizable systems (see for example Hannan (1970), p. 9), however in some physical situations where the r.ve.s $X(n)$ are either positive or discrete the preceeding assumptions about the r.ve.s $\epsilon(n)$ are inappropriate (see Lewis (1980), p. 152).

Several researchers, addressing themselves to this problem, have been constructing univariate stationary M.A. models and univariate stationary autoregressive moving average (A.R.M.A.) models where the r.v.s $X(n)$ have exponential (exp.) or gamma distributions (dist.s) and discrete models where $X(n)$ assumes values in a common set. Lawrence-Lewis (1977) present stationary M.A. models where the r.v.s $X(n)$ have exp. dist.s; Gaver-- Lewis (1978) consider stationary A.R.M.A. models where the r.v.s $X(n)$ have gamma dist.s. Jacobs-Lewis (1978,a,b,c, 1983) construct A.R.M.A. models where the r.v. s $X(n)$ are discrete and assume values in a common finite set. The aformentioned models can be used in the various fields of applied probability and time series analysis. In particular, these models have been used to model and to analyze univariate point processes with correlated interarrival times. Jacobs (1978) uses the exp. models in queues with correlated service and correlated interarrival times. Other authors have concentrated their efforts on the analysis of univariate point processes in the context of time series analysis, for example Bartlett (1963, 1966), Brillinger (1972, 1978), Jacobs and Lewis (1977) and Lawrance and Lewis (1977) to mention a few. More details concerning the univariate geo. M.A. process and the corresponding point processes may be found in Langberg-Stoffer (1985).

In this paper we present two classes of finite and infinite M.A. sequences of bivariate r.ve.s. The first class has exp. marginals while the second class has geometric (geo.) marginals. Within each class of M.A., the sequences are classified according to their order of dependence on the past. For the sake of clarity we restrict ourselves to bivariate M.A. sequences. However these models can be extended in a straight forward way. We use the theory of positive dependence to show that in a variety of cases the two classes of M.A. sequences are associated (AS.). We then apply the association to obtain bounds for the bivariate point processes and to establish some moment inequalities.
In Section 2 we define the bivariate exp. and geo. dist.s, which are the underlying dist.s of our two classes, and present a variety of examples of such dist.s. Further in Section 2 we define the concept of association and present a variety of bivariate exp. and geo. dist.s that are AS. In Section 3 we construct the two classes of M.A.s proving that they have exp. or geo. marginals and showing that if the underlying dist. is AS., so is the related M.A. sequence. Finally in Section 3 we present the autocovariance matrices for both classes of M.A. sequences. In Section 4 we define bivariate point processes whose interarrival times are described by the bivariate exp. or geo. M.A. processes discussed in Section 3. We show in Section 4 that if the interarrival times are AS. the corresponding point processes inherit positive dependence properties. We then exploit positive dependence to obtain bounds on the joint probabilities of the point processes. We conclude Section 4 with moment inequalities for the bivariate processes and their interarrival times.

2. Preliminaries.

In this section we present definitions and prove some basic results to be used in the sequel. First we present a definition of a bivariate geometric distribution (B.G.D.).

Definition 2.1. Let $N(1)$, $N(2)$ be r.v.s assuming values in the set $\{1,2,\ldots\}$. We say that $(N(1), N(2))$ has a B.G.D. if $N(1)$, $N(2)$ have geometric distributions.

Examples 2.2. (a) Let N be a geometric r.v. Then (N,N) has a B.G.D. (b) Let $N(1)$, $N(2)$ be independent geometric r.v.s. Then $(N(1),N(2))$ has a B.G.D. (c) Let $N(1),N(2),N(3)$ be independent geometric r.v.s. Then $(\min(N(1),N(3)), \min(N(2),N(3)))$ has a B.G.D.; the Esary-Marshall (1974) B.G.D. (d) Let $P(0,0)$, $P(0,1)$, $P(1,0)$, and $P(1,1)$ be in $[0,1]$ s.t. (i) $P(0,0)+P(0,1)+P(1,0)+P(1,1)=1$, (ii) $P(0,1)+P(1,1)$, $P(1,0)+P(1,1)<1$, and let $N(1),N(2)$ be r.v.s assuming values in the set $\{1,2,\ldots\}$ determined by:

$$(2.3)\quad P(N(1)>a,\ N(2)>b) = \begin{cases} [P(1,1)]^a[P(0,1)+P(1,1)]^{b-a}, & b \geq a \\ [P(1,1)]^b[P(1,0)+P(1,1)]^{a-b}, & b \leq a \end{cases}, \quad a,b=1,2,\ldots .$$

Then $(N(1),N(2))$ has a B.G.D.; the Block (1977) fundamental B.G.D. (see also Block-Paulson (1984)). (e) Let $(M(1),M(2))$ have a B.G.D. and let $\underset{\sim}{N}(j) = (N(j,1),N(j,2))$, $j=1,2,\ldots$, be an i.i.d. sequence of random vectors (r.ve.s), independent of $(M(1),M(2))$, with a B.G.D. Then $(\sum_{j=1}^{M(1)} N(j,1), \sum_{j=1}^{M(2)} N(j,2))$ has a B.G.D.

In the following remark we show that Examples (Ex.s)(a),(b) and (c) in Ex.(2.2) (but not (e)) are particular cases of Ex. (2.2)(d).

Remarks 2.4. (a) Let $P(1,0)=P(0,1)=0$. Then we obtain the B.G.D. introduced in Ex. 2.2(a). (b) Let $P(1,1)=(P(1,1)+P(1,0))(P(1,1)+P(0,1))$. Then we obtain the B.G.D. introduced in Ex. 2.2(b). (c) Let $P(1,1)\geq(P(1,1)+P(1,0))(P(1,1)+P(0,1))$, and let $M(1)$, $M(2)$, $M(3)$ be independent geometric r.v.s with parameters $P(1,1)(P(1,1) + P(1,0))^{-1}$, $P(1,1)(P(1,1) + P(0,1))^{-1}$, $[P(1,1)]^{-1}(P(1,1) + P(1,0))(P(1,1) + P(0,1))$, respectively. Then $(N(1),N(2))$, given by (2.3), is stochastically equal to the Esary-Marshall B.G.D.: $(\min(M(1),M(3)),\min(M(2),M(3)))$.

Next we present a definition of a <u>bivariate exponential</u> (<u>exp</u>.) <u>distribution</u> (<u>B.E.D.</u>)

<u>Definition 2.5</u>. Let $E(1)$, $E(2)$ be r.v.s assuming values in $(0,\infty)$. We say that $(E(1),E(2))$ has a B.E.D. if $E(1),E(2)$ have exp. distributions.

