(12)

LJI-R-85-350
September 15, 1985

LaJolla INSTITUTE

CENTER FOR STUDIES OF NONLINEAR DYNAMICS
3252 HOLIDAY COURT, SUITE 208
LA JOLLA, CALIFORNIA 92037 • PHONE (619) 587-6000
AFFILIATED WITH THE UNIVERSITY OF CALIFORNIA, SAN DIEGO

THEORETICAL STUDIES AND DATA ANALYSIS OF WAVE PROPAGATION IN RANDOM MEDIA

Final Technical Report

Sponsored by
Defense Advanced Research Projects Agency (DoD)
ARPA Order No. 4852

Under Contract No. MDA903-83-C-0515 issued by
Department of Army, Defense Supply Service-Washington,
Washington, DC 20310

Effective Date of Contact: December 1, 1983
Expiration Date: September 30, 1985

Principal Investigator: Stanley M. Flatté
Telephone (619) 587-6013

La Jolla Institute
Center for Studies of Nonlinear Dynamics*
3252 Holiday Ct., Suite 208
La Jolla, CA 92037

"The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Defense Advanced Research Projects Agency or the U.S. Government."

*Affiliated with the University of California, San Diego

LJI-R-85-350
September 15, 1985

THEORETICAL STUDIES AND DATA ANALYSIS OF WAVE PROPAGATION IN RANDOM MEDIA

Final Technical Report

Sponsored by
Defense Advanced Research Projects Agency (DoD)
ARPA Order No. 4852

Under Contract No. MDA903-83-C-0515 issued by
Department of Army, Defense Supply Service-Washington,
Washington, DC 20310

Effective Date of Contact: December 1, 1983
Expiration Date: September 30, 1985

Principal Investigator: Stanley M. Flatté
Telephone (619) 587-6013

La Jolla Institute
Center for Studies of Nonlinear Dynamics*
3252 Holiday Ct., Suite 208
La Jolla, CA 92037

*Affiliated with the University of California, San Diego

THEORETICAL STUDIES AND DATA ANALYSIS OF WAVE PROPAGATION IN RANDOM MEDIA

FINAL TECHNICAL REPORT

December 1, 1983 to September 30, 1985

Contract No. MDA903-83-C-0515

Principal Investigator: Stanley M. Flatté

SUMMARY

The objective of this work was to develop a general theoretical framework for calculating fluctuations of signals on waves propagated through random media (WPRM) and to apply this framework to sound through the ocean; light through the atmosphere; radio waves through the ionosphere, solar wind, or interstellar plasma; and any other similar case of waves propagating through continuous media. Comparison with real data is an important aspect of the effort.

This report will consist of a summary (with list of references), followed by copies of the journal articles resulting from this contract that have been published, or have been submitted for publication. Some work in progress will not be included in detail here, since it will be discussed in the reports of future DARPA contracts.

The two most common signals sent on a carrier are the phase and amplitude of a nearly monochromatic wave. If enough bandwidth is available, one can send a pulse, and one can speak of the intensity and arrival time of that pulse. The technical problem is then to explain the statistical behavior of the intensity and arrival time in terms of medium fluctuations, where the medium is described statistically, usually by a power spectrum covering a large dynamic range of scales.

The eventual practical applications of an understanding of WPRM to science and to the defense department are myriad. The phase of a light wave from an astronomical

object or a satellite is used by a telescope to focus to a detector; the quality of the focus depends on the state of the atmosphere - a random medium. A ground-based laser with large optics attempts to focus on a small spot for a period of time; the atmosphere spreads out the spot by its action on the phase of the wave. Determination of pulsar parameters depends on observations of radio pulses through a distorting random medium-interstellar plasma. Communication with spacecraft by radio pulses using radio telescopes depends on coding schemes and antenna control that must contend with effects due to the solar wind and the earth's ionosphere. Communication with earth satellites must contend with the ionosphere, and the effects of a disturbed ionosphere (due, e.g., to nuclear explosions) must be predicted. Probing ocean processes, from large-scales (Gulf stream) on down (internal waves and microstructure) depends on understanding acoustic propagation through a random medium; for example, a favored method is to send pulses over long range and observe their arrival-time variations. Detection of submarines by passive acoustics is limited by ocean fluctuations that control the maximum antenna size and integration time that can be used for coherent signal integration. Active sonar for communication has a limited bandwidth due to ocean fluctuations. Determination of the characteristics of earthquakes, thought to be important for earthquake prediction, is done solely by seismic detection. The earth through which the seismic waves travel has randomness, and this limits the information that can be gleaned from seismic signals. The same effect limits our ability to distinguish between underground nuclear explosions and earthquakes, and thus affects our political stance vis-a-vis nuclear test bans. On the other hand, the distortions of seismic waves due to the earth's random properties can be used as a probe of those properties and hence can lead to a better understanding of earth structure.

The path-integral method for treating wave propagation has been successfully used by the principal investigator for the analysis of many experiments in ocean acoustics.[1] This method is therefore utilized extensively in our work. Another school of wave propagation theory, which began in the early 1960's in order to explain experiments in light transmission through the turbulent atmosphere, utilizes a different technique involving partial differential equations for the moments of the wave field.[2] The moment equations and the path integral have now been used by enough researchers that the value of both approaches is appreciated, but the relations between the two methods has remained confusing to many people.

A first step in WPRM is characterizing the statistics of the random medium. In this report we will speak of the spectrum of medium fluctuations, which would be obtained by dragging an index-of-refraction sensor through the medium and taking a Fourier transform of the resulting time (=space) series. This spectrum is characterized by a power law (e.g. -5/3 for Kolmogorov turbulence) that implies much more variation at large scales than at small scales. It is likely that a sensor dragged in different directions will observe different spectra even on the average. In that case we speak of an anisotropic spectrum. For example, the strength of the ocean spectrum is much higher in the vertical than the horizontal for the same wave-number. Finally it is important to know that the spectrum cannot continue indefinitely at either large or small scales. At the "outer scale" the spectrum cuts off due usually to finite container size -- the height of the atmosphere or the depth of the ocean. At the "inner scale" the spectrum cuts off due to physical processes; for example, viscosity becomes important at scales of order a few millimeters in the atmosphere and ocean.

This technical report covers the two year period of our contract effort. The next several paragraphs summarize the technical results we have obtained: more details are given in following sections. The work has been carried out under the direction of Dr. Stanley Flatté, and involves effort by Dr. Flatté, senior scientist Dr. Frank Henyey, two post-doctoral researchers Drs. Dennis Creamer and Rod Frehlich, and a graduate student (in the UCSD Electrical Engineering and Computer Science Department) Johanan Codona.

Our progress in understanding the travel time of pulses in random media has resulted in a paper published in Physical Review Letters,[3] as well as results that will lead to later publication. Pulses sent through a fluctuating medium arrive earlier or later than they would in the absence of fluctuations, depending on the particular realization of the medium. The variance of arrival time can be calculated by straightforward methods in the geometrical optics limit. Our dramatic new result is for the average arrival time, which we find advanced in weak media. Heretofore, researchers were of the opinion that pulses were delayed on the average. This effect is of little importance in communication applications where the average arrival time is usually less important than the variance. However, the effect can be important in probing a random medium for large-scale variations by their effect on average travel time. Our result implies a possible confusion between a changing turbulence level and a change in the average index of refraction on a large scale. For example, ocean acoustic tomography attempts to measure the warming of a 100-km-square area of the ocean by an expected change in travel time of about 20 ms. However we find a change in average travel time of about 10 ms, due to an internal-wavefield that has no average warming at all. We have recently studied the range dependence of this effect, and have found that it grows as the square of the range. This implies that experiments being planned in the 1000-4000 km region will have major difficulties sorting out the effects of internal waves from the effects of large-scale structure. Most importantly, the determinations of internal-wave effects will NOT be contaminated by the large-scale effects.[4]

The above understanding of travel-time effects arose from studying the mutual coherence function (MCF) of the complex wave function of the wave field arriving at a receiver. We have developed quantitative treatments of the MCF in an anisotropic medium with curved deterministic rays,[5] and have applied these treatments to data from a 35-km, 5-kHz ocean-acoustic experiment, with good success.[6] The medium fluctuations in that experiment were measured by instruments that were independent of the acoustic information, so no free parameters were available to the theory.

Moving from arrival time, which is related to phase, we discuss amplitude or intensity. We have developed a method for calculating the spatial correlation function of intensity on a transverse plane through a receiver. This is a long-standing problem that is of great importance in using wave propagation for probing the structure of a random medium, because measuring intensity is often the observation that can be made most easily. In addition, an amplitude-modulated signal will be degraded by intensity fluctuations due to the medium. The standard theory develops a series solution for the intensity spatial spectrum. The first few terms are an accurate representation of the small-wave-number end of the spectrum. In order to calculate the high-wave-number region many terms of the series had to be evaluated. We have determined a different series expansion, whose first few terms give the high-wave-number section of the spectrum. Hence the evaluation of the full spectrum is simplified considerably. We have also made considerable progress toward evaluating the intensity spectrum for an arbitrary source distribution, going beyond the standard procedure of considering the special cases of a point source or an incident plane wave. Our general case will include a source that is extended over a large aperture. An example of a coherent source of large aperture would be a large-aperture laser beam. An example of an incoherent source is a planet, or an illuminated satellite, or an infrared plume from an ascending booster. A paper describing our results is in review at the journal of Radio Science.[7]

Intensity correlations at two different frequencies are of interest for a variety of reasons. We have derived the intensity cross-spectrum for scintillations caused by a plane wave passing through a random phase screen. A common approximation for a case of this sort is the Gaussian-Field approximation, in which the cross-spectrum is modelled as the transform of the square of the second moment. We have shown how this approximation breaks down when the outer scale is large compared with the diameter of the scattering disk (the transverse region of significant wave energy). A paper describing these results has been submitted to Radio Science.[8] Furthermore, the thesis of J.L. Codona will contain much of the two-frequency, extended-source results.[9]

Because researchers favoring the moment-equation method or the path-integral method typically knew only one of the methods in any depth, the relation between the two methods has been a mystery to many. We have expended quite a bit of effort to understand this relation. We have shown that the two methods are mathematically equivalent, in much the same way that the Heisenberg and Schrödinger approaches to quantum mechanics were shown to be mathematically equivalent. A better analogy for those familiar with quantum mechanics is the equivalence of the Schrödinger and Feynman approaches to quantum mechanics. The equivalence extends to the equivalence term by term of the series solutions for the intensity spectra mentioned earlier. a paper describing these results has been accepted for publication in the Journal of Mathematical Physics.[10]

In nearly all cases, in order to compare theory to experimental data in WPRM, we must use a model spectrum for the medium fluctuations. We have developed phenomenological spectra, as a function of wave vector, that allow for an anisotropic component added to a turbulent isotropic component.[11] This model is meaningful both for the ocean, where the anisotropic component represents internal waves, and the ionosphere, where the anisotropy is due to electrons preferentially moving along magnetic field lines. We are in the process of calculating intensity spectra in the weak fluctuation region using these model spectra. We have data from an ocean-acoustic experiment that will be used for comparison purposes; the experiment utilized 10-70 kHz sound over several hundred meters under the Arctic ice.[12] We should note that the weak-fluctuation regime is one in which the intensity series solution for the low-wave-number regime is the only relevant one.

For more than one hundred years, eclipse observers have noted "mysterious" bands of shadows moving on the ground just before and after an eclipse. Many exotic theories of these shadow bands have been put forward, but most observers agree that they are probably due to atmospheric scintillation that becomes visible when the crescent of the eclipsed moon becomes thin enough (a few minutes before and after an eclipse). Johanan Codona, the graduate student associated with this project, has made the first systematic application of WPRM theory to eclipse shadow-band observations.[13] He explains the orientation and contrast of the bands as a function of time, and describes the effects of eclipse geometry and the importance of wind direction. He relates shadow-band observations to stellar-scintillation observations. One important conclusion he draws is that as the illuminated crescent gets thinner, the shadow-band observations probe higher into the atmosphere. Recently published data[14] from the eclipse of February 16, 1980 in India agrees with Codona's predictions. Further data from the annular eclipse of May 30, 1984 in Georgia should soon be forthcoming.

We have begun the analysis of seismic data from the Center for Seismic Studies. Two nuclear explosions in the Soviet Union with good detections on the NORESS array, which has about twenty elements spaced out to a few kilometers have been obtained. The first look shows rather small travel-time fluctuations, somewhat at odds with the large amplitude fluctuations that have been suggested previously. Some of the preliminary data are shown in a subsequent section. We have looked at the data as a function of frequency up to about 20 Hz, and have seen no obvious systematic differences in travel time between the different frequencies, except for an unusual change in the arrival structure between 5 and 10 Hz.

The theory of wave propagation through three-dimensional, continuous, random media is based largely on the parabolic wave equation. This equation did not appear in a classical physics context until about 1950. Yet, after 1926 it was used in a quantum-mechanical context, where it is called the Schrodinger Equation. We have studied the history of the development of this equation, and have developed some pedagogically useful ideas on how to introduce the Schrodinger equation to students in a manner that is less mysterious than is used in the present curriculum. A paper on this subject has been submitted to the American Journal of Physics.[15]

We have presented selected portions of our work at meetings of the American Physical Society at Providence (November, 1984), the Acoustical Society of America at Austin (April, 1985), and the Union of Radio Scientists at Vancouver (June 1985), as well as in an invited talk at the International Symposium/Workshop on Multiple Scattering of Waves in Random Media and Random Rough Surfaces at the Pennsylvania State University (July, 1985).[16]

Our longer-term goals will include the implementation of our new theoretical results into computer codes for calculation of general phase and amplitude fluctuations. Two directions are contemplated that will require large-scale computing. The first is propagating waves through individual realizations of random media to compare with our theoretical results and to extrapolate those results into parameter regimes in which the theory is not valid. This propagation can be done via a parabolic wave equation, so that it is a marching solution. The second involves evaluating the theoretical formulas which involve either multidimensional ordinary integrals, or in some cases, path integrals. We have begun some computer work on simulation of WPRM using our VAX, and we plan to implement the code on an IBM-PC that has two FFT hardware boards that should allow uninterrupted calculations at about twice the speed of a VAX. These efforts are in preparation for proposals to do simulation calculations on a CRAY.

Finally, it is desirable to make the comparison in a unified way between these theoretical approaches and data from experiments in seismology, ocean acoustics, atmospheric optics, and radio waves through the ionosphere, the solar wind, or the interstellar medium. Our results will be important building blocks in making that hope a reality.

REFERENCES

* Starred references were supported wholly or in part by this DARPA contract.

1. S.M. Flatté, et al., Sound Transmission through a Fluctuating Ocean, Cambridge University Press, 1979.

2. V.I. Tatarskii, The Effects of the Turbulent Atmosphere on Wave Propagation, Israel Program for Scientific Translation: Jerusalem, Israel, 1971.

3.* J.L. Codona, D.B. Creamer, S.M. Flatté, R.G. Frehlich and F.S. Henyey, "Average Arrival Time of Wave Pulses through Continuous Random Media," Physical Review Letters vol. 55, 9-12, July, 1985.

4. S.M. Flatté, "The Effect of Internal Waves on the Travel-Time of Acoustic Pulses over Long Ranges in the Ocean," in preparation.

5.* R. Dashen, S.M. Flatté, and S.A. Reynolds, "Path-Integral treatment of acoustic mutual coherence functions for rays in a sound channel," J. Acoust. Soc. Am. vol. 77, 1716-22, May, 1985.

6. S.A. Reynolds, S.M. Flatté, R. Dashen, B. Buehler, and P. Maciejewski, "AFAR measurements of acoustic mutual coherence functions of time and frequency," J. Acoust. Soc. Am. vol. 77, 1723-31, May, 1985.

7.* J.L. Codona, D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F.S. Henyey, "Solution for the Fourth Moment of Waves Propagating in Random Media," submitted for publication in Radio Science, May, 1985.

8.* J.L. Codona, et al, "Two-Frequency Intensity Cross-Spectrum" submitted for publication in Radio Science (September, 1985).

9.* J.L. Codona, "Electromagnetic Wave Propagation through Random Media", Ph.D. Thesis, University of California at San Diego (Fall, 1985).

10.* J.L. Codona, et al., "Comparison of Path-Integral and Moment-Equation Techniques for Wave Propagation in Random Media," to be published in the Journal of Mathematical Physics.

11.* D. Creamer and S.M. Flatté, in preparation, 1985.

12. M. Schulkin, G.R. Garrison, and T. Wen, "High-frequency Acoustic Variability in the Arctic, preprint from the University of Washington Applied Physics Laboratory.

13.* J.L. Codona, "The Scintillation Theory of Eclipse Shadowbands," in preparation.

14. L.A. Marschall, R. Mahon, and R.C. Henry, "Observations of Shadow Bands at the Total Solar Eclipse of 16 February 1980," Applied Optics, **23**, 4390-93, 1984.

15.* S.M. Flatté, "On the Derivation of the Schrodinger Equation", submitted for publication in the American Journal of Physics (1985).

16.* S.M. Flatté, "Some Recent Results on Wave Propagation Through Continuous Random Media", To appear in the first issue of the Journal of Wave-Material Interaction (January, 1986).

APPENDIX A

Average Arrival Time of Wave Pulses through Continuous Random Media

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,

R. G. Frehlich, and Frank S. Henyey

Average Arrival Time of Wave Pulses through Continuous Random Media

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,[a] R. G. Frehlich, and Frank S. Henyey
Center for Studies of Nonlinear Dynamics,[b] *La Jolla Institute, La Jolla, California 92037*
(Received 9 October 1984)

It is pointed out that a continuous random medium can cause an average advance of the arrival time of a pulse. This advance will occur for unsaturated and partially saturated propagation, but not in full saturation (which corresponds to the discrete-scatterer case). The effect, which is associated with Fermat's principle of least time, can be observed by measuring the difference between intensity-weighted and unweighted average arrival times.

PACS numbers: 03.40.Kf, 05.40.+j, 42.20.Cc, 43.30.+m

Wave packets, or pulses, are frequently used to probe inhomogeneous media. If the wave speed in the medium varies on scales small compared with the total distance traveled by the pulse, then the medium is treated by statistical methods. "Macroscopic" examples include sound through ocean internal waves,[1,2] light through atmospheric turbulence,[3] and radio waves through plasmas such as the ionosphere,[4] the solar wind,[5] or the interstellar medium.[6] Microscopic examples include various "sounds" through liquid helium,[7] and waves through inhomogeneous condensed matter.[8] These continuous media may be distinguished from media consisting of discrete scatterers such as would occur in light transmission through fog,[9] or in wave transmission through a gaslike medium with *random*-point particles.[10]

This Letter points out that a fluctuating continuous medium can cause an average advance of the pulse arrival time. All previous analyses have dealt with situations in which pulses are delayed on the average.[11,12] By convention,[1-12] the ensemble average of a random medium is taken as the medium reference state, and the small fluctuations about this reference state are thus by definition a zero-mean random process. The arrival-time advance or delay is relative to the travel time through the reference states. Thus, for example, results through turbulent air or plasma are relative to quiescent air or plasma, not vacuum.

The behavior of a wave propagating through a random medium is controlled by relationships between the wave number (k) of the propagating wave, the range (R), and the strength and size of the medium fluctuations.[1,2] *Unsaturated* behavior corresponds to one stationary-phase path (ray), and occurs if the medium fluctuations are weak enough. In *fully saturated* behavior the original ray breaks up into many new microrays which are statistically independent of each other. Propagation through a medium of discrete scatterers falls in this category. *Partially saturated* behavior occurs in a strongly fluctuating medium with a power-law spectrum, which has enough small-scale fluctuations to cause the breakup into many microrays, and enough large-scale fluctuations to make the microray bundle behave like a single ray in its wandering from the unperturbed ray. Experiments in waves propagating through continuous random media typically fall into this category. We deal only with the important case in which the transverse wandering from the unperturbed ray is small compared with the range of propagation.

Briefly our results are as follows: If the travel time of a pulse is averaged over an ensemble of the random medium, with each pulse weighted by its intensity, then the average pulse is delayed, regardless of the type of propagation behavior, in agreement with previous results.[11,12] However, if the average travel time is obtained without weighting by pulse intensity, then a pulse advance is expected for both unsaturated and partially saturated behavior, while a pulse delay remains for the fully saturated case. The difference between intensity-weighted and unweighted travel time probes the variance of the first derivative of the refractive index, smoothed over a microray bundle.

To explain our effect qualitatively we first take a simple special case. Consider a point source and point receiver separated by range R, and a homogeneous medium in the absence of fluctuations, so that the unperturbed ray from source to receiver is a straight line. The random medium is concentrated in a "phase screen" at a distance z from the source. This screen has the effect of advancing the time of a wave front by a random amount $t(x)$ where x is the position on the screen, and $t(x)$ is a stationary Gaussian random process with zero mean. (We take x as one-dimensional for simplicity.)

Weak fluctuations.—In the geometrical-optics limit only one ray exists from source to receiver. The travel time for a path through point x is

$$T(x) = T_0 + 0.5c_0^{-1}Ax^2 - t(x), \tag{1}$$

where $A^{-1} \equiv z(R-z)/R$. By Fermat's principle the ray is at the point x_r such that $T(x_r)$ is a minimum.

For the case of weak fluctuations we may expand $t(x)$ as

$$t(x) = t_0 + t'x + 0.5t''x^2. \quad (2)$$

The position of the ray follows to first order as

$$x_r = A^{-1}c_0 t'. \quad (3)$$

The travel time of the ray is then

$$T(x_r) = T_0 + 0.5c_0 A^{-1} t'^2 - t_0 - c_0 A^{-1} t'^2. \quad (4)$$

This case requires that, typically

$$|c_0 A^{-1} t'^2| << |t_0|. \quad (5)$$

But t (and hence t_0 and t') are (by construction) random variables with zero mean. Therefore the t_0 term will disappear in the *average* travel time and the only effect of the fluctuations will come from the t' terms. These terms arise because the ray has moved away from its unperturbed position. The first t' term is positive, corresponding to a pulse delay, and represents the effects of geometry; the perturbed path is physically longer than the unperturbed one. The second t' term is negative, corresponding to a pulse advance; we call this the Fermat term; the ray sought out a region of the medium with a pulse advance. The Fermat term is twice as large in magnitude as the geometry term. The average travel time is

$$\langle T \rangle = T_0 - 0.5c_0 A^{-1} \langle t'^2 \rangle, \quad (6)$$

so that the pulse on the average arrives early.

There is a subtlety to this result. In the weak-fluctuation limit the intensity is controlled by the focusing due to the curvature of the wave front as it exists from the phase screen. It is not difficult to show that the intensity I is, to first order,

$$I = 1 + A^{-1}c_0 t''. \quad (7)$$

Consider the intensity-weighted average travel time

$$\langle IT(x_r) \rangle = T_0 - 0.5c_0 A^{-1} \langle t'^2 \rangle - c_0 A^{-1} \langle t_0 t'' \rangle, \quad (8)$$

where the last term comes from the correlation between the intensity and the travel time. For any random function $t(x)$ whose Fourier components are uncorrelated (i.e., the correlation function is translation-invariant)

$$\langle t_0 t'' \rangle = -\langle t'^2 \rangle. \quad (9)$$

Therefore

$$\langle IT(x_r) \rangle = T_0 + 0.5c_0 A^{-1} \langle t'^2 \rangle. \quad (10)$$

In other words, the intensity-weighted average travel time is *delayed* by fluctuations by exactly the amount that the unweighted average is advanced! The focusing effect exactly cancels the Fermat term, leaving a resultant equal to the geometry effect alone. This occurs because a positive fluctuation, which delays the pulse, acts as a converging lens to increase the intensity.

The simple example of a phase screen in the weak-fluctuation geometrical-optics limit has illustrated our point. We will now make some remarks on generalizations to extended media and strong fluctuations which we have treated rigorously but do not have space within the Letter format to describe in detail. We then describe a rigorous extension of these results by means of a path-integral method to include diffractive effects in a power-law medium.

There is no difficulty in extending the above results from a phase screen to extended media in which (6) and (10) are replaced by

$$\langle T \rangle - T_0 = -0.5c_0^{-1} \int dz\, A^{-1}(z) \times \left[\int dz'\, \rho_{xx}(z,z') \right], \quad (11)$$

$$\langle IT \rangle - T_0 = +0.5c_0^{-1} \int dz\, A^{-1}(z) \times \left[\int dz'\, \rho_{xx}(z,z') \right], \quad (12)$$

$$\rho_{xx}(z,z') = \langle \partial_x \mu(z) \partial x \mu(z') \rangle, \quad (13)$$

where $\partial_x \mu(z)$ is the transverse gradient of the refractive index due to the fluctuations at location z along the unperturbed ray. These results require the Markov approximation [that is, the quantity in square brackets in (11) is a local function of z]. If an incident plane wave rather than a point source is used, all three terms (geometry, Fermat, and focusing) are reduced by a factor of 3. If the Markov approximation is not made, the ratio between Fermat and geometry remains -2, while all terms are modified by terms of order L_p/R, where L_p is the longitudinal correlation length of the medium fluctuations.

Strong fluctuations.—If the medium fluctuations are strong enough, one can show in the limit of small wavelength that both the Fermat and focusing terms become negligible. This occurs when the intensity fluctuations become of order unity. From (7) this corresponds to

$$A^{-1}c_0 \langle t''^2 \rangle^{1/2} \gtrsim 1. \quad (14)$$

If this condition is satisfied, the unperturbed ray breaks up into many microrays that are not minima, but extrema, in accord with Hamilton's principle of stationary action. The converging (or diverging) lenses now are so strong that caustics occur, destroying the correlation between intensity and pulse delay. The Fermat term disappears because the microrays are not at global minima; hence the geometry term strongly dominates, resulting in the validity of (10) even for unweighted average travel time.

If $t(x)$ has a power-law spectrum, then the above argument must be modified to separate the effects of large-scale and small-scale fluctuations. The small-scale fluctuations, if they alone satisfy (14), break the unperturbed ray into a bundle of microrays of size L_μ. Fluctuations larger than L_μ correlate the travel times of all the microrays as though they were a single ray, and cause the average pulse arrival time to behave according to the weak-fluctuation rule (6), where $\langle t'^2\rangle$ is to be interpreted as coming only from scales larger than L_μ. It then becomes crucial to estimate L_μ. For the phase-screen case, we find

$$L_\mu \approx c_0 A^{-1}\langle t''^2\rangle^{1/2}. \quad (15)$$

The difficulty here is defining the average travel time of a pulse that itself has complicated structure due to microrays.

Path-integral result.—A major limitation of the above treatment is its restriction to cases of very small wavelength, and the related requirement of defining an inner scale for the medium fluctuations (in order for $\langle t''^2\rangle$ or even $\langle t'^2\rangle$ to be finite). The path-integral method can treat the case of finite wavelength for intensity-weighted travel time.

To arrive at average pulse travel time we note that

$$\langle IT\rangle - T_0 = -(i/c_0)\partial_\kappa\langle\psi^*(k_0+k)\psi(k_0)\rangle|_{k=0}, \quad (16)$$

where k_0 is a central wave number of the propagating wave, k is a deviation wave number whose excursion represents the pulse bandwidth, and ψ is the reduced wave function at the receiver. Let us treat the phase-screen case for simplicity. The second moment as a function of wave number is known to be[13,14]

$$\langle\psi^*(k_0+k)\psi(k_0)\rangle = \exp[-0.5k^2c_0^2\langle t_0^2\rangle]Q(k), \quad (17)$$

$$Q(k) = N\int du\,\exp\{\tfrac{1}{2}[iAk_0^2k^{-1}u^2 - k_0^2c_0^2D(u)]\}, \quad (18)$$

where $D(u)$ is the structure function of the medium,

$$D(u) \equiv \langle[t(u)-t(0)]^2\rangle, \quad (19)$$

and N is a normalization such that $Q(0)=1$. It is easy to see that $Q(k)$ has an increasing imaginary part as k grows from zero.

At very small k, $Q(k)$ is controlled by the behavior of $D(u)$ at small u, which must be quadratic if an inner scale exists. Thus for small u

$$D(u) = \langle t'^2\rangle u^2. \quad (20)$$

Then[14]

$$Q(k) = [1 - ic_0^2A^{-1}\langle t'^2\rangle k]^{-1/2}, \quad (21)$$

and the result for the intensity-weighted average travel time is (10) exactly.[15]

If we attempt to model $D(u)$ at infinitesimal u as a fractional power law, then the average travel time diverges.

This result for finite wave number predicts a delay equal to the geometry term. Most importantly this derivation has made no distinction between the weak- and strong-fluctuation cases, and can be easily generalized to the extended medium. We can conclude that $\langle IT\rangle - T_0$ is equal to the geometry term alone regardless of the fluctuation strength, a result already suggested by the geometrical-optics calculation.

When (18) is generalized to an extended medium by a path-integral formalism the same conclusions are easily shown, subject to the additional assumption of the Markov approximation.

An important modification of the above result occurs if, in the absence of fluctuations, the medium has focusing properties. In ocean sound propagation this is due to the sound channel. In radio-wave propagation from pulsars this might be due to very large-scale medium fluctuations that are effectively frozen during the time of observation. The modification can be simply expressed by generalizing $A^{-1}(z)$ in (11) and (12) for a curved ray.[2] The key result is that $A^{-1}(z)$ can be negative for various regions along the ray, and hence the *geometry term can be negative for curved rays.* This complication is crucial to the comparison between calculation and experiment in the ocean, though probably not in other media. Note that this effect provides another, different mechanism by which fluctuations in a medium may cause an average pulse advance. Finally, dispersive propagation, such as occurs for radio waves through plasma, can be treated with the same techniques.

This work was supported by the Defense Advanced Research Projects Agency.

---

[a]On leave from the University of California, Santa Cruz, Cal. 95064.

[b]Affiliated with the University of California, San Diego, Cal. 92093.

[1]S. M. Flatté, R. Dashen, W. M. Munk, K. M. Watson, and F. Zachariasen, *Sound Transmission Through a Fluctuating Ocean* (Cambridge Univ. Press, New York, 1979).

[2]S. M. Flatté, Proc. IEEE, 71, 1267–1294 (1983).

[3]*Laser Beam Propagation in the Atmosphere*, edited by J. Strohbehn, Topics in Applied Physics, Vol. 25 (Springer-Verlag, New York, 1978).

[4]C. L. Rino, Radio Sci. 17, 855–67 (1982).

[5]G. L. Tyler, J. F. Vesecky, M. A. Plume, H. T. Howard, and A. Barnes, Astrophys. J. 249, 318–332 (1981).

[6]B. J. Rickett, Annu. Rev. Astron. Astrophys. 15, 479–504 (1977).

[7]R. J. Donnelly, J. Low Temp. Phys. 52, 189 (1983).
[8]J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8, pp. 395–422.
[9]A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).
[10]K. M. Watson, Phys. Rev. 118, 886–98 (1960).
[11]J. P. Williamson, Proc. Roy. Soc. London, Ser. A 342, 131–147 (1975).
[12]C. H. Liu and K. C. Yeh, J. Opt. Soc. Am. 70, 168–172 (1980).
[13]L. C. Lee and J. R. Jokipii, Astrophys. J. 201, 532 (1975).
[14]R. Dashen, J. Math. Phys. 20, 894–920 (1979).
[15]Confusion has arisen becuase (20) for all u implies $t''=0$. The resolution is that limits taken with care imply that (9), and hence (10), remain true.

APPENDIX B

Path-integral treatment of acoustic mutual coherence functions for rays in a sound channel

Roger Dashen, Stanley M. Flatté, and Stephen A. Reynolds

Path-integral treatment of acoustic mutual coherence functions for rays in a sound channel

Roger Dashen
Institute for Advanced Study, Princeton, New Jersey 08540

Stanley M. Flatté[a] and Stephen A. Reynolds[a]
Center for Studies of Nonlinear Dynamics,[b] *La Jolla Institute, La Jolla, California 92037*

(Received 8 October 1984; accepted for publication 23 January 1985)

The mutual coherence function (MCF) of the acoustic wavefunction from a point source is derived by the path-integral technique for transmission in the presence of a sound channel. Separations in time, transverse horizontal position, vertical position, and acoustic frequency are treated. Approximate coherence times, lengths, and bandwidths due to internal-wave fluctuations are derived. The MCF of frequency is explicitly evaluated for fluctuations due to internal waves. The shape of an ensemble-averaged pulse is derived from the MCF of frequency.

PACS numbers: 43.30.Bp, 43.30.Ft, 43.20.Bi, 43.60.Cg

INTRODUCTION

The mutual coherence function (MCF) contains important statistical properties of the acoustic field that has traversed a medium filled with random fluctuations. For example, the coherence time and coherence lengths that determine the maximum effective integration times and array lengths that can be utilized in sonar systems are contained in the MCF. We show in this paper how to derive a number of results about the MCF which were indicated in our earlier works on the subject.[1,2] In particular, we show how to derive the MCF from the path-integral technique for transmission in the presence of a *deterministic sound channel*.

The MCF of frequency controls the coherent bandwidth and also describes the behavior of the ensemble-averaged pulse for a pulse-transmission experiment. In this paper, we derive the path-integral expression for the MCF of frequency, and then give explicit rules for calculating it in the special case of internal-wave medium fluctuations. We are able to explicitly evaluate the case of internal waves because the accepted spectrum of internal waves implies a nearly quadratic structure function and path integrals for quadratic actions are known.

An appropriate Fourier transform of the MCF is the ensemble-averaged pulse (EAP) at the receiver for a narrow transmitted pulse. With no sound channel the EAP rises sharply and has an exponential tail. We show how the presence of a sound channel can cause a precursor region to the pulse, and can cause the tail at late time to decrease in intensity.

I. PATH-INTEGRAL EXPRESSION FOR THE ACOUSTIC WAVEFUNCTION

We begin with the wave equation for the pressure as a function of space and time in the presence of a spatially varying wave speed. We follow the notation of the review article by Flatté[3]: The sound speed can be expressed as

$$C(\mathbf{x},t) = C_0[1 + U_0(z) + \mu(\mathbf{x},t)]\,, \tag{1}$$

where C_0 is a reference sound speed, $U_0(z)$ is a dimensionless function of the depth z representing the deterministic sound channel, and $\mu(\mathbf{x},t)$ is a random, zero-mean function of position representing the effect of medium fluctuations such as internal waves. The wave equation for an acoustic wave is unaffected by the time dependence of μ because μ has only components with very low frequency.

The acoustic pressure φ obeys the wave equation

$$\partial_{tt}\varphi - C^2\nabla^2\varphi = 0\,. \tag{2}$$

The parabolic approximation consists of considering solutions in which waves are traveling only at small angles to a particular direction; in the ocean this direction is in the horizontal, labeled by x. Thus we try

$$\varphi = \exp[i(qx - \sigma t)]\,\psi(\mathbf{x},t)\,, \tag{3}$$

where q and σ are the wavenumber and frequency of an acoustic wave traveling along the x axis at speed C_0: that is, $q = \sigma/C_0$. The "reduced" wavefunction ψ is slowly varying in space and time compared with q and σ, and satisfies a parabolic equation[4,5]

$$2iq\partial_x\psi = [-\partial_{yy} - \partial_{zz} + 2q^2(U_0 + \mu)]\,\psi\,. \tag{4}$$

Equation (4) is a Schrodinger equation, and thus its solution can be directly expressed in terms of a Feynman path integral[6]

$$\psi = N\int D\mathbf{z}\,\exp\left(iqS_0(\mathbf{z}) - iq\int_0^R \mu[x,\mathbf{z}(x),t\,]dx\right), \tag{5}$$

where the path integration (indicated by D) is over all paths $\mathbf{z}(x) = [y(x),z(x)]$ connecting the source to the receiver. The phase associated with the path in the absence of fluctuations is

$$qS_0 = q\int_0^R [\tfrac{1}{2}(\partial_x y)^2 + \tfrac{1}{2}(\partial_x z)^2 - U_0(z)]dx\,, \tag{6}$$

and N is a normalization factor chosen by convention so that $\psi \equiv 1$ for $\mu = 0$.

---

[a] On leave from the University of California, Santa Cruz, CA 95064.
[b] Affiliated with the University of California, San Diego.

II. PATH-INTEGRAL DERIVATION OF THE MCF—THE MONOCHROMATIC CASE

The MCF for the single-frequency case is $\langle \psi^*(2)\psi(1)\rangle$, where the angle brackets indicate averaging over the ensemble of random μ functions. The MCF measures the coherence between the acoustic fields at two different points, labeled by 1 and 2. These points may be separated in space, time, or both. We treat spatial separations that lie in the (y,z) plane.

The use of (5) and (6) results in

$$\langle \psi^*(2)\psi(1)\rangle = \Big\langle |N|^2 \int Dz_1\, Dz_2 \exp\Big(iqS_0(1) - iqS_0(2) - iq\int_0^R \mu(z_1,t_1)dx + iq\int_0^R \mu(z_2,t_2)dx\Big)\Big\rangle . \quad (7)$$

The ensemble average applies only to the μs, so we may write

$$\langle \psi^*(2)\,\psi(1)\rangle = |N|^2 \int Dz_1\, Dz_2 \exp[iqS_0(1) - iqS_0(2) - \tfrac{1}{2} V_{12}] , \quad (8)$$

where

$$V_{12} = q^2\Big\langle\Big(\int_0^R \mu(z_1,t_1)dx - \int_0^R \mu(z_2,t_2)dx\Big)^2\Big\rangle \quad (9)$$

and we have used the fact that

$$\langle \exp(i\alpha)\rangle = \exp(-\tfrac{1}{2}\langle\alpha^2\rangle) , \quad (10)$$

if α is a zero-mean, Gaussian random variable, such as any combination of μs is assumed to be. Even if α is not a GRV, (10) can still be true if the higher moments of α are small.

The problem of finding a useful result for the MCF now reduces to evaluating[7] the double path integral in (8). We first express the unperturbed phases in terms of the path variables using (6):

$$S_0(1) - S_0(2) = \int_0^R [\tfrac{1}{2}(\partial_x y_1)^2 - \tfrac{1}{2}(\partial_x y_2)^2 + \tfrac{1}{2}(\partial_x z_1)^2 - \tfrac{1}{2}(\partial_x z_2)^2 - U_0(z_1) + U_0(z_2)]dx , \quad (11)$$

and we expand the deterministic sound channel function to second order in the displacement of the paths away from the equilibrium ray. That is, we define $z_r(x)$ as the function that satisfies the ray equation (in the parabolic approximation):

$$\partial_{xx} z_r + U_0'(z_r) = 0 \quad (12)$$

with boundary conditions

$$z_r(0) = z_s, \quad z_r(R) = [z(1) + z(2)]/2 . \quad (13)$$

Define two new path variables by

$$v_1(x) \equiv z_1(x) - z_r(x), \quad v_2(x) \equiv z_2(x) - z_r(x) , \quad (14)$$

and expand $U_0(z_1)$ around the point $z_r(x)$:

$$U_0(z_1) \approx U_0(z_r) + U_0' v_1 + \tfrac{1}{2} U_0'' v_1^2 , \quad (15)$$

where it is understood that U_0' and U_0'' are evaluated at $z_r(x)$. This expression will be valid as long as the effective bundle of acoustic energy stays well confined around the unperturbed ray, $z_r(x)$. If more than one solution of (12) exists (deterministic multipath), then we treat one unperturbed ray at a time. Addition of the results depends on the coherence between deterministic rays, which can often be well estimated. At caustics our method breaks down.

In terms of the new displacement path variables (11) becomes

$$S_0(1) - S_0(2) = \int_0^R [\tfrac{1}{2}(\partial_x y_1)^2 - \tfrac{1}{2}(\partial_x y_2)^2 + \tfrac{1}{2}(\partial_x v_1)^2 - \tfrac{1}{2}(\partial_x v_2)^2 + \partial_x z_r(\partial_x v_1 - \partial_x v_2) - U_0'(v_1 - v_2) - \tfrac{1}{2} U_0''(v_1^2 - v_2^2)]dx . \quad (16)$$

Now change variables once again, to

$$\alpha(x) = y_1(x) - y_2(x), \quad \beta(x) = [y_1(x) + y_2(x)]/2 ,$$
$$u(x) = v_1(x) - v_2(x) = z_1(x) - z_2(x) , \quad (17)$$
$$w(x) = [v_1(x) + v_2(x)]/2 .$$

Then the unperturbed phase term becomes

$$S_0(1) - S_0(2) = \int_0^R (\partial_x \alpha \partial_x \beta + \partial_x u \partial_x w + \partial_x z_r \partial_x u - U_0' u - U_0'' uw)dx . \quad (18)$$

Now the crucial observation is that V_{12} is a function mainly of the difference between two paths z_1 and z_2, and not a function of the average $(z_1 + z_2)/2$. The path integral (8) is being done over four scalar functions α, β, u, and w. The above observations correspond to noting that V_{12} is not a function of β and w.

Consider the β path integral. The only β term in (18) gives

$$I_\beta = \int D\beta \exp\Big(iq\int_0^R (\partial_x \alpha \partial_x \beta)dx\Big) . \quad (19)$$

First, integrate by parts within the x integral, keeping α and β fixed:

$$I_\beta = \int D\beta \exp\Big(iq\int_0^R [-(\partial_{xx}\alpha)\beta]dx\Big) . \quad (20)$$

Now the β integral is direct, in analogy with the definition of the Dirac delta function, and yields a restriction on α:

$$\partial_{xx}\alpha = 0 . \quad (21)$$

The boundary conditions for α can be expressed easily for a point source:

$$\alpha(0) = 0, \quad \alpha(R) = y(1) - y(2) . \quad (22)$$

Thus the solution for $\alpha(x)$ is

$$\alpha(x) = \alpha(R)(x/R) , \quad (23)$$

which is the separation between two straight lines from the source to the two receivers at positions 1 and 2. Thus the α path integral is effectively done despite the dependence of V_{12} on α.

Consider the w path integral. First we can integrate the $\partial_x z_r \partial_x u$ term by parts within the x integral yielding $(-\partial_{xx} z_r)u$. But (12) shows that this cancels the $U_0' u$ term. Integrating the $\partial_x u \partial_x w$ terms by parts finally yields

$$S_0(1) - S_0(2) = \int_0^R (-\partial_{xx} u - U_0'' u)w\, dx , \quad (24)$$

and the w part of the path integral yields, again in analogy with the Dirac δ function:

$$\partial_{uu} u + U_0'' u = 0 . \quad (25)$$

The boundary conditions for u can be expressed in the same way as (22)

$$u(0) = 0, \quad u(R) = z(1) - z(2) . \quad (26)$$

The solution of (25) and (26) is easily shown to be the separation between two nearby rays in the sound channel, which start at the source and end at the two receivers.

Thus the path integral is done, and the only requirement is to evaluate V_{12} at the separations that have been determined solely from the integrations involving the unperturbed phase terms. A stationary-phase approximation has been invoked to define the unperturbed ray in the absence of fluctuations. However, the fact that no stationary-phase approximation has been invoked for propagation through the fluctuations should be emphasized. But V_{12} for $z_1(x)$ and $z_2(x)$ being nearby rays is defined as the well-known phase structure function:

$$V_{12}(\alpha, u = \text{ray separation}) \equiv D(1,2) . \quad (27)$$

Thus the result is

$$\langle \psi^*(2)\psi(1)\rangle = \exp[-\tfrac{1}{2} D(1,2)] . \quad (28)$$

At this point it is worth noting that a number of endpoint terms that resulted from the integrations by parts have been subsumed in the normalization, which is required to give unity at zero separation of the two receivers.

The result (28) has been obtained previously in many different ways.[8–11] The path-integral derivation provides a means of seeing the physics of the approximations in a new way. The only approximations have been the parabolic approximation and that V_{12} is not a function of β or w. The result is valid in the presence of a deterministic sound channel, regardless of whether the wave fluctuations are unsaturated or not. The limitations on the result come from the approximation being violated by inhomogeneity and anisotropy.[3]

III. EVALUATION OF THE MONOCHROMATIC MCF FOR INTERNAL WAVES

Under the Markov approximation, the phase structure function can be expressed as

$$D(1,2) = 2q^2 \int_0^R dx \langle \mu^2 \rangle L_P f(\alpha,u,t) , \quad (29)$$

where $\langle \mu^2 \rangle$ and L_P have been defined previously[2,3] and $f(\alpha,u,t)$ is the phase correlation function (PCF) defined by Esswein and Flatté.[12] The PCF has been evaluated for internal waves by Esswein and Flatté,[12] using a combination of analytical and numerical techniques. Since α and u are both functions of range, x, (29) must in general be evaluated by a numerical integration code.

At small separations, approximations to the PCF are possible. For example,

$$f(0,0,t) = \tfrac{1}{2}\{\omega^2\} t^2 , \quad (30)$$

$$f(0,u,0) = \tfrac{1}{2}\{k_V^2\} u^2 \ln(u_0/u) , \quad (31)$$

where $\{\omega^2\}$ is an average internal-wave frequency that is dependent on the local depth and position of the ray $z_r(x)$, and likewise the quantities $\{k_V^2\}$ and u_0. All three quantities are evaluated in Esswein and Flatté. Because (31) is nearly quadratic the logarithm may rather accurately be replaced by a constant. The best constant to use can be shown[1,2] to be $\ln \Phi$, where

$$\Phi^2 = q^2 \int dx \langle \mu^2 \rangle L_P . \quad (32)$$

Evaluations of Φ for some particular examples are given by Esswein and Flatté.[13]

The evaluation of $f(\alpha,0,0)$ for internal waves is complicated by the coupling between horizontal internal-wave structure and both vertical structure and frequency.[12] Numerical evaluations have been done[12] and seem to follow approximately the law

$$f(\alpha,0,0) \approx \tfrac{1}{2} B\alpha^p , \quad (33)$$

where the power p is empirically about 1.5. Thus our final approximate results for the monochromatic MCF are

$$\langle \psi^*(t)\psi(0)\rangle = \exp[-0.5(t/t_0)^2] , \quad (34)$$

$$\langle \psi^*(\Delta z)\psi(0)\rangle = \exp[-0.5(\Delta z/z_0)^2] , \quad (35)$$

$$\langle \psi^*(\Delta y)\psi(0)\rangle = \exp[-0.5(\Delta y/y_0)^p] , \quad (36)$$

where the coherence time t_0, vertical coherence length z_0 and horizontal coherence length y_0 are given by

$$t_0^{-2} = q^2 \int_0^R dx \langle \mu^2 \rangle L_P \{\omega^2\} , \quad (37)$$

$$z_0^{-2} = q^2 \ln \Phi \int_0^R dx \langle \mu^2 \rangle L_P \{k_V^2\} , \quad (38)$$

$$y_0^{-p} = q^2 \int_0^R dx \langle \mu^2 \rangle L_P B , \quad (39)$$

and the evaluations of all the quantities in (37)–(39) can be done by the methods of Esswein and Flatté.[12]

Comparisons of (34) and (37) with experiment are done in a companion paper.[14] It would be very desirable to compare (35) and (36) with experimental data in an appropriate parameter range in which internal waves should dominate.

IV. PATH-INTEGRAL DERIVATION OF THE MCF OF ACOUSTIC FREQUENCY

We will treat in detail only the MCF for frequency separations with no simultaneous space or time separations. Then (8) is replaced by

$$\langle \psi^*(q_2)\psi(q_1)\rangle = |N|^2 \int Dz_1 Dz_2 \exp[iq_1 S_0(1) - iq_2 S_0(2) - \tfrac{1}{2} V_{12}] , \quad (40)$$

where $q_i = \sigma_i / C_0$ and

$$V_{12} = \left\langle \left(q_1 \int_0^R \mu(z_1, 0) dx - q_2 \int_0^R \mu(z_2, 0) dx \right)^2 \right\rangle . \quad (41)$$

The unperturbed phase is expressed in the manner of (11) as

$$q_1 S_0(1) - q_2 S_0(2) = \int_0^R [\tfrac{1}{2} q_1 (\partial_x y_1)^2 - \tfrac{1}{2} q_2 (\partial_x y_2)^2 + \tfrac{1}{2} q_1 (\partial_x z_1)^2 - \tfrac{1}{2} q_2 (\partial_x z_2)^2 - q_1 U_0(z_1) + q_2 U_0(z_2)] dx . \quad (42)$$

Applying the same changes of variables indicated in (14) and (17), we find

$$q_1S_0(1) - q_2S_0(2) = \bar{q}[S_0(1) - S_0(2)] + \frac{1}{2}\Delta q \int_0^R [(\partial_x \beta)^2 + \tfrac{1}{4}(\partial_x \alpha)^2 + (\partial_x w)^2 + \tfrac{1}{4}(\partial_x u)^2 + 2\partial_x w \partial_x z_r + (\partial_x z_r)^2 - 2U_0(z_r) - 2U_0' w - U_0'' w^2 - \tfrac{1}{4} U_0'' u^2] dx, \quad (43)$$

where $\bar{q} = (q_1 + q_2)/2$, $\Delta q = q_1 - q_2$, and $S_0(1) - S_0(2)$ is expressed in these variables in (18). Again we observe that V_{12} is not a function of β and w, and this allows the path integrals over β and w to be done. However, in this unequal frequency case the path integrals are not analogous to δ functions; instead they are Gaussian integrals because of the quadratic forms in (43).

Consider the β integral. Collecting β terms in (43) we have

$$I_\beta = \int D\beta \exp\left[\frac{i}{2}\Delta q \int_0^R \left((\partial_x \beta)^2 + \frac{2\bar{q}}{\Delta q}\partial_x \alpha \partial_x \beta\right) dx\right]. \quad (44)$$

This path integral is done by completing the square, and, aside from factors that will be subsumed in the normalization, yields

$$I_\beta = \exp\left(-\frac{i}{2}\frac{\bar{q}^2}{\Delta q}\int_0^R (\partial_x \alpha)^2 dx\right). \quad (45)$$

Now consider the w path integral. This can be expressed as

$$I_w = \int Dw \exp\left[\frac{i}{2}\Delta q \int_0^R \left(wLw + \frac{2\bar{q}}{\Delta q} wLu\right) dx\right], \quad (46)$$

where L is an operator given by

$$L = -\partial_{xx} - U_0''. \quad (47)$$

Note that integration by parts has been used in several places to obtain (46). Evaluation of I_w is done by completing the square and yields, apart from normalization terms,

$$I_w = \exp\left(-\frac{i}{2}\frac{\bar{q}^2}{\Delta q}\int_0^R uLu\, dx\right). \quad (48)$$

Combining (43), (45), and (48), we find (40) becomes

$$\langle \psi^*(q_2)\psi(q_1)\rangle = N\int D\alpha\, Du \exp\left(-\frac{i}{2}\frac{q_1q_2}{\Delta q} \times \int_0^R [(\partial_x \alpha)^2 + uLu] dx - \frac{1}{2}V_{12}\right). \quad (49)$$

First, let us note that if $\Delta q = 0$, then this path integral is essentially a stationary-phase integral around the extremum of the range integral in the exponential. The solutions for α and u are then (23) and (25) and (26), respectively. This MCF expression (49) is a path integral over two scalar (or one vector) path, and it cannot be simplified without some assumptions about the nature of V_{12}. Let us express V_{12} in terms of $\bar{q}$ and Δq:

$$V_{12} = \bar{q}^2\left\langle\left(\int \mu(z_1)dx - \int \mu(z_2)dx\right)^2\right\rangle + \bar{q}\Delta q\left[\left\langle\left(\int \mu(z_1)dx\right)^2\right\rangle - \left\langle\left(\int \mu(z_2)dx\right)^2\right\rangle\right] + \frac{1}{4}(\Delta q)^2\left\langle\left(\int \mu(z_1)dx + \int \mu(z_2)dx\right)^2\right\rangle. \quad (50)$$

The lack of dependence of V_{12} on the centroid of the paths (β, w) allows us to neglect the $\bar{q}\Delta q$ term and to estimate the last term as $(\Delta q/\bar{q})^2\Phi^2$, where

$$\Phi^2 = \bar{q}^2\left\langle\left(\int \mu(z_r)dx\right)^2\right\rangle. \quad (51)$$

Since Φ^2 is a function only of the unperturbed ray z_r, and not the paths, the path-dependent part of V_{12} is only the first term of (50). Using the Markov approximation at this point we have (32) and

$$V_{12} = (\Delta q/\bar{q})^2\Phi^2 + 2\bar{q}^2\int_0^R \langle\mu^2\rangle L_P f(\alpha,u,0)dx. \quad (52)$$

The dependence on paths is contained in $f(\alpha,u,0)$, and the expression for the MCF is now

$$\langle \psi^*(q_2)\psi(q_1)\rangle = \exp\left[-\frac{1}{2}\left(\frac{\Delta q}{\bar{q}}\right)^2\Phi^2\right]Q(\Delta q), \quad (53)$$

$$Q(\Delta q) = N\int D\alpha\, Du \exp\left(-\frac{i}{2}\frac{q_1q_2}{\Delta q}\int_0^R [(\partial_x \alpha)^2 + uLu]dx - \bar{q}^2\int_0^R \langle\mu^2\rangle L_P f(\alpha,u,0)dx\right). \quad (54)$$

The evaluation of the path integral, $Q(\Delta q)$, depends on our understanding of $f(\alpha,u,0)$. We make use of the approximations given in (31) and (33), and our knowledge of the magnitudes of B and $\{k_V^2\}$, to justify the neglect of the dependence of f on the horizontal variable α (because B is very small). The α integral then yields

$$Q(\Delta q) = N\int Du \exp\left(-\frac{i}{2}\frac{q_1q_2}{\Delta q}\int_0^R uLu\, dx - \frac{1}{2}\bar{q}^2 \ln \Phi \int_0^R \langle\mu^2\rangle L_P\{k_V^2\}u^2 dx\right). \quad (55)$$

This is a one-dimensional path integral with quadratic action, whose explicit evaluation will be given in the next section. Note that the normalization N is set so that $Q(\Delta q) = 1$ if $\mu = 0$ everywhere.

V. EVALUATION OF THE MCF OF FREQUENCY

The evaluation of $Q(\Delta q)$ from the path integral (55) can be done in at least two useful ways. The first involves fixing Δq and solving an ordinary differential equation for Q. The second involves an eigenvalue method that finds the singularities in $Q(\Delta q)$.

It will be useful to express Q in the following way:

$$Q(\Delta q) = N\int Du \exp\left(-\frac{i}{2}\bar{q}^2\int_0^R uMu\, dx\right), \quad (56)$$

$$M(x) = -\partial_{xx} - U_0'' - i\Delta q \ln \Phi \langle\mu^2\rangle L_P\{k_V^2\}. \quad (57)$$

where we have used the fact that $q_1 q_2 \simeq \bar{q}^2$. The solution of this quadratic path integral is given in terms of the solution of a differential equation[6] for a function $S(x,\Delta q)$. The equation is

$$MS = 0, \quad (58)$$

with initial condition $S(0,\Delta q) = 0$. Then

$$Q(\Delta q) = [S(R,0)/S(R,\Delta q)]^{1/2}. \quad (59)$$

Note that S is complex.

The form (59) is useful for finding the behavior of the MCF at small frequency separations. (Remember that the frequency separation $\Delta\sigma = \Delta q C_0$, and we define $\bar{\sigma} \equiv \bar{q} C_0$.) Consider M as made up of two parts

$$M = L - \Delta\sigma M_1, \quad (60)$$

$$M_1 = iC_0^{-1} \ln \Phi \langle \mu^2 \rangle L_P\{k_V^2\}. \quad (61)$$

Considered in position space these operators are infinite-dimensional matrices, and the inverse of L can be found by the equation

$$(-\partial_{xx} - U_0'')L^{-1}(x,x') = \delta(x'-x), \quad (62)$$

hence,

$$L^{-1}(x,x') = g(x,x'), \quad (63)$$

where $g(x,x')$ is the Green's function defined in Ref. 3; it depends on the sound channel through U_0''.

At small $\Delta\sigma$, we may solve (58) by a perturbation expansion. This yields

$$\ln[Q(\Delta\sigma)] = \frac{\Delta\sigma}{2} \mathrm{Tr}(L^{-1}M_1) + \frac{(\Delta\sigma)^2}{4} \mathrm{Tr}(L^{-1}M_1 L^{-1}M_1), \quad (64)$$

where Tr indicates the trace. The traces become integrals in the infinite-dimensional case, so that we may finally express Q as

$$\ln[Q(\Delta\sigma)] \simeq i\Delta\sigma\tau_1 - \tfrac{1}{2}(\Delta\sigma\tau_0)^2, \quad (65)$$

$$\tau_1 = \frac{\ln\Phi}{2C_0} \int_0^R \langle \mu^2 \rangle L_P\{k_V^2\}\, g(x,x) dx, \quad (66)$$

$$\tau_0^2 = \left(\frac{\ln\Phi}{2C_0}\right)^2 \int_0^R dx \langle \mu^2 \rangle L_P\{k_V^2\} \times \int_0^R dx' \langle \mu^2 \rangle L_P\{k_V^2\} [g(x,x')]^2, \quad (67)$$

where it is understood that the first $\langle \mu^2 \rangle L_P\{k_V^2\}$ is evaluated at x and the second at x'.

Although (65) is only exact as $\Delta\sigma \to 0$, it is worthwhile to model the MCF as

$$\langle \psi^*(\sigma_1)\psi(\sigma_2) \rangle = \exp\left[-\frac{1}{2}\left(\frac{\Delta\sigma}{\bar{\sigma}}\right)^2 \Phi^2\right] \times \exp[i\Delta\sigma\tau_1 - \tfrac{1}{2}(\Delta\sigma\tau_0)^2], \quad (68)$$

where the second exponential comes from the Q function. In this form it is clear that τ_1 is a shift of the time origin for the frequency determination, and τ_0 is to be compared with Φ/σ to find which term is more important in determining the width of the MCF. That is, τ_0^{-1} and σ/Φ combine to determine the coherent bandwidth of the transmission.

TABLE I. Parameters of $Q(\Delta\sigma)$ for two examples with constant $F(x)$ and U_0''. $F(x)$ is set at 10^{-2} ms/km^2. Units for the τs are ms. Units for the eigenvalues are (ms)$^{-1}$. The eigenvalues were calculated by a general computer code (CC); they should agree with (72) for these simple examples, which they do reasonably well.

U_0''(km^{-2})	0		0.034	
τ_1	1.02		−2.61	
τ_0	0.65		2.76	
	CC	(72)	CC	(72)
λ_1	0.80	0.81	−2.60	−2.59
λ_2	3.22	3.24	−0.18	−0.18
λ_3	7.24	7.29	3.84	3.85
λ_4	12.87	12.89	9.47	9.49
λ_5	20.1	20.1	16.71	16.47
λ_{10}	80.4	80.6	77.0	77.2
λ_{15}	181	181	178	178
λ_{20}	322	322	318	319

The behavior of the MCF near zero $\Delta\sigma$ is only part of the information available. To learn more, we return to (58) and solve it by an eigenvalue method:

$$LS_n(x) = \lambda_n F(x) S_n(x), \quad (69)$$

$$F(x) = C_0^{-1} \ln \Phi \langle \mu^2 \rangle L_P\{k_V^2\}, \quad (70)$$

with boundary conditions $S_n(0) = S_n(R) = 0$. Then

$$Q(\Delta\sigma) = \prod_n (1 - i\Delta\sigma/\lambda_n)^{-1/2}. \quad (71)$$

There are many numerical methods for solving (69). We find the most effective to be expanding $S_n(x)$ in Fourier modes along with the eigenvalues λ_n. In applying this method one must be sure that enough Fourier modes to accurately represent the S_n up to the desired maximum n have been taken.

Both $F(x)$ and U_0'' affect the values of λ_n. As an example, take U_0'' and $F(x) =$ constant. Then

$$\lambda_n = F^{-1}[(\pi/R)^2 n^2 - U_0''], \quad n = 1,2,\ldots, . \quad (72)$$

Note that all of the λ_n are positive if $U_0'' = 0$ (no sound channel). In general, a nonzero U_0'' can pull the lowest lying λ_n below zero. The number of λ_n below zero is equal to the

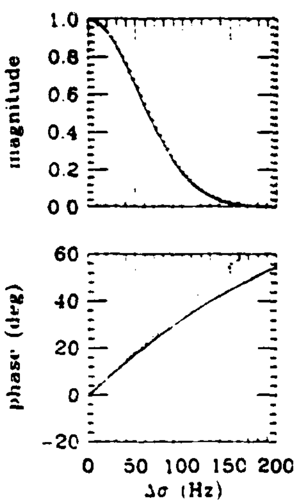


FIG. 1. The MCF of frequency for no sound channel and constant $F(x) = 10^{-2}$ (ms)/km^2. The range is taken to be 35 km. The solid curve is calculated by a general computer code. The dotted curve is the approximation (68) with (66) and (67), which give $\tau_1 = 1.02$ ms and $\tau_0 = 0.65$ ms.

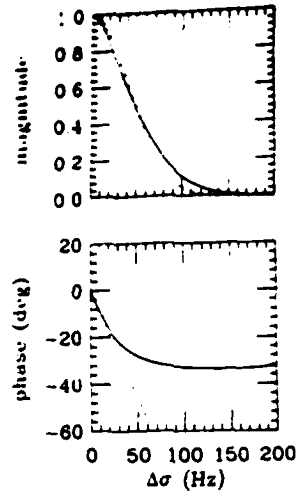


FIG. 2. The same as Fig. 1, except the sound channel is represented by constant $U_0'' = 0.034$ (km)$^{-2}$, $\tau_1 = -2.61$ ms, and $\tau_0 = 2.76$ ms.

number of caustics the unperturbed ray has passed from source to receiver.[15] As a particular λ_n passes through zero, the receiver is passing through a caustic.

Consider two examples in which $\Phi/\sigma = 2.8$ ms, $F = 10^{-2}$ ms/km^2, and either $U_0'' = 0$, or 0.034 km^{-2}. These values are comparable in magnitude to those in the AFAR experiment[14] which, however, has an F and U_0'' that vary with range. The range is taken to be 35 km. Table I gives the values of the eigenvalues for the examples, as calculated from our numerical solutions to (69). They agree with the analytic expectation (72). (The same computer code has been used in calculating the eigenvalues for the AFAR experiment.[14]) Figure 1 shows the $Q(\Delta\sigma)$ for this case. The values of τ_1 and τ_0 calculated from (66) and (67) are given in Table I. These values fit the phase slope and the $1/e^{1/2}$ magnitude half-width rather well.

When U_0'' is set to 0.034 km^{-2}, the resulting $Q(\Delta\sigma)$ has quite a different phase behavior; the phase slope has changed sign (Fig. 2). The significance of this slope is discussed in the next section. The values of τ_1 and τ_0 calculated from (68) and (67) are given in Table I, for this case, and again, they fit the phase slope and magnitude half-width rather well.

VI. PULSE TRANSMISSION

Consider the Fourier transform of the MCF as a function of $\Delta\sigma$; label the conjugate variable as τ, which we call pulse time. If an experiment sends many pulses from a fixed source to a fixed receiver, then the shape of the ensemble-averaged pulse (EAP) is the Fourier transform of the MCF.

What are the meanings of the various parameters we have used to characterize the MCF, such as τ_0, τ_1, and the set of λ_n? First, τ_1 is a shift in the mean arrival time of the EAP. That is, the medium fluctuations cause a mean shift of the pulse from when it would arrive in the absence of fluctuations, and that shift is τ_1. Unfortunately, very few experiments (none in the ocean) can measure the pulse time in the absence of fluctuations, and the absolute sound speed is not known well enough to calculate it, so a prediction of τ_1 cannot be directly tested. Second, τ_0 is a contribution to the average width of the pulse, as is Φ/σ. Both act as standard deviations in a Gaussian EAP.

The tails of the EAP at both early and late times are controlled by the smallest λ_n. This can be seen by noting that (71) is a product of terms, so its Fourier transform is a convolution of the Fourier transforms of all the terms separately. In a convolution, the tail is controlled by the term with the widest Fourier transform. Therefore we need the Fourier transform of

$$Q_m(\Delta\sigma) = (1 - i\Delta\sigma/\lambda_m)^{-1/2}, \tag{73}$$

where λ_m is the eigenvalue with the smallest magnitude with the appropriate sign. The Fourier transform is

$$\begin{aligned} P_m(\tau) &= (\lambda_m/\pi\tau)^{1/2} e^{-\lambda_m \tau}, \quad \lambda_m \tau > 0, \\ &= 0, \quad \lambda_m \tau < 0, \end{aligned} \tag{74}$$

where we have normalized so that $\int \langle I(\tau)\rangle d\tau = 1$. To find the shape of the tail we convolve this function with the rest of the MCF: that is, with a Gaussian of standard deviation Φ/σ, and all the other eigenvalue terms, resulting in a tail given by

$$P_c(\tau) = \exp\left[\frac{1}{2}\left(\frac{\lambda_m \Phi}{\sigma}\right)^2\right]\left[\prod_{n\neq m}\left(1 - \frac{\lambda_m}{\lambda_n}\right)^{1/2}\right] \times \left(\frac{\lambda_m}{\pi\tau}\right)^{1/2} e^{-\lambda_m \tau}, \quad \lambda_m \tau > 0. \tag{75}$$

The smallest positive eigenvalue dominates the very late τ behavior. The smallest negative eigenvalue gives a precursor and dominates the very early τ behavior. If there are no negative eigenvalues (a caustic has not been passed) there is no precursor.

Consider the examples given in Figs. 1 and 2. The corresponding EAPs are shown in Figs. 3 and 4 along with the predicted tail and precursor from (75). Note that the approximation for Q given by (68) would give the position and gross width of the pulse, but would not give the precursor in Fig. 4. Note also that if the controlling eigenvalue has a large magnitude (such as $\lambda_3 = +3.85$ ms^{-1} for the positive τ tail) then the Gaussian remains in control to much larger τ. The time at which (75) would become relevant is approximately $\tau \approx \lambda_m(\Phi/\sigma)^2$, or 30 ms in the case of Fig. 4.

The EAP calculation is valid whether the transmission is unsaturated or not. In the unsaturated region a narrow transmitted pulse will be received as narrow, and the precursor or tail of the EAP is formed by there being a probability

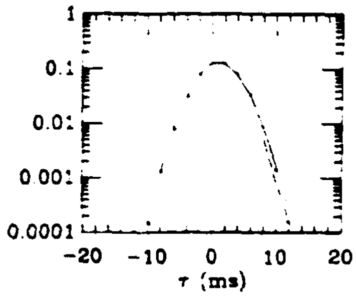


FIG. 3. Ensemble-averaged pulse calculated for the example given in Fig. 1. The solid curve is calculated by a general code. The dotted curve is from (68). The dashed curve is the prediction of the tail from (75) where the controlling eigenvalue is λ_1, whose value is 0.81 ms^{-1}.

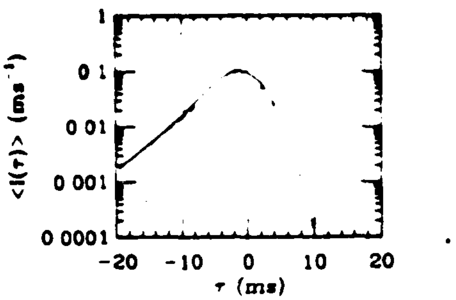


FIG. 4. Ensemble-averaged pulse calculated for the example given in Fig. 2. The curves have the same significance as in Fig. 3. The controlling eigenvalue is λ_2 (for negative τ) whose value is $-0.18\ \mathrm{ms}^{-1}$.

for the pulse to be early or late, as well as a probability for pulses arriving at different times to have different intensities.[5] In the saturated region each pulse may have its individual precursor or tail.