<u>Examples 2.6</u>. (a) Let E be an exp. r.v. Then (E,E) has a B.E.D. (b) Let $E(1),E(2)$ be independent exp. r.v.s. Then $(E(1),E(2))$ has a B.E.D. (c) Let $E(1),E(2),E(3)$ be independent exp. r.v.s. Then $(\min(E(1),E(3)), \min(E(2),E(3)))$ has a B.E.D.; the Marshall-Olkin (1967) B.E.D. (d) Let $(N(1),N(2))$ have a B.G.D. and let $\underset{\sim}{E}(j) = (E(j,1), E(j,2))$, $j = 1,2,\ldots$, be an i.i.d. sequence of r.ve.s, independent of $(N(1),N(2))$, with a B.E.D. Then $(\sum_{j=1}^{N(1)} E(j,1), \sum_{j=1}^{N(2)} E(j,2))$ has a B.E.D. (e) Let $0 \leq \alpha \leq 1$. Then $(E(1),E(2))$ determined by: $P(E(1) > x, E(2) > y) = e^{-x-y-\alpha xy}$, $x,y > 0$, has a B.E.D.; a Gumble (1960) B.E.D. (f) Let $|\alpha| \leq 1$. Than $(E(1),E(2))$ determined by: $P(E(1) \leq x, E(2) \leq y) = (1-e^{-x})(1-e^{-y})(1+\alpha e^{-x-y})$, $x,y > 0$, has a B.E.D.; a Gumble (1960) B.E.D. (g) Let $\alpha \geq 1$. Then $(E(1),E(2))$ determined by: $P(E(1) > x, E(2) > y) = e^{-(x^{\alpha}+y^{\alpha})^{1/\alpha}}$, $x,y > 0$, has a B.E.D. (h) Let (X,Y) be a r.ve. with continuous marginal distributions F,G respectively. Then $(-\ln(1-F(X)), -\ln(1-G(Y)))$ has a B.E.D.

Ex. 2.6(d) has been used by several researchers to generate bivariate distributions (for example Arnold (1975), Downton (1970), and Hawkes (1972) to mention few). In the following remark we illustrate how some of the B.E.D. are obtained from Ex. 2.6(d).

Remarks 2.7. (a) Let $N(1) = N(2)$ and let $E(j,1)$, $E(j,2)$ be independent r.v.s, $j = 1,2,\ldots$. Then we obtain the B.E.D. introduced by Downton (1970). (b) Let $(N(1), N(2))$ be as in Ex. 2.2(d) and let $E(j,1)$, $E(j,2)$ be independent r.v.s, $j = 1,2,\ldots$. Then we obtain the B.E.D. introduced by Hawkes (1972)-Paulson (1973). (c) Let $(N(1),N(2))$ be as in Ex. 2.2(c) and let $E(j,1) = E(j,2)$, $j = 1,2,\ldots$. Then we obtain the Marshall-Olkin (1967) B.E.D. given in Ex. 2.6(c) (for details see Marshall-Olkin (1967)).

Finally we present a concept of positive dependence.

Definition 2.8. Let $\underset{\sim}{T} = (T(1),\ldots,T(n))$, $n = 1,2,\ldots$, be a multivariate r.ve. We say that the r.v.s $T(1),\ldots,T(n)$ are associated (AS.) if for all pairs of measurable bounded functions f,g: $R^n \to R$ both nondecreasing in each argument $\mathrm{cov}(f(\underset{\sim}{T}),g(\underset{\sim}{T})) \geq 0$.

Remarks 2.9. (a) Note that independent r.v.s are AS. (Barlow-Proschan (B-P) (1975), Th. 2.2 p. 31) and that nondecreasing functions of AS. rv.s are AS. (B-P P_3 p. 30). Thus the components of the r.ve. given in Ex. 2.2(c) and the components of the r.ve. given in Ex. 2.6(c) are AS. (b) Let $(E(1),E(2))$ be as in Ex. 2.6(e) (with $\alpha > 0$) or in 2.6(f) (with $-1 \leq \alpha < 0$). Since $P\{E(1) > x, E(2) > y\} < P\{E(1) > x\} P\{E(2) > y\}$ for $x,y > 0$, $E(1)$, $E(2)$ are not AS. (d) Let (X,Y) be as in Ex. 2.6(h). If X,Y are AS. (not AS.) then $-\ln(1 - F(X))$, $-\ln(1-G(Y))$ are AS. (not AS.) (by P_3 of B-P p. 30).

The following lemma provides sufficient conditions for some of the bivariate dist.s presented in Ex.s 2.2 and 2.6 to be AS.

Lemma 2.10. Let $\underset{\sim}{Q} = (Q(1),Q(2))$ be a r.ve. with components assuming values in the set $\{1,2,\ldots\}$ and let $\underset{\sim}{R}(j) = (R(j,1),R(j,2))$, $j = 1,2,\ldots$ be an i.i.d. sequence of r.ve.s with nonnegative components independent of $\underset{\sim}{Q}$. Assume that $Q(1),Q(2)$ are AS. and that $R(1,1)$, $R(1,2)$ are AS. Then $\sum_{j=1}^{Q(1)} R(j,1), \sum_{j=1}^{Q(2)} R(j,2)$ are AS.

<u>Proof</u>: Let $f,g: R^2 \to R$ be measurable bounded functions nondecreasing in each argument and let $U(\ell) = \sum_{j=1}^{Q(\ell)} R(j,\ell)$, $\ell = 1,2$. First note that

$$\operatorname{cov}\{f(U(1),U(2)),g(U(1),U(2)\} = E\{\operatorname{cov}(f(U(1)U,(2)),g(U(1),U(2))|\underset{\sim}{Q}\}$$

$$+ \operatorname{cov}\{Ef(U(1),U(2))|\underset{\sim}{Q},\ Eg(U(1),U(2))|\underset{\sim}{Q}\}.$$

Now $Ef(U(1),U(2))|\underset{\sim}{Q}$, $Eg(U(1),U(2))|\underset{\sim}{Q}$ are nondecreasing functions of $Q(1),Q(2)$. Since $Q(1),Q(2)$ are AS.

$$\operatorname{cov}\{Ef(U(1),U(2))|\underset{\sim}{Q},\ Eg(U(1),U(2))|\underset{\sim}{Q})\} \geq 0.$$