VII. SUMMARY AND CONCLUSION

The derivation of the mutual coherence function (MCF) of time, transverse space, and frequency by the path-integral technique has been given. Allowance for a deterministic sound channel and the presence of reasonable inhomogeneity and anisotropy in the fluctuation field has been included. The MCF has been evaluated for fluctuations dominated by internal waves, which have a vertical structure function that is nearly quadratic. Reasonably simple expressions in terms of environmental measurements for acoustic coherence times and lengths, and coherent bandwidths have been given.

The Fourier transform of the MCF of frequency is the ensemble-averaged pulse (EAP). The possibility of a precursor or a tail on the EAP is shown to depend on both the deterministic sound channel and the fluctuations, and methods have been given to calculate these effects.

All of the parameters that can be used to approximate the various MCFs and the EAP can be evaluated on a VAX computer in a few seconds for a typical ocean-acoustic experiment.

ACKNOWLEDGMENTS

We are grateful for a number of useful conversations with Dennis Creamer and Frank Henyey of CSND. This work was supported by the Office of Naval Research and the Defense Advanced Research Projects Agency.

[1] R. Dashen, "Path integrals for waves in random media," SRI Tech. Rep. JSR 76-1; J. Math. Phys. 20, 894–920 (1979).

[2] S. M. Flatté, R. Dashen, W. H. Munk, K. M. Watson, and F. Zachariasen, *Sound Transmission Through a Fluctuating Ocean*, edited by S. M. Flatté (Cambridge U. P., New York, 1979).

[3] S. M. Flatté, "Wave propagation through random media: Contributions from ocean acoustics," Proc. IEEE 71, 1267–1294 (1983).

[4] F. D. Tappert, in *Lecture Notes in Physics "Wave propagation and underwater acoustics,"* edited by J. B. Keller and J. S. Papadakis, (Springer-Verlag, New York, 1977), Vol. 70, pp. 224–287.

[5] J. L. Codona, D. B. Creamer, S. M. Flatté, R. Frehlich, and F. Henyey (in preparation, 1984).

[6] R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[7] Note that some inaccuracies in the derivation in Ref. 3 are corrected in our version here.

[8] J. A. Fejer, Proc. R. Soc. London Ser. A 220, 455–471 (1953).

[9] V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation* (Israel Program for Scientific Translation, Jerusalem, 1971).

[10] B. Uscinski, *The Elements of Wave Propagation in a Random Medium* (McGraw Hill, New York, 1977).

[11] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).

[12] R. Esswein and S. M. Flatté, J. Acoust. Soc. Am. 70, 1387–1396 (1981).

[13] R. Esswein and S. M. Flatté, J. Acoust. Soc. Am. 67, 1523–1531 (1980).

[14] S. A. Reynolds, S. M. Flatté, R. Dashen, B. Buehler, and P. Maciejewski, J. Acoust. Soc. Am. 77, 1723–1731 (1985).

[15] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978), p. 442.

APPENDIX C

Solution for the Fourth Moment of Waves

Propagating in Random Media

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,

R. G. Frehlich, and Frank S. Henyey

LJI-P-84-295R1

Solution for the Fourth Moment of Waves Propagating in Random Media

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté[a],
R.G. Frehlich, and Frank S. Henyey

Center For Studies of Nonlinear Dynamics[b]
La Jolla Institute
3252 Holiday Court, Suite 208
La Jolla, California 92037

Abstract

A series expression is developed for the fourth moment of a beamed field incident on a random phase screen or an extended medium. The series has a symmetry that allows its first few terms to generate useful approximations at both low and high spatial frequency. The parabolic wave equation, the Markov approximation, and Gaussian refractive index fluctuations are assumed. The result for the phase screen is obtained by Green's-function techniques. The extended-medium result is derived in an analogous manner using path integral methods. The same results are also derived by moment-equation methods. The behavior of the leading terms is compared to previous results for plane-wave and point-source geometries.

PACS numbers: 03.40.Kf, 05.40.+j, 42.20.Cc, 92.10.Vz

a. On leave from the University of California, Santa Cruz, CA 95064
b. Affiliated with the University of California, San Diego.

September 9

1. INTRODUCTION

Many years of research have been devoted to the study of wave propagation in random media (WPRM). The first comprehensive review of the field was by Tatarskii [1971], followed by Prokhorov et al [1975], Ishimaru [1978], and Fante [1975,1980]. The propagation of radio waves through the ionosphere is reviewed by Yeh and Liu [1982]. The phenomenon of interstellar scintillation is reviewed by Rickett [1977] and Rickett et al [1984]. Sound propagation through the ocean and path integral techniques are discussed by Flatte et al [1979] and Flatte [1983].

We consider waves propagating from an arbitrary source distribution in a random medium. We assume the statistics of the medium are locally homogeneous, and we make the Markov approximation; i.e. the field fluctuations induced within a correlation length along the propagation direction are weak. For a more complete discussion see Codona et al, [1985]. The wave propagation is characterized by narrow angular scattering due to the small random fluctuations in refractive index. It is then convenient to write the complex monochromatic scalar field as $E(\mathbf{x},z)e^{ikz}$ where z is the propagation direction, $\mathbf{x}$ is the transverse coordinate and k is the wavenumber of the wave with no refractive index fluctuations.

The random nature of the fields is conveniently described by statistical moments evaluated in the transverse plane located at distance R. Ensemble averages of random variables are denoted by <>. The first moment

$$\Gamma_1(\mathbf{x},R) = \langle E(\mathbf{x},R)\rangle \tag{1}$$

or average of the field and the second moment

$$\Gamma_2(\mathbf{x}_1,\mathbf{x}_2,R) = \langle E(\mathbf{x}_1,R)E^*(\mathbf{x}_2,R)\rangle \tag{2}$$

or mutual coherence function are well understood [Tatarskii, 1971]. However, there are few analytic results for the fourth moment

$$\Gamma_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,R) = \langle E(\mathbf{x}_1,R)E^*(\mathbf{x}_2,R)E(\mathbf{x}_3,R)E^*(\mathbf{x}_4,R)\rangle \tag{3}$$

Previous theoretical work concentrated on plane-wave and point-source geometry. We present three main results for arbitrary source distribution.

A series expression for the fourth moment is derived as an expansion of the Green's function for the fourth moment, thus avoiding the difficulties associated with the source distribution. For the thin-screen problem, the expansion quantity is a combination of phase structure functions. For the extended random media, the expansion quantity is an analogous combination of phase structure function densities. The Green's function is expressed as a multiple path integral. The resulting series of path integrals is evaluated with a useful identity.

Our second result is the generation of two series for the intensity correlation or intensity spectrum. The fourth moment $\Gamma_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,R)$ has the obvious symmetries that it is unchanged by interchanging $\mathbf{x}_1$ and $\mathbf{x}_3$ or by interchanging $\mathbf{x}_2$ and $\mathbf{x}_4$. Each term of the series expansion does not share the symmetry of the entire expression. Thus two separate series are obtained by invoking symmetry. In principle, either series could be summed to give Γ_4. We demonstrate, however, that it is better to consider both series in order to describe the fourth moment with the fewest number of terms. This assertion is demonstrated for the second moment of intensity or intensity correlation, $C(\mathbf{x}_1,\mathbf{x}_2,R)$, which is a special case of the fourth moment, i.e.

$$C(\mathbf{x}_1,\mathbf{x}_2,R) = \langle I(\mathbf{x}_1,R)I(\mathbf{x}_2,R)\rangle = \Gamma_4(\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2) = \Gamma_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2,\mathbf{x}_1) \tag{4}$$

Note that the symmetry of the fourth moment has been explicitly indicated. A clear presentation of the behavior of the intensity correlation series obtained from the fourth moment expansion requires the introduction of a spatial spectrum of intensity fluctuations for a spatially nonstationary random process. We adopt the definition

$$\Phi(\boldsymbol{\mu},\mathbf{q},R) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} C(\boldsymbol{\mu},\boldsymbol{\rho},R)e^{-i\mathbf{q}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\rho} \tag{5}$$

where

$$\boldsymbol{\mu} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) \qquad \boldsymbol{\rho} = \mathbf{x}_1 - \mathbf{x}_2 \tag{6}$$

(Note the free format of the argument list of functions). The spectrum has the property

$$\int_{-\infty}^{\infty} \Phi(\boldsymbol{\mu},\mathbf{q},R)\, d\mathbf{q} = C(\boldsymbol{\mu},0,R) = \langle I(\boldsymbol{\mu},R)^2\rangle \tag{7}$$

It should be noted that the spatial spectrum may depend on the centroid $\boldsymbol{\mu}$.

Since there are two series for the intensity correlation there are also two series for the intensity spectrum. The leading terms of one series for $\Phi(\rho, q, R)$ describe the small q behavior while the other series is valid at high q. The rate of convergence of each series provides a criterion for merging the two results to produce a complete expression for the intensity spectrum. In general, an analogous treatment of the intensity correlation series is not possible since the leading terms of both series do not converge to the variance as the spatial separation approaches zero.

Our third result is the demonstration of the equivalence of path integral and moment-equation methods. Early theoretical work on WPRM concentrated on geometrical optics and the method of small perturbations [Barabanenkov, 1971; Tatarskii, 1971]. These two approaches were limited to weak scattering conditions. This restriction was removed with the introduction of differential equations for the moments of the field [Prokhorov, 1975]. Functional techniques of high energy physics (path integrals and operator methods) provided another point of view to WPRM [Klyatskin, 1973; Dashen, 1979]. The moment equation method and functional techniques are equivalent [Codona et al, 1985] and must generate identical results when expansions are performed in the same quantity. This equivalence is demonstrated by deriving the same fourth moment series expression using moment-equation methods.

The thin-screen case is considered in section 2. The second and fourth moment are analyzed with Green's function techniques and the behavior of the intensity correlation is investigated. The same analysis for the extended medium case is presented in section 3. Here we use the path integral representation of the Green's function. Identical results for the fourth moment are derived with moment equation methods in section 4. The main results of the paper are summarized in section 5.

2. GREEN'S FUNCTION APPROACH TO THE THIN SCREEN PROBLEM

2.1 Introduction

One of the first WPRM problems considered was the propagation of plane waves through a random phase screen [Mercier, 1962; Salpeter, 1967; Bramley, 1967; Gochelashvily and Shishov, 1971, 1972, 1975; Rumsey, 1975; Rino, 1979a, b; Uscinski and Macaskill, 1983a, b]. The propagation of radio waves through the ionosphere and the solar wind are two applications of this model. The theory of scintillation from a point source viewed through a random phase screen has been investigated by Lee [1977]. The case of a Gaussian beam focussed on the observation plane has been considered by Gochelashvily [1974]. Previous work concentrated on plane-wave and point-source geometries. We analyze the more general problem of an arbitrary beam incident on a phase screen using Green's function methods. The following analysis is presented in a fashion that permits a clear extension to the more complex problem of wave propagation in extended random media. We review Green's function methods with a discussion of the second moment. A series expression for the fourth moment is presented as an expansion of the Green's function for the fourth moment. The behavior of the resulting series for the intensity correlation is then discussed.

Consider the scalar wave field, $E(\mathbf{r},z)$, incident on a thin random phase screen situated at the plane z=0. The field, $E(\mathbf{r},0+)$, emerging from this screen is given by

$$E(\mathbf{r},0+) = E(\mathbf{r},0)e^{i\theta(\mathbf{r})} \tag{8}$$

where $E(\mathbf{r},0)$ is the field just before the interface of the screen and the phase fluctuations are

$$\theta(\mathbf{r}) = k\int_0^{0+} n(\mathbf{r},z)dz \tag{9}$$

where $n(\mathbf{r},z)$ is the random fluctuations in refractive index. Assume $\theta(\mathbf{r})$ is a zero mean Gaussian random variable with homogeneous statistics and correlation function

$$C_\theta(\mathbf{s}) = \langle\theta(\mathbf{r})\theta(\mathbf{r}+\mathbf{s})\rangle = \int_{-\infty}^{\infty}\Phi_\theta(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{s}}\,d\mathbf{q} \tag{10}$$

where $\Phi_\theta(\vec{q})$ is the spectrum of phase fluctuations. The structure function of phase fluctuations, $D_\theta(\vec{\rho})$, is defined by

$$D_\theta(\vec{\rho}) = \langle[\theta(\vec{r}) - \theta(\vec{r} + \vec{\rho})]^2\rangle = 2[C_\theta(0) - C_\theta(\vec{\rho})] = 2\int_{-\infty}^{\infty}[1 - \cos(\vec{q}\cdot\vec{\rho})]\Phi_\theta(\vec{q})\, d\vec{q} \tag{11}$$

For narrow angular scattering the scalar field satisfies the parabolic wave equation

$$2ik\frac{\partial E}{\partial z} + \nabla^2 E = 0 \tag{12}$$

The solution of the field at a distance R from the screen is (Mercier, 1962)

$$E(\vec{r},R) = \int_{-\infty}^{\infty} E(\vec{r}',0)G(\vec{r};\vec{r}',R)\, d\vec{r}' \tag{13}$$

where the Green's function is

$$G(\vec{r};\vec{r}',R) = e^{i\theta(\vec{r}')}G^f(\vec{r};\vec{r}',R) \tag{14}$$

with the free space Green's function

$$G^f(\vec{r};\vec{r}',R) = \frac{k}{2\pi iR}\exp[\frac{ik}{2R}(\vec{r} - \vec{r}')^2] \tag{15}$$

2.2 Green's-Function Approach to the Second Moment

Since the random fields have a Green's function solution, the moments of the field also have a Green's function representation. Consider the second moment of the field or mutual coherence function, $\Gamma_2(\vec{r}_1,\vec{r}_2,R)$, given by

$$\Gamma_2(\vec{r}_1,\vec{r}_2,R) = \langle E(\vec{r}_1,R)E^*(\vec{r}_2,R)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_2(\vec{r}'_1,\vec{r}'_2,0)\langle G(\vec{r}_1;\vec{r}'_1,R)G^*(\vec{r}_2;\vec{r}'_2,R)\rangle\, d\vec{r}'_1\, d\vec{r}'_2 \tag{16}$$

The Green's function for the second moment is identified as the expression inside the <> , i.e.

$$G_2(\vec{r}_i;\vec{r}'_i,R) = G_2^f(\vec{r}_i;\vec{r}'_i,R)\langle\exp[i(\theta(\vec{r}'_1) - \theta(\vec{r}'_2))]\rangle \tag{17}$$

where

$$G_2^f(\vec{r}_i;\vec{r}'_i,R) = G^f(\vec{r}_1;\vec{r}'_1,R)G^{f*}(\vec{r}_2;\vec{r}'_2,R) = \frac{k^2}{(2\pi R)^2}\exp[\frac{ik}{2R}[(\vec{r}_1 - \vec{r}'_1)^2 - (\vec{r}_2 - \vec{r}'_2)^2]] \tag{18}$$

is the free space Green's function for the second moment. Here, $\vec{r}'_i$ and $\vec{r}_i$ denote the set of $\vec{r}'$ and $\vec{r}$ coordinates respectively. The expectation over the random phase is performed with the identity

$$\langle e^{i\theta} \rangle = e^{-\frac{1}{2}\langle \theta^2 \rangle} \tag{19}$$

which is valid for zero mean Gaussian random variables. The Green's function for the second moment is then

$$G_2(\vec{r}_i;\vec{r}'_i,R) = G_2^f(\vec{r}_i;\vec{r}'_i,R)\exp[-\frac{1}{2}D_\theta(\vec{r}'_1-\vec{r}'_2)] \tag{20}$$

2.3 Green's-function Approach to the Fourth Moment

Previous work on the fourth moment has concentrated on plane-wave conditions [Zavorotnyi, 1979a]. We address the full fourth moment with an arbitrary source distribution. Following the previous analysis of the second moment, the Green's function for the fourth moment is given by

$$G_4(\vec{r}_i;\vec{r}'_i,R) = G_4^f(\vec{r}_i;\vec{r}'_i,R) \tag{21}$$

$$\exp[-\frac{1}{2}[D_\theta(\vec{r}'_1-\vec{r}'_2)+D_\theta(\vec{r}'_3-\vec{r}'_4)+D_\theta(\vec{r}'_1-\vec{r}'_4)+D_\theta(\vec{r}'_2-\vec{r}'_3)-D_\theta(\vec{r}'_2-\vec{r}'_4)-D_\theta(\vec{r}'_1-\vec{r}'_3)]]$$

where $G_4^f(\vec{r}_i;\vec{r}'_i,R)$ is the free space Green's function for the fourth moment. It is convenient to apply the unitary coordinate transformation [Rumsey, 1975]

$$\begin{aligned}
2\vec{\alpha} &= \vec{r}_1+\vec{r}_2+\vec{r}_3+\vec{r}_4 & 2\vec{r}_1 &= \vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta} \\
2\vec{\beta} &= \vec{r}_1+\vec{r}_2-\vec{r}_3-\vec{r}_4 & 2\vec{r}_2 &= \vec{\alpha}+\vec{\beta}-\vec{\gamma}-\vec{\delta} \\
2\vec{\gamma} &= \vec{r}_1-\vec{r}_2-\vec{r}_3+\vec{r}_4 & 2\vec{r}_3 &= \vec{\alpha}-\vec{\beta}-\vec{\gamma}+\vec{\delta} \\
2\vec{\delta} &= \vec{r}_1-\vec{r}_2+\vec{r}_3-\vec{r}_4 & 2\vec{r}_4 &= \vec{\alpha}-\vec{\beta}+\vec{\gamma}-\vec{\delta}
\end{aligned} \tag{22}$$

The set $(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}')$, will denote the same transformation on the coordinate set $(\vec{r}'_1,\vec{r}'_2,\vec{r}'_3,\vec{r}'_4)$. The Green's function for the fourth moment then becomes

$$G_4(\mathbf{x}_i;\mathbf{x}'_i,R) = G_4^0(\mathbf{x}_i;\mathbf{x}'_i,R) \tag{23}$$

$$\exp\left[-\tfrac{1}{2}[D_\theta(\gamma' + \delta') + D_\theta(\gamma' - \delta') + D_\theta(\beta' + \delta') + D_\theta(\beta' - \delta') - D_\theta(\beta' + \gamma') - D_\theta(\beta' - \gamma')]\right]$$

where

$$G_4^0(\mathbf{x}_i;\mathbf{x}'_i,R) = \frac{k^4}{(2\pi R)^4}\exp\left[\frac{ik}{2R}[(\mathbf{x}_1 - \mathbf{x}'_1)^2 - (\mathbf{x}_2 - \mathbf{x}'_2)^2 + (\mathbf{x}_3 - \mathbf{x}'_3)^2 - (\mathbf{x}_4 - \mathbf{x}'_4)^2]\right]$$

$$= \frac{k^4}{(2\pi R)^4}\exp\left[i\frac{k}{R}[(\alpha - \alpha')\cdot(\delta - \delta') + (\beta - \beta')\cdot(\gamma - \gamma')]\right] \tag{24}$$

is the free space Green's function for the fourth moment. This expression is intractable, both analytically and numerically. Marians [1975], numerically calculated the intensity spectrum of plane-waves incident on a two-dimensional phase screen.

The plane-wave case was considered by Zavorotnyi et al [1977]. They noted that a combination of structure functions was small in the important regions of integration. A series expression was then obtained as a Taylor series expansion. For the general case, we identify that same expansion quantity as

$$Q = -\tfrac{1}{2}[D_\theta(\beta' + \delta') + D_\theta(\beta' - \delta') - D_\theta(\beta' + \gamma') - D_\theta(\beta' - \gamma')] \tag{25}$$

$$= 2\int_{-\infty}^{\infty} \Phi_\theta(\kappa)[\cos(\delta'\cdot\kappa) - \cos(\gamma'\cdot\kappa)]e^{i\beta'\cdot\kappa}\, d\kappa$$

We will investigate the behavior of Q for a structure function that is power law above the inner scale l_0. The exponent in (23) is large unless two of the first four structure functions are small. (The other two can nearly cancel the last two structure functions). The only way the exponent can be small, while allowing the cancellations, is for

$$\delta' = O(s_0) \quad \text{and} \quad \beta' = O(s_0) \quad or \quad \gamma' = O(s_0) \tag{26}$$

where s_0 is the field correlation distance defined by $D_\theta(s_0) = 1$. The remaining variable (β' or γ') is typically of order of the scattering disk

$$W = \Theta_0 R = \frac{R}{s_0 k} \tag{27}$$

where Θ_0 is the width of the angular spectrum. When the inner scale is larger than W, there is little scintillation. Therefore, we consider the case $s_0 \ll W$, $l_0 \ll W$, $\gamma = O(s_0)$, and $\beta' = O(W)$. A Taylor series expansion about β' reduces Q to

$$Q = \frac{1}{2}(\delta'^2 + \gamma'^2) D_\theta''(\beta') \tag{28}$$

For a power-law structure function

$$D_\theta(\beta') = (\beta' / s_0)^p \tag{29}$$

Q becomes

$$Q = \frac{p(p-1)}{2}\left(\frac{\delta'^2 + \gamma'^2}{\beta'^2}\right)\left(\frac{\beta'}{s_0}\right)^p = O\left(\frac{s_0}{W}\right)^{2-p} \tag{30}$$

On the other hand, the other terms in the exponent are $O(D_\theta(s_0)) = O(1)$, which was the condition which caused γ' and δ' to be order s_0. Thus, if $s_0 < W$, a Taylor series in Q is appropriate, but a Taylor series in the entire exponent requires many more terms.

The other possibility, $\beta' = O(s_0)$, and $\gamma' = O(W)$ requires an expansion in another variable, Q', obtained from Q by interchanging β' with γ'. This alternate expansion is the fundamental reason that two different series are required.

We now return to the Taylor series expansion in Q, with the result

$$G_4(\mathbf{x}_i;\mathbf{r}_i,R) = \sum_{n=0}^{\infty} G_{4_n}(\mathbf{x}_i;\mathbf{r}_i;z) = G_4^0(\mathbf{x}_i;\mathbf{r}_i,R)\exp\left[-\frac{1}{2}[D_\theta(\gamma' + \delta') + D_\theta(\gamma' - \delta')]\right] \tag{31}$$

$$\left[1 + \sum_{n=1}^{\infty} \frac{2^n}{n!} \int_{-\infty}^{\infty} d\boldsymbol{\kappa}_1 \cdots \int_{-\infty}^{\infty} d\boldsymbol{\kappa}_n \exp\left[i\sum_{j=1}^{n} \boldsymbol{\kappa}_j \cdot \boldsymbol{\beta}'\right] \prod_{j=1}^{n} \Phi_\theta(\boldsymbol{\kappa}_j)[\cos(\boldsymbol{\delta}'\cdot\boldsymbol{\kappa}_j) - \cos(\boldsymbol{\gamma}'\cdot\boldsymbol{\kappa}_j)]\right]$$

This series should converge quickly when the quantity Q is small over the important region of integration. Note that the symmetry of the full fourth moment expression (23) does not hold for each term of the expansion. The equivalent moment-equation derivation is presented in section 4.1.

The first term of (31) reduces to

$$G_{4_0}(\mathbf{x}_i;\mathbf{r}_i,R) = G_4^0(\mathbf{x}_i;\mathbf{r}_i,R)e^{-\frac{1}{2}[D_\theta(\mathbf{r}_1-\mathbf{r}_2)+D_\theta(\mathbf{r}_3-\mathbf{r}_4)]} = G_2(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2,R)G_2(\mathbf{x}_3,\mathbf{x}_4,\mathbf{r}_3,\mathbf{r}_4,R) \tag{32}$$

Thus, for spatially coherent sources, the first term of the fourth moment is a product of two second moments, i.e.

$$\Gamma_{4_0}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,R) = \Gamma_2(\mathbf{x}_1,\mathbf{x}_2,R)\Gamma_2(\mathbf{x}_3,\mathbf{x}_4,R) \tag{33}$$

The next term of the Green's function expansion is

$$G_{4_1}(\mathbf{x}_i;\mathbf{x}'_i,R) = 2G_4^0(\mathbf{x}_i;\mathbf{x}'_i,R)\exp[-\frac{1}{2}[D_\theta(\gamma'+\delta')+D_\theta(\gamma'-\delta')]] \tag{34}$$

$$\int_{-\infty}^{\infty}\Phi_\theta(\kappa)[\cos(\delta'\cdot\kappa)-\cos(\gamma'\cdot\kappa)]e^{i\beta'\cdot\kappa}\,d\kappa$$

These two terms contain the useful first-order description of the fourth moment. The rate of convergence of the series is determined by the higher order terms. For the plane-wave case, the fourth-moment expressions generated from (33) and (34) are identical to the asymptotic results of Zavorotnyi [1979a Eq (7)].

2.4 Intensity Correlation

There are few measurements of the full fourth moment of WPRM [Gurvich et al, 1978, 1979a]. However, the intensity correlation, a special case of the fourth moment, is commonly observed. We will now demonstrate that our fourth-moment series (31) generates two different expressions for the intensity correlation (4), one valid at low spatial frequency, the other valid at high spatial frequency.

The low frequency version of the intensity correlation is obtained from the fourth moment (3) by setting $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{x}_3 = \mathbf{x}_4$ or by setting $\delta = 0$ and $\gamma = 0$. The n=0 term produces

$$C_I^0(\alpha,\beta,R) = \frac{k^4}{(2\pi R)^4}\int_{-\infty}^{\infty}\Gamma_4(\alpha',\beta',\gamma',\delta',0)\exp[-i\frac{\kappa}{R}[(\alpha-\alpha')\cdot\delta'+(\beta-\beta')\cdot\gamma']] \tag{35}$$

$$\exp[-\frac{1}{2}[D_\theta(\gamma'+\delta')+D_\theta(\gamma'-\delta')]]\,d\alpha'\,d\beta'\,d\gamma'\,d\delta'$$

For spatially coherent sources

$$C_I^0(\alpha,\beta,R) = \langle I(\frac{\alpha+\beta}{2},R)\rangle\langle I(\frac{\alpha-\beta}{2},R)\rangle = \langle I(\mathbf{x}_1,z)\rangle\langle I(\mathbf{x}_3,z)\rangle \tag{36}$$

or

$$\Phi_0^{IJ}(\vec{\rho},\vec{q},R) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} <I(\vec{\rho}+\frac{\vec{\beta}}{2},R)><I(\vec{\rho}-\frac{\vec{\beta}}{2},R)>\, e^{-i\vec{\beta}\cdot\vec{q}}\, d\vec{\beta} \tag{37}$$

which normally describes the smallest spectral scale of the process. Indeed, for plane-wave conditions $\Phi_0^{IJ}(\vec{\rho},\vec{q},R) = <I>^2\delta(\vec{q})$ since the average intensity, $<I>$, is a constant.

The corresponding terms for n=1 may be written as

$$C_1^{IJ}(\vec{\alpha},\vec{\beta},R) = \frac{2k^4}{(2\pi R)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)\exp\left[-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}' + (\vec{\beta}-\vec{\beta}')\cdot\vec{\gamma}']\right] \tag{38}$$

$$\exp\left[-\frac{1}{2}[D_\theta(\vec{\gamma}'+\vec{\delta}') + D_\theta(\vec{\gamma}'-\vec{\delta}')]\right]\Phi_\theta(\vec{\kappa})[\cos(\vec{\delta}'\cdot\vec{\kappa}) - \cos(\vec{\gamma}'\cdot\vec{\kappa})]e^{i\vec{\beta}'\cdot\vec{\kappa}}\; d\vec{\alpha}'\, d\vec{\beta}'\, d\vec{\gamma}'\, d\vec{\delta}'\, d\vec{\kappa}$$

and

$$\Phi_1^{IJ}(\vec{\alpha},\vec{q},R) = \frac{2k^2}{(2\pi)^4 R^2}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}' = -\vec{q}\frac{R}{k},\vec{\delta}',0)\exp[-i\frac{k}{R}(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}']e^{i(\vec{\kappa}-\vec{q})\cdot\vec{\beta}'} \tag{39}$$

$$\exp\left[-\frac{1}{2}[D_\theta(\vec{q}\frac{R}{k}+\vec{\delta}') + D_\theta(\vec{q}\frac{R}{k}-\vec{\delta}')]\right]\Phi_\theta(\vec{\kappa})[\cos(\vec{\delta}'\cdot\vec{\kappa}) - \cos(\vec{\gamma}'\cdot\vec{\kappa})]\; d\vec{\alpha}'\, d\vec{\beta}'\, d\vec{\delta}'\, d\vec{\kappa}$$

This term describes the refractive focussing by irregularities of the size of the scattering disk (Rickett et al, 1984). For the case of plane waves, (39) reduces to the familiar result

$$\Phi_1^{IJ}(\vec{\rho},\vec{q},R) = 4\Phi_\theta(\vec{q})\exp[-D_\theta(\vec{q}\frac{R}{k})]\sin^2[q^2\frac{R}{2k}] \tag{40}$$

which also describes the low q behavior of $\Phi(\vec{\rho},\vec{q},R)$. The Born, or weak-scattering approximation is obtained by ignoring the exponential term of (40). Similarly, the Born approximation for the general case is obtained from (39) by ignoring the last exponential term.

The high frequency version of the intensity correlation is obtained from the fourth moment (3) by setting $\vec{x}_1 = \vec{x}_4$ and $\vec{x}_2 = \vec{x}_3$ or by setting $\vec{\delta} = 0$ and $\vec{\beta} = 0$. The n=0 term produces

$$C_0^{IJ}(\vec{\alpha},\vec{\gamma},R) = \frac{k^4}{(2\pi R)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)\exp\left[-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}' + (\vec{\gamma}-\vec{\gamma}')\cdot\vec{\beta}']\right] \tag{41}$$

$$\exp\left[-\frac{1}{2}[D_\theta(\vec{\gamma}'+\vec{\delta}') + D_\theta(\vec{\gamma}'-\vec{\delta}')]\right] d\vec{\alpha}'\, d\vec{\beta}'\, d\vec{\gamma}'\, d\vec{\delta}'\, d\vec{\kappa}$$

For spatially coherent sources

$$C_0^{N}(\mathbf{x}_1,\mathbf{x}_2,R)=\Gamma_2(\mathbf{x}_1,\mathbf{x}_2,R)\Gamma_2^*(\mathbf{x}_1,\mathbf{x}_2,R) \tag{42}$$

This expression is the high frequency approximation, i.e. the intensity correlation is the square of the mutual coherence function, and may also be derived by assuming that the complex electric fields are zero mean Gaussian random variables.

The n=1 term reduces to

$$C_1^{N}(\vec{\alpha},\vec{\gamma},R)=\frac{2k^4}{(2\pi R)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)\exp\left[-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}'+(\vec{\gamma}-\vec{\gamma}')\cdot\vec{\beta}']\right] \tag{43}$$

$$\exp\left[-\frac{1}{2}[D_\theta(\vec{\gamma}'+\vec{\delta}')+D_\theta(\vec{\gamma}'-\vec{\delta}')]\right]\Phi_\theta(\vec{\kappa})[\cos(\vec{\delta}'\cdot\vec{\kappa})-\cos(\vec{\gamma}'\cdot\vec{\kappa})]e^{i\vec{\beta}'\cdot\vec{\kappa}}\,d\vec{\alpha}'\,d\vec{\beta}'\,d\vec{\gamma}'\,d\vec{\delta}'\,d\vec{\kappa}$$

and

$$\Phi_1^{N}(\vec{\alpha},\vec{q},R)=\frac{2k^2}{(2\pi)^4R^2}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}'=-\vec{q}\frac{R}{k},\vec{\gamma}',\vec{\delta}',0)\exp[-i\frac{k}{R}(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}']e^{-i(\vec{\gamma}+\vec{\kappa}\frac{R}{k})\cdot\vec{q}} \tag{44}$$

$$\exp\left[-\frac{1}{2}[D_\theta(\vec{\gamma}'+\vec{\delta}')+D_\theta(\vec{\gamma}'-\vec{\delta}')]\right]\Phi_\theta(\vec{\kappa})[\cos(\vec{\delta}'\cdot\vec{\kappa})-\cos(\vec{\gamma}'\cdot\vec{\kappa})]\,d\vec{\alpha}'\,d\vec{\gamma}'\,d\vec{\delta}'\,d\vec{\kappa}$$

Rickett et al [1984] argue that the physical mechanism for this term is the modulation of the small scale structure (42) by the large scale refractive process (38).

We have shown how the fourth-moment series generates two expressions for the intensity correlation: one ($\vec{\delta}=0,\vec{\gamma}=0$) useful at low spatial frequencies, and the other ($\vec{\delta}=0,\vec{\beta}=0$) useful at high spatial frequencies. The region of validity depends on the statistics of the phase fluctuations and the initial source distribution. The case of plane waves incident on a random phase screen with a power-law spectrum has been investigated by Gochelashvily and Shishov [1975]. Their calculations of the first few terms of the intensity spectrum [Figure 1] imply the two series converge quickly when the quantity Q is small over the important region of integration. The rate of convergence is difficult to determine a priori; the contribution to the intensity spectrum from the higher terms is the best indication of convergence. A finite number of terms from the two series can be merged by a weighted sum based on this rate of convergence. In strong scattering conditions, these two series merge quickly and only a few terms are required to describe the complete spectrum. The intensity correlation series is more difficult to interpret since the errors of the expansion accumulate in the region of small spacing but in the spectral domain, these errors appear in the central regions of the spectrum [cf Figure 1].

3. PATH INTEGRAL TECHNIQUES FOR EXTENDED RANDOM MEDIA

3.1 Introduction

We now consider the more complex problem of wave propagation in a random media that is locally homogeneous with statistics that vary slowly in the direction of propagation. Laser propagation in the atmosphere, radio propagation through the interstellar medium, and sound propagation through the ocean with no deterministic background are common examples of this phenomena. Many theoretical methods have been applied to these problems. Moment-equation methods are reviewed by Prokhorov et al [1975]. Recently, Macaskill [1983] and Frankenthal et al [1984] have applied the two-scale embedding procedure (Frankenthal et al [1982], Beran et al [1982]) to produce a solution for the fourth moment. The application of path integral techniques to problems of wave propagation in random media was introduced by Klyatskin and Tatarskii [1970], Zavorotnyi et al [1977] and Dashen [1979]. Path integral methods have been successfully applied to the difficult problem of WPRM for anisotropic, inhomogeneous medium with deterministic background of refractive index [Flatté et al, 1979]. A functional operator form of the path integral was used by Tatarskii and Zavorotnyi [1980], to extend thin-screen analysis to the problem of wave propagation in extended random media for the plane-wave case. We use the path integral representation of the Green's function to illustrate this connection because one obtains a clear presentation of the role of the source distribution. We believe that the operator formalism is equivalent to the path integral method and produces the same results.

The path integral technique is introduced by a review of the second-moment derivation. The reduction of path integrals to familiar Reimann integrals is performed by a useful identity [cf (62)]. Using this identity, we present a series expression for the fourth moment that is analogous to the thin-screen results of section 2.2. The behavior of the resulting correlation series is then discussed.

For narrow angular scattering, the scalar field satisfies the parabolic wave equation

$$2ik\frac{\partial E}{\partial z} + \nabla^2 E + 2k^2 n(\mathbf{x}, z)E = 0 \tag{45}$$

where $n(\mathbf{x}, z)$ denotes the refractive index fluctuations. We define the correlation of refractive index fluctuations, $B_n(\mathbf{s}, t, z)$, as

$$B_n(\mathbf{s}, t, z) = \langle n(0, z) n(\mathbf{s}, z + t)\rangle \tag{46}$$

The path integral formulation for the Green's function was developed by Feynman [1948], and may be written as

$$G(\mathbf{x}'; \mathbf{x}, R) = \int D\mathbf{r}(z) \exp\left[i\frac{k}{2}\int_0^R [\dot{\mathbf{r}}(z)]^2 dz - ik\int_0^R n[\mathbf{r}(z), z] dz \right] \tag{47}$$

where $D\mathbf{r}(z)$ denotes the infinite dimensional integration over all possible paths, $\mathbf{r}(z)$, connecting the points $(\mathbf{x}', 0)$ and $(\mathbf{x}, R)$ and $\dot{\mathbf{r}}(z) = \frac{d\mathbf{r}}{dz}$. The most important paths are those near the geometrical path from $(\mathbf{x}', 0)$ to $(\mathbf{x}, R)$, given by

$$\mathbf{r}_G(z) = \mathbf{x}'(1 - \frac{z}{R}) + \mathbf{x}\frac{z}{R} \tag{48}$$

Transformations of these geometrical paths will be denoted with the subscript G.

If there are no refractive index fluctuations, the Green's function becomes the free space Green's function.

$$G^f(\mathbf{x}; \mathbf{x}', R) = \int D\mathbf{r}(z) \exp\left[i\frac{k}{2}\int_0^R [\dot{\mathbf{r}}(z)]^2 dz \right] = \frac{k}{2\pi i R} \exp\left[\frac{ik}{2R}(\mathbf{x} - \mathbf{x}')^2 \right] \tag{49}$$

3.2 Second Moment by Path Integral Techniques

The Green's function for the moments of the field are easily expressed in terms of the path integral. The Green's function for the first and second moment were derived by Dashen

[1979]. These results were obtained under the Markov approximation [Zavorotnyi, 1978]. We review the derivation for the Green's function for the second moment, which is given by

$$G_2(\vec{x}_1,\vec{x}_2;\vec{x}'_1,\vec{x}'_2,R) = \langle G(\vec{x}_1;\vec{x}'_1,R)G^*(\vec{x}_2;\vec{x}'_2,R)\rangle \tag{50}$$

$$= \int\int D\vec{r}_1(z)\, D\vec{r}_2(z) \exp\left[i\frac{k}{2}\int_0^R ([\dot{\vec{r}}_1(z)]^2 - [\dot{\vec{r}}_2(z)]^2)dz \right] \left\langle \exp\left[-ik\int_0^R [n(\vec{r}_1(z),z) - n(\vec{r}_2(z),z)]dz \right] \right\rangle$$

Applying (19) and the Markov approximation results in

$$G_2(\vec{x}_i;\vec{x}'_i,R) = \frac{k^2}{(2\pi R)^2} \int\int D\vec{r}_1(z)\, D\vec{r}_2(z) \tag{51}$$

$$\exp\left[i\frac{k}{2}\int_0^R ([\dot{\vec{r}}_1(z)]^2 - [\dot{\vec{r}}_2(z)]^2)dz - \frac{1}{2}\int_0^R d[\vec{r}_1(z) - \vec{r}_2(z), z]dz \right]$$

and the phase structure function density is given by

$$d(\vec{\beta},z) = 2k^2 \int_{-\infty}^{\infty} [B_n(0,t,z) - B_n(\vec{\beta},t,z)]dt$$

$$= 4\pi k^2 \int_{-\infty}^{\infty} [1 - \cos(\vec{\beta}\cdot\vec{q})]\Phi_n(\vec{q},q_z = 0,z)\, d\vec{q} \tag{52}$$

where

$$\Phi_n(\vec{q},q_z,R) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_n(\vec{x},z,R) \exp[-i(\vec{x}\cdot\vec{q} + zq_z)]\, d\vec{x}\, dz \tag{53}$$

is the refractive index spectrum. Change path variables to the centroid and difference coordinates

$$\vec{\mu}(z) = \frac{1}{2}[\vec{r}_1(z) + \vec{r}_2(z)] \tag{54}$$

$$\vec{\beta}(z) = \vec{r}_1(z) - \vec{r}_2(z) \tag{55}$$

The double path integral can be evaluated by expressing the paths as deviations from the two geometrical paths defined by (48), i.e.

$$\vec{\mu}_1(z) = \vec{\mu}(z) - \vec{\mu}_C(z) \tag{56}$$

$$\vec{\beta}_1(z) = \vec{\beta}(z) - \vec{\beta}_C(z) \tag{57}$$

where

$$\boldsymbol{\rho}_C(z) = \boldsymbol{\rho}(0)(1-\frac{z}{R}) + \boldsymbol{\rho}(R)\frac{z}{R} = \frac{1}{2}(\mathbf{r}'_1+\mathbf{r}'_2)(1-\frac{z}{R}) + \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\frac{z}{R} \tag{58}$$

and

$$\tilde{\boldsymbol{\rho}}_C(z) = \tilde{\boldsymbol{\rho}}(0)(1-\frac{z}{R}) + \tilde{\boldsymbol{\rho}}(R)\frac{z}{R} = (\mathbf{r}'_1-\mathbf{r}'_2)(1-\frac{z}{R}) + (\mathbf{r}_1-\mathbf{r}_2)\frac{z}{R} \tag{59}$$

Then $\boldsymbol{\rho}_1(0) = \boldsymbol{\rho}_1(R) = \tilde{\boldsymbol{\rho}}_1(0) = \tilde{\boldsymbol{\rho}}_1(R) = 0$ and the second-moment Green's function becomes

$$G_2(\mathbf{r}_i;\mathbf{r}'_i,R) = \exp\left[\frac{i}{2k}[(\mathbf{r}'_1-\mathbf{r}_1)^2-(\mathbf{r}'_2-\mathbf{r}_2)^2]\right]\int\int D\boldsymbol{\rho}_1(z)\, D\tilde{\boldsymbol{\rho}}_1(z)$$

$$\exp\left[ik\int_0^R \dot{\boldsymbol{\rho}}_1(z)\cdot\dot{\tilde{\boldsymbol{\rho}}}_1(z)dz - \frac{1}{2}\int_0^R d[\tilde{\boldsymbol{\rho}}_1(z)+\tilde{\boldsymbol{\rho}}_C(z),z]dz\right] \tag{60}$$

Integrating the first path term in the exponential by parts and substituting the free space Green's function results in

$$G_2(\mathbf{r}_i;\mathbf{r}'_i,R) = \frac{(2\pi R)^2}{k^2} G^f(\mathbf{r}_1;\mathbf{r}'_1,R)G^{f*}(\mathbf{r}_2,\mathbf{r}'_2,R)\int\int D\boldsymbol{\rho}_1(z)\, D\tilde{\boldsymbol{\rho}}_1(z)$$

$$\exp\left[-ik\int_0^R \boldsymbol{\rho}_1(z)\cdot\ddot{\tilde{\boldsymbol{\rho}}}_1(z)dz - \frac{1}{2}\int_0^R d(\tilde{\boldsymbol{\rho}}_1(z)+\tilde{\boldsymbol{\rho}}_C(z),z)dz\right] \tag{61}$$

This path integral is evaluated with the identity

$$\int\int D\boldsymbol{\rho}(z)\, D\tilde{\boldsymbol{\rho}}(z) F[\tilde{\boldsymbol{\rho}}(z)]\exp\left[ik\int_0^R \boldsymbol{\rho}(z)\cdot[\ddot{\tilde{\boldsymbol{\rho}}}(z)-\mathbf{b}(z)]dz\right] = \frac{k^2}{(2\pi R)^2}F[\mathbf{g}(z)] \tag{62}$$

where $\mathbf{g}(z)$ is the solution of $\ddot{\mathbf{g}}(z) = \mathbf{b}(z)$ that satisfies the appropriate boundary conditions. Identity (62) reduces (61) to

$$G_2(\mathbf{r}_i;\mathbf{r}'_i,R) = G^f(\mathbf{r}_1;\mathbf{r}'_1,R)G^{f*}(\mathbf{r}_2;\mathbf{r}'_2,R)\exp\left[-\frac{1}{2}\int_0^R d[\tilde{\boldsymbol{\rho}}_C(z),z]dz\right] \tag{63}$$

and $\mathbf{g}(z) = 0$ is the solution of $\ddot{\mathbf{g}}(z) = 0$, that satisfies the required boundary conditions.

3.3 Fourth Moment by Path Integral Techniques

Using the path-integral technique, we derive a series expression for the fourth moment in a manner analogous to the thin-screen derivation. The Green's function for the fourth moment is given by the multiple path integral

$$G_4(\vec{x}_i;\vec{x}'_i,R) = \langle G(\vec{x}_1;\vec{x}'_1,R)G^*(\vec{x}_2,\vec{x}'_2,R)G(\vec{x}_3;\vec{x}'_3,R)G^*(\vec{x}_4,\vec{x}'_4,R)\rangle = \int D\vec{r}_1\, D\vec{r}_2\, D\vec{r}_3\, D\vec{r}_4$$

$$\exp\left[i\frac{k}{2}\int_0^R[\dot{\vec{r}}_1^2-\dot{\vec{r}}_2^2+\dot{\vec{r}}_3^2-\dot{\vec{r}}_4^2]dz\right]\langle\exp[-ik\int_0^R[n(\vec{r}_1,z)-n(\vec{r}_2,z)+n(\vec{r}_3,z)-n(\vec{r}_4,z)]dz]\rangle \qquad (64)$$

If $\int_0^R[n(\vec{r}_1,z)-n(\vec{r}_2,z)+n(\vec{r}_3,z)-n(\vec{r}_4,z)]dz$ is a zero mean Gaussian random variable (19) and the Markov approximation is valid, the last term of (64) becomes $\exp[-\frac{1}{2}\int_0^R[d(\vec{r}_1-\vec{r}_2,z)+d(\vec{r}_3-\vec{r}_4,z)+d(\vec{r}_1-\vec{r}_4,z)+d(\vec{r}_2-\vec{r}_3,z)-d(\vec{r}_2-\vec{r}_4,z)-d(\vec{r}_1-\vec{r}_3,z)]dz]$.

Change path variables to

$$2\vec{\alpha}(z)=\vec{r}_1(z)+\vec{r}_2(z)+\vec{r}_3(z)+\vec{r}_4(z) \qquad 2\vec{r}_1(z)=\vec{\alpha}(z)+\vec{\beta}(z)+\vec{\gamma}(z)+\vec{\delta}(z)$$

$$2\vec{\beta}(z)=\vec{r}_1(z)+\vec{r}_2(z)-\vec{r}_3(z)-\vec{r}_4(z) \qquad 2\vec{r}_2(z)=\vec{\alpha}(z)+\vec{\beta}(z)-\vec{\gamma}(z)-\vec{\delta}(z)$$

$$2\vec{\gamma}(z)=\vec{r}_1(z)-\vec{r}_2(z)-\vec{r}_3(z)+\vec{r}_4(z) \qquad 2\vec{r}_3(z)=\vec{\alpha}(z)-\vec{\beta}(z)-\vec{\gamma}(z)+\vec{\delta}(z)$$

$$2\vec{\delta}(z)=\vec{r}_1(z)-\vec{r}_2(z)+\vec{r}_3(z)-\vec{r}_4(z) \qquad 2\vec{r}_4(z)=\vec{\alpha}(z)-\vec{\beta}(z)+\vec{\gamma}(z)-\vec{\delta}(z) \qquad (65)$$

Following the second moment derivation, we express the paths as deviations from the transformed geometrical paths, i.e.

$$\vec{\alpha}_1(z)=\vec{\alpha}(z)-\vec{\alpha}_C(z) \qquad (66)$$

$$\vec{\beta}_1(z)=\vec{\beta}(z)-\vec{\beta}_C(z)$$

$$\vec{\gamma}_1(z)=\vec{\gamma}(z)-\vec{\gamma}_C(z)$$

$$\vec{\delta}_1(z)=\vec{\delta}(z)-\vec{\delta}_C(z)$$

where

$$\vec{\alpha}_C=\vec{\alpha}(0)(1-\frac{z}{R})+\vec{\alpha}(R)\frac{z}{R} \qquad (67)$$

$$\vec{\beta}_C=\vec{\beta}(0)(1-\frac{z}{R})+\vec{\beta}(R)\frac{z}{R}$$

$$\vec{\gamma}_C=\vec{\gamma}(0)(1-\frac{z}{R})+\vec{\gamma}(R)\frac{z}{R}$$

$$\vec{\delta}_C=\vec{\delta}(0)(1-\frac{z}{R})+\vec{\delta}(R)\frac{z}{R}$$

In order to simplify the large expressions, $[\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}']$ will replace $[\vec{\alpha}(0),\vec{\beta}(0),\vec{\gamma}(0),\vec{\delta}(0)]$ and $[\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\delta}]$ will replace $[\vec{\alpha}(R),\vec{\beta}(R),\vec{\gamma}(R),\vec{\delta}(R)]$. Then,

$$G_4(\vec{x}_i;\vec{x}'_i,R)=\frac{(2\pi R)^4}{k^4}G_1^4(\vec{x}_i;\vec{x}'_i,R)\int D\vec{\alpha}_1\,D\vec{\beta}_1\,D\vec{\gamma}_1\,D\vec{\delta}_1\exp\left[ik\int_0^R(\dot{\vec{\alpha}}_1\cdot\dot{\vec{\delta}}_1+\dot{\vec{\beta}}_1\cdot\dot{\vec{\gamma}}_1)dz\right] \tag{68}$$

$$\exp\left[-\frac{1}{2}\int_0^R[d(\vec{\gamma}_1+\vec{\delta}_1+\vec{\gamma}_C(z)+\vec{\delta}_C(z),z)+d(\vec{\gamma}_1-\vec{\delta}_1+\vec{\gamma}_C(z)-\vec{\delta}_C(z),z)+d(\vec{\beta}_1+\vec{\delta}_1+\vec{\beta}_C(z)+\vec{\delta}_C(z),z)\right.$$

$$\left.+d(\vec{\beta}_1-\vec{\delta}_1+\vec{\beta}_C(z)-\vec{\delta}_C(z),z)-d(\vec{\beta}_1+\vec{\gamma}_1+\vec{\beta}_C(z)+\vec{\gamma}_C(z),z)-d(\vec{\beta}_1-\vec{\gamma}_1+\vec{\beta}_C(z)-\vec{\gamma}_C(z),z)]dz\right]$$

Substituting (integration by parts)

$$ik\int_0^R(\dot{\vec{\alpha}}_1\cdot\dot{\vec{\delta}}_1+\dot{\vec{\beta}}_1\cdot\dot{\vec{\gamma}}_1)dz=-ik\int_0^R(\vec{\alpha}_1\cdot\ddot{\vec{\delta}}_1+\vec{\beta}_1\cdot\ddot{\vec{\gamma}}_1)dz \tag{69}$$

and performing the integration over the paths $\vec{\alpha}_1$ and $\vec{\delta}_1$ using (62) results in

$$G_4(\vec{x}_i;\vec{x}'_i,R)=\frac{(2\pi R)^2}{k^2}G_1^4(\vec{x}_i;\vec{x}'_i,R)\iint D\vec{\beta}_1\,D\vec{\gamma}_1\exp\left[-ik\int_0^R\vec{\beta}_1\cdot\ddot{\vec{\gamma}}_1dz\right]$$

$$\exp\left[-\frac{1}{2}\int_0^R[d(\vec{\gamma}_1+\vec{\gamma}_C(z)+\vec{\delta}_C(z),z)+d(\vec{\gamma}_1+\vec{\gamma}_C(z)-\vec{\delta}_C(z),z)+\varepsilon\{d(\vec{\beta}_1+\vec{\beta}_C(z)+\vec{\delta}_C(z),z)\right. \tag{70}$$

$$\left.+d(\vec{\beta}_1+\vec{\beta}_C(z)-\vec{\delta}_C(z),z)-d(\vec{\beta}_1+\vec{\gamma}_1+\vec{\beta}_C(z)+\vec{\gamma}_C(z),z)-d(\vec{\beta}_1-\vec{\gamma}_1+\vec{\beta}_C(z)-\vec{\gamma}_C(z),z)\}]dz\right]$$

Note that (70) is similar to (23) of the thin-screen derivation. The parameter ε has been added to label the quantity

$$Q=-\frac{1}{2}\int_0^R[d(\vec{\beta}_1+\vec{\beta}_C(z)+\vec{\delta}_C(z),z)+d(\vec{\beta}_1+\vec{\beta}_C(z)-\vec{\delta}_C(z),z)$$

$$-d(\vec{\beta}_1+\vec{\gamma}_1+\vec{\beta}_C(z)+\vec{\gamma}_C(z),z)-d(\vec{\beta}_1-\vec{\gamma}_1+\vec{\beta}_C(z)-\vec{\gamma}_C(z),z)\}]dz$$

$$=4\pi k^2\int_0^R\int_{-\infty}^{\infty}\Phi_n(\vec{q},q_z=0,z)\exp[i\vec{q}\cdot[\vec{\beta}_1+\vec{\beta}_C(z)]][\cos(\vec{q}\cdot\vec{\delta}_C(z))-\cos(\vec{q}\cdot[\vec{\gamma}_1+\vec{\gamma}_C(z)])]\,d\vec{q}\,dz \tag{71}$$

as the analogous expansion quantity to the thin screen derivation. For a point source, this expression is small in the contributing regions of path space [Dashen, 1979]. Performing the Taylor series in Q results in

$$G_4(\vec{x}_i;\vec{x}'_i,R) = \frac{(2\pi R)^2}{k^2} G_4^c(\vec{x}_i;\vec{x}'_i,R) \int\int D\vec{\beta}_1 \, D\vec{\gamma}_1$$

$$\exp\left[-\frac{1}{2}\int_0^R [d(\vec{\gamma}_1(z)+\vec{\gamma}_C(z)+\vec{\delta}_C(z),z) + d(\vec{\gamma}_1(z)+\vec{\gamma}_C(z)-\vec{\delta}_C(z),z)]dz\right]$$

$$\left[\exp[-ik\int_0^R \vec{\beta}_1(z)\cdot\dot{\vec{\gamma}}_1(z)dz] + \sum_{n=1}^{\infty}\frac{(4\pi k^2)^n}{n!}\varepsilon^n \int_0^R dz_1 \cdots \int_0^R dz_n \int_{-\infty}^{\infty} d\vec{q}_1 \cdots \int_{-\infty}^{\infty} d\vec{q}_n\right.$$

$$\exp\left[-ik\int_0^R \vec{\beta}_1(z)\cdot[\dot{\vec{\gamma}}_1(z) - \sum_{j=1}^{n}\frac{\vec{q}_j}{k}\delta(z-z_j)]dz + i\sum_{j=1}^{n}\vec{\beta}_C(z_j)\cdot\vec{q}_j\right]$$

$$\left.\prod_{j=1}^{n}\Phi_n(\vec{q}_j,q_z=0,z_j)[\cos(\vec{q}_j\cdot\vec{\delta}_C(z_j)) - \cos(\vec{q}_j\cdot(\vec{\gamma}_1(z_j)+\vec{\gamma}_C(z_j)))]\right] \tag{72}$$

Applying identity (62) yields

$$G_4(\vec{x}_i;\vec{x}'_i,R) = G_4^c(\vec{x}_i;\vec{x}'_i,R)\left[\exp\left[-\frac{1}{2}\int_0^R [d(\vec{\gamma}_C(z)+\vec{\delta}_C(z),z) + d(\vec{\gamma}_C(z)-\vec{\delta}_C(z),z)]dz\right] +\right.$$

$$\sum_{n=1}^{\infty}\frac{(4\pi k^2)^n}{n!}\varepsilon^n \int_0^R dz_1 \cdots \int_0^R dz_n \int_{-\infty}^{\infty} d\vec{q}_1 \cdots \int_{-\infty}^{\infty} d\vec{q}_n \tag{73}$$

$$\exp\left[i\sum_{j=1}^{n}\vec{\beta}_C(z_j)\cdot\vec{q}_j - \frac{1}{2}\int_0^R [d(\sum_{m=1}^{n}\frac{\vec{q}_m}{k}h(z;z_m)+\vec{\gamma}_C(z)+\vec{\delta}_C(z),z) + d(\sum_{m=1}^{n}\frac{\vec{q}_m}{k}h(z;z_m)+\vec{\gamma}_C(z)-\vec{\delta}_C(z),z)]dz\right]$$

$$\left.\prod_{j=1}^{n}\Phi_n(\vec{q}_j,q_z=0,z_j)[\cos(\vec{q}_j\cdot\vec{\delta}_C(z_j)) - \cos(\vec{q}_j\cdot(\sum_{m=1}^{n}\frac{\vec{q}_m}{k}h(z_j,z_m)+\vec{\gamma}_C(z_j)))]\right] = \sum_{n=0}^{\infty}G_{4_n}(\vec{x}_i;\vec{x}'_i,R)\varepsilon^n$$

where $h(z;z_1)$ is the solution of

$$\ddot{h}(z;z_1) = \delta(z-z_1) \tag{74}$$

that is

$$h(z;z_1) = z(\frac{z_1}{R}-1) \qquad z<z_1$$

$$h(z;z_1) = z_1(\frac{z}{R}-1) \qquad z>z_1 \tag{75}$$

The free space Green's function for the fourth moment, $G_4^f(\mathbf{x}_i;\mathbf{x}'_i,R)$, is given by (24). The terms of the series are identified as the coefficients of the parameter ε^n. This construction will be used to prove the equivalence of the path-integral series to the iterated moment-equation series of section 4.2

The first term of (73)

$$G_{4_0}(\mathbf{x}_i;\mathbf{x}'_i,R) = G_4^f(\mathbf{x}_i;\mathbf{x}'_i,R)e^{-\frac{1}{2}\int_0^R [d(\boldsymbol{\eta}_G(z)+\boldsymbol{\delta}_G(z),z)+d(\boldsymbol{\eta}_G(z)-\boldsymbol{\delta}_G(z),z)]dz} \tag{76}$$

$$= G_2(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}'_1,\mathbf{x}'_2,R)G_2(\mathbf{x}_3,\mathbf{x}_4;\mathbf{x}'_3,\mathbf{x}'_4,R)$$

is analogous to the thin-screen result (32).

The n=1 term is more complex.

$$G_{4_1}(\mathbf{x}_i;\mathbf{x}'_i,R) = 4\pi k^2 G_4^f(\mathbf{x}_i;\mathbf{x}'_i,R)\int_0^R\int_{-\infty}^{\infty}\Phi_n(\mathbf{q},q_z=0,z_1)[\cos(\mathbf{q}\cdot\boldsymbol{\delta}_G(z_1)) - \cos(\mathbf{q}\cdot[\frac{\mathbf{q}}{k}h(z_1,z_1)+\boldsymbol{\eta}_G(z_1)])]$$

$$e^{i\mathbf{q}\cdot\boldsymbol{\delta}_G(z_1)}\exp\left[-\frac{1}{2}\int_0^R[d(\frac{\mathbf{q}}{k}h(z;z_1)+\boldsymbol{\eta}_G(z)+\boldsymbol{\delta}_G(z),z)+d(\frac{\mathbf{q}}{k}h(z;z_1)+\boldsymbol{\eta}_G(z)-\boldsymbol{\delta}_G(z),z)]dz\right]d\mathbf{q}dz_1 \tag{77}$$

The higher order terms of the fourth-moment Green's function are obtained from (73) but become more intractable. However, the first order description of the fourth moment is given by the leading terms of the series expansion. We now compare these expressions to previous results for the fourth moment. The point-source result is obtained from the Green's function by setting $\mathbf{x}'_i$ to zero. This reproduces the iterated series of Shishov [1972 Eq (13)] that was derived using moment-equation methods in a spherical coordinate system. Appling the plane-wave initial condition generates the series expression of Shishov [1971 Eq (20)]. Gurvich et al [1979b] proved this series was convergent. It can be shown that the point-source series is also convergent. Applying the plane-wave case to (77) with $\boldsymbol{a}=0$ and $\boldsymbol{\delta}=0$, reproduces the strong scattering results of Fante [1975 Eq (B1)] and Zavorotnyi [1979b Eq (3)].

3.4 Intensity Correlation

Furutsu [1972] derived expressions for the intensity correlation from a Gaussian beam propagating in a random medium with a square-law structure function. This case describes random wander of a beam [Wandzura, 1980]. Weak scattering results for intensity correlation from a Gaussian beam are presented by Ishimaru [1969]. We now consider the intensity correlation following the thin screen analysis of section 2.3.

The fourth-moment series generates two expressions for the intensity correlation; one is obtained by setting $\vec{\delta}=0$ and $\vec{\gamma}=0$ (the low spatial frequency region) and the other by setting $\vec{\delta}=0$ and $\vec{\beta}=0$ (the high spatial frequency region). The resulting expressions for the n=0 term are

$$C_0^{\gamma}(\vec{\alpha},\vec{\beta},R)=\frac{k^4}{(2\pi R)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)\exp[-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}'+(\vec{\beta}-\vec{\beta}')\cdot\vec{\gamma}']] \tag{78}$$

$$\exp\left\{-\frac{1}{2}\int_0^R[\,d((\vec{\gamma}'+\vec{\delta}')(1-\frac{z}{R}),z)+d((\vec{\gamma}'-\vec{\delta}')(1-\frac{z}{R}),z)]dz\right\}d\vec{\alpha}'\,d\vec{\beta}'\,d\vec{\gamma}'\,d\vec{\delta}'$$

and

$$C_0^{\beta}(\vec{\alpha},\vec{\gamma},R)=\frac{k^4}{(2\pi R)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)\exp[-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}'+(\vec{\gamma}-\vec{\gamma}')\cdot\vec{\beta}']] \tag{79}$$

$$\exp\left\{-\frac{1}{2}\int_0^R[\,d((\vec{\gamma}'+\vec{\delta}')(1-\frac{z}{R})+\vec{\gamma}\frac{z}{R},z)+d((\vec{\gamma}'-\vec{\delta}')(1-\frac{z}{R})+\vec{\gamma}\frac{z}{R},z)]dz\right\}d\vec{\alpha}'\,d\vec{\beta}'\,d\vec{\gamma}'\,d\vec{\delta}'$$

For spatially coherent sources, these expressions reduce to

$$C_0^{\gamma}(\vec{\alpha},\vec{\beta},R)=\langle I(\frac{\vec{\alpha}+\vec{\beta}}{2},R)\rangle\langle I(\frac{\vec{\alpha}-\vec{\beta}}{2},R)\rangle=\langle I(\vec{x}_1,z)\rangle\langle I(\vec{x}_3,z)\rangle \tag{80}$$

and

$$C_0^{\beta}(\vec{x}_1,\vec{x}_2,R)=\Gamma_2(\vec{x}_1,\vec{x}_2,R)\Gamma_2^*(\vec{x}_1,\vec{x}_2,R) \tag{81}$$

which are analogous to the thin-screen results (36) and (42).

The corresponding expressions for the n=1 term are given by

$$C_1^{\gamma}(\vec{\alpha},\vec{\beta},R)=\frac{4\pi k^6}{(2\pi R)^4}\int_0^R\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)e^{-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}'+(\vec{\beta}-\vec{\beta}')\cdot\vec{\gamma}']}e^{i\vec{q}\cdot[\vec{\beta}'(1-\frac{z_1}{R})+\vec{\beta}\frac{z_1}{R}]}\Phi_n(\vec{q},z_1)$$

$$\exp\left[-\frac{1}{2}\int_0^R [d(\frac{\vec{q}}{k}h(z;z_1)+(\vec{\gamma}'+\vec{\delta}')(1-\frac{z}{R}),z)+d(\frac{\vec{q}}{k}h(z;z_1)+(\vec{\gamma}'-\vec{\delta}')(1-\frac{z}{R}),z)]dz\right]$$

$$[\cos(\vec{q}\cdot\vec{\delta}'(1-\frac{z_1}{R}))-\cos(\vec{q}\cdot(\frac{\vec{q}}{k}h(z_1,z_1)+\vec{\gamma}'(1-\frac{z_1}{R})))]\,d\vec{\alpha}'\,d\vec{\beta}'\,d\vec{\gamma}'\,d\vec{\delta}'\,d\vec{q}\,dz_1 \qquad (82)$$

and

$$C_1^{hf}(\vec{\alpha},\vec{\gamma},R)=\frac{4\pi k^8}{(2\pi R)^4}\int_0^R\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}',\vec{\delta}',0)e^{-i\frac{k}{R}[(\vec{\alpha}-\vec{\alpha}')\cdot\vec{\delta}'+\vec{\beta}'\cdot(\vec{\gamma}-\vec{\gamma}')]}e^{i\vec{q}\cdot\vec{\beta}'[1-\frac{z_1}{R}]}\Phi_n(\vec{q},z_1)$$

$$\exp\left[-\frac{1}{2}\int_0^R [d(\frac{\vec{q}}{k}h(z;z_1)+(\vec{\gamma}'+\vec{\delta}')(1-\frac{z}{R})+\vec{\gamma}\frac{z}{R},z)+d(\frac{\vec{q}}{k}h(z;z_1)+(\vec{\gamma}'-\vec{\delta}')(1-\frac{z}{R})+\vec{\gamma}\frac{z}{R},z)]dz\right]$$

$$\left[\cos[\vec{q}\cdot\vec{\delta}'(1-\frac{z_1}{R})]-\cos[\vec{q}\cdot(\frac{\vec{q}}{k}h(z_1,z_1)+\vec{\gamma}'(1-\frac{z_1}{R})+\vec{\gamma}\frac{z_1}{R})]\right]d\vec{\alpha}'\,d\vec{\beta}'\,d\vec{\gamma}'\,d\vec{\delta}'\,d\vec{q}\,dz_1 \quad (83)$$

The appropriate Fourier transform (cf (5)) produces the corresponding expressions for the intensity spectrum. The Born approximation is secured from (82) by ignoring the last exponential term.

The low-frequency series for the intensity spectrum has been investigated for a power-law spectrum of refractive index fluctuations and plane-wave [Gochelashvily and Shishov, 1971, 1974] and point-source [Gochelashvily et al, 1974] geometries. The qualitative behavior of the intensity spectrum is similar to the thin screen result [Figure 1]. In weak scattering, the first two terms describe the complete intensity spectrum. In strong scattering the intensity spectrum is characterized by two components. The low frequency region, $\Phi_1^{lf}(\vec{q},R)$, is due to refractive focussing by irregularities of the size of the scattering disk (Rickett et al, 1984). The high frequency behavior was first determined by summing the low-frequency series, which yields the high frequency approximation $\Phi_0^{hf}(\vec{q},R)$. The derivation presented here generates this same result plus higher correction terms without performing a complicated summation. As in the thin screen case, the physical mechanism for the first correction term (83) is the modulation of the small scale structure (81) by the large scale refractive process (82).

Uscinski [1982] derived an approximate expression for the intensity spectrum for the case of plane waves by summing a perturbation series for the fourth moment equation.

Macaskill [1983] derived the same result using the two-scale expansion [Beran et al, 1982]. In our notation, their expression for the intensity spectrum is

$$\Phi(\vec{q},R) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\exp\left[-\int_0^R [\, d(\vec{x},z) + d(\frac{\vec{q}}{k}(R-z),z) - \frac{1}{2}d(\vec{x}+\frac{\vec{q}}{k}(R-z),z) - \frac{1}{2}d(\vec{x}-\frac{\vec{q}}{k}(R-z),z)]dz\right]e^{-i\vec{q}\cdot\vec{x}}\, d\vec{x} \qquad (84)$$

The extreme low and high spatial frequency behavior is identical to that obtained from (82) and (81). However, the predictions for the intermediate frequency region are different. More theoretical calculations are required in order to determine the accuracy of the two methods.