Next let $Q = \max\ (Q(1),Q(2))$, $f(U(1),U(2))|\underset{\sim}{Q}$, $g(U(1),U(2)|\underset{\sim}{Q}$ are nondecreasing functions of $R(1,1),\ldots,R(Q,1)$, $R(1,2),\ldots,R(Q,2)$, by P_3 and Th. 2.2 of B-P p. 30-31 these r.v.s. are AS. Thus:

$$\operatorname{cov}\{f(U(1),U(2)|\underset{\sim}{Q},\ g(U(1),U(2))|\underset{\sim}{Q}\} \geq 0.$$

Consequently $\operatorname{cov}\{f(U(1),U(2)),\ g(U(1),U(2))\} \geq 0$. ||

<u>Remark 2.11</u>. In particular we conclude from Lemma 2.10 that: (i) The components of the B.G.D. given in Ex. 2.2(e) are AS. provided $M(1),M(2)$ are AS. and $N(1,1)$, $N(1,2)$ are AS. (ii) The components of the B.E.D. given in Ex. 2.6(d) is AS. provided $N(1),N(2)$ are AS. and $E(1,1)$, $E(1,2)$ are AS.

<u>3. Model Constructions</u>.

In this section we construct two classes of finite and infinite M.A. sequences of bivariate r.ve.s. We denote the first class of sequences by $\{\underset{\sim}{X}(n,m) = (X(n,m,1),\ X(n,m,2)),\ n = 0,\pm 1,\pm 2,\ldots\}$ $m = 1,2,\ldots,\infty$, and the second class of sequences by $\{\underset{\sim}{G}(n,m) = (G(n,m,1),G(n,m,2)),\ n = 0,\pm 1,\pm 2,\ldots\}$ $m = 1,2,\ldots,\infty$. We show that each r.ve. $\underset{\sim}{X}(n,m)$ has a B.E.D. with a vector mean that does not depend on n or m and that each r.ve. $\underset{\sim}{G}(n,m)$ has a B.G.D. with a vector mean independent of n or m. Within each class of sequences the order of dependence on the past is indicated by the parameter m. For each positive integer m, $\underset{\sim}{X}(n,m)$ $(\underset{\sim}{G}(n,m))$ depends only on the previous m r.ve.s: $\underset{\sim}{X}(n-1,m),\ldots,\underset{\sim}{X}(n-m,m)$ $(\underset{\sim}{G}(n-1,m),\ldots,\underset{\sim}{G}(n-m,m))$ while the r.ve. $\underset{\sim}{X}(n,\infty)(\underset{\sim}{G}(n,\infty))$ depends on all the preceeding r.ve.s: $\underset{\sim}{X}(n-1,\infty)$, $\underset{\sim}{X}(n-2,\infty),\ldots,(\underset{\sim}{G}(n-1,\infty), \underset{\sim}{G}(n-2,\infty),\ldots)$. After constructing the various models we present sufficient conditions for the r.v.s. $X(n(j),m,\ell)(G(n(j),m,\ell))$, $\ell=1,2$, $j=1,\ldots,k$ to be AS.(where $k=1,2,\ldots$, and $n(1)<n(2)<\ldots<n(k)\in\{0,\pm 1,\pm 2,\ldots\}$). We conclude this section by computing the autocovariance matrices for the two classes of sequences and present sufficient conditions for the sequences to be stationary. For the stationary sequences we give the spectral density matrices.

First we construct the "exp." class. Some notation is needed.

<u>Notation 3.1</u>. Throughout, n ranges over the integers and m,j over the positive integers. Let $\underset{\sim}{E}(n) = (E(n,1), E(n,2))$ be i.i.d. r.ve.s with a B.E.D. and mean vector $(\lambda^{-1}(1),\lambda^{-1}(2))$ $(\lambda(1),\lambda(2)>0)$. Let $(\beta(n,j,1),\beta(n,j,2))$ be bivariate vectors with components in [0,1] and let B(n,j) be a 2×2 diagonal matrix with $\beta(n,j,1)$, $\beta(n,j,2)$ on the main diagonal. Further let $(I(n,j,1),I(n,j,2))$ be independent bivariate r.ve.s independent of all the r.ve.s $\underset{\sim}{E}(n)$ s.t. $I(n,j,1)$, $I(n,j,2)$ are Bernoullis r.v.s with parameters $1-\beta(n,j,1)$, $1-\beta(n,j,2)$,respectively, and let V(n,j,q) be a 2×2 random diagonal matrix with

$$\prod_{\ell=q}^{j} I(n,\ell,1),\ \prod_{\ell=q}^{j} I(n,\ell,2) \text{ on the main diagonal } q\in\{1,\ldots,j\}.$$

<u>Finally</u>, <u>let a sum</u> (<u>product</u>) <u>over an empty set of indices be equal to zero</u> (<u>one</u>). We present now the class of "exp" sequences. For $m=1,2,\ldots$, and $n=0,\pm 1,\pm 2,\ldots$, let

(3.2) $$\underset{\sim}{X}(n,m) = \sum_{r=0}^{m-1} V(n,r,1)B(n,r+1)\underset{\sim}{E}(n-r) + V(n,m,1)\underset{\sim}{E}(n-m),$$

and

(3.3) $$\underset{\sim}{X}(n,\infty) = \sum_{r=0}^{\infty} V(n,r,1)B(n,r+1)\underset{\sim}{E}(n-r).$$

We show in Corollary 3.8 and Lemma 3.9 that for all n,m, $\underset{\sim}{X}(n,m)$ and $\underset{\sim}{X}(n,\infty)$ have B.E.D.s. Next we construct the "geometric" class. Some notation is needed.

<u>Notation 3.4</u>. Let p(1),p(2) be real numbers in (0,1] and let $(\alpha(n,1),\alpha(n,2))$ be bivariate vectors s.t. $p(\ell) \leq \alpha(n,\ell) \leq 1$, $\ell = 1,2$. Further let $\underset{\sim}{N}(n) = (N(n,1),N(n,2))$ be independent r.ve.s with B.G.D.s and mean vector $(p^{-1}(1)\alpha(n,1),\ p^{-1}(2)\alpha(n,2))$ and let $\underset{\sim}{M}(n) = (M(n,1),M(n,2))$ be i.i.d. r.ve.s, independent of all $\underset{\sim}{N}(n)$, with a B.G.D. and mean vector $(p^{-1}(1),p^{-2}(2))$. Finally let $(J(n,j,1),J(n,j,2))$ be independent r.ve.s independent of all previous r.ve.s s.t. $J(n,j,1),J(n,j,2)$ are Bernoulli r.v.s with parameters $1-\alpha(n,1)$, $1-\alpha(n-2)$, respectively,and let $U(n,j,q)$ be a 2×2 random diagonal matrix with $\prod_{\ell=q}^{j} J(n,\ell,1)$, $\prod_{\ell=q}^{j} J(n,\ell,2)$ on the main diagonal $q \in \{1,\ldots,j\}$.