The intensity correlation from a point source embedded in a random medium with irregularities that are constant in one transverse dimension is obtained from the point-source result by the substitution $\Phi_n(\vec{q}) = \Phi_n(q_x)\delta(q_y)$. The leading order terms of the intensity correlation series are

$$C_0^{N}(\beta,R) = \langle I\rangle^2 \exp[-\int_0^R d(\beta\frac{z}{R})dz] \qquad (85)$$

$$C_1^{\prime}(\beta,R) = 8\pi k^2 \langle I\rangle^2 \int_0^R\int_{-\infty}^{\infty}\Phi_n(q,z_1)\sin^2[\frac{q^2}{2k}h(z_1,z_1)] \exp\left[-\int_0^R d[\frac{q}{k}h(z;z_1),z]dz + i\beta q\frac{z_1}{R}\right]dq\, dz_1 \qquad (86)$$

and

$$C_1^{N}(\beta,R) = 8\pi k^2 \langle I\rangle^2 \int_0^R\int_{-\infty}^{\infty}\Phi_n(q,z_1)\sin^2[q\beta\frac{z_1}{2R} + \frac{q^2}{2k}h(z_1,z_1)] \exp\left[-\int_0^R d[\beta\frac{z}{R} + \frac{q}{k}h(z;z_1),z]dz + i\beta q\frac{z_1}{R}\right]dq\, dz_1 \qquad (87)$$

Ignoring the exponential term of (86) reproduces the weak scattering (Rytov approximation) result of Tur and Beran [1983].

Fante [1983] investigated the effect of the inner scale of turbulence on scintillation for the case of strong scattering of plane waves by calculating the expression $C_1^{\prime}(\vec{\beta},R)$. The contribution of this term to the total intensity variance is appreciable over a wide range of parameter space. Therefore, higher terms are required for an accurate description of the intensity correlation.

4. MOMENT EQUATION APPROACH

4.1 Moment Equation Method for the Thin Screen

We now derive the fourth-moment result (23), using the moment-equation method. This method is based on partial differential equations for the transverse moments that are derived from the parabolic equation for the random fields. These differential equations may then be solved by transform methods [Rumsey, 1975] and the method of characteristics [Kiang and Liu, 1982]. There are, in general, many transforms that will simplify the problem. We choose one [Shishov, 1971] that permits an analogous derivation for the fourth moment of waves propagating through extended random media.

The fourth moment satisfies the differential equation

$$\frac{\partial \Gamma_4}{\partial z}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4,z) = \frac{i}{2k}(\nabla_1^2 - \nabla_2^2 + \nabla_3^2 - \nabla_4^2)\Gamma_4(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4,z) \tag{88}$$

with initial condition

$$\Gamma_4(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4,0+) = \Gamma_4(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4,0)\langle e^{i[\theta(\vec{x}_1)-\theta(\vec{x}_2)+\theta(\vec{x}_3)-\theta(\vec{x}_4)]}\rangle = \Gamma_4(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4,0)$$

$$\exp[-\frac{1}{2}[D_\theta(\vec{x}_1-\vec{x}_2)+D_\theta(\vec{x}_3-\vec{x}_4)+D_\theta(\vec{x}_1-\vec{x}_4)+D_\theta(\vec{x}_2-\vec{x}_3)-D_\theta(\vec{x}_2-\vec{x}_4)-D_\theta(\vec{x}_1-\vec{x}_3)]] \tag{89}$$

Change variables to the coordinate system (22). Then

$$\frac{\partial \Gamma_4}{\partial z}(\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\delta},z) = \frac{i}{k}(\nabla_\alpha \cdot \nabla_\delta + \nabla_\beta \cdot \nabla_\gamma)\Gamma_4(\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\delta},z) \tag{90}$$

This equation is solved in the Fourier transform domain

$$M(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\delta},R)e^{-i[\vec{\kappa}\cdot\vec{\alpha}+\vec{q}\cdot\vec{\beta}]}\,d\vec{\alpha}\,d\vec{\beta} \tag{91}$$

which transforms (90) to

$$\frac{\partial M}{\partial z}(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) = -(\frac{\vec{\kappa}}{k}\cdot\nabla_{\delta} + \frac{\vec{q}}{k}\cdot\nabla_{\gamma})M(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) \tag{92}$$

Changing variables to

$$\vec{a} = k\vec{\delta} - \vec{\kappa}z \qquad \vec{b} = k\vec{\gamma} - \vec{q}z \qquad \tau = z$$

$$\vec{\delta}(\tau) = \frac{\vec{a} + \vec{\kappa}\tau}{k} \qquad \vec{\gamma}(\tau) = \frac{\vec{b} + \vec{q}\tau}{k} \qquad z = \tau$$

$$\nabla_{\delta} = k\nabla_{a} \qquad \nabla_{\gamma} = k\nabla_{b} \qquad \frac{\partial}{\partial z} = -\vec{q}\cdot\nabla_{b} - \vec{\kappa}\cdot\nabla_{a} + \frac{\partial}{\partial\tau} \tag{93}$$

transforms (92) to

$$\frac{\partial M}{\partial\tau}(\vec{\kappa},\vec{q},\vec{a},\vec{b},\tau) = 0 \tag{94}$$

Since M is independent of τ, the solution in the original variables is

$$M(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) = M(\vec{\kappa},\vec{q},\vec{\gamma} - \vec{q}\frac{R}{k},\vec{\delta} - \vec{\kappa}\frac{R}{k},0+) \tag{95}$$

The fourth moment is the inverse Fourier transform of this expression, i.e.

$$\Gamma_4(\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\delta},R) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\Gamma_4(\vec{\alpha}',\vec{\beta}',\vec{\gamma}-\vec{q}R/k,\vec{\delta}-\vec{\kappa}R/k,0)e^{i[\vec{\kappa}\cdot(\vec{\alpha}-\vec{\alpha}')+\vec{q}\cdot(\vec{\beta}-\vec{\beta}')]} \tag{96}$$

$$\exp\left[-\frac{1}{2}[D_\theta(\vec{\gamma}'+\vec{\delta}')+D_\theta(\vec{\gamma}'-\vec{\delta}')+D_\theta(\vec{\beta}'+\vec{\delta}')+D_\theta(\vec{\beta}'-\vec{\delta}')-D_\theta(\vec{\beta}'+\vec{\gamma}')-D_\theta(\vec{\beta}'-\vec{\gamma}')]\right]d\vec{\alpha}'\,d\vec{\beta}'\,d\vec{q}\,d\vec{\kappa}$$

The change of variables $\vec{\gamma}'=\vec{\gamma}-\vec{q}R/k$ and $\vec{\delta}'=\vec{\delta}-\vec{\kappa}R/k$ produces the Green's function result, (23). The moment- equation method is based on the formulation of the free space Green's function in the Fourier transform domain. This transform technique will now be applied to the more difficult problem of wave propagation in an extended random medium.

4.2 Moment Equation Method for Extended Random Media

The moment-equation method was a *major* contribution to the theory of wave propagation in extended random media [Prokhorov et al 1975, Tatarskii, 1971]. Using moment-equation methods, a series expression for the fourth moment was derived for plane-wave [Shishov, 1971] and point-source [Shishov, 1972] conditions. For an arbitrary source distribution, we present an analogous expression that is identical to the path integral results of section 3.3, thus demonstrating the equivalence of the two methods.

The transverse moments of the electric field satisfy differential equations [de Wolf, 1967; Brown, 1967; Shishov, 1968; Dolin, 1968; Chernov, 1969; Tatarskii, 1969; Lee, 1974]. We follow the techniques of Shishov [1971] to solve the differential equation for the fourth moment

$$\frac{\partial \Gamma_4}{\partial z} - \frac{i}{2k}[\nabla_1^2 - \nabla_2^2 + \nabla_3^2 - \nabla_4^2]\Gamma_4 + V\,\Gamma_4 = 0 \tag{97}$$

where

$$2V = d(\vec{x}_1 - \vec{x}_2, z) + d(\vec{x}_1 - \vec{x}_4, z) + d(\vec{x}_2 - \vec{x}_3, z) + d(\vec{x}_3 - \vec{x}_4, z) - d(\vec{x}_1 - \vec{x}_3, z) - d(\vec{x}_2 - \vec{x}_4, z) \tag{98}$$

In the coordinate system of (22), the fourth moment satisfies

$$\frac{\partial \Gamma_4}{\partial z} - \frac{i}{k}[\nabla_{\vec{\alpha}} \cdot \nabla_{\vec{\delta}} + \nabla_{\vec{\beta}} \cdot \nabla_{\vec{\gamma}}]\Gamma_4 + V\,\Gamma_4 = 0 \tag{99}$$

with

$$2V = d(\vec{\gamma} + \vec{\delta}, R) + d(\vec{\gamma} - \vec{\delta}, R) + \varepsilon\Big[d(\vec{\beta} + \vec{\delta}, R) + d(\vec{\beta} - \vec{\delta}, R) - d(\vec{\beta} + \vec{\gamma}, R) - d(\vec{\beta} - \vec{\gamma}, R)\Big] \tag{100}$$

where the parameter ε identifies

$$S = \frac{1}{2}\Big[d(\vec{\beta} + \vec{\delta}, z) + d(\vec{\beta} - \vec{\delta}, z) - d(\vec{\beta} + \vec{\gamma}, z) - d(\vec{\beta} - \vec{\gamma}, z)\Big] \tag{101}$$

as the same combination of structure-function densities we used in the path-integral expansion variable Q of (71). The Fourier transform (91) converts (99) to the integral equation

$$\frac{\partial M}{\partial z} + \frac{\vec{\kappa}}{k} \cdot \nabla_{\vec{\delta}} M + \frac{\vec{q}}{k} \cdot \nabla_{\vec{\gamma}} M + \frac{1}{2}[d(\vec{\gamma} + \vec{\delta}) + d(\vec{\gamma} - \vec{\delta})]M = \varepsilon G(\vec{\kappa}, \vec{q}, \vec{\gamma}, \vec{\delta}, z) \tag{102}$$

where G is the convolution of M with the transform of S, i.e.

$$G(\vec{\kappa}, \vec{q}, \vec{\gamma}, \vec{\delta}, z) = 4\pi k^2 \int_{-\infty}^{\infty} \Phi_n(\vec{q}_1, z)[\cos(\vec{\delta} \cdot \vec{q}_1) - \cos(\vec{\gamma} \cdot \vec{q}_1)] M(\vec{\kappa}, \vec{q} - \vec{q}_1, \vec{\gamma}, \vec{\delta}, z)\, d\vec{q}_1 \tag{103}$$

The change of variables (93) transforms (102) to

$$\frac{\partial M}{\partial r} + \frac{1}{2}[d(\vec{\gamma}(r) + \vec{\delta}(r), r) + d(\vec{\gamma}(r) - \vec{\delta}(r), r)]M = \varepsilon G(\vec{\kappa}, \vec{q}, \vec{\gamma}(r), \vec{\delta}(r), r) \tag{104}$$

which is analogous to (94) of the thin screen derivation. This is an ordinary differential equation in r, with solution

$$M(\vec{\kappa},\vec{q},\vec{b},\vec{a},r) = M(\vec{\kappa},\vec{q},\frac{\vec{b}}{k},\frac{\vec{a}}{k},0)\exp\left[-\frac{1}{2}\int_0^r [d(\vec{\gamma}(z)+\vec{\delta}(z),z) + d(\vec{\gamma}(z)-\vec{\delta}(z),z)]dz\right]$$

$$+\varepsilon\int_0^r \exp\left[-\frac{1}{2}\int_{z_1}^r [d(\vec{\gamma}(z)+\vec{\delta}(z),z)+d(\vec{\gamma}(z)-\vec{\delta}(z),z)]dz\right] G(\vec{\kappa},\vec{q},\vec{\gamma}(z_1),\vec{\delta}(z_1),z_1)\, dz_1 \quad (105)$$

Changing variables back to $\vec{\delta},\vec{\gamma},R$ results in

$$M(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) = M(\vec{\kappa},\vec{q},\vec{\gamma}_3,\vec{\delta}_3,0)\exp\left[-\frac{1}{2}\int_0^R [d(\vec{\gamma}_2+\vec{\delta}_2,z) + d(\vec{\gamma}_2-\vec{\delta}_2,z)]dz\right]$$

$$+\varepsilon\int_0^R \exp\left[-\frac{1}{2}\int_{z_1}^R [d(\vec{\gamma}_2+\vec{\delta}_2,z)+d(\vec{\gamma}_2-\vec{\delta}_2,z)]dz\right] G(\vec{\kappa},\vec{q},\vec{\gamma}_1,\vec{\delta}_1,z_1)\, dz_1 \quad (106)$$

where

$$\vec{\gamma}_1 = \vec{\gamma} - \frac{\vec{q}}{k}(R-z_1) \qquad \vec{\gamma}_2 = \vec{\gamma} - \frac{\vec{q}}{k}(R-z) \qquad \vec{\gamma}_3 = \vec{\gamma} - \frac{\vec{q}}{k}R$$

$$\vec{\delta}_1 = \vec{\delta} - \frac{\vec{\kappa}}{k}(R-z_1) \qquad \vec{\delta}_2 = \vec{\delta} - \frac{\vec{\kappa}}{k}(R-z) \qquad \vec{\delta}_3 = \vec{\delta} - \frac{\vec{\kappa}}{k}R \quad (107)$$

This expression can be written as the integral equation

$$M(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) = Z(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R)$$

$$+\varepsilon\int_0^R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\vec{\kappa},\vec{q},\vec{q}',\vec{\gamma},\vec{\gamma}',\vec{\delta},\vec{\delta}',R,z_1)M(\vec{\kappa},\vec{q}',\vec{\gamma}',\vec{\delta}',z_1)\, d\vec{q}'\, d\vec{\gamma}'\, d\vec{\delta}'\, dz_1 \quad (108)$$

where

$$Z(\vec{\kappa},\vec{q},\vec{\gamma},\vec{\delta},R) = M(\vec{\kappa},\vec{q},\vec{\gamma}_3,\vec{\delta}_3,0)\exp\left[-\frac{1}{2}\int_0^R [d(\vec{\gamma}_2+\vec{\delta}_2,z)+d(\vec{\gamma}_2-\vec{\delta}_2,z)]dz\right] \quad (109)$$

and

$$K(\kappa,q,q',\gamma,\gamma',\delta,\delta',R,z_1)=4\pi k^2\Phi_n(q-q',z_1)\delta(\delta'-\delta_1)\delta(\gamma'-\gamma_1) \tag{110}$$

$$\exp\left[-\frac{1}{2}\int_{z_1}^{R}[d(\gamma_2+\delta_2,z)+d(\gamma_2-\delta_2,z)]dz\right]\left[\cos[\delta'\cdot(q-q')]-\cos[\gamma'\cdot(q-q')]\right]$$

The formal solution is

$$M(\kappa,q,\gamma,\delta,R)=\sum_{n=0}^{\infty}M_n(\kappa,q,\gamma,\delta,R)\varepsilon^n \tag{111}$$

where

$$M_n(\kappa,q,\gamma,\delta,R)=\int_0^R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\kappa,q,q',\gamma,\gamma',\delta,\delta',R,z_1)M_{n-1}(\kappa,q',\gamma',\delta',z_1)\,dq'\,d\gamma'\,d\delta'\,dz_1 \tag{112}$$

and $M_0(\kappa,q,\gamma,\delta,R)=Z(\kappa,q,\gamma,\delta,R)$. This solution is a power series in ε. The path-integral expansion (73) is also a power series in the same parameter ε. Since this problem has a unique solution and since the path-integral and moment equation are equivalent representations (Codona et al, 1985), the two series are equal, term by term. We will show this equivalence explicitly for the n=0 term. The fourth moment is given by the inverse transform of (110), i.e.

$$\Gamma_{4_0}(\alpha,\beta,\gamma,\delta,R)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\Gamma_{4_0}(\alpha',\beta',\gamma-q\frac{R}{k},\delta-\kappa\frac{R}{k},0)e^{i[\kappa\cdot(\alpha-\alpha')+q\cdot(\beta-\beta')]} \tag{114}$$

$$\exp\left[-\frac{1}{2}\int_0^R[d(\gamma+\delta-(q+\kappa)(R-z)/k,z)+d(\gamma-\delta-(q-\kappa)(R-z)/k,z)]dz\right]d\alpha'd\beta'dq\,d\kappa$$

The change of variables $\gamma'=\gamma-qR/k$ and $\delta'=\delta-\kappa R/k$ produces the same result obtained from the n=0 term, (76), of the path-integral derivation. Unfortunately, for higher order terms, the equality of the two expansions is less obvious since the functional form of the two series is different: the moment equation series is essentially a multiple convolution, while the path integral result is not. However, with careful algebraic manipulation, the equality of the two series can be explicitly demonstrated.

The moment-equation method was first applied to the simple geometries of plane waves and point sources. Since the moments are then independent of the centroid coordinate, the differential equations simplify. Early work concentrated on moment-equation methods because of this simplification. Moment-equation techniques have also been used to investigate the validity of the Markov approximation [Klyatskin, 1969; Klyatskin and Tatarskii, 1971] and the effects of non-gaussian refractive index fluctuations under the Markov approximation [Klyatskin, 1975]. The Green's function formulation provides a clear connection between the thin screen and extended medium. The operator form of the path integral has also been useful for evaluating the corrections to the Markov approximation for the higher intensity moments [Zavorotnyi, 1978].

5. SUMMARY

A series expression has been derived for the fourth moment of waves incident on a phase screen or propagating through extended random media. These results can be derived using moment-equation techniques or functional methods (path integral or operator). The asymmetric terms of the expansion generate two expressions for the intensity correlation; one that approximates the low frequency region of the spatial spectrum and the other appropriate for the high frequencies. The rate of convergence of the two approximations can be used to produce a complete expression for the intensity spectrum valid for any initial source distribution. The calculations required for a complete expression may be excessive. However, in strong scattering conditions, the leading order behavior of the intensity spectrum is well described by a few terms of the series. The expressions presented here are applicable to many problems of WPRM that involve arbitrary source distributions. These include

a) The intensity statistics from beamed lasers, navigational beacons, radar, spacecraft and satellite radio transmissions, astronomical sources and other extended wave sources.

b) The effects of slowly varying refractive-index statistics along the propagation path, e.g. the turbulence profile of planetary atmospheres.

c) Comparison of thin-screen and extended-media results.

d) The relationship between intensity statistics and the refractive-index spectrum.

e) Imaging through random media.

Acknowledgements: This research was funded by DARPA.

References

Barabanenkov, Y.N., Y.A. Kravtsov, S.M. Rytov, and V.I. Tatarskii, Status of the theory of propagation of waves in randomly inhomogeneous medium, Sov. Phys. Usp., 13, 551-680, 1971.

Beran, M.J., A.M. Whitman, and S. Frankenthal, Scattering calculations using the characteristic rays of the coherence function, J. Acoust. Soc. Am., 71, 1124-1130, 1982.

Bramley, E.N., Diffraction of an angular spectrum of waves by a phase-changing screen, J. Atmos. Terr. Phys., 29, 1-28, 1967.

Brown, W.P., Propagation in random media- cumulative effect of weak inhomogeneities IEEE Trans. Ant. Prop., AP-15, 81-89, 1967.

Chernov, L.A., Equations for the statistical moments of the field in a randomly-inhomogeneous medium, Akust. Zh., 15, 594-603, 1969. English, Sov. Phys. Acoustics, 15, 511-517, 1970.

Codona, J.L., D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F. Henyey, Moment-equation and path-integral techniques for wave propagation in random media, Accepted by J. Math. Phys., 1985.

Dashen, R., Path integrals for waves in random media, J. Math. Phys., 20, 894-920, 1979.

de Wolf, D.A., Multiple scattering in a random continuum, Radio Science, 2, 1379-1392, 1967.

Dolin, L.S., Equations for the correlation functions of a wave beam in a randomly-inhomogeneous medium, Izv. VUZ. Radiofiz., 11, 840-849, 1968. English, Radiophysics and Quantum Electronics, 11, 486-491, 1968.

Fante, R.L., Electromagnetic beam propagation in turbulent media, Proc. IEEE, 63, 1669-1692, 1975.

Fante, R.L., Electromagnetic beam propagation in turbulent media: An update, Proc. IEEE, 68, 1424-1443, 1980.

Fante, R.L., Inner-scale size effect on the scintillations of light in the turbulent atmosphere, J. Opt. Soc. Am., 73, 277-281, 1983.

Feynman, R.P., Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys., 20, 367-387, 1948.

Flatté, S.M., R. Dashen, W.H. Munk, K.M. Watson, and F. Zachariasen, Sound Transmission Through a Fluctuating Ocean, published by the Cambridge University Press in their series on Mechanics and Applied Mathematics, 1979.

Flatté, S.M., Wave propagation through random media: Contributions from ocean acoustics, Proc. IEEE, 71, 1267-1294, 1983.

Frankenthal, Shimshon, M.J. Beran, and A.M. Whitman, Caustic corrections using coherence theory, J. Acoust. Soc. Am., 71, 348-358, 1982.

Frankenthal, Shimshon, Alan M. Whitman, and Mark J. Beran, Two-scale solutions for intensity fluctuations in strong scattering, J. Opt. Soc. Am., 585-597, 1984.

Furutsu, K., Statistical theory of wave propagation in a random medium and the irradiance distribution function, J. Opt. Soc. Am., 62, 240-254, 1972.

Gochelashvily, K.S., and V.I. Shishov, Laser beam scintillation beyond a turbulent layer, Optica Acta, 18, 313-320, 1971.

Gochelashvily, K.S., and V.I. Shishov, Focused irradiance fluctuations beyond a layer of turbulent atmosphere, Optica Acta, 19, 327-332, 1972.

Gochelashvily, K.S., Propagation of focused laser radiation in a turbulent medium, Kvant. Elektron., 1, 848-857, 1974. English, Sov. J. Quant. Electron., 4, 465-470, 1974.

Gochelashvily, K.S., and V.I. Shishov, Saturated fluctuations in the laser radiance intensity in a turbulent medium, Zh. Eksp. Teor. Fiz., 66, 1237-1247, 1974. English, Sov. Phys. JETP, 39, 605-609, 1974.

Gochelashvily, K.S., V.G. Pevgov, and V.I. Shishov, Saturation of fluctuations of the intensity of laser radiation at large distances in a turbulent atmosphere Fraunhofer zone of transmitter Kvant. Elektron., 1, 1156-1165, 1974. English, Sov. J. Quant. Electron., 4, 632-637, 1974.

Gochelashvily, K.S., and V.I. Shishov, Saturation of laser irradiance fluctuations beyond a turbulent layer, Optical and Quantum Electronics, 7, 524-536, 1975.

Gurvich, A.S., and V. Kan, Measurement of four-point field coherence function in region of random focusing of laser emission, Izv. Vysh. Ucheb. Zaved. Radiofiz., 21, 398-407, 1978. English, Radiophysics and Quantum Electronics, 21, 274-281, 1978.

Gurvich, A.S., V. Kan, V.I. Tatarskii, and V.U. Zavorotnyi, Four-point field coherence function in a turbulent medium, Optica Acta, 26, 543-553, 1979a.

Gurvich, A.S., B.S. Elenov, Vl. V. Pokasov, K.K. Sabel'fel'd, and V.I. Tatarskii, Spatial structure of strong fluctuations of light intensity in a turbulent medium, Izv. Vysh. Ucheb. Zaved. Radiofiz., 22, 198-207, 1979b. English, Radiophysics and Quantum Electronics, 22, 135-142, 1979b.

Ishimaru, Akira, Fluctuations of a focused beam wave for atmospheric turbulence probing, Proc. IEEE, 57, 407-414, 1969.

Ishimaru, Akira, Wave Propagation and Scattering in Random Media, Academic Press, 1978.

Kiang, Yean-Woei, and C.H. Liu, Propagation of beam waves through a turbulent layer, Radio Science, 17, 1211-1219, 1982.

Klyatskin, V.I. Applicability of the approximation of a markov random process in problems relating to the propagation of light in a medium with random inhomogeneities, Zh. Eksp. Teor. Fiz., 57, 952-958, 1969. English, Soviet Physics JETP, 30, 520-523, 1970.

Klyatskin, V.I., and V.I. Tatarskii, The parabolic equation approximation for propagation of waves in a medium with random inhomogeneities, Zh. Eksp. Teor. Fiz., 58, 624-634, 1970. English, Soviet Physics JETP, 31, 335-339, 1970.

Klyatskin, V.I., and Tatarskii, A new method of successive approximations in the problem of the propagation of waves in a medium having random large-scale inhomogeneities, Izv. Vysh. Ucheb. Zaved. Radiofiz., 14, 1400-1415, 1971. English, Radiophysics and Quantum Electronics, 14, 1100-1111, 1971.

Klyatskin, V.I., Statistical theory of light propagation in a randomly-inhomogeneous medium (Functional methods) (Review), Izv. Vysh. Ucheb. Zaved. Radiofiz., 16, 1629-1644, 1973. English, Radiophysics and Quantum Electronics, 16, 1261-1271, 1973.

Klyatskin, V.I., and V.I. Tatarskii, Statistical theory of light propagation in a turbulent medium (Review), Izv. Vysh. Ucheb. Zaved. Radiofiz., 15, 1433-1455, 1972. English, Radiophysics and Quantum Electronics, 15, 1095-1111, 1972.

Lee, L.C., Wave propagation in a random medium: A complete set of the moment equations with different wave numbers, J. Math. Phys., 15, 1431-1435, 1974.

Lee, L.C., Theory of thin-screen scintillations for a spherical wave, Astrophysical Journal, 218, 468-476, 1977.

Macaskill, C., An improved solution to the fourth moment equation for intensity fluctuations, Proc. R. Soc. Lond. A, 386, 461-474, 1983.

Marians, M. Computed scintillation spectra for strong turbulence, Radio Sci., 10, 115-119, 1975.

Mercier, R.P., Diffraction by a screen causing large random phase fluctuations, Proc. Camb. Phil. Soc., 58, 382-400, 1962.

Prokhorov, A.M., F.V.Bunkin, K.S. Gochelashvily, and V.I.Shishov, Laser irradiance propagation in turbulent media, Proc. IEEE, 63, 790-811 1975.

Rickett, B.J., Interstellar scattering and scintillation of radio waves, Ann. Rev. Astron. Astrophys., 15, 479-504, 1977.

Rickett, B.J., W.A. Coles and G. Bourgois, Slow scintillation in the interstellar medium, Astron. Astrophys., 134, 390-395, 1984.

Rino, C.L., A power law phase screen model for ionospheric scintillation 1. Weak scatter, Radio Science, 14, 1135-1145, 1979a.

Rino, C.L., A power law phase screen model for ionospheric scintillation 2. Strong scatter, Radio Science, 14, 1147-1155, 1979b.

Rumsey, V.H., Scintillations due to a concentrated layer with a power-law turbulence spectrum, Radio Science, 10, 107-114, 1975.

Salpeter, E.E., Interplanetary scintillations I. Theory, Astrophys. J., 147, 433-448, 1967.

Shishov, V.I., Theory of wave propagation in random media, Izv. Vuz. Radiofiz., 11, 866-875, 1968. English, Radiophysics Quantum Electronics, 11, 500-505, 1968.

Shishov, V.I., Strong fluctuations of the intensity of a plane wave propagating in a random refractive medium, Zh. Eksp. Teor. Fiz., 61, 1399-1409, 1971. English, Sov. Phys. JETP, 34, 744-748, 1972.

Shishov, V.I., Strong fluctuations of the intensity of a spherical wave propagating in a randomly refractive medium, Izv. Vysh. Ucheb. Zaved. Radiofiz., 15, 904-912, 1972. English, Radiophysics and Quantum Electronics, 15, 689-695, 1972.

Tatarskii, V.I., Light propagation in a medium with random refractive index inhomogeneities in the Markov random process approximation, Zh. Eksp. Teor. Fiz., 56, 2106-2117, 1969. English, Soviet Physics JETP, 29, 1133-1138, 1969.

Tatarskii, V.I., *The Effects of the Turbulent Atmosphere on Wave Propagation*, National Technical Information Service, Springfield, Va., 1971.

Tatarskii, V.I., and V.U. Zavorotnyi, Strong fluctuations in light propagation in a randomly inhomogeneous medium, E. Wolf, Progress in Optics XVIII, North-Holland, 1980.

Tur, Moshe, and Mark J. Beran, Wave propagation in random media: a comparison of two theories, J. Opt. Soc. Am., 73, 1343-1349, 1983.

Uscinski, B.J., Intensity fluctuations in a multiple scattering medium. Solution of the fourth moment equation, Proc. R. Soc. Lond. A., 380, 137-169, 1982.

Uscinski, B.J., and C. Macaskill, Intensity fluctuations due to a deeply modulated phase screen- I. Theory, J. Atmos. Terr. Phys., 45, 595-605, 1983a.

Uscinski, B.J., and C. Macaskill, Intensity fluctuations due to a deeply modulated phase screen-II. Results, J. Atmos. Terr. Phys., 45, 607-615, 1983b.

Wandzura, S.M., Meaning of quadratic structure functions, J. Opt. Soc. Am., 70, 745-747, 1980.

Yeh, Kung Chie, and Chao-Han Liu, Radio wave scintillations in the ionosphere, Proc. IEEE, 70, 324-360, 1982.

Zavorotnyi, V.U., V.I. Klyatskin, and V.I. Tatarskii Strong fluctuations of the intensity of electromagnetic waves in randomly inhomogeneous media, Zh. Eksp. Teor. Fiz., 73, 481-497, 1977. English, Soviet Physics JETP, 46, 252-260, 1977.

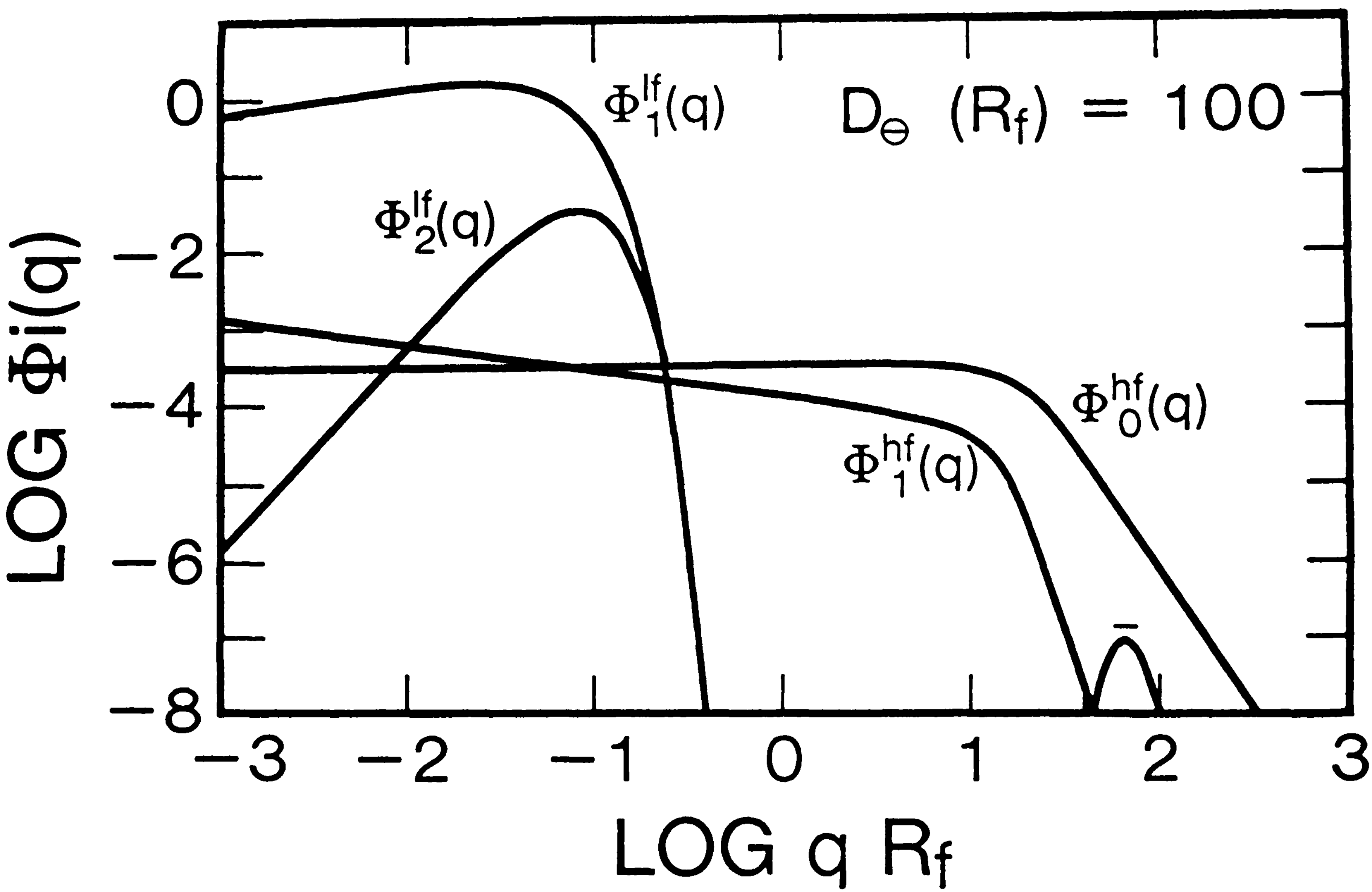
Zavorotnyi, V.U., Strong fluctuations of electromagnetic waves in a random medium with finite longitudinal correlation of the inhomogeneities, Zh. Eksp. Teor. Fiz., 75, 56-65, 1978. English, Soviet Physics JETP, 48, 27-31, 1978.

Zavorotnyi, V.U., Four-point coherence function of the field behind a phase screen for strong fluctuations of wave intensity, Izv. Vysh. Ucheb. Zaved. Radfiz., 22, 462-469, 1979a . English, Radiophysics and Quantum Electronics, 22, 318-323, 1979a.

Zavorotnyi, V.U., Four-point function for field-coherence in a turbulent medium in a region of strong fluctuations of wave intensity, Izv. Vysh. Ucheb. Zaved. Radfiz., 22, 979-988, 1979b. English, Radiophysics and Quantum Electronics, 22, 677-683, 1979b.

Figure Captions

Figure 1. The leading terms of the intensity spectrum versus normalized spatial frequency, qR_f, where $R_f=(R/k)^{\frac{1}{2}}$ is the Fresnel scale. The curves are calculated from expressions given by Gochelashvily and Shishov [1975] for the case of plane waves incident on a random phase screen with a Kolmogorov spectrum of phase fluctuations and $D_\theta(R_f)=100$. The (-) sign indicates that $\Phi_1^{NJ}(q)$ is negative at high frequency.



LOG Φi(q)
0
−2
−4
−6
−8
−3
−2
−1
0
1
2
3
LOG q Rf
Φ1lf(q)
Φ2lf(q)
D⊖ (Rf) = 100
Φ0hf(q)
Φ1hf(q)

APPENDIX D

Two-Frequency Intensity Cross-Spectrum

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,

R. G. Frehlich, and Frank S. Henyey

LJI-P-85-321

Two-Frequency Intensity Cross-Spectrum

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,*

R.G. Frehlich, and Frank S. Henyey

Center For Studies of Nonlinear Dynamics†

La Jolla Institute

3252 Holiday Court, Suite 208

La Jolla, California 92037

Abstract

The intensity cross-spectrum (spatial Fourier transform of the two-frequency intensity correlation) for scintillations caused by a plane wave passing through a random phase screen is considered. Two series solutions (one valid for low and the other for high spatial frequencies) are obtained which are the generalizations of previous results for the monochromatic intensity spectrum. We show that the Gaussian-field approximation (modelling the cross-spectrum as the transform of the square of the second moment) breaks down when the outer scale is large compared with the diameter of the scattering disk.

*On leave from the University of California, Santa Cruz, CA 95064

†Affiliated with the University of California, San Diego.

1. Introduction

Wave propagation through random media gives rise to intensity fluctuations that are wavelength dependent. Examples of this phenomenon are chromatic stellar scintillation [Jakeman et al, 1978], pulsar scintillation [Rickett,1969], interplanetary scintillation of compact radio sources [Cole & Slee,1980], and multifrequency laser propagation [Gurvich et al, 1979; Azar et al, 1985]. In weak scattering, the intensity fluctuations are correlated over a wide range of frequencies. However, in strong scattering, the intensity fluctuations are decorrelated after a relatively small change in frequency [Popov & Soglasnov, 1985; Cordes et al, 1985]. Further complexity arises when the intensity correlation between spatially separated points are considered, and different wavelengths at these separated points are allowed [Azar et al, 1985]. Finite-bandwidth, finite-aperture receivers are examples of such cases. These problems may be analyzed by studying the two-frequency, two-point, intensity cross-correlation, or its Fourier transform: the two-frequency intensity cross-spectrum.

A common approach for dealing with *intensity* correlations in strong scattering conditions is to argue that the real and imaginary parts of the field are the sum of many *independent* contributions and therefore become zero-mean Gaussian random processes. We call this the Gaussian field (GF) approximation. The GF approximation implies that the correlation of intensity fluctuations is equal to the magnitude squared of an appropriate second moment of the field. In particular, the two-frequency two-point intensity cross-correlation would be the magnitude squared of the two-frequency second moment, which contains the factor $\exp[-\frac{1}{2}(\Delta\sigma/\sigma)^2\Phi^2]$, where σ is the center frequency, $\Delta\sigma$ is the frequency difference, and Φ is the rms phase shift[Ginzburg & Erukhimov, 1971]. For many practical experiments Φ is large, making the predicted intensity decorrelation bandwidth extremely small, and inconsistent with experiment [Gurvich et al, 1979; Flatté, 1983]. Theoretical approaches in the 1970's gave various reasons for neglecting the rms-phase-shift factor completely [Shishov, 1973; Lee, 1976; Dashen, 1979] Dashen's argument, which applies to both the thin-screen and extended-medium cases, hinges on the relative size of the outer scale (largest sized medium fluctuations) to the diameter of the scattering disk. He argues that if the diameter of the scattering disk is much smaller than the outer scale, then, for small enough frequency differences, the phase-shift factor can be ignored.

In this paper, we analyze intensity decorrelations, in frequency and transverse spatial separation, of a plane wave propagating through a random phase screen. A series representation of the two-frequency intensity cross-spectrum (spatial Fourier transform of two-frequency intensity correlation) is derived. An approximation to the cross-spectrum is presented that is valid for all values of the outer scale. These results extend the thin-screen results of Dashen to arbitrary frequency differences and provide a description of the transition from a small (relative to the outer scale) to a large scattering disk. The rms-phase-shift factor is replaced by a new, more accurate, factor, and the conditions under which this factor may be omitted entirely are detailed. In Section 2 we introduce the second and fourth moments of the field and the two-point, two-frequency (intensity) correlation function. In Section 3, we consider the Fourier transform of this correlation function, the two-frequency intensity cross-spectrum. We derive two different series representations for this cross-spectrum. The first series describes the low-spatial-frequency behaviour (i.e. it converges quickly in this region). The other series describes the high-spatial-frequency behaviour. These series extend (for the thin screen) the results for the monochromatic intensity correlation [Codona et. al., 1985] to two frequencies. In section 4 we discuss the relationship of our approximation to the GF approximation. It is shown that the GF approximation (with the mean-square phase shift factor) is valid only over a negligible portion of the intensity cross-spectrum. Finally, in Section 5 we summarize our approximations to the two-frequency intensity cross-spectrum.

2. Definitions and Notation

We consider plane waves, normally incident on a phase screen, that freely propagate a distance R beyond the screen to an observation plane. The refractive index fluctuations, $\mu(\vec{x})$, in the screen induce a random phase change, $\Theta(\vec{x})=k\int\mu(\vec{x},z)dz$, as the field passes through the screen. Here z is the direction of the propagating wave, $\vec{x}$ is the co-ordinate transverse to this direction, and k is the wavenumber. The phase change Θ is assumed to be a zero-mean Gaussian random variable with homogeneous statistics. We consider wave propagation which is characterized by narrow angular scattering by the phase screen due to the small fluctuations of the refractive index. The complex scalar wave field can be expressed as $E(\vec{x},z;k)e^{ikz}$, where the field E has the value on emerging from the screen

$$E(\vec{x},0+;k) = E(\vec{x},0;k)\, e^{i\Theta(\vec{x})} \tag{1}$$

expressed in terms of the incident field $E(\vec{x},0;k)$. In the space after the screen the field satisfies the parabolic equation

$$\frac{\partial E}{\partial z} = \frac{i}{2k} \frac{\partial^2 E}{\partial \vec{x}^2} \tag{2}$$

For a plane wave, the incident field is a constant, which we set to unity. The solution of the parabolic equation at the observation plane with the proper initial condition is then

$$E(\vec{x},R;k) = \frac{k}{2i\pi R} \int_{-\infty}^{\infty} e^{i\Theta(\vec{x}')} \exp\left(\frac{ik(\vec{x}-\vec{x}')^2}{2R}\right) d^2x' \tag{3}$$

Averages of the fields are performed by the use of the identity

$$< e^{i\Theta} > = e^{-\frac{1}{2} < \Theta^2 >} \tag{4}$$

valid for any zero-mean Gaussian random variable. This, for example, gives the average of the field (first moment) as

$$< E(\vec{x},R;k) > = \exp\left[-\frac{1}{2}\Phi^2\right] \tag{5}$$

where Φ^2 is the mean square phase shift.

The random nature of the fields is conveniently described by statistical moments of the field evaluated in the observation plane. The moments of particular concern for the study of frequency decorrelation are the second and fourth moments. The general second moment is given by

$$\Gamma_2(\vec{x}_1,\vec{x}_2,R;k_1,k_2) \equiv < E(\vec{x}_1,R;k_1) E^*(\vec{x}_2,R;k_2) > \tag{6}$$

$$= \frac{k_1 k_2}{(2\pi R)^2} \iint_{-\infty}^{\infty} \exp\left[-(\frac{\delta k}{\bar{k}})^2 \Phi^2 - k_1 k_2 D_\mu(\vec{x}_1{}' - \vec{x}_2{}')\right] \exp\left[\frac{i}{2R}[k_1(\vec{x}_1 - \vec{x}_1{}')^2 - k_2(\vec{x}_2 - \vec{x}_2{}')^2]\right] d^2x_1{}' d^2x_2{}'$$

where the fields are given by (3). The identity (4) is used to obtain

$$e^{-\frac{1}{2} < |\int (k_1 \mu(\vec{x}_1,z') - k_2 \mu(\vec{x}_2,z'))dz'|^2 >} = e^{-\frac{1}{2}(\frac{\delta k}{\bar{k}})^2 \Phi^2 + k_1 k_2 D_\mu(\vec{x}_1 - \vec{x}_2)} \tag{7}$$

with the wavenumbers expressed in terms of sum and difference variables

$$\bar{k} = \frac{k_1 + k_2}{2} \quad , \quad \delta k = k_1 - k_2 \tag{8}$$

The second moment is expressed in terms of Φ^2, which is the mean-square phase shift for a wave with the mean wave number, $\bar{k}$, and the phase structure function, $k_1 k_2 D_\mu$, associated with a wavenumber which is the geometric mean. The wavenumber dependence of the phase structure function is explicily displayed by expressing it in terms of a structure function for the integrated index of refraction fluctuations, D_μ,

$$D_\mu(\vec{x}_1 - \vec{x}_2) \equiv < [\int (\mu(\vec{x}_1, z') - \mu(\vec{x}_2, z'))\, dz']^2 > \tag{9}$$

It is convenient to express the second moment in terms of sum and difference variables for the co-ordinates

$$\vec{\alpha} = \frac{\vec{x}_1 + \vec{x}_2}{2} \quad , \quad \vec{\beta} = \vec{x}_1 - \vec{x}_2 \tag{10}$$

with similar expressions for the (primed) co-ordinates on the screen. Then the second moment is given by

$$\Gamma_2(\vec{\alpha}, \vec{\beta}, R; k_1, k_2) = \frac{k_1 k_2}{(2\pi R)^2} \exp\left[-\left(\frac{\delta k}{\bar{k}}\right)^2 \frac{\Phi^2}{2}\right] \iint_{-\infty}^{\infty} e^{-\frac{1}{2} k_1 k_2 D_\mu(\vec{\beta}')}$$

$$\exp\left[i\frac{\delta k}{2R}[(\vec{\alpha} - \vec{\alpha}')^2 + (\vec{\beta} - \vec{\beta}')^2] + i\frac{\bar{k}}{R}(\vec{\alpha} - \vec{\alpha}') \cdot (\vec{\beta} - \vec{\beta}')\right] d^2\alpha' d^2\beta' \tag{11}$$

The α' integration is immediate since the phase Γ_2 is independent of $\vec{\alpha}$, which follows from the translation invariance of the problem for an incident plane wave. Performing the α' integration we secure for the second moment

$$\Gamma_2(\vec{\beta}, R; k_1, k_2) = \frac{k_1 k_2}{2\pi \delta k R} \exp[-\left(\frac{\delta k}{\bar{k}}\right)^2 \frac{\Phi^2}{2}] \int_{-\infty}^{\infty} e^{-\frac{1}{2} k_1 k_2 D_\mu(\vec{\beta}')} \exp\left[\frac{i k_1 k_2}{2R\, \delta k} (\vec{\beta} - \vec{\beta}')^2\right] d^2\beta' \tag{12}$$

The general fourth moment involves the field at four different spatial points. From the solution for the field, (3), this moment also involves the integration over four points on the screen. It is convenient to transform the co-ordinates of the screen to

$$\begin{bmatrix} \vec{\alpha}' \\ \vec{\beta}' \\ \vec{\gamma}' \\ \vec{\delta}' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{x}_1' \\ \vec{x}_2' \\ \vec{x}_3' \\ \vec{x}_4' \end{bmatrix} \tag{13}$$

A similar transformation exists for the co-ordinates of the observation plane. For the two-frequency, two-point intensity correlation function, which is a special case of the fourth moment,

$$C_I(\vec{x}_1,\vec{x}_2,R;k_1,k_2) \equiv \langle I(\vec{x}_1,R;k_1)I(\vec{x}_2,R;k_2)\rangle \tag{14}$$

$$= \langle E(\vec{x}_1,R;k_1)E^*(\vec{x}_1,R;k_1)E(\vec{x}_2,R;k_2)E^*(\vec{x}_2,R;k_2)\rangle$$

two of the transformed quantities on the observation plane, γ and δ, are zero. Using the solutions (3), the above transformation, and (8), we find that the integration over the centroid, α', gives

$$C_I(\vec{\beta},R;k_1,k_2) = \left(\frac{k_1k_2}{2\pi\bar{k}R}\right)^2 \int\!\!\int_{-\infty}^{\infty} e^{-\frac{1}{2}V_4} \exp\left[-i\frac{\bar{k}}{R}[1-(\frac{\delta k}{2\bar{k}})^2]\vec{\gamma}'\cdot(\vec{\beta}-\vec{\beta}')\right] d^2\beta' d^2\gamma' \tag{15}$$

where $\vec{\beta}=\vec{x}_1-\vec{x}_2$ and

$$V_4 = k_1^2\, D_\mu((1-\frac{\delta k}{2\bar{k}})\vec{\gamma}') + k_2^2\, D_\mu((1+\frac{\delta k}{2\bar{k}})\vec{\gamma}') \tag{16}$$

$$-k_1k_2[D_\mu(\vec{\beta}'+\vec{\gamma}')+D_\mu(\vec{\beta}'-\vec{\gamma}')-D_\mu(\vec{\beta}'+\frac{\delta k}{2\bar{k}}\vec{\gamma}')-D_\mu(\vec{\beta}'-\frac{\delta k}{2\bar{k}}\vec{\gamma}')]$$

3. Intensity Cross-Spectrum

The intensity cross spectrum is given by the $\vec{\beta}$ Fourier transform of (15), which produces the delta function, $\delta(\vec{\kappa}-\frac{\bar{k}}{R}\vec{\gamma}'(1-(\delta k/2\bar{k})^2))$, where $\vec{\kappa}$ is the transform variable. Integration over γ' yields

$$\Phi_I(\vec{\kappa},R;k_1,k_2) = \int_{-\infty}^{\infty} e^{i\vec{\kappa}\cdot\vec{\beta}'} e^{-\frac{1}{2}V_4} d^2\beta' \tag{17}$$

where

$$V_4 = k_1^2 D_\mu(R_s s_0\vec{\kappa}(1+\frac{\delta k}{2\bar{k}})) + k_2^2 D_\mu(R_s s_0\vec{\kappa}(1-\frac{\delta k}{2\bar{k}})) \tag{18}$$

$$-k_1k_2[D_\mu(\vec{\beta}'+R_s s_0\vec{\kappa}) + D_\mu(\vec{\beta}'-R_s s_0\vec{\kappa}) - D_\mu(\vec{\beta}'+\frac{\delta k}{2\bar{k}}R_s s_0\vec{\kappa}) - D_\mu(\vec{\beta}'-\frac{\delta k}{2\bar{k}}R_s s_0\vec{\kappa})]$$

We have introduced two physical scales of the medium which are defined at a wavenumber

$$k_0 = \frac{k_1 k_2}{\bar{k}} \tag{19}$$

These scales are the coherence length of the field, s_0,

$$k_0^2 D_\mu(s_0) = 1 \tag{20}$$

and the size of the scattering disk, R_s,

$$R_s = \frac{R}{k_0 s_0} \tag{21}$$

The coherence length is important in that the cross-spectrum is negligible for spatial frequencies $\kappa > s_0^{-1}$, so that the combination $s_0 \kappa$ is always less than unity. The scattering disk size, R_s^{-1}, sets the scale that separates high and low spatial frequencies. Notice that the combination

$$R_F^2 \equiv R_s s_0 \tag{22}$$

appears in (18). R_F is known conventionally as the Fresnel radius; it determines the intensity spatial scale in weak scattering. Unfortunately, the final integral in (17) cannot be done exactly. Our approach is to find two separate approximations for high and low spatial frequency. First we consider low spatial frequencies.

When $\kappa \ll 1/R_s$, the $\vec{\beta}'$ Fourier transform has substantial contributions from everywhere in the integration plane, and over most of this region, $\beta' \gg R_s s_0 \kappa$. It can be seen from (18) that V_4 will be dominated by the first two terms on the right hand side of that equation since the β' dependent terms nearly cancel. In this circumstance, the dominant behaviour is $\vec{\beta}'$ independent. Since the $\vec{\beta}'$ dependent terms in V_4 are small, the exponential may be expanded in a Taylor series

$$e^{-\frac{1}{2}V_4} = \exp\left[-\frac{1}{2}\left[k_1^2 D_\mu(\vec{\gamma}'+\vec{\delta}') + k_2^2 D_\mu(\vec{\gamma}'-\vec{\delta}')\right]\right] \tag{23}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}(k_1 k_2)^n \left[D_\mu(\vec{\beta}'+\vec{\gamma}')+D_\mu(\vec{\beta}'-\vec{\gamma}')-D_\mu(\vec{\beta}'+\vec{\delta}')-D_\mu(\vec{\beta}'-\vec{\delta}')\right]^n$$

where $\vec{\gamma}' = R_s s_0 \vec{\kappa}$ and $\vec{\delta}' = (\delta k/2\bar{k}) R_s s_0 \vec{\kappa}$. Inserting the above expansion into (17) yields explicit expressions for the terms in the low-spatial-frequency series,

$$\Phi_I(\vec{\kappa}, R; k_1, k_2) = \sum_{n=0}^{\infty} \Phi_{I,0}^{(n)}(\vec{\kappa}, R; k_1, k_2) \tag{24}$$

The structure function D_μ can be expressed in terms of a spectrum by

$$D_\mu(\vec{s}) = 2\int_{-\infty}^{\infty} \Phi_\mu(\vec{\kappa}')\left[1-e^{i\vec{\kappa}'\cdot\vec{s}}\right]d^2\kappa' \tag{25}$$

with Φ_μ as the power spectrum of a quantity which is the refractive index fluctuations integrated through the screen. The two leading terms in the low-spatial-frequency series are

$$\Phi_{I,W}^{(0)}(\vec{\kappa},R;k_1,k_2) = \delta(\vec{\kappa}) \tag{26}$$

and

$$\Phi_{I,W}^{(1)}(\vec{\kappa},R;k_1,k_2) = \frac{4k_1k_2}{\bar{k}^2}\exp\left[-\frac{1}{2}\left[k_1^2\, D_\mu((1+\frac{\delta k}{2\bar{k}})R_e\, s_0\vec{\kappa})+k_2^2\, D_\mu((1-\frac{\delta k}{2\bar{k}})R_e\, s_0\vec{\kappa})\right]\right] \tag{27}$$

$$\Phi_\mu(\vec{\kappa})\left[\sin^2(R_F^2\kappa^2/2) - \sin^2(\frac{\delta k}{\bar{k}}\, R_F^2\kappa^2/4)\right]$$

The correction (27) modifies the behaviour of the leading order expression, (26), for $0<\kappa \ll 1/R_s$. Notice that the Fresnel radius squared, $R_F^2 \equiv R_e s_0$ appears in the $\sin^2$ terms, revealing these terms as the usual Fresnel filter. For weak scattering ($R_s < R_F$) the Fresnel filter cuts off the spectrum at large spatial frequencies. However, in strong scattering, the exponential term in (27) provides the cutoff.

We now turn our attention to high spatial frequencies. It is convenient to separate V_4 into $V_4 \equiv V_4^0 + V_4^{R0} + V_4^{R1}$, with

$$V_4^0(\vec{\beta}',\vec{\delta}') \equiv k_1 k_2[D_\mu(\vec{\beta}'+\vec{\delta}') + D_\mu(\vec{\beta}'-\vec{\delta}')] \tag{28a}$$

$$V_4^{R0}(\vec{\gamma}',\vec{\delta}') = k_1^2\, D_\mu(\vec{\gamma}'+\vec{\delta}')+k_2^2\, D_\mu(\vec{\gamma}'-\vec{\delta}')-2k_1k_2D_\mu(\vec{\gamma}') \tag{28b}$$

and

$$V_4^{R1}(\vec{\beta}',\vec{\gamma}') = k_1k_2[2D_\mu(\vec{\gamma}')-D_\mu(\vec{\beta}'+\vec{\gamma}')-D_\mu(\vec{\beta}'-\vec{\gamma}')] \tag{28c}$$

where $\vec{\gamma}' = R_e s_0\vec{\kappa}$ and $\vec{\delta}' = (\delta k/2\bar{k})R_e\, s_0\vec{\kappa}$. Notice that V_4^{R0} is independent of β'. The exact expression for the cross-spectrum is then

$$\Phi_I(\vec{\kappa},R;k_1,k_2) = e^{-\frac{1}{2}V_4^{R0}}\int_{-\infty}^{\infty} e^{i\vec{\kappa}\cdot\vec{\beta}'}e^{-\frac{1}{2}[V_4^0+V_4^{R1}]}d^2\beta' \tag{29}$$

In the monochromatic case ($\delta k=0,\vec{\delta}'=0$) at high spatial frequencies and strong scattering, the terms corresponding to V_4^0 dominate the integral for the spectrum. [Gochelashvily & Shishov, 1975; Rumsey, 1975]. Rumsey showed that, over the important region of integration, the β' dependence of V_4^{R1} is negligible compared to that of V_4^0, giving the intensity spectrum as the Fourier transform of $\exp[-k_1k_2D_\mu(\vec{\beta}')]$. This is because the quantity $s_0\kappa$ is of the order of unity, so that $\gamma'\approx R_s$, while the important region of integration is $\beta'\approx s_0 \ll R_s$. When δk is not zero (but small), the high-spatial-frequency behaviour of the cross-spectrum is still controlled by V_4^0 for the same reasons. Note that these terms do *not* dominate the behaviour of V_4 for all $\vec{\beta}'$ and $\vec{\kappa}$, but only for $\beta' \ll R_s s_0\kappa$. Unlike the monochromatic case, when β' is zero V_4 is non-zero. However V_4 still increases with increasing β', so that the Fourier transform integrand, $e^{-V_4/2}$, is largest when β' is small. The approximate domain of $\vec{\beta}'$, in which the Fourier integrand is large, is where $\beta' \leq s_0$. In this regime the V_4^R's are roughly independent of β', so that the integrand is dominated by V_4^0. The limiting value, $\beta'\approx s_0$, leads to the requirement

$$\kappa \gg R_s^{-1} \tag{30}$$

wherein we expect the β' behaviour of $e^{-V_4/2}$ to be controlled by the $D_\mu(\vec{\beta}'+\vec{\delta}')+D_\mu(\vec{\beta}'-\vec{\delta}')$ terms. This is verified explicitly in Appendix 1. Thus V_4^{R1} is small compared to V_4^0 allowing us to expand the final exponential in a Taylor series yielding a corresponding series for the intensity cross-spectrum

$$\Phi_I(\vec{\kappa},R;k_1,k_2) = \sum_{n=0}^{\infty} \Phi_{I,N}^{(n)}(\vec{\kappa},R;k_1,k_2) \tag{31}$$

The leading term is

$$\Phi_{I,N}^{(0)}(\vec{\kappa},R;k_1,k_2) = e^{-V_4^{R0}} \int_{-\infty}^{\infty} e^{i\vec{\kappa}\cdot\vec{\beta}'} \exp\left[-\frac{k_1k_2}{2}\left[D_\mu(\vec{\beta}'+\frac{\delta k}{2\bar{k}}R_s s_0\vec{\kappa})+D_\mu(\vec{\beta}'-\frac{\delta k}{2\bar{k}}R_s s_0\vec{\kappa})\right]\right]d^2\beta' \tag{32}$$

For the monochromatic case, the corresponding term is commonly called the high-spatial-frequency approximation.

Writing the structure function in terms of the spectrum as in (25), the second term in the series expansion of $\exp[-\frac{1}{2}V_4^{R1}]$ is

$$-\frac{1}{2}V_4^{R1} = 4\frac{k_1k_2}{\bar{k}^2}\int_{-\infty}^{\infty}\Phi_\mu(\vec{\kappa}')e^{i\vec{\kappa}'\cdot R_s s\vec{\sigma}}\sin^2\left[\frac{\vec{\kappa}'\cdot\vec{\beta}'}{2}\right]d^2\kappa' \tag{33}$$

This yields the first correction to the leading approximation, (32),

$$\Phi_{I}^{(1)}(\vec{\kappa},R;k_1,k_2) = 4\frac{k_1k_2}{\bar{k}^2}\; e^{-V_4^{R0}} \int\!\!\int_{-\infty}^{\infty} \Phi_\mu(\vec{\kappa}\,')\, e^{i\vec{\kappa}\cdot(\vec{\beta}\,'+R_e s_0\vec{\kappa}\,')} \tag{34}$$

$$\exp\left[-\frac{k_1k_2}{2}\left[D_\mu(\vec{\beta}\,'+\frac{\delta k}{2\bar{k}}R_e\, s_0\vec{\kappa})+D_\mu(\vec{\beta}\,'-\frac{\delta k}{2\bar{k}}R_e\, s_0\vec{\kappa})\right]\right]\sin^2\left[\frac{\vec{\kappa}\,'\cdot\vec{\beta}\,'}{2}\right]d^2\beta' d^2\kappa'$$

where V_4^{R0} is given in (28b). This correction term is important in determining the convergence properties of the series (31) for high spatial frequencies.

4. Comparison to the Gaussian-Field Approximation

In strong scattering, the coherence length of the field, s_0, is much smaller than the size of the scattering disk. This implies that the field at the observation point is the sum of very many contributions from the scattering disk. If these contributions were independent, then by the central limit theorem the field would obey Gaussian statistics. If the field is a zero-mean Gaussian random process then the intensity correlation function is

$$C_{GF}(\vec{x}_1,\vec{x}_2,R;k_1,k_2) = \langle I(\vec{x}_1,R;k_1)\rangle\langle I(\vec{x}_2,R;k_2)\rangle + |\Gamma_2(\vec{x}_1,\vec{x}_2,R;k_1,k_2)|^2 \tag{35}$$

In the thin screen model and an incident plane wave, the mean intensities in the first term are independent of position and, by definition, are unity. Because of translation invariance the second term is only a function of $\vec{\beta}=\vec{x}_1-\vec{x}_2$. Inserting Γ_2 from (12) and transforming over β yields the GF approximation to the intensity cross-spectrum

$$\Phi_{GF}(\vec{\kappa},R;k_1,k_2) = \delta(\vec{\kappa}) \;+ \exp\left[-(\frac{\delta k}{\bar{k}})^2\Phi^2\right]\int_{-\infty}^{\infty} e^{-i\vec{\kappa}\cdot\vec{\beta}'}$$

$$\exp\left[-\frac{k_1k_2}{2}\left[D_\mu(\vec{\beta}\,'+\frac{\delta k}{2\bar{k}}R_e s_0\vec{\kappa}) + D_\mu(\vec{\beta}\,'-\frac{\delta k}{2\bar{k}}R_e s_0\vec{\kappa})\right]\right]d^2\beta' \tag{36}$$

We wish to compare this GF result with our previously derived approximations: (26) + (27) for small κ, (32) + (34) for large κ. We see that (36) resembles (26) + (32) except that the factor $\exp[-V_4^{R0}]$ has been replaced by the phase-shift factor $\exp[-(\delta k/2\bar{k})^2\Phi^2]$. However there is a problem with the low-spatial-frequency behaviour of the GF approximation. On quite general grounds it can be argued that, for low spatial frequencies, the cross-spectrum consists of a delta function piece plus a term that must vanish as the spatial wavenumber goes to zero [Tatarskii, 1971]. If (36) is applied for small κ, the delta function is correct but the second term goes to a non-zero constant for small κ. Compare this situation with our results (26) and (27). The result (26) correctly gives the delta function at the origin. Our correction (27) substantially modifies the spectrum for non-zero κ, but satisfies the requirement of vanishing as $\kappa \rightarrow 0$. For a power-law medium, such that $\Phi_\mu \approx \kappa^{-\alpha}$, with $\alpha < 4$, it is seen that (27) vanishes as $\kappa^{4-\alpha}$.

To evaluate the GF approximation at high spatial frequencies, we are interested in whether $\exp[-(\delta k/2\bar{k})^2 \Phi^2]$ is a good approximation to $\exp[-V_4^{R0}]$. Consider a pure power-law phase structure function

$$k_0^2 D_\mu(\vec{s}) = \left[\frac{s}{s_0}\right]^p \qquad s < l_{outer} \tag{37a}$$

$$k_0^2 D_\mu(\vec{s}) = \Phi^2 \qquad s > l_{outer} \tag{37b}$$

where s_0 is the coherence length of the field, and l_{outer} is the outer scale of the spectrum of phase irregularities. In most cases of practical interest, the outer scale is much larger than the size of the scattering disk.

Since the phase structure function saturates to Φ^2 for scales larger than the outer scale, (32) gives the same result as (36) only for spatial frequencies such that $\kappa > (l_{outer}/R_s)s_0^{-1}$. The only non-negligible portion of the spectrum is for $\kappa < 1/s_0$, leading to the conclusion that the GF approximation is only valid for a significant portion of the spectrum when the outer scale, l_{outer}, is small compared to the size of the scattering disk. However, in virtually every case of practical interest, the scattering disk is *small* relative to any estimates of an outer scale. In these cases, the GF approximation becomes valid only after the cross-spectrum has dropped to a negligible value.

5. Comparison to Neglecting the Phase-Shift Factor

As mentioned previously, various arguments have been given for using (36) for the cross-spectrum but neglecting the phase shift factor, $e^{-(\delta k/\bar{k})^2 \Phi^2}$:

$$\Phi_{GF}(\vec{\kappa},R;k_1,k_2) \approx \int_{-\infty}^{\infty} e^{i\vec{\kappa}\cdot\vec{\beta}'} \exp\left[-\frac{k_1 k_2}{2}[D_\mu(\vec{\beta}' + \frac{\delta k}{2\bar{k}} R_s s_0 \vec{\kappa}) + D_\mu(\vec{\beta}' - \frac{\delta k}{2\bar{k}} R_s s_0 \vec{\kappa})]\right] d^2\beta' \tag{38}$$

Dashen [1979] considered this problem theoretically for both a phase screen and for an extended random medium and presented results valid for infinitesimal frequency differences. There are two different regimes in strong scattering: "partial" and "full" saturation. When the size of the scattering disk is larger than the outer scale of medium fluctuations (full saturation) the contributions are independent and the GF approximation is valid to first order. However, when the scattering disk is smaller than the outer scale (partial saturation), fluctuations with a size larger than that of the disk contribute a random, *coherent* phase to the field so the many contributions are not independent. Since this coherency only affects the rms phase, this lack of independence will not affect any monochromatic intensity statistic but will be important for multifrequency statistics. For a medium characterized by a fluctuation spectrum with a spectral index less than four, the small-scale fluctuations can cause large (saturated) intensity fluctuations, but the rms phase is dominated by the large-scale fluctuations. For small enough frequency differences, this phase cannot affect the intensity, providing a first-order justification for dropping the exponential factor containing the rms phase. Further experimental confirmation of this result was provided from ocean-acoustic data [Flatté, 1983].

Our more accurate expression (32) implies that neglecting the phase-shift factor is valid over the portion of the spectrum for which $V_4^{RO} \ll 1$. Using the largest significant κ value (s_0^{-1}) in (28b) leads to the requirement

$$k_1^2 D_\mu(R_s(1+\frac{\delta k}{2\bar{k}}))+k_2^2 D_\mu(R_s(1-\frac{\delta k}{2\bar{k}}))-2k_1k_2D_\mu(R_s) \ll 1 \tag{39}$$

For a power law structure function this expression can be simplified by performing a small δk expansion (valid for $\delta k < 2\bar{k}$) leading to

$$(\frac{\delta k}{2\bar{k}})^2 U \ll 1 \tag{40}$$

where U is defined by

$$U \equiv k_0^2 D_\mu(R_s)\left[\frac{R_s}{s_0}\right]^p \tag{41}$$

This requirement implies that, as the scattering becomes stronger, the region (in δk) for which (38) is valid becomes smaller. This is not a severe restriction because the frequency decorrelation bandwidth decreases as the scattering becomes stronger, limiting the values of δk which enter (40). We set this limiting value to be the decorrelation bandwidth, which we estimate by looking at the asymptotic (large κ) form of (38).

For large κ, there are two important regions of integration in (38): near $\beta = 0$, where the integrand is largest, and near $\vec{\beta} = \pm(\delta k/2\bar{k})R_e s_0 \kappa$, where one of the structure functions vanish. In Appendix 2, it is shown that the second region dominates the asymptotic behaviour, leading to

$$\Phi_I \approx \exp-\left[\frac{k_1 k_2}{2} D_\mu(\frac{\delta k}{\bar{k}} R_e s_0 \vec{\kappa})\right] \cos\left[\frac{\delta k}{2\bar{k}} R_F^2 \kappa^2\right] \int_{-\infty}^{\infty} e^{i\vec{\kappa}\cdot\vec{\beta}'} e^{-\frac{k_1 k_2}{2} D_\mu(\vec{\beta}')} d^2\beta' \tag{42}$$

The integral term in (41) is just the angular spectrum and its important scale is s_0. The cosine term oscillates rapidly for $\kappa \gg \kappa_b = \sqrt{\delta k/2\bar{k}}\,/R_F$. The frequency decorrelation bandwidth can be estimated as that δk for which $\kappa_b s_0 \approx 1$. Using (41) and (21) this leads to

$$\delta k_b \approx \pi \bar{k}\, U^{-1/p} \tag{43}$$

Inserting this bandwidth in (40) yields the criterion for neglecting the phase-shift factor

$$U^{1-2/p} \ll 1 \tag{44}$$

Thus, for $p < 2$, the approximation of neglecting the mean-square phase shift improves as the strength of scattering increases.