We now present the class of "geometric" sequences. For $m = 1,2,\ldots$, $n = 0,\pm 1,\pm 2,\ldots$, let

(3.5) $$\underset{\sim}{G}(n,m) = \sum_{r=0}^{m} U(n,r,1)\underset{\sim}{N}(n-r) + U(n,m+1,1)\underset{\sim}{M}(n-m)$$

and

(3.6) $$\underset{\sim}{G}(n,\infty) = \sum_{r=0}^{\infty} U(n,r,1)\underset{\sim}{N}(n-r).$$

Next we show that $\underset{\sim}{X}(n,m)$ $(\underset{\sim}{G}(n,m))$ has a B.E.D. (B.G.D.). The following lemma is needed.

<u>Lemma 3.7</u>. Let $\underset{\sim}{Y}(n,m,q) = \sum_{r=0}^{m-1} V(n,r+q-1,q)B(n,r+q)\underset{\sim}{E}(n-r-q+1) + V(n,m+q-1,q)$ $\underset{\sim}{E}(n-m-q+1)$, $(\underset{\sim}{W}(n,m,q) = \sum_{r=0}^{m} U(n,r+q-1,q)\underset{\sim}{N}(n-r-q+1) + U(n,m+q,q)\underset{\sim}{M}(n-m-q+1))$, $n = 0,\pm 1,\pm 2,\ldots$, $m,q = 1,2,\ldots$. Then for all n,m, and q, $\underset{\sim}{Y}(n,m,q)(\underset{\sim}{W}(n,m,q))$ has a

B.E.D. (B.G.D.) with mean vector $(\lambda^{-1}(1),\lambda^{-1}(2))\ ((p^{-1}(1),p^{-1}(2)))$.

Proof: We prove the result of the lemma by an induction argument on m. For $m=1$ $(m=0)$ $\underset{\sim}{Y}(n,1,q) = B(n,q)\underset{\sim}{E}(n-q+1) + V(n,q,q)\underset{\sim}{E}(n-q)\ (\underset{\sim}{W}(n,0,q) = \underset{\sim}{N}(n-q+1) + U(n,q,q)\underset{\sim}{M}(n-q+1))$. By computing the characteristic functions of the components of $\underset{\sim}{Y}(n,1,q)\ (\underset{\sim}{W}(n,1,q))$ one can verify that the results of the lemma hold for all n,q. Let us assume that the results of the lemma hold for m and all n,q. Note that $\underset{\sim}{Y}(n,m+1,q) = B(n,q)\underset{\sim}{E}(n-q+1) + V(n,q,q)[\sum_{r=0}^{m-1} V(n,r+q,q+1)B(n,r+q+1)\underset{\sim}{E}(n-q-r) + V(n,m+q,q+1)\underset{\sim}{E}(n-m-q)]\ (\underset{\sim}{W}(n,m+1,q) = \underset{\sim}{N}(n-q+1) + U(n,q,q)[\sum_{r=0}^{m} U(n,r+q,q+1)\underset{\sim}{N}(n-q-r) + U(n,m+q+1,q+1)\underset{\sim}{M}(n-m-q)])$. By the induction assumption the r.ve. in the brackets has a B.E.D. (B.G.D.) with mean vector $(\lambda^{-1}(1),\lambda^{-1}(2))$ $((p^{-1}(1),p^{-1}(2)))$. Since this r.ve. is independent of $\underset{\sim}{E}(n-q+1)\ (\underset{\sim}{N}(n-q+1))$ it follows, as in the case $m=1$ $(m=0)$, that $\underset{\sim}{Y}(n,m+1,q)\ (\underset{\sim}{W}(n,m+1,q))$ has a B.E.D. (B.G.D.) with mean vector $(\lambda^{-1}(1),\lambda^{-1}(2))\ ((p^{-1}(1),p^{-1}(2)))$ for all n,q. ||

Note that $\underset{\sim}{X}(n,m)$ $(\underset{\sim}{G}(n,m))$, given by (3.2) ((3.5)), is equal to $\underset{\sim}{Y}(n,m,1)$ $(\underset{\sim}{W}(n,m,1))$. Thus we conclude from Lemma 3.7 that

Corollary 3.8. For all n,m $\underset{\sim}{X}(n,m)$ $(\underset{\sim}{G}(n,m))$ has a B.E.D. (B.G.D.) with mean vector $(\lambda^{-1}(1),\lambda^{-1}(2))\ ((p^{-1}(1),p^{-1}(2)))$.

We show now that $\underset{\sim}{G}(n,\infty)$, given by (3.6), has a B.G.D. Also we show that if for all n and $i=1,2$

$$\lim_{m\to\infty} \prod_{\ell=1}^{m} (1-\beta(n,\ell,i)) = 0 \tag{3.9}$$

then $X(n,\infty)$, given by (3.3), has a B.E.D.

Lemma 3.10. (a) For all n, $\underset{\sim}{G}(n,\infty)$ has a B.G.D. with mean vector $(p^{-1}(1),p^{-1}(2))$.
(b) If Condition (3.9) holds then for all n, $\underset{\sim}{X}(n,\infty)$ has a B.E.D. with mean vector $(\lambda^{-1}(1),\lambda^{-1}(2))$.

Proof: Let n be a positive integer. Since $\lim_{m\to\infty} (1-\alpha(n,\ell))^m \le \lim_{m\to\infty}(1-p(\ell))^m = 0$, $\ell=1,2$, $\underset{\sim}{G}(n,m) \xrightarrow[m\to\infty]{p} \underset{\sim}{G}(n,\infty)$. By (3.9), $\underset{\sim}{X}(n,m) \xrightarrow[m\to\infty]{p} \underset{\sim}{X}(n,\infty)$. Thus in particular $\underset{\sim}{X}(n,m) \xrightarrow[m\to\infty]{D} \underset{\sim}{X}(n,\infty)$ and $\underset{\sim}{G}(n,m) \xrightarrow[m\to\infty]{D} \underset{\sim}{G}(n,\infty)$. Consequently the results of the lemma follow from Corollary 3.8. ||

Note that for (3.9) to hold it suffices that for all n and $i = 1,2$, $\inf\{\beta(n,\ell,i),\ \ell = 1,2,\ldots\} > 0$.

Next we investigate some dependency aspects of both classes.

Remarks 3.11. (a) For a fixed m, the sequences $\{\underset{\sim}{X}(n,m),\ n = 0,\pm1,\pm2,\ldots\}$, $\{\underset{\sim}{G}(n,m),\ n = 0,\pm1,\pm2,\ldots\}$ are m-dependent (i.e. for n(1), n(2) integers s.t. $|n(1)-n(2)| > m$ the r.ve.s $\underset{\sim}{X}(n(1),m)$, $\underset{\sim}{X}(n(2),m)$ ($\underset{\sim}{G}(n(1),m)$, $\underset{\sim}{G}(n(2),m)$) are independent). (b) Clearly if we choose m to be a function, say ψ, of n ($\psi(n) \in \{1,2,\ldots\}$ for all n) then the dependency of $\underset{\sim}{X}(n, \psi(n))$ ($\underset{\sim}{G}(n,\psi(n))$) on the past varies with n. (c) It is easy to see that for all n the r.ve. $\underset{\sim}{X}(n,\infty)$ ($\underset{\sim}{G}(n,\infty)$) depends on all the preceeding r.ve.s $\underset{\sim}{X}(q,\infty)$, $-\infty < q < n$ ($\underset{\sim}{G}(q,\infty)$, $-\infty < q < n$).

We now investigate one positive dependence aspect of both classes.

Lemma 3.12. Suppose that E(1,1),E(1,2) are AS. Then for all positive integers m,k, and all integers $n(1) < n(2) < \ldots < n(k)$ the r.v.s $X(n(j),m,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$ are AS.

Proof: By Th. 2.2 p. 31 and P_4 p. 30 of B-P, the r.v.s. $E(r,\ell)$, $I(n(j),q,\ell)$, $\ell = 1,2$, $q = 1,\ldots,m$, $j = 1,\ldots,k$, $r = n(1),\ldots,n(k)$ are AS. Since the r.v.s $X(n(j),m,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$ are nondecreasing functions of the previous collection of AS. r.v.s the result of the lemma follows by P_3 p. 30 of B-P. ||

In a similar way one can prove the following lemma.

Lemma 3.13. Suppose that M(1,1), M(1,2) are AS. and that for all n, N(n,1), N(n,2) are AS. Then for all positive integers m,k, and all integers $n(1) < n(2) < \ldots < n(k)$ the r.v.s $G(n(j),m,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$ are AS.

Next we prove similar results for the sequences $\{\underset{\sim}{X}(n,\infty),\ n = 0,\pm1,\pm2,\ldots\}$ and $\{\underset{\sim}{G}(n,\infty),\ n = 0,\pm1,\pm2,\ldots\}$.

Lemma 3.14. (a) Let us assume that M(1,1), M(1,2) are AS. and that for all n, N(n,1), N(n,2) are AS. Then for all positive integers k and all integers

$n(1) < n(2) < \ldots < n(k)$, the r.v.s $G(n(j),\infty,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$ are AS. (b) Let us assume that $E(1,1)$, $E(1,2)$ are AS and that Condition (3.9) holds. Then for all positive integers k and all integers $n(1) < n(2) < \ldots < n(k)$, the r.v.s $X(n(j),\infty,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$ are AS.

Proof: By similar arguments to the ones given in the proof of Lemma 3.10 we conclude that the two sequences of r.ve.s: $(G(n(1),m,1), G(n(1),m,2),\ldots, G(n(k),m,1), G(n(k),m,2))$ and $(X(n(1),m,1), X(n(1),m,2),\ldots,X(n(k),m,1), X(n(k),m,2))$, $m = 1,2,\ldots$, converge in distribution as $m \to \infty$ to the r.ve.s: $(G(n(1),\infty, 1), G(n(1),\infty,2),\ldots,G(n(k),\infty,1), G(n(k),\infty,2)$ and $(X(n(1),\infty,1), X(n(1),\infty,2),\ldots,X(n(k),\infty,1), X(n(k),\infty,2))$, respectively. By Lemma 3.12 the r.v.s $X(n(j),m,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$, are AS. for all m and by Lemma 3.13 the r.v.s $G(n(j),m,\ell)$, $\ell = 1,2$, $j = 1,\ldots,k$, are AS. for all m. Consequently the results of the lemma follow by P_4 of Esary-Proschan-Walkup (1967). ||

Next we compute the autocovariance matrices for both classes of sequences. Some notation is needed.

Notation 3.15. Let $\Gamma(e)$, $\Gamma(g)$, and $\Gamma(g,n)$ be the covariance matrices of $\underset{\sim}{E}(1)$, $\underset{\sim}{M}(1)$ and $\underset{\sim}{N}(n)$ respectively. For $\ell = 0,1,\ldots$, let $\Gamma(e,n,\ell,m)$ and $\Gamma(g,n,\ell,m)$ be the autocovariance matrices of $\underset{\sim}{X}(n,m)$, and $\underset{\sim}{G}(n,m)$, respectively, $n,\ell = 0,\pm1,\pm2,\ldots$, $m = 1,2,\ldots,\infty$. Further let $A(n,j)$ be a 2×2 diagonal matrix with $(1-\alpha(n,1))^j$, $(1-\alpha(n,2))^j$ on the diagonal, let I be the 2×2 identity matrix, and χ the indicator function. By some simple calculations we obtain for $n = 0,\pm1,\pm2,\ldots$, $m = 1,2,\ldots,\infty$, and $\ell = 1,2,\ldots$ (but not zero), that

(3.16) $\Gamma(e,n,\ell,m) =$

$$\sum_{r=0}^{m-\ell-1} B(n,r+1)\Big[\prod_{j=1}^{r}(I-B(n,j))\Big]\Gamma(e)\Big[\prod_{j=1}^{r+\ell}(I-B(n+\ell,j)\Big]B(n+\ell,r+\ell-1)$$
$$+ B(n,m-\ell+1)\Big[\prod_{j=1}^{m-\ell}(I-B(n,j))\Big]\Gamma(e)\Big[\prod_{j=1}^{m}(I-B(n+\ell,j))\Big].$$

We may obtain the off-diagonal elements of $\Gamma(e,n,0,m)$ from Equation (3.16) by setting $B(n,m+1) = I$ and $\ell = 0$. The diagonal elements are the variances of $X(n,m,1)$ and $X(n,m,2)$, namely, $\lambda^{-2}(1)$ and $\lambda^{-2}(2)$, respectively.
In a similar way we obtain for $n = 0,\pm 1,\pm 2,\ldots$, $m = 1,2,\ldots,\infty$, and $\ell = 1,2,\ldots$ (but not zero), that

$$(3.17) \qquad \Gamma(g,n,\ell,m) = \sum_{r=0}^{m-\ell} A(n,r)\Gamma(g,n-r)A(n+\ell,r+\ell)$$

$$+ \chi_{\{0\}}(\ell)\; A(n,m+1)\Gamma(g)A(n,m+1).$$

We may obtain the off diagonal elements of $\Gamma(g,n,0,m)$ from Equation (3.17) by setting $\ell = 0$. The diagonal elements are the variances of $G(n,m,1)$, $G(n,m,2)$, namely, $(1-p(1))p^{-2}(1)$ and $(1-p(2))p^{-2}(2)$, respectively.