It has been shown that the contribution to the scintillation index from the high-spatial-frequency correction term is proportional to $U^{1-2/p}$ [Prokhorov et al, 1975]. Thus if the correction term to the scintillation index is small, it is appropriate to ignore the phase-shift factor over the main portion of the spectrum. On the other hand, if correction terms, such as (34), are not neglected, then it is inappropriate to neglect the phase-shift factor, since it generates corrections of the same order-of-magnitude.

6. Summary

In the previous sections we derived two different series for the intensity cross-spectrum: one describes the low-frequency behaviour while the other describes the high-frequency behaviour. Keeping the first two terms in each series gives expressions which approximate the cross-spectrum in the two respective regions. The relevant formula are given in (26),(27),(32), and (34). The phase-shift factor in (32) containing V_4^{R0} correctly describes the effects of any coherent contributions from the scattering disk for all values of the outer scale and for arbitrary frequency differences.

The GF approximation should *not* be used when the outer scale is much larger than the diameter of the scattering disk, and the reduced approximation, (40), should only be used when $U^{1-2/p} \ll 1$. In very strong scattering (large U), when the scintillation index has nearly saturated to unity, (40) provides a good approximation over the main portion of the spectrum.

Acknowledgements: This research was funded by DARPA.

References

Azar, Z., H.M. Loebenstein, G. Appelbaum, E. Azoulay, U. Halavee, M. Tamir, and M. Tur, Aperture averaging of the two-wavelength intensity covariance funtion in atmospheric turbulence, *Applied Optics*, **24**, 2401-2407, 1985.

Codona, J.L., D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F. Henyey, Solution for the fourth moment of waves propagating in random media, submitted for publication, 1985.

Cole, T.W. and O.B. Slee, Spectra of interplanetary scintillation, *Nature*, **285**, 93-95, (1980).

Cordes, J.M., and J.M. Weisberg, and V. Boriakoff, Small-scale electron density turbulence in the interstellar medium, *Ap. J.*, **288**, 221-227, 1985.

Dashen, Roger, Path integrals for waves in random media, *J. Math. Phys.* **20**, 894-920, 1979.

Flatté, S.M., Wave propagation through random media: Contributions from ocean acoustics, *Proc. IEEE*, **71**, 1267-1294, 1983.

Ginzburg, V.L., and L.M. Erukhimov, On the influence of electronic concentration fluctuations of the dispersion measure and rotation measure, *Astrophys. Space Sci.*, **11**, 362-372, 1971.

Gochelashvily, K.S., and V.I. Shishov, Saturation of laser irradiance fluctuations beyond a turbulent layer, *Optical and Quantum Electronics*, **7**, 524-536, 1975.

Gurvich, A.S., and V. Kan, Fluctuation of intensity at two wavelengths in a turbulent medium, *Izv. Vysh. Ucheb. Zaved. Radiofiz.*, **22**, 843-847, 1979. English, *Radiophysics and Quantum Electronics*, **22**, 585-587, 1979.

Gurvich, A.S., V. Kan, and Vl. V. Pokasov, Two-frequency fluctuations of light intensity in a turbulent medium, *Optica Acta*, **26**, 555-562, 1979.

Jakeman, E., G. Parry, E.R. Pike, and P.N. Pusey, The twinkling of stars, *Contemp. Phys.*, **19**, 127-145, 1978.

Lee, L. C., Strong scintillations in astrophysics. IV. Cross-correlation between different frequencies and finite bandwidth effects, *Ap. J.* **206**, 744-752, 1976.

Popov, M.V. and V.A. Soglasnov, Observations of interstellar scattering of radio emission of pulsar PSR 0329+54 at 102.5 MHz, *Astron. Zh.*, **61**, 727-732, 1984. English, *Sov. Astron.*, **28**, 424-427, 1984.

Prokhorov, A.M., F.V. Bunkin, K.S. Gochelashvily, and V.I. Shishov, Laser irradiance propagation in turbulent media, *Proc. IEEE*, **63**, 790-811, 1975.

Rickett, B.J. Frequency structure of pulsar intensity variations, *Nature*, **221**, 158-159, 1969.

Rumsey, V.H., Scintillations due to a concentrated layer with a power-law turbulence spectrum, *Radio Science*, **14**, 107-114, 1975.

Shishov, V. I., Frequency correlation of scintillations, *Izv. Vysh. Ucheb. Zaved. Radiofiz.* **16**, 423-433 (1973). English, *Radiophysics and Quantum Electronics* **16**, 319-327, 1973.

Tatarski, V. I., The Effects of the Turbulent Atmosphere on Wave Propagation, National Technical Information Service, Springfield, Va., 1971.

Appendix 1

We show that $\vec{\beta'}$ dependence of V_4^{R1} is negligible compared to that of V_4^0 where, from (28),

$$V_4^0 = k_1 k_2 [D_\mu(\vec{\beta'}+\vec{\delta'}) + D_\mu(\vec{\beta'}-\vec{\delta'})] \tag{A1a}$$

and

$$V_4^{R1} = k_1 k_2 [2D_\mu(\vec{\gamma'} - D_\mu(\vec{\beta'}+\vec{\gamma'}) - D_\mu(\vec{\beta'}-\vec{\gamma'})] \tag{A1b}$$

where $\vec{\gamma} = R_e s_0 \vec{\kappa}$ and $\vec{\delta} = (\delta k/2\bar{k}) R_e s_0 \vec{\kappa}$. We have argued in section 3 that the dominant region of integration is for small $\beta' \ll R_e s_0 \kappa$. In that case, the two terms in (A1b) may be expanded as

$$D_\mu(R_e s_0 \vec{\kappa} + \beta') + D_\mu(R_e s_0 \vec{\kappa} - \beta') = 2D_\mu(R_e s_0 \vec{\kappa}) + \left[\vec{\beta'} \cdot \frac{\partial}{\partial \vec{s}}\right]^2 D_\mu(\vec{s})|_{\vec{s}=R_e s_0 \vec{\kappa}} + \cdots \tag{A2}$$

The second term may be bounded by

$$\left[\left[\vec{\beta'} \cdot \frac{\partial}{\partial \vec{s}}\right]^2 D_\mu(\vec{s})|_{\vec{s}=R_e s_0 \vec{\kappa}}\right] \leq \beta'^2 |D_\mu''(R_e s_0 \kappa)| \tag{A3}$$

giving the leading $\vec{\beta'}$ dependence of V_4^{R1} as

$$V_4^R \approx k_1 k_2 \beta'^2 D_\mu''(R_e s_0 \kappa) \tag{A4}$$

In verifying that V_4^0 controls the $\vec{\beta'}$ dependence we will examine three cases

$$\begin{aligned} &Case\,1: \quad \frac{\delta k}{2\bar{k}} R_e s_0 \kappa \ll \beta' \\ &Case\,2: \quad \beta' \ll \frac{\delta k}{2\bar{k}} R_e s_0 \kappa \\ &Case\,3: \quad \frac{\delta k}{2\bar{k}} R_e s_0 \kappa \approx \beta' \end{aligned} \tag{A5}$$

In the first case, β is large compared to $\delta k R_e s_0 \kappa / 2\bar{k}$ giving the leading dependence of V_4^0 as

$$V_4^0 \approx 2k_1 k_2 D_\mu(\beta') \tag{A6}$$

so that we require

$$\beta'^2 D_\mu''(R_e s_0 \kappa) \ll D_\mu(\beta') \tag{A7}$$

Using a model power-law structure function, $k_e^2 D_\mu(\vec{s}) = (s/s_0)^p$, this requirement becomes

$$\left[\frac{\beta'}{s_0}\right]^{2-p} \left[R_e \kappa\right]^{p-2} \ll \frac{2}{p\,(p-1)} \tag{A8}$$

Since p is typically in the range $0 < p \leq 2$, we find an upper bound for the left-hand side by setting $\beta' = s_0$ and $\kappa = s_0^{-1}$ resulting in the requirement

$$\frac{p\,(p-1)}{2} \ll U^{(2-p)/p} \tag{A9}$$

where U is the strength of scattering parameter, $U \equiv k_0^2\, D_\mu(R_e)$, with k_0 defined in (19). Therefore, the V_4^0 term controls the $\vec{\beta}$ dependence when the strength of scattering parameter, U, is large. When p is 1 or 2, our argument breaks down. For $p=1$ the approximation still holds by another argument, while for $p=2$, there are no scintillations.

In the second case, β' is small compared to $\delta k R_e\, s_0 \kappa / 2\bar{k}$, so that a Taylor expansion of the V_4^0 term yields the leading β' dependence

$$V_4^0 \approx k_1 k_2 \beta'^2 D_\mu''\left(\frac{\delta k}{2\bar{k}} R_e\, s_0 \kappa\right) \tag{A10}$$

This leads to the requirement

$$D_\mu''(R_e\, s_0 \kappa) \ll D_\mu''\left(\frac{\delta k}{2\bar{k}} R_e\, s_0 \kappa\right) \tag{A11}$$

which for a power-law structure function is equivalent to

$$1 \ll \left[\frac{\delta k}{2\bar{k}}\right]^{p-2} \tag{A12}$$

Since p is less than 2, this implies that δk is small compared to twice the mean of the wavenumbers.

Finally, in the third case, β' is about the same size as $\delta k R_e\, s_0 \kappa / 2\bar{k}$ so that the requirement for neglecting the β' dependence of V_4^R becomes

$$\left[\frac{\delta k}{2\bar{k}} R_e\, s_0 \kappa\right]^2 D_\mu'' \ll D_\mu\left(\frac{\delta k}{\bar{k}} R_e\, s_0 \kappa\right) \tag{A13}$$

which, for a power-law structure function, leads to

$$(\delta k/\bar{k})^{2-p} \ll \frac{4}{p(p-1)} \tag{A14}$$

and is easily satisfied for small wavenumber differences.

In all three case considered above, the combination of structure functions in V_4^0 dominate the β' dependence if U is large and $\delta k/2\bar{k}$ is small.

Appendix 2

As mentioned in section 4, in (38) there are two important regions of integration for large κ. One is for small β where the integrand is largest. The other is where one of the structure functions vanish, $\vec{\beta}=\pm(\delta k/2\bar{k})R_{s s_0}\kappa$. Near $\beta=0$ the exponent in the integrand, V_4^0 may be approximated

$$V_4^0 \approx k_1 k_2 \left[2 D_\mu(\frac{\delta k}{2\bar{k}} R_{s s_0} \vec{\kappa}) + \left[\vec{\beta}' \cdot \frac{\partial}{\partial \vec{s}} \right]^2 D_\mu(\vec{s}) \Big|_{\vec{s}=(\delta k/2\bar{k})R_{s s_0}\vec{\kappa}} \right] \qquad \text{(B1)}$$

which leads to rapidly decaying component of the cross-spectrum. For the other region of integration we approximate the integrand of (37) in the vicinity of $\vec{\beta}'=(\delta k/2\bar{k})R_{s s_0}\vec{\kappa}$ to be dominated by the "fast" variation of the structure function near its zero and the "slow" variation of the other portion of the exponent. Including the $\vec{\beta}=0$ contribution gives the cross-spectrum asymptotically as

$$\Phi_I \approx e^{-\frac{k_1 k_2}{s} D_\mu(\frac{\delta k}{2\bar{k}} R_{s_0}\vec{\kappa})} \int_{-\infty}^{\infty} e^{-i\vec{\kappa}\cdot\vec{\beta}'} \exp\left[\left[\vec{\beta}' \cdot \frac{\partial}{\partial \vec{s}} \right]^2 D_\mu(\vec{s}) \Big|_{\vec{s}=(\delta k/2\bar{k})R_{s s_0}\vec{\kappa}} \right] \qquad \text{(B2)}$$

$$+2 e^{-\frac{k_1 k_2}{s} D_\mu(\frac{\delta k}{\bar{k}} R_{s_0}\vec{\kappa})} \cos\left[\frac{\delta k}{2\bar{k}} R_F^2 \kappa^2 \right] \int_{-\infty}^{\infty} e^{-i\vec{\kappa}\cdot\vec{\beta}'} e^{-\frac{k_1 k_2}{2} D_\mu(\vec{\beta})} d_2\beta$$

For a power-law structure function, the first term falls off like

$$C_1 \kappa^{2-p} e^{-C_2 \kappa^{4-p}} \qquad \text{(B3)}$$

where C_1 and C_2 are constants. The integral in the second term is just the angular spectrum, which for $p<2$, falls off like κ^{-p-2}. Since this is a much slower fall off than the $\beta=0$ contribution, the asymptotic behaviour of the cross-spectrum is dominated by the points, $\vec{\beta}=\pm(\delta k/2\bar{k})R_F^2\vec{\kappa}$.

APPENDIX E

Moment-Equation and Path-Integral Techniques for Wave Propagation in Random Media

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,
R. G. Frehlich, and Frank S. Henyey

LJI-P-84-294R1

Moment-Equation and Path-Integral Techniques for Wave Propagation in Random Media

by

Johanan L. Codona, Dennis B. Creamer, Stanley M. Flatté,*
R.G. Frehlich, and Frank S. Henyey

Center For Studies of Nonlinear Dynamics†
La Jolla Institute
3252 Holiday Court, Suite 208
La Jolla, California 92037

March 1985

*On leave from the University of California, Santa Cruz, CA 95064
†Affiliated with the University of California, San Diego.

7-11-'85

Abstract

Differential equations for all moments of the field of a wave propagating through a random medium are derived under the parabolic approximation and the Markov approximation, but including anisotropy in the random medium and a deterministic background refractive index. Mathematical equivalence is demonstrated between these moment equations and path-integral expressions for the moments obtained under the same approximations. A discussion of approximations that are weaker than Markov is given.

I. Introduction

Many problems in wave propagation through random media concern phenomena in which there is no significant backscatter, so that a parabolic approximation may be made to the wave equation.[1] In these cases a further approximation, called the Markov approximation,[2] leads to relatively tractable mathematical expressions for moments of the field that can be used for practical calculations. Two quite different formalisms have been used in this context: the moment-equation and path-integral techniques.

A path-integral expression for a general moment of the field of a wave propagating through an inhomogeneous, anisotropic medium in the presence of a deterministic background refractive index has been derived,[3] and the expression has been used for specific calculations.[4,5,6]

Moment equations in coordinate representation have been derived for homogeneous isotropic media in the absence of a deterministic background.[2] Treatments of inhomogeneity, anisotropy, and deterministic background by moment-equation techniques have heretofore been confined to special cases involving the first and second moments.[7,8]

We present here general moment equations in coordinate representation that account for inhomogeneity, anisotropy, and deterministic background, but require the Markov approximation. We derive these equations using the time-ordered-product method of Van Kampen,[9] which also provides a derivation of equations that are valid under conditions more general than the Markov approximation. The modified equations are more complicated than those that require the Markov approximation: a special case was previously derived by Besieris and Tappert.[10]

We also show that our new general moment equations derived under the Markov approximation are mathematically equivalent to the path-integral expressions for the moments that have been previously presented. Thus, the two popular formalisms, under the Markov approximation, are not different in content.

The plan of the paper is as follows: in Section II we establish notation, present our new moment equations, and present path-integral expressions for the moments in similar notation. In Section III we establish the mathematical equivalence between the two techniques. In Section IV we present the derivation of our moment equations, and, along the way, derive the modified equations. In Section V, for completeness, we rederive the path-integral expressions for the moments. In Section VI we comment on the use of different coordinate systems (such as cylindrical or spherical) in the writing of moment equations. A summary concludes the paper.

II. Notation and Markov-Approximation Results

Consider waves travelling predominantly in the z direction. Let $\vec{x}$ be a transverse coordinate (e.g. two-dimensional, but in fact general), and k be a reference wave number ($k = 2\pi\omega/C_0$), where ω is the wave frequency and C_0 is a reference wave speed). Express the full wave field as

$$u(\vec{x}, z, t) = \psi(\vec{x}, z) \exp\left[ik(z - C_0 t) \right] \tag{1}$$

Let the wave speed (a function of position only) be

$$C(\vec{x}, z) = C_0 \left[1 - 2\,U_0(\vec{x}) - 2\mu(\vec{x}, z) \right]^{-1/2} \approx C_0 \left[1 + U_0(\vec{x}) + \mu(\vec{x}, z) \right] \tag{2}$$

where U_0 represents the deterministic background and μ represents the fluctuating random medium, assumed to be a realization of a zero-mean Gaussian process.

Then, the parabolic equation (in rectangular coordinates) for the reduced wave function ψ is:

$$ik\partial_z \psi = -\frac{1}{2} \nabla^2 \psi + k^2 U_0(\vec{x})\psi + k^2 \mu(\vec{x}, z)\psi \tag{3}$$

where ∇^2 is the transverse Laplacian.

A moment Γ is the ensemble expectation value of a product of ψ's and ψ^*'s where each ψ or ψ^* is evaluated at a different position $\vec{x}_j$ and wavenumber k_j. We write, in abbreviated form,

$$\Gamma_{mn} = < \psi_1^* \cdots \psi_m^* \psi_{m+1} \cdots \psi_{m+n} > \tag{4}$$

Define an operator L_0 such that

$$L_0 = \sum_{j=1}^{m+n} \pm \frac{1}{k_j} \left(-\frac{1}{2} \nabla_j^2 + k_j^2 U_{0j} \right) \tag{5}$$

The terms that apply to the ψ's use the plus sign and those that apply to the ψ^*'s use the minus sign. The subscript j requires that ∇_j^2 operate only on $\vec{x}_j$ and $U_{0j} \equiv U_0(\vec{x}_j)$.

Define the important combination of fluctuation quantities as

$$M(z) = \sum_{j=1}^{m+n} \pm k_j \mu(\vec{x}_j, z) \tag{6}$$

Our general moment equation under the Markov approximation can be written

$$\partial_z \Gamma_{mn}(z) = -i L_0 \Gamma_{mn}(z) - \frac{1}{2} \int_{-\infty}^{\infty} dz' < M(z) M_{shift}(z') > \Gamma_{mn}(z) \qquad (7)$$

where $M_{shift}(z')$ is obtained by evaluating $M(z)$ with all the $\vec{x}_j$ at z shifted by the transverse distance that a deterministic ray through $(\vec{x}_j, z)$ moves in travelling from z to z' (see Figure 1). In other words $M_{shift}(z')$ is evaluated at point B: i.e. $\vec{x}_j = \vec{x}_{ray}(z')$ where the ray is forced to go through $\vec{x}_j(z)$. The particular ray is determined not only by the local position $(\vec{x}_j, z)$, but also by the initial conditions on the moment; for example, the location of a point source, or the direction of a plane wave.[11] The unphysical assumption of delta-correlated medium fluctuations along the propagation direction would imply that $M_{shift}(z')$ would be evaluated at point C: i.e. $\vec{x}_j(z)$ (and z'). In the isotropic case (or in the case of propagation along a principal axis of the anisotropy) the difference between evaluating $M_{shift}(z')$ at $\vec{x}_j(z)$ and $\vec{x}_{ray}(z')$ is negligible, and the delta-correlated assumption is adequate. In the anisotropic case, the necessity of defining the unperturbed ray makes (7) somewhat complicated to apply for general initial conditions. However, since (7) is a linear equation, superposition can be used whether the source is a point, an incident plane wave, or an arbitrary coherent or incoherent sum of point sources. Equation (7), which is one of the principal results of this paper, is derived in Section IV.

We now turn to the path integral method. Equation (3) has the formal solution

$$\psi = \int D\vec{x}(z) e^{iS} \qquad (8)$$

where $\int D\vec{x}(z)$ means integration over paths, $\vec{x}(z)$ is a transverse vector indicating the position of the path at z, and

$$S = k \int_0^R dz \left(\frac{1}{2} \left(\frac{d\vec{x}}{dz} \right)^2 - U_0(\vec{x}) - \mu(\vec{x}, z) \right) \qquad (9)$$

In order to obtain a given moment, expressions like (8) (or its complex conjugate) are multiplied together, and the ensemble average is taken:

$$\Gamma_{mn} = \int \prod_{j=1}^{m+n} D\vec{x}_j(z) < e^{\sum_j \pm i S_j} > \qquad (10)$$

The Markov approximation yields (See Section V):

$$\Gamma_{mn} = \int \prod_{j=1}^{m+n} D\vec{x}_j(z) \exp \int_0^R dz \left[\sum_j \pm ik_j \left(\frac{1}{2} \left(\frac{d\vec{x}_j}{dz} \right)^2 - U_{0j} \right) - \frac{1}{2} \int_{-\infty}^{\infty} dz' < M(z) M_{shift}(z') > \right] \tag{11}$$

We show in the next section that the moment equations (7) and the path integral expressions (11) are mathematically equivalent.

III. Equivalence of Path Integral and Moment Equations under the Markov Approximation

We follow the technique that Feynman[12] used to show that his path-integral expression for nonrelativistic quantum mechanics is equivalent to the Schrödinger equation. The key to this demonstration is an understanding of how the important paths behave transversely as they move in z from a particular point. Feynman found that these paths resembled random walks in that

$$|\vec{x}(z') - \vec{x}(z)| \sim (z'-z)^{1/2} \tag{12}$$

as z' gets close to z. Given this behavior, it is easy to expand (11) in a Taylor series and obtain a differential equation which will turn out to be (7). We give the demonstration of (12) in the Appendix.

The path integral is defined as the limit of an integration over a set of "phase screens." These screens are at values $z_N = N\delta z$. The derivative $\frac{d\vec{x}}{dz}$ at $z = z_N$ is defined as $\left(\vec{x}(z_N + \delta z) - \vec{x}(z_N)\right)/\delta z = \frac{\delta \vec{x}}{\delta z}$. The limit $\delta z \to 0$ is taken after the integrals are evaluated. The differential equation is obtained by considering the integral over the very last phase screen. The last integral in (11) can be written in terms of $\vec{x}_j{}' = \vec{x}_j(R - \delta z)$ and $\vec{x}_j = \vec{x}_j(R)$. Also, we define $\delta\vec{x}_j \equiv \vec{x}_j - \vec{x}_j{}'$. Then Γ_{mn} can be expressed as:

$$\Gamma_{mn}(\{\vec{x}\}, R) = \tag{13}$$

$$N \int \prod_j d\vec{x}_j{}' \exp\left(\delta z \left[\sum_j \pm i k_j \left(\frac{1}{2} \left(\frac{\delta \vec{x}_j}{\delta z} \right)^2 - U_{0j} \right) \right.\right.$$

$$\left.\left. - \frac{1}{2} \int dz' < M(R)\, M_{shift}(z') > \right] \right) \Gamma_{mn}(\{\vec{x}'\}, R - \delta z)$$

where $\{\vec{x}\}$ denotes the set of $m+n$ $\vec{x}_j$'s. The first term in the exponent, $\pm \frac{ik_j}{2} \frac{(\delta \vec{x}_j)^2}{\delta z}$, is O(1) for small δz, because of (12). The exponent of the remaining terms can be expanded, since they have an explicit δz, as well as higher order terms. This results in

$$\Gamma_{mn}(\{\vec{x}\},R) = \tag{14}$$

$$N \int \prod_j \left(d\delta\vec{x}_j \exp\left\{ \pm ik_j(\delta\vec{x}_j)^2/2\delta z \right\} \right)$$

$$\left[1 - \delta z \left(\sum_j \pm ik_j\, U_{0j}(R) + \frac{1}{2} \int_{-\infty}^{\infty} dz' < M(R)\, M_{shift}(z') > \right) \right] \Gamma_{mn}(\{\vec{x}'\}, R - \delta z) + O(\delta z^{\frac{3}{2}})$$

We now have a relationship between the moment at R and the moment at $R - \delta z$, which we derived from our path-integral expression. But since the moment is a differentiable function we can find another relationship by Taylor expansion as follows:

$$\Gamma_{mn}(\{\vec{x}'\}, R - \delta z) = \left[1 - \delta z\, \partial_R - \sum_j \delta\vec{x}_j \cdot \nabla_j + \frac{1}{2} \left(\sum_j \delta\vec{x}_j \cdot \nabla_j \right)^2 \right] \Gamma_{mn}(\{\vec{x}\}, R) \tag{15}$$

$$+ O(\delta z^{\frac{3}{2}})$$

Substituting (15) into (14) we find

$$\Gamma_{mn}(\{\vec{x}\},R) = N \int \prod_j \left(d\delta\vec{x}_j \;\; \exp\left\{ \pm ik_j(\delta\vec{x}_j^{\,2})/2\delta z \right\} \right) \tag{16}$$

$$\left\{ 1 - \sum_j \delta\vec{x}_j \cdot \nabla_j + \frac{1}{2} \left(\sum_j \delta\vec{x}_j \cdot \nabla_j \right)^2 - \delta z\, \partial_R \right.$$

$$\left. - \delta z \left(\sum_j \pm ik_j\, U_{0j}(R) + \frac{1}{2} \int_{-\infty}^{\infty} dz' < M(R) M_{shift}(z') > \right) \right\} \Gamma_{mn}(\{\vec{x}\}, R) + O(\delta z^{\frac{3}{2}})$$

The term linear in $\sum_j \delta\vec{x}_j \cdot \nabla_j$ is odd in $\delta\vec{x}_j$ and therefore gives zero due to the $\delta\vec{x}_j$ integral. The term that is quadratic in $\delta\vec{x}_j$ can be integrated by parts, yielding, to order δz:

$$\Gamma_{mn}(\{\vec{x}\},R) = \mathrm{N}\int\prod_j\left(d\delta\vec{x}_j\exp\left\{\pm ik_j(\delta\vec{x}_j)^2/2\delta z\right\}\right)$$

$$\cdot\left\{1+\delta z\left[-\partial_R-\frac{1}{2}\sum_j\frac{1}{\pm ik_j}\nabla_j^2-\sum_j\pm ik_j\,U_{0j}(R)\right.\right.$$

$$\left.\left.-\frac{1}{2}\int_{-\infty}^{\infty}dz'<M(R)M_{shift}(z')>\right]\right\}\Gamma_{mn}(\{\vec{x}\},R) \quad . \tag{17}$$

The only way (17) can be true for all δz is for the coefficient of δz within the curly brackets operating on Γ_{mn} to give zero. Therefore, setting $R\rightarrow z$,

$$\partial_z\Gamma_{mn}(\{\vec{x}\},z) = -i\sum\pm\frac{1}{k_j}\left(-\frac{1}{2}\nabla_j^2+k_j^2U_{0j}\right)\Gamma_{mn}(\{\vec{x}\},z)$$

$$-\frac{1}{2}\int_{-\infty}^{\infty}dz'<M(z)M_{shift}(z')>\Gamma_{mn}(\{\vec{x}\},z) \tag{18}$$

which is identical to (7), as required. Thus, we have derived the moment equation (7) from the path integral expression (11). This shows that the path-integral expression (11) is a solution of the moment equation (7) and hence the two techniques are equivalent.

IV. Moment-Equation Derivation

We derive our moment equations by the method of Van Kampen.[9] The advantage of his method is that the physical basis for each approximation is readily apparent. He bases his method on techniques that were developed for quantum mechanics.

We shall find that the Markov approximation requires that the dimensionless number $L_p^2 M_t^2$ be small where L_p is the medium correlation length in the direction of the wave propagation, and M_t is the "typical" value of M, defined by (6) and called the "interaction strength." For the first moment $M = k\mu$, but for higher moments M is the sum and difference of a number of $k\mu$'s at different positions, and with different values of k.

We start with the parabolic wave equation (3) and the definition of L_0 and M, and write:

$$i\,\partial_z \psi_1^* \psi_2^* \cdots \psi_{m+n} = (L_0 + M)\psi_1^* \psi_2^* \cdots \psi_{m+n} \tag{19}$$

The "interaction representation" is defined by:

$$(\psi_1^* \psi_2^* \cdots \psi_{m+n})_I = e^{iL_0 z}\, \psi_1^* \psi_2^* \cdots \psi_{m+n} \tag{20}$$

and

$$M_I(z) = e^{iL_0 z} M(z) e^{-iL_0 z} \tag{21}$$

With these definitions, (19) becomes

$$i\,\partial_z (\psi_1^* \psi_2^* \cdots \psi_{m+n})_I = M_I(z)(\psi_1^* \psi_2^* \cdots \psi_{m+n})_I \tag{22}$$

This equation is linear and has the formal solution

$$(\psi_1^* \psi_2^* \cdots \psi_{m+n})_I = T\exp\left(-i\int_0^z M_I(z')\,dz'\right)\Gamma_{mn}(0) \tag{23}$$

$\Gamma_{mn}(0)$ is the initial condition. The "time-ordering" symbol T requires explanation. One notices that M_I is an operator, not just a function of space. $M_I(z_1)$ and $M_I(z_2)$ do not, in general, commute. If they did the solution of (22) would be given by (23) without the T symbol. The T symbol means that a product of operators to the right is not applied in the usual order, but in such a way that operators with smaller values of z' are to be applied first. Thus there is an ordering in z. (The T-symbol was invented for solving problems in quantum mechanics where the analog of the longitudinal direction z is the time.) For example,

$$T\exp\left(-i\int_0^z M_l(z')\,dz'\right) = \tag{24}$$

$$\left[T\exp\left(-i\int_{z_1}^{z} M_l(z')\,dz'\right)\right]\left[T\exp\left(-i\int_0^{z_1} M_l(z')\,dz'\right)\right]$$

for $0 \le z_1 \le z$. Another example is

$$T\,\frac{(-i)^k}{k!}\left[\int_0^z M_l(z')dz'\right]^k = (-i)^k \int M_l(z_k)\cdots M_l(z_2)\,M_l(z_1)\,dz_1\,dz_2\cdots dz_k \tag{25}$$

where the integration region on the right side of (25) is $0 < z_1 < z_2 \cdots < z_k < z$, which is $k!$ times smaller than that of the left side, cancelling the factor of $k!$. Using either (24) or (25), one readily checks that (23) is a formal solution of (22).

We are assuming that M is a Gaussian process. The result that the expectation of the exponential of a zero-mean Gaussian random variable is the exponential of half the variance follows from combinatorial factors and remains true for a time-ordered exponential. Thus

$$(\Gamma_{mn})_l = T\exp\left[-\frac{1}{2} < \left(\int_0^z M_l(z')\,dz'\right)^2 >\right]\Gamma_{mn}(0) \tag{26}$$

Although this is a formal expression for Γ_{mn}, it is not immediately useful for calculations, since there is no simple algorithm for evaluating a time-ordered exponential (in contrast to a normal exponential). Van Kampen proceeds by differentiating (26):

$$\partial_z(\Gamma_{mn})_l = \tag{27}$$

$$-T < M_l(z)\int_0^z dz'\,M_l(z') > \exp\left[-\frac{1}{2} < \left(\int_0^z M_l(z'')\,dz''\right)^2 >\right]\Gamma_{mn}(0)$$

The $M_l(z)$ has the largest z, so it is written in the proper ordered position. The $M_l(z')$ that it is correlated with, however, might occur anywhere relative to the $M_l(z'')$'s in the exponential. If $L_p^2 M_l^2 \ll 1$, very little error is made by assuming that the first two M_l's are in the proper order, so that the T symbol can be brought through the first expectation value, yielding:

$$\partial_z(\Gamma_{mn})_I = -<M_I(z)\int_0^z dz'\, M_I(z')>(\Gamma_{mn})_I \tag{28}$$

This may be shown by expanding the exponential operators in (26) or (27) and discussing the order of M's in each term. The Nth term in the expansion has $2N$ occurrences of M_I, and is of a magnitude

$$<(\int_0^z M_I\, dz')^2>^N/N! \tag{29}$$

where typical eigenvalues of the operators are implied. The terms beyond

$$N \approx 4<(\int_0^z M_I\, dz')^2> \tag{30}$$

become negligible compared to the original exponential in (26), so we have to deal with at most N pairs of M_I's from source to range z. The two M_I's in a correlated pair must be within L_p of each other to give a nonzero correlation. The number of pairs may be estimated as

$$N \approx 4L_p\, z\, M_t^2 \tag{31}$$

where M_t^2 is a typical value of M_I^2. (See Figure 2 for a schematic representation.) Our approximation reduces to saying it is unlikely to find a third occurrence of an M_I in between a pair that are within L_p of each other. This probability is roughly

$$\text{Probability} \approx \frac{L_p N}{z} \approx L_p^2 M_t^2 \quad . \tag{32}$$

Thus if the fluctuations are weak enough ($L_p M_t \ll 1$), the approximation is valid, and (28) is justified.