Finally we give sufficient conditions for the exponential and geometric sequences to be stationary and present their spectral density matrices.
First we address ourselves to the exponential case. Let us assume that $\beta(n,j,1)$, $\beta(n,j,2)$, given in Notation 3.1, do not depend on n. Then clearly the bivariate exponential sequences given by (3.2) and (3.3) are stationary. Let us denote the autocovariance function of the stationary $\underset{\sim}{X}(n,m)$ by $\Gamma(e,\ell,m)$.
We obtain $\Gamma(e,\ell,m)$ from Equation (3.16) by simply suppressing the index corresponding to n in Equation (3.16). Note that for $m < \infty$ $\sum_{\ell=-\infty}^{\infty} \|\Gamma(e,\ell,m)\|$ is a finite sum and is thus finite, and for $m = \infty$, $\sum_{\ell=-\infty}^{\infty} \|\Gamma(e,\ell,m)\| < \infty$ by Consition (3.9). Consequently the 2×2 spectral density matrix, $f(e,\omega,m)$, of the stationary process $\underset{\sim}{X}(n,m)$ is given by:

$$(3.18) \qquad f(e,\omega,m) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} \Gamma(e,\ell,m)e^{-i\omega\ell}, \quad \omega \in [-\pi,\pi],$$

$$m = 1,\ldots,\infty,$$

with inverse relationship

$$(3.19)\qquad \Gamma(e,\ell,m) = \int_{-\pi}^{\pi} f(e,\omega,m)e^{i\omega\ell}d\omega,\ \ \ell = 0,\pm1,\pm2,\ldots,$$

$$m = 1,2,\ldots,\infty$$

(where $\Gamma(e,-\ell,m)$ is the transpose of $\Gamma(e,\ell,m)$).

A sufficient condition for the geometric sequences, given by Equations (3.5) and (3.6), to be stationary is that $\alpha(n,1)$, $\alpha(n,2)$, given in Notation 3.4 are equal to $\alpha(1)$, $\alpha(2)$ respectively for all n. Let us denote the autocovariance function of the stationary sequence $\underset{\sim}{G}(n,m)$ by $\Gamma(g,\ell,m)$. We obtain $\Gamma(g,\ell,m)$ from Equation (3.17) by suppressing the index corresponding to n in Equation (3.17). Since $\alpha(1) \geq p(1)$ and $\alpha(2) \geq p(2)$, $\sum_{\ell=-\infty}^{\infty}\|\Gamma(g,\ell,m)\| < \infty$ for $m = 1,2,\ldots,\infty$. Consequently the 2×2 spectral density matrix, $f(g,\omega,m)$, of the stationary process $\underset{\sim}{G}(n,m)$ is given by:

$$(3.20)\qquad f(g,\omega,m) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}\Gamma(g,\ell,m)e^{-i\omega\ell},\ \omega \in [-\pi,\pi],$$

$$m = 1,2,\ldots,\infty,$$

with inverse relationship

$$(3.21)\qquad \Gamma(g,\ell,m) = \int_{-\pi}^{\pi} f(g,\omega,m)e^{i\omega\ell}d\omega,\qquad \ell = 0,\pm1,\pm2,\ldots$$

$$m = 1,2,\ldots,\infty,$$

(where $\Gamma(g,-\ell,m)$ is the transpose of $\Gamma(g,\ell,m)$).

4. Inequalities.

For $m = 1,2,\ldots,\infty$, let $\{\underset{\sim}{R}(s,t,m) = (R(s,m,1), R(t,m,2)),\ s,t \geq 0\}$, $\underset{\sim}{R}(0,0,m) = (0,0)$, be a bivariate point process with interarrival times equal to $\underset{\sim}{X}(1,m)$, $\underset{\sim}{X}(2,m),\ldots$, given by Equation (3.2) or (3.3). Furthermore let $\{\underset{\sim}{S}(a,b,m) = (S(a,m,1), S(b,m,2)),\ a,b = 0,1,\ldots\}$, $\underset{\sim}{S}(0,0,m) = (0,0)$, be a bivariate point process with interarrival times equal to $\underset{\sim}{G}(1,m),\underset{\sim}{G}(2,m),\ldots$, given by Equation (3.5) or (3.6). We show that if the interarrival times of the process

$\underset{\sim}{R}(\underset{\sim}{S})$ are AS. then the process $\underset{\sim}{R}(\underset{\sim}{S})$ inherits positive dependence properties. Then, we use the positive dependence properties and the special structure of the interarrival times to obtain lower bounds for the joint probabilities of the bivariate point processes. Finally, we utilize the positive dependence to obtain moment inequalities for the processes $\underset{\sim}{R}$ and $\underset{\sim}{S}$ and for their interarrival times.

First we define two concepts of positive dependence.

Definition 4.1. Let $q = 2,3,\ldots,$ and let $\underset{\sim}{X} = (X_1,\ldots,X_q)$ be a r.ve. We say that $\underset{\sim}{X}$ is positively upper orthant dependent (P.U.O.D.) [positively lower orthant dependent (P.L.O.D.)] if for all real numbers $t_1,\ldots,t_q$

$$P\{X_j > t_j,\ j = 1,\ldots,q\} \geq \prod_{j=1}^{q} P\{X_j > t_j\}$$

$$[P\{X_j \leq t_j,\ j = 1,\ldots,q\} \geq \prod_{j=1}^{q} P\{X_j \leq t_j\}].$$

Remarks 4.2. (a) In the bivariate case $(q=2)$ $\underset{\sim}{X}$ is P.U.O.D. iff $\underset{\sim}{X}$ is P.L.O.D. (b) For $q > 2$ the two concepts of positive dependence are not equivalent. (c) If $X_1,\ldots,X_q$ are AS. then clearly $\underset{\sim}{X}$ is P.U.O.D. and P.L.O.D. (d) Let $f_1,\ldots,f_q\colon (-\infty,\infty) \to [0,\infty)$ be measurable nondecreasing (nonincreasing) functions and let $\underset{\sim}{X}$ be P.U.O.D. (P.L.O.D.). Then

$$E\prod_{j=1}^{q} f_j(X_j) \geq \prod_{j=1}^{q} Ef_j(X_j) \tag{4.3}$$

(see Lehmann (1966)). For the sake of completeness we present the following definition.

Definition 4.4. Let X,Y be r.v.s. We say that X is stochastically less than or equal to Y, and write $X \overset{s}{\leq} Y$ is for every real number $t, P(X>t) \leq P(Y>t)$.

Remark 4.5. Let $f\colon (-\infty,\infty) \to [0,\infty)$ be a measurable nondecreasing function and let $X \overset{s}{\leq} Y$. Then $Ef(X) \leq Ef(Y)$ (see Lehmann (1966)).

Now we show that if the interarrival times of $\underset{\sim}{R}(\underset{\sim}{S})$ are AS., then $\underset{\sim}{R}(\underset{\sim}{S})$ inherits positive dependency properties.

<u>Lemma 4.6</u>. Let $m = 1,2,\ldots,\infty$, and let us assume that for $q = 1,2,\ldots$, the r.v.s $\{X(j,m,\ell),\ j = 1,\ldots q,\ \ell = 1,2\}$ are AS. Then for all positive real numbers $s_1,\ldots,s_q$ and $\ell_1,\ldots,\ell_q = 1,2$ the r.ve.s $(R(s_j,m,\ell_j),\ j = 1,\ldots,q)$ are P.U.O.D. and P.L.O.D.

<u>Proof</u>: Let $n_1,\ldots,n_q$ be positive integers, let $f_j = \chi_{(\sum_{r=1}^{n_j} X(r,m,\ell_j) < s_j)}$ and let $g_j = \chi_{(\sum_{r=1}^{n_j} X(r,m,\ell_j) \geq s_j)}$, $j = 1,\ldots,q$. The functions $f_1,\ldots,f_q$ $(g_1,\ldots,g_q)$ are nonincreasing (nondecreasing) functions of AS. r.v.s. By B-P P_3 p. 30 $f_1,\ldots,f_q$ $(g_1,\ldots,g_q)$ are AS. r.v.s. Consequently $P\{R(s_j,m,\ell_j) > n_j,\ j = 1,\ldots,q\}$ $= E \prod_{j=1}^{q} f_j \geq \prod_{s=1}^{q} Ef_j = \prod_{j=1}^{q} P\{R(s_j,m,\ell_j) > n_j\}$ and $P\{R(s_j,m,\ell_j) \leq n_j,\ j = 1,\ldots,q\} =$ $E \prod_{j=1}^{q} g_j \geq \prod_{j=1}^{q} E\, g_j = \prod_{j=1}^{q} P\{R(s_j,m,\ell_j) \leq n_j\}$. ||

<u>Lemma 4.7</u>. Let $m = 1,2,\ldots,\infty$, and let us assume that for $q = 1,2,\ldots$, the r.v.s $\{G(j,m,\ell),\ j = 1,\ldots,q,\ \ell = 1,2\}$ are AS. Then for all positive integers $n_1,\ldots,n_q$ and $\ell_1,\ldots,\ell_q = 1,2$ the r.v.s $\{S(n_j,m,\ell_j),\ j = 1,\ldots,q\}$ are AS.

<u>Proof</u>: Note that $S(n_j,m,\ell_j) = \sum_{r=1}^{n_j} G(r,m,\ell_j)$, $j = 1,\ldots,q$. Thus the $S(n_j,m,\ell_j)$'s are nondecreasing functions of AS. r.v.s and hence are AS. ||
Using these two last results one can obtain a variety of probability inequalities. For more details see Tong (1980).

In Lemmas 4.6 and 4.7 we bounded from below joint probabilities of $\underset{\sim}{R}(\underset{\sim}{S})$ by a product of marginal probabilities of $R(\cdot,m,\ell)$ $(S(\cdot,m,\ell))$. For the stationary models we bound from below some of these marginal probabilities by Poisson and negative binomial probabilities.

Lemma 4.8. Let us assume that $\beta(n,1,1)$, $\beta(n,1,2)$ are equal, respectively, to $\beta(1), \beta(2)$ for all n. Let $m = 1,2,\ldots,\infty$, $\ell = 1,2$, $s > 0$ and let $L(\ell,s)$ be a Poisson r.v. with mean $\lambda(\ell)\beta^{-1}(\ell)s$. Then $R(s,m,\ell) \overset{s}{\leq} L(\ell,s)$.

Proof: From Equation (3.2) or (3.3) we obtain that $X(q,m,\ell) \geq \beta(\ell)E(q,\ell)$, $q = 1,2,\ldots$. Now for $r = 1,2,\ldots$ $P\{R(s,m,\ell) < r\} = P\{\sum_{q=1}^{r} X(q,m,\ell) > s\} \geq P\{\sum_{q=1}^{r} \beta(\ell)E(q,\ell) > s\} = P\{L(\ell) < r\}$. ||

Lemma 4.9. Let us assume that $\alpha(n,1)$, $\alpha(n,2)$ are equal to $\alpha(1)$, $\alpha(2)$, respectively for all n. Let $m = 1,2,\ldots,\infty$, $\ell = 1,2$, $r = 1,2,\ldots$, and let $Q(\ell,r)$ be a negative binomial r.v. with parameters $(r, p(\ell)\alpha^{-1}(\ell))$. Then $S(r,m,\ell) \overset{s}{\geq} Q(\ell,r)$.

Proof: From Equation (3.5) or (3.6) we obtain that $G(q,m,\ell) \geq N(q,\ell)$, $q = 1,2,\ldots$. Now for $a = 1,2,\ldots$ $P\{S(r,m,\ell) \geq r+a\} = P\{\sum_{q=1}^{r} G(q,m,\ell) \geq r+a\} \geq P\{\sum_{q=1}^{r} N(q,\ell) \geq r+a\} \geq P\{Q(\ell,r) \geq r+a\}$. ||

Using the AS. of $\underset{\sim}{E}(1)$ $(\underset{\sim}{G}(1))$ we obtain the following "residual" inequalities.

Lemma 4.10. Let the assumption and notation of Lemma 4.8 hold and assume that $E(1,1)$, $E(1,2)$ are AS. Then for $t_1 \geq s_1$, $t_2 \geq s_2$, n_1, n_2, $r \in \{1,2,\ldots\}$, $n_1 \geq r$, $n_2 \geq r$

$$P\{R(t_\ell,m,\ell) \leq n_\ell,\ \ell = 1,2 \mid R(s_\ell,m,\ell) < r, \ell = 1,2\}$$
$$\geq \prod_{\ell=1}^{2} P\{L(\ell, t_\ell - s_\ell) \leq n_\ell - r\}.$$

Proof: Since $X(q,m,\ell) \geq \beta(\ell)E(q,\ell)$, $q = 1,2,\ldots$, $\ell = 1,2$, we obtain that $P\{R(t_\ell,m,\ell) \leq n_\ell,\ R(s_\ell,m,\ell) < r,\ \ell = 1,2\} = P\{\sum_{q=1}^{n_\ell+1} X(q,m,\ell) > t_\ell,\ \sum_{q=1}^{r} X(q,m,\ell) > s_\ell,$ $\ell = 1,2\} \geq P\{\sum_{q=r+1}^{n_\ell+1} X(q,m,\ell) > t_\ell - s_\ell,\ \sum_{q=1}^{r} X(q,m,\ell) > s_\ell,\ \ell = 1,2\} \geq P\{\sum_{q=r+1}^{n_\ell+1} \beta(\ell)$ $E(q,\ell) > t_\ell - s_\ell,\ \sum_{q=1}^{r} X(q,m,\ell) > s_\ell,\ \ell = 1,2\}$. Note that $\sum_{q=r+1}^{n_\ell+1} \beta(\ell)E(q,\ell)$, $\ell = 1,2$ are nondecreasing functions of the AS. rv.s $\{E(q,\ell),\ q = r+1,\ldots, n_1+n_2+1,\ \ell = 1,2\}$

and thus are AS., and that they are independent of $\sum_{q=1}^{r} X(q,m,\ell), \ell = 1,2$. Consequently $P\{R(t_\ell,m,\ell) \le n_\ell,\ R(s_\ell,m,\ell) < r,\ \ell = 1,2\} \ge P\{\sum_{q=1}^{r} X(q,m,\ell) > s_\ell,\ \ell = 1,2\} \prod_{\ell=1}^{2}$

$P\{\sum_{q=r+1}^{n_\ell+1} \beta(\ell)E(q,\ell) > t_\ell - s_\ell\} = P\{\sum_{q=1}^{r} X(q,m,\ell) > s_\ell,\ \ell = 1,2\} \prod_{\ell=1}^{2} P\{L(\ell,t_\ell - s_\ell) \le n_\ell - r\}$.

Now the result of the lemma follows. ||

In a similar way one can show the following result.

<u>Lemma 4.11</u>. Let the assumption and notation of Lemma 4.9 hold and assume that $N(1,1)$, $N(1,2)$ are AS. Then for n_1, n_2, k, r_1, r_2, a_1, a_2 positive integers s.t. $r_1, r_2 > k$, $n_1 > a_1$, $n_2 > a_2$ both conditional probabilities $P\{S(r_\ell,m,\ell) > n_\ell + r_\ell,\ \ell = 1,2 \mid S(k,m,\ell) > a_\ell + k,\ \ell = 1,2\}$, and $P\{S(r_\ell,m,\ell) > n_\ell + r_\ell,\ \ell = 1,2 \mid S(k,m,\ell) = a_\ell + k,\ \ell = 1,2\}$ are not smaller than $\prod_{\ell=1}^{2} P\{Q(\ell,r_\ell - k) \ge n_\ell - a_\ell + r_\ell - k\}$.

Finally we address ourselves to some moment inequalities.

<u>Lemma 4.12</u>. Let the assumptions and notation of Lemma 4.6 hold and let $k_1,\ldots,k_q$ be positive integers. Then $E \prod_{j=1}^{q} [R(s_j,m,\ell_j)]^{k_j} \ge \prod_{j=1}^{q} E[R(s_j,m,\ell_j)]^{k_j}$.

<u>Proof</u>: The result of the lemma follows from Lemma 4.6 and Inequality (4.3) (with $f_j(x) = x^{k_j}$, $x \ge 0$, $j = 1,\ldots,q$). ||

<u>Lemma 4.13.</u> Let us assume the assumptions and notation of Lemma 4.8 hold and let k be a positive integer. Then $ER(s,m,\ell)^k \le EL(\ell,s)^k$.

<u>Proof</u>: The result of the lemma follows by Lemma 4.8 and Remark 4.5 (with $f(x) = x^k$, $x > 0$).

In a similar way we can prove the following result.

<u>Lemma 4.14</u>. Let us assume that the assumptions and notation of Lemmas 4.7 and 4.9 hold and let $k_1,\ldots,k_q$ be positive integers. Then $E \prod_{j=1}^{q} [S(n_j,m,\ell_j)]^{k_j}$ $\ge \prod_{j=1}^{q} E[Q(\ell_j,n_j)]^{k_j}$.

<u>Lemma 4.15</u>. Let us assume that the assumption and notation of Lemma 4.6 hold and let $k_1,\ldots,k_q$ be positive integers. Then $E \prod_{j=1}^{q} [X(j,m,\ell_j)]^{k_j} \ge$

$\prod_{j=1}^{q} \{k_j![\lambda(\ell_j)]^{-k_j}\}$.

Proof: The result follows by Corollary 3.8 and the association of the r.v.s $\{X(j,m,\ell_j),\ j=1,\ldots,q\}$. ||

Similarly we obtain the following.

Lemma 4.16. Let us assume that the assumption and notation of Lemma 4.7 hold and let $k_1,\ldots,k_q$ be positive integers. Then $E \prod_{j=1}^{q} [G(j,m,\ell_j)]^{k_j} \geq \prod_{j=1}^{q} E[G(1,m,\ell_j)]^{k_j}$.

Note that $G(1,m,\ell_j)$ is by Corollary 3.8, a geometric r.v. with mean $p^{-1}(\ell_j)$, $j=1,\ldots,q$.

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Two classes of finite and infinite moving average sequences of bivariate random vectors are considered. The first class has bivariate exponential marginals while the second class has bivariate geometric marginals. The theory of positive dependence is used to show that in various cases the two classes consist of associated random variables. Association is then applied to establish moment inequalities and to obtain approximations to some joint probabilities of the related bivariate point processes.

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