We call (28) "first order perturbation theory." In typical situations, z is much larger than L_P, and the lower limit can be replaced by $-\infty$, making the equation independent of the source position. Moreover, the integral from $-\infty$ to z can be replaced by half the integral from $-\infty$ to ∞, when the correlation is a much slower function of $\frac{1}{2}(z+z')$ than of $z-z'$. The result is used, not in the interaction representation, but in the original representation. The exponentials of (20) and (21) are removed, giving

$$\partial_z \Gamma_{mn}(z) = \tag{33}$$

$$- iL_0\Gamma_{mn}(z) - \int_{-\infty}^{z} dz' < M(z)\, e^{-iL_0(z-z')} M(z') e^{iL_0(z-z')} > \Gamma_{mn}(z)$$

For the second moment, this equation is related to an expression of Besieris and Tappert.[10] Although their work was for the second moment, we can generalize it directly; therefore in the rest of our comments we treat the general moment Γ_{mn} where Besieris and Tappert treated only Γ_{11}. Their equation 3.2 was expressed in a Fourier-transformed domain, but can be expressed in our notation as

$$\partial_z \Gamma_{mn}(z) = -iL_0\Gamma_{mn}(z) - \int_{-\infty}^{z} dz' < M(z)\, e^{-iL_0(z-z')} M(z') > \Gamma_{mn}(z') \,. \tag{34}$$

This equation is equivalent to (33) to order $L_p^2 M_t^2$, and it should be noted that both (33) and (34) are invalid if $L_p^2 M_t^2$ is not small. Unlike (33), (34) implies a "memory" effect in which the gradient of the moment depends explicitly on the moment at all previous z's. The Markov approximation leads to (7), which eliminates the memory effect and requires only a *correlation function of the medium* along a *specified* (shifted) direction. Besieris and Tappert pointed out that a weaker approximation, called the "long-time Markov" approximation leads to a local (non-memory) equation (their equation 3.3), that in our notation is expressed as

$$\partial_z \Gamma_{mn} = -iL_0\Gamma_{mn}(z) - \left\{ \int_{-\infty}^{z} dz' < M(z)\, e^{-iL_0(z-z')} M(z') > \right\} \Gamma_{mn}(z) \,, \tag{35}$$

where the L_0 operator acts only on $M(z')$, not on $\Gamma_{mn}(z)$, in the last term. We have derived (35) by use of the Wigner-function notation of Besieris and Tappert. We are only considering situations in which the parabolic wave equation is valid. It has been shown that in that case the long-time Markov approximation is valid,[10] and therefore (35) is as valid as (33).

Because L_0 is an operator, the integrals in (33-35) involve the medium correlation function in all directions, or, in the Fourier-transform domain, require a scattering kernel as a function of scattering angle. The Markov approximation to (33) consists of simplifying the deterministic propagation operator $e^{-iL_0(z-z')}$ for $z - z'$ on the order of L_p. Instead of correlating $M(z)$ with all possible transverse positions of $M(z')$, the

Markov approximation corresponds to choosing only one transverse position for $M(z')$. (See Figure 1, where point A represents an arbitrary transverse position.) If the wave represented by $\Gamma_{mn}(z)$ were the unperturbed solution, then deterministic propagation would move the phase in the direction of the unperturbed ray. If the wave energy is travelling close to the unperturbed ray this operator retains its behavior to first approximation. As a result, deterministic propagation approximates a shift along the unperturbed ray to point B, i.e., $\mathbf{r}(z') = \mathbf{r}_{ray}(z')$, where the ray is forced to go through $\mathbf{r}(z)$. Hence $e^{-iL_0(z-z')}M(z')e^{iL_0(z-z')}$ can be approximated by $M_{shift}(z')$. This is the appropriate definition of the Markov approximation (rather than assuming the medium is delta-correlated along the z axis) and it immediately yields (7) from (33). In practice, instead of using the actual unperturbed ray, the tangent to the ray at z is often used.

If the delta-correlated assumption were made, it would correspond to evaluating $M_{shift}(z')$ at point C, which is strictly valid only if there is a single unperturbed ray travelling along the z-axis. If the medium fluctuations are isotropic, the correlation of any point at z' with the point P at z will give the same result because of the parabolic approximation, and hence the delta-correlated assumption is as good as any other choice. However, for an anisotropic medium it is important that point B (and hence (7)) be used, even when the Markov approximation is invoked. Note that (7) can be used in the presence of a deterministic background refractive index.

The difference between (33) and (7) can be caused by directions different from the unperturbed ray becoming important. It is in this sense that (33)-(35), which never refer to unperturbed rays, are more general than (7), which does. A transverse wavenumber k_T, coming, for example, from M, causes the angle to change by $\delta\theta = k_T / k$. A transverse error in position of about $k_T L_P / k$ is made by assuming the direction of the unperturbed ray. Thus, in order for the Markov approximation to be valid, it is required that $k_T L_P / k \ll L_T$, where L_T is the transverse scale of concern. Since $L_T \approx 1/k_T$, the Markov approximation fails at sufficiently small $k \approx L_P/L_T^2$. The parameter $\alpha = kL_T^2/L_P$ introduced by Beran and McCoy[13] and discussed further in Flatté [4] reflects these considerations. For small α, one can use (33) or its equivalent.

V. Path-Integral Derivation

We recapitulate the derivation of the path-integral expression (7) from (10). Using the assumed Gaussian behavior of the fluctuations, we obtain from (10)

$$\Gamma_{mn} = \int \prod_{j=1}^{n} D\vec{x}_j(z)\, e^{\sum_j \pm i S_0}\, e^{V} \tag{36}$$

where S_0 is the part of S in (9) that does not involve μ, and

$$V = -\frac{1}{2}\int dz\; dz' < M(z) M(z') > \tag{37}$$

The expression (36) is an exact representation of the moment of the solution of the parabolic equation with Gaussian fluctuations. It is not used in practice as it stands because V depends on the paths at two values of z, namely z and z'.

The Markov approximation for the path integral comes from assuming that the paths do not stray far in transverse space over a distance L_P; they all move approximately parallel to the unperturbed ray. Thus, in the Markov approximation

$$V = -\frac{1}{2}\int dz\; dz' < M(z) M_{shift}(z') > \tag{38}$$

which only requires knowledge of the path at z. The final result (11) follows directly.

VI. Coordinate Systems

Moment equations can be formulated in a variety of coordinate systems, while path integrals require a rectangular coordinate system. There has been a fair amount of effort expended on using polar coordinate systems, especially for point source problems.

The same results (for point sources among others) can be obtained in either polar or rectangular coordinates. Thus, the results of Shishov[14] on the intensity correlation, derived in spherical polar coordinates, can be seen to be identical (after an appropriate transformation) to the results of Codona *et al.*,[15] derived in rectangular coordinates. It was necessary for Shishov to make small angle approximations in addition to the parabolic approximation of dropping the second derivative in the propagation direction, whereas Codona *et al.* only require the single parabolic approximation.

VII. Summary

We have derived moment equations in coordinate representation under the Markov approximation that apply in anisotropic, inhomogeneous media with deterministic background. The derivation shows the relationship between these moment equations and modified equations that are valid under approximations weaker than Markov; the second-moment equation of Besieris and Tappert is a special case of these modified equations.

In a hierarchy of approximations we begin with the parabolic wave equation itself. A path integral with non-local exponent can be written as an exact solution, although it is not yet useful in practice. The next level is the approximation that the interaction strength over a correlation length is small—this "first-order perturbation theory" leads to the modified moment equations, and in homogeneous, isotropic media, to the standard moment equations and path-integral expressions. In anisotropic, inhomogeneous media, however, a further approximation is necessary to obtain the moment equations and path integral expressions. This further approximation is that the significant flow of wave energy, or the important paths, are parallel to the unperturbed ray; we call this the Markov approximation because its violation implies the appearance of correlations between successive scatterings. We have shown that the moment equations and the path-integral expressions for the moments are mathematically equivalent under the Markov approximation. Thus the two formalisms have exactly the same physical content. In an anisotropic medium, the moment equation involves a shift operation to calculate the medium correlation function along the unperturbed ray; this form of the moment equation has not been given before.

We have also pointed out that all appropriate formulae can be derived in a rectangular coordinate system (even for point sources).

Acknowledgement: This work was supported by DARPA.

Appendix

We must show that the scaling $|\delta \bar{x}_j| \sim (\delta z)^{1/2}$ holds for integrals of the form

$$\int \prod d\delta \bar{x}_j \exp\left(\sum \pm ik_j \frac{\delta x_j^2}{2\delta z}\right) F(\bar{x}_j) \quad . \tag{A.1}$$

If F is expandable in a power series (even if the radius of convergence is zero) this result follows immediately. One expands F and integrates term by term, obtaining a power series in $(\delta z)^{1/2}$. By standard methods in the theory of asymptotic expansions, only the low order terms need to be retained as $\delta z \to 0$.

For singular functions, a demonstration is not as simple. One may worry about cancellations between terms in the exponent, since the signs might differ.

We will content ourselves with a demonstration in the case likely to arise in practice. It is common to model a random medium as having a power law structure function. Thus as two x's become equal, a singularity $|x_i - x_j|^p$ with $p > 0$ might occur in the integrand. In order to have possible cancellations in the exponent, we assume that $k_i = k_j = k$, and the exponential factor is $\exp\left(ik\,(\delta x_i^2 - \delta x_j^2)/2\delta z\right)$. We assume, for simplicity, that x_i and x_j are one-dimensional; higher dimensional singularities are effectively weaker.

Define $\nu = (\delta x_i + \delta x_j)/2$, $\mu = \delta x_i - \delta x_j$, $\alpha = x_i - x_j$. The singularity from the previous step, $x_j' = x_j - \delta x_j$ is $|\mu - \alpha|^p$. The integral to be evaluated is

$$\int d\mu\, d\nu\, e^{ik\mu\nu/\delta z}\; f(\alpha,\mu,\nu)\, |\mu - \alpha|^p \tag{A.2}$$

We would like to ignore the μ dependence in f. However, spurious large-μ contributions would arise, even though we are only interested in contributions from μ close to α. To drop the μ dependence of f and also to simplify the analysis, we introduce a convergence factor $\exp(-a(\mu^2 + \nu^2)/\delta z^{1-\epsilon})$. As long as $\mu, \nu \sim \delta z^{1/2}$, this factor does not change the integral as $\delta z \to 0$ (we are assuming $\epsilon > 0$). Conversely, if the integral in the limit $\delta z \to 0$ does not depend on a and ϵ, then μ and ν are of order $\delta z^{1/2}$.

The integral is then

$$I = \int d\mu\, d\nu\, e^{ik\mu\nu/\delta z}\; \exp\left[-a(\mu^2 + \nu^2)/\delta z^{1-\epsilon}\right] |\mu - \alpha|^p\, f(\alpha,\nu) \tag{A.3}$$

The μ integral can be done:

$$I = C_1 \int d\nu\, e^{ik\alpha\nu/\delta z} \exp\left[-a(\alpha^2+\nu^2)/\delta z^{1-\epsilon}\right] \tag{A.4}$$

$$f(\alpha,\nu)\; a^{-(1+p)/2}\, \delta z^{(1+p)(1-\epsilon)/2}\, M\left(\frac{p+1}{2}, \frac{1}{2}, -\frac{(\nu+2i\,\delta z^{\epsilon}\,a\,\alpha)^2}{4a\,\delta z^{1+\epsilon}}\right)$$

where M is a confluent hypergeometric function and C_1 is a constant independent of $\delta z, \epsilon$, and a (as are C_2 and C_3, below). The hypergeometric function has a part that behaves as the exponential of its argument for large (positive) values of its argument, a part that falls as a power (since $(p+1)/2$ is positive) and a part at small values of the argument. These last two parts can be combined into a bounded part. We show that the exponential part gives the leading behavior and the bounded part is a higher power of δz.

The contribution I_1 from the exponential asymptotic part of M is

$$I_1 = C_2 \int d\nu\, e^{ik\alpha\nu/\delta z} \exp\left[-a(\alpha^2+\nu^2)/\delta z^{1-\epsilon}\right] \tag{A.5}$$

$$f(\alpha,\nu)\; a^{-(1+p)/2}\, \delta z^{(1+p)(1-\epsilon)/2} \exp\left(\frac{(2\delta z^{\epsilon}\,a\,\alpha - ik\,\nu)^2}{4a\,\delta z^{1-\epsilon}}\right)$$

The exponential from M cancels much of the first two exponentials:

$$I_1 = C_2 \int d\nu \exp\left\{-\nu^2\left[\frac{a}{\delta z^{1-\epsilon}} + \frac{k^2}{4a\,\delta z^{1+\epsilon}}\right]\right\} f(\alpha,\nu)\; a^{-(1+p)/2}\, \delta z^{(1+p)(1-\epsilon)/2} \tag{A.6}$$

which can be done explicitly. Only the first term in the exponential survives as $\delta z \to 0$. The result is independent of a and ϵ, and is

$$I_1 = C_3\, \delta z\; f(\alpha,0)\, |\alpha|^p \tag{A.7}$$

exactly as would be obtained from the Taylor series expansion for I.

We now turn to the contribution I_2 from the bounded part of M. We show I_2 has a higher power of δz than I_1. We can set ϵ to any positive value. At large ϵ we depend on the fact that $e^{ik\alpha\nu/\delta z}$ averages to zero for $\nu \sim \delta z^{1+\delta}$ for any positive δ, but it would be necessary to examine the detailed behavior of M to use this fact. On the other hand, for small enough ϵ, it suffices to bound the integral by the integral of the absolute value of the integrand. The convergence factor provides a cutoff at $\nu \sim \delta z^{1-\epsilon/2}$. Thus $\int d\nu \exp\left[-a\nu^2/\delta z^{1-\epsilon}\right] f(\alpha,\nu)$ gives a contribution scaling like $\delta z^{1-\epsilon/2}$. Thus I_2 is bounded by an expression which scales as

$$I_2 \sim \delta x^{(1-\epsilon)/2+(1+p)(1-\epsilon)/2} = \delta x^{(1+p/2)(1-\epsilon)} \quad (A.8)$$

As long as we have chosen ϵ small enough, the exponent of δx is larger than 1, and I_2 can be neglected relative to I_1. Thus we have established the necessary scaling of μ and ν even in the singular case.

References

1. V.A. Fock, "Theory of Radio-Wave Propagation in an Inhomogeneous Atmosphere for a Raised Source," *Bulletin de l'Academie des Sciences de l'URSS*, 70, 1950. Translated as Ch. 14 of *Electromagnetic Diffraction and Propagation Problems*, by V.A. Fock, Permagamon Press, Oxford, 1965.
2. V.I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation*, p. 223, Israel Program For Scientific Translation, Jerusalem, 1971.
3. R. Dashen, "Path Integrals for Waves in Random Media," *J. Math. Phys.*, vol. 20, 894-920, May 1979.
4. S.M. Flatté, R. Dashen, W.H. Munk, K.M. Watson, and F. Zachariasen, *Sound Transmission Through a Fluctuating Ocean*, Cambridge University Press, New York, 1979.
5. S.M. Flatté, "Wave Propagation Through Random Media: Contributions from Ocean Acoustics," *Proceedings of the IEEE*, vol. 71, 1267-1294, 1983.
6. S. M. Flatté, D.R. Bernstein, and R. Dashen, "Intensity Moments by Path Integral Techniques for Wave Propagation Through Random Media, with Application to Sound in the Ocean," *Physics of Fluids*, vol. 26, 1701-1713, 1983.
7. M.J. Beran, A.M. Whitman, and S. Frankenthal, "Scattering Calculations Using the Characteristic Rays of the Coherence Function," *J. Acoust. Soc. Am.*, vol. 71, 1124, 1982.
8. R.N. Baer, J.S. Perkins, E.B. Wright, and J.J. McCoy, "Stochastic Propagation of the Mutual Coherence Function in the Deep Ocean," *J. Acoust. Soc. Am.*, vol. 75, 1407, 1984.
9. N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, North Holland, New York, 1981.
10. I.M. Besieris and F.D. Tappert, "Kinetic Equations for the Quantized Motion of a Particle in a Randomly Perturbed Potential Field," *J. Math. Phys.*, vol. 14, 1829, 1973.

11. In the ocean there may be many deterministic rays connecting the source and receiver. In that case (7) is still correct, but an additional index needs to be added to Γ_{mn} to indicate which deterministic ray is connected to each transverse variable $\underline{x}_j$. An entire matrix of moments would be followed, though each element of the matrix can be followed independently. If the deterministic rays are far enough from each other as to be uncorrelated, or if the source or receiver distinguishes between rays (e.g. by travel time or angle), then each ray can be treated independently. Near a caustic, at least two rays are very close to each other, but this does not cause any difficulty since the rays are also adequately coincident in angle.

12. R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals,* McGraw-Hill, New York, 1965.

13. M.J. Beran and J.J. McCoy, "Propagation through an anisotropic random medium," *J. Math. Phys.*, vol. 15, 1901-1912, 1974.

14. V.I. Shishov, "Strong Fluctuations of the Intensity of a Plane Wave Propagating in a Random Medium," *Soviet Physics JETP,* vol. 34, 744, 1972.

15. J.L. Codona, D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F.S. Henyey, *Solution for the Fourth Moment of Waves Propagating in Random Media,* submitted for publication, 1985.

Figure Captions

Figure 1a. Moment-equation expression of the Markov approximation. The correlation should be taken between a point at z (point P) and an arbitrary point at z' (point A). Instead it is taken with the point B, obtained by extrapolating along the unperturbed ray from P. The assumption of delta-correlated medium fluctuations leads to the incorrect formulation of correlations between points P and C. The dashed lines indicate the idea of a scattering as a function of angle from point P.

Figure 1b. Path-integral expression of the Markov approximation. The general path at z' (point A) is approximated by the path at z extrapolated along the unperturbed ray (point B).

Figure 2a. Typical z values of the interactions from a Taylor series term in (9) are indicated by x's. Dashed lines show which interactions are correlated. It is assumed that $L_p^2 M_t^2 \ll 1$.

Figure 2b. A portion of a contribution to (9) which is improperly ordered in "first order perturbation theory." Such contributions are small if $L_p^2 M_t^2 \ll 1$.

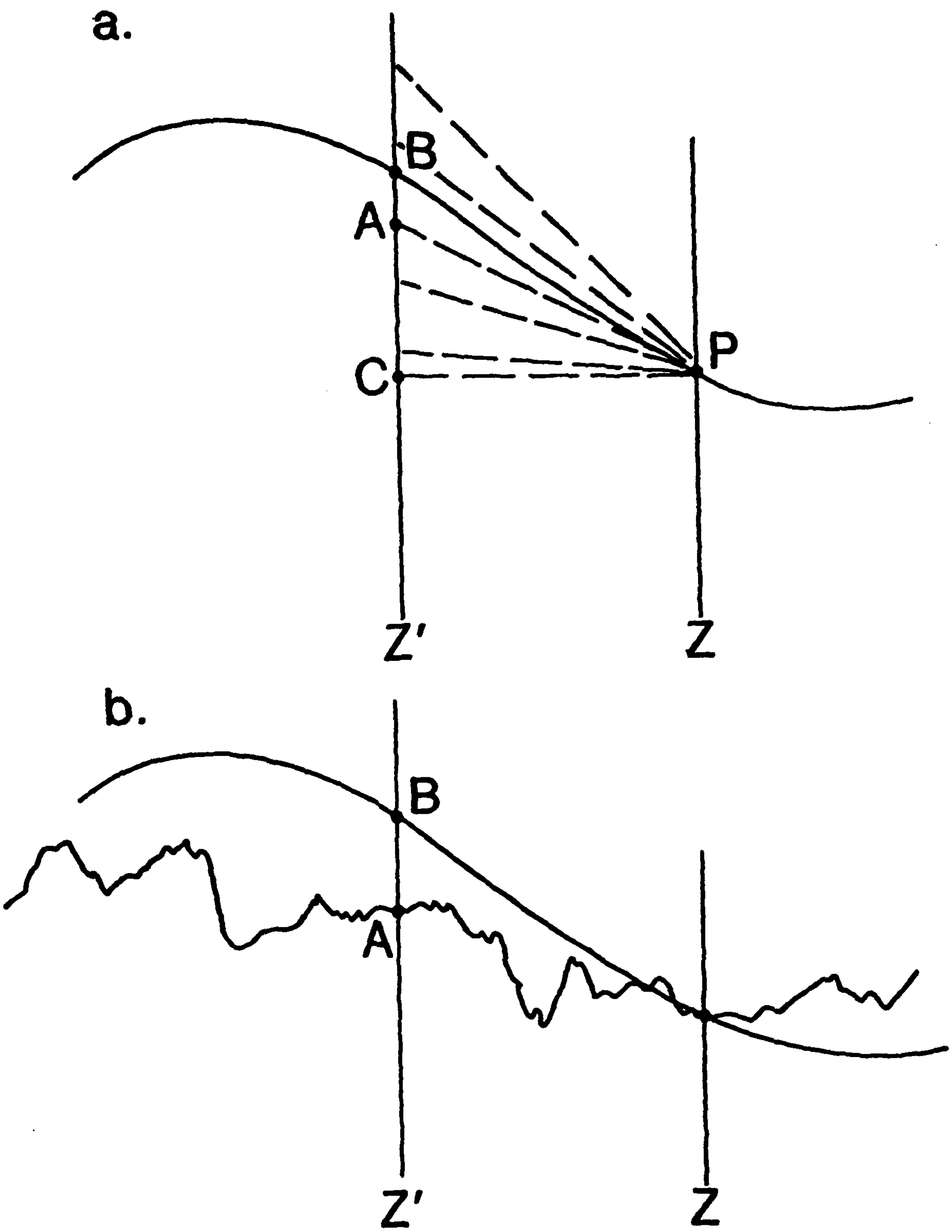


Figure 1

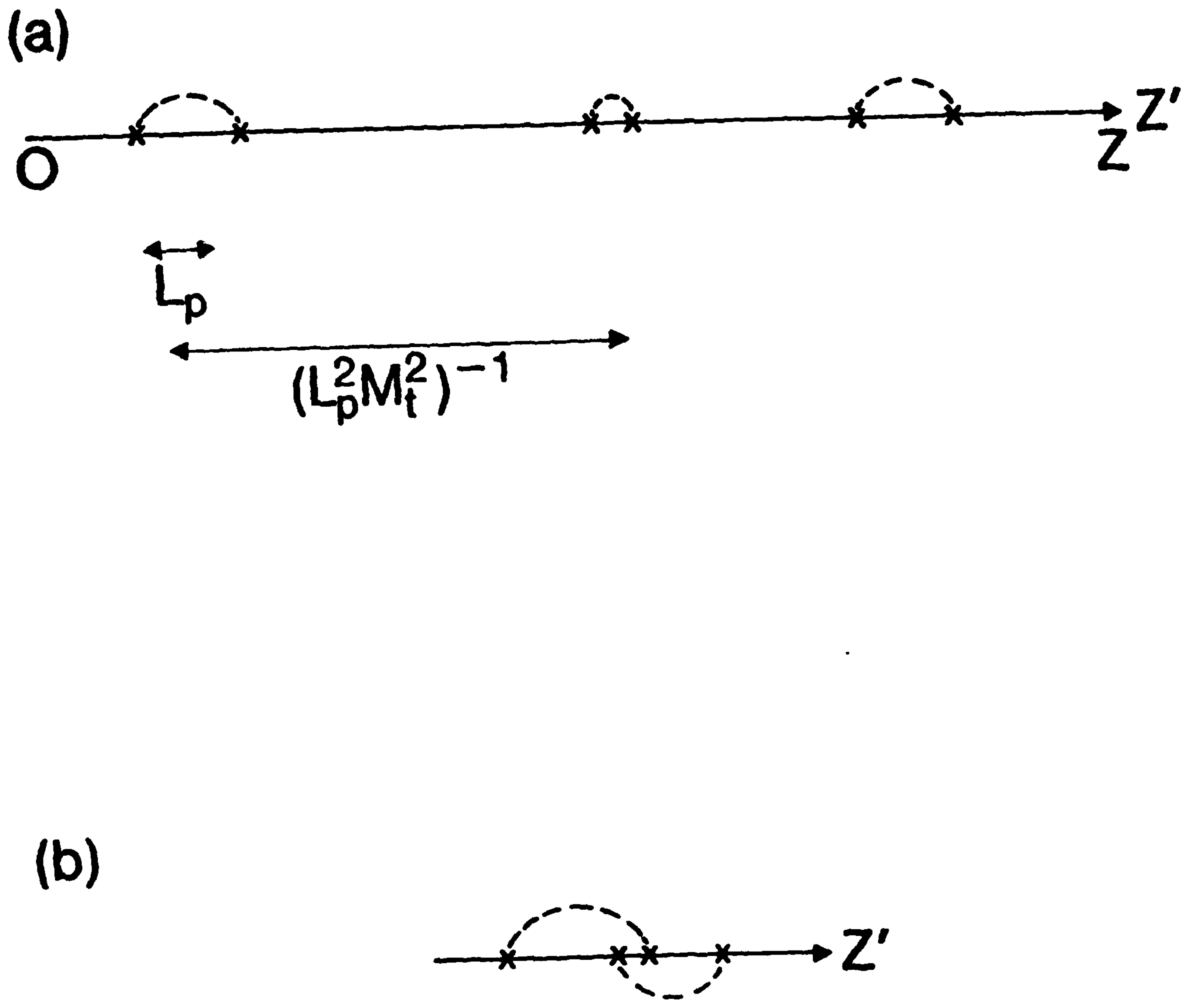


Figure 2

APPENDIX F

On the Derivation of the Schrödinger Equation

Stanley M. Flatté

LJI-P-84-293

On the Derivation of the Schrödinger Equation

by

Stanley M. Flatté*

Center For Studies of Nonlinear Dynamics†
La Jolla Institute
La Jolla, California 92037

December 1984

*On leave from the University of California, Santa Cruz, CA 95064
†Affiliated with the University of California, San Diego.

4-28-'86

Abstract

A relativistic derivation of the Schrödinger equation is given from principles known to physicists in 1926. Schrödinger's preference for Hamilton's optical-mechanical analogy over the relativistic route is discussed. The derivation is given of a classical analog to the Schrödinger equation called the parabolic wave equation, which describes waves propagating in a narrow angular cone; the fact that this classical version was not discovered until about 1950 is discussed.

Introduction

Two great triumphs of twentieth century physics, relativity and nonrelativistic quantum mechanics, have been part of the undergraduate physics curriculum for only a few decades. Both these subjects have their paradoxical aspects with which students must grapple.

It is generally agreed that quantum mechanics is more difficult to accept than relativity, because of its apparent violations of cherished ideas such as causality or locality. However, one aspect of quantum mechanics that should not remain mysterious is the derivation of Schrödinger's wave equation. On a suggestion of Debye, Schrödinger set himself to writing a wave equation for an electron around a proton.[1] Once given the idea that the electron might be represented by a wavefunction, this is a problem in classical physics; yet Schrödinger came out with a "wave equation" that no one had seen before.

The purposes of this article may be stated as follows: 1) to show a simple derivation of the Schrödinger equation starting from a classical wave equation and some physical assumptions that would have been plausible to physicists in 1926; 2) to discuss why Schrödinger preferred to use Hamilton's optical-mechanical analogy rather than follow the relativistic route to his equation; and 3) to discuss how physicists have used an analog to the Schrödinger equation called the parabolic wave equation in completely classical contexts, and why nineteenth-century physicists, who certainly had the mathematical tools, did not write down a Schrödinger equation in solving some classical wave-propagation problems. Finally, the relationships between the full wave equation, the parabolic wave equation, Huygens' construction, the Schrödinger equation, and Feynman's path integral are briefly discussed.

I. The Relativistic Equation

Electromagnetic waves obey the classical wave equation

$$\nabla^2 F - \frac{1}{c^2}\partial_{tt} F = 0 \tag{1}$$

where F may be a component of the electric or magnetic field. If we desire the equation that can represent matter as a wave, following de Broglie, we must deal with the problem of rest mass. Schrödinger himself solved this problem in 1925, but did not publish until 1926 for reasons we will discuss later.[1] By that time Klein[2] and Gordon[3] had derived the same equation by generalizing Schrödinger's nonrelativistic equation.

The matter-wave equation should be able to describe something like a particle at rest with finite energy mc^2. The connection between energy and frequency $\omega_0 = E/\hbar$ was not only known but used extensively by 1926, so it is plausible to want a wave function

$$u = \exp\left[-i\,\frac{mc^2}{\hbar}t\right] \tag{2}$$

(with no space dependence) to satisfy the fundamental equation. This wave function was explicitly suggested by de Broglie in 1925, but he failed in his attempts to find the appropriate equation.[4] It is plausible to simply add a covariant term to (1) in order to have (2) satisfy the equation. One is then led to try

$$\nabla^2 u - \frac{1}{c^2}\partial_{tt} u - \frac{m^2c^2}{\hbar^2}u = 0 \tag{3}$$

as the fundamental, relativistic, equation for free, massive particles. This is the Klein-Gordon equation. The wavefunction u has something to do with a free particle, but we will avoid as much as possible discussing interpretations of the wavefunction. Equation (3) is often justified as the wave analog of the relativistic energy-momentum relation, $E^2 = p^2c^2 + m^2c^4$, transformed using de Broglie's relations, but the above derivation is preferable pedagogically. The addition of the new m^2 term is crucial to all that follows, since without it all waves would move at speed c, and there would then be no possibility of a nonrelativistic approximation.

II. Derivation of the Free-Particle Schrödinger Equation

We desire solutions to (3) to represent particles moving through space. We have one solution (Eq. (2)), but it is not too interesting since it describes a free particle at rest whose wavefunction fills all of space and oscillates very fast. Let us search for more interesting solutions that vary in both space and time. We try

$$u = \psi(x,t)\exp\left[-\frac{i\,mc^2}{\hbar}t\right] . \tag{4}$$

Putting (4) into (3) results in

$$\nabla^2\psi - \frac{1}{c^2}\left[-2i\,\frac{mc^2}{\hbar}\partial_t - \left(\frac{mc^2}{\hbar}\right)^2 + \partial_{tt}\right]\psi - \frac{m^2c^2}{\hbar^2}\psi = 0 . \tag{5}$$

This is a general equation for ψ. Since it is an equation that is linear in ψ, it makes sense to consider each Fourier component of ψ separately:

$$\psi \sim \exp\left[i(kx - \omega t)\right] \tag{6}$$

where, of course, $k = 2\pi/\lambda$ and λ is the wavelength. If we restrict ourselves to low-frequency solutions in which

$$\omega \ll \frac{mc^2}{\hbar} , \tag{7}$$

then the ∂_{tt} term in (5) is negligible, and (5) becomes

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi \tag{8}$$

which is the free-particle Schrödinger equation. Note that the first derivative in time and its imaginary coefficient come naturally from the second derivative because the time dependence of ψ is a *difference* frequency from the fundamental ω_0.

The low-frequency requirement (7) can be stated in two ways. The first is obvious:

$$\hbar\omega \ll mc^2 ; \tag{9}$$

that is, interpreting the frequency of ψ as an energy, that energy must be much less that the rest-mass energy. A second interpretation is afforded if we imagine making a wave packet of waves like (8). Equation (8) implies

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} \tag{10}$$

and therefore the group velocity of these waves is

$$v_g = \frac{\hbar k}{m} \tag{11}$$

Using (10) and (11) one quickly shows that (9) gives

$$v_g^2 \ll c^2 \tag{12}$$

In other words, the Schrödinger equation (8) is a nonrelativistic approximation to the Klein-Gordon equation (3). Finally, (9) and (10) can be combined to give another version of the validity requirement:

$$\lambda \gg \frac{\hbar}{mc} . \tag{13}$$

In other words, the Schrödinger equation is a *large-wavelength* approximation to the full wave equation; that is, the wavelength must be large compared to the Compton wavelength of the electron.

III. Addition of an External Potential

The addition of electromagnetic potentials to (3) should follow the rules of relativistic invariance. It was known in 1926 that φ and $\mathbf{A}/c$, the scalar and vector potentials, are components of a four-vector. The minimal coupling implies that (3) becomes

$$\left[(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2 - \frac{1}{c^2}(\partial_t - \frac{ie}{\hbar}\varphi)^2 - \frac{m^2c^2}{\hbar^2}\right]u = 0 \tag{14}$$

Consider the case in which $\mathbf{A} = 0$. In that case the equation (5) for ψ becomes

$$\nabla^2\psi - \frac{1}{c^2}\left[-2i\frac{mc^2}{\hbar}\partial_t + \frac{2mc^2}{\hbar^2}e\varphi + \partial_{tt} - \frac{2ie}{\hbar}\varphi\partial_t - \frac{e^2}{\hbar^2}\varphi^2\right]\psi = 0 \tag{15}$$

The size of atoms (known in 1926) implies that the relevant Coulomb potentials felt by an electron are small compared with mc^2; that is

$$e\varphi \ll mc^2 \ . \tag{16}$$

Therefore the terms in (14) satisfy a hierarchy such that the last three terms are negligible. Again this corresponds to ψ having low frequency; that is (9) and (12) are satisfied. The resulting equation is

$$i\hbar\, \partial_t \psi = \left[-\frac{\hbar^2}{2m} \nabla^2 + e\varphi \right] \psi \ , \tag{17}$$

that is, the full nonrelativistic Schrödinger equation with external potential.

IV. The Fine Structure of Hydrogen

Schrödinger wrote down equation (14) in 1925, but did not mention it in a publication until late 1926. He gave as his reason that an exact solution for the energy levels of hydrogen disagrees with experiment in the fine structure.[1] Since fine-structure *energy-level differences* are quite small, this amounts to saying that the neglected three terms in (15) are incorrect. The actual numbers are elegantly worked out in Schiff's text,[5] where it is shown that the hydrogen fine structure from the last three terms on the left of (15) is about twice as large as experiment, and has a slightly different dependence on the orbital quantum number.

Schrödinger himself pointed out that the discrepancy probably had something to do with Goudsmit and Uhlenbeck's hypothesis of electron spin.[6] This is probably the main reason that Schrödinger did not derive his nonrelativistic time-dependent equation along the lines of our discussion. And of course it is one of the reasons why Dirac's contribution, which seemed to find the existence of electron spin in a "natural" way, is so admired.

However, as will be discussed later, the derivation of the Schrödinger equation from a classical wave equation is much to be desired pedagogically. Given the many paradoxes of quantum-mechanical matter waves in 1926, including the existence of spin, the interpretation of the wavefunction, and the inability to create a dispersion-free wave-packet, the hydrogen-fine-structure discrepancy was hardly a reason to abandon the whole approach.

V. Antimatter and Electron Spin

Dirac's prediction of antimatter from his relativistic equation for the electron with spin is rightly considered one of the triumphs of modern theoretical physics. It is possible to make the same point from the Klein-Gordon equation, by simply pointing out that the nature of equation (3)—second order in time, with m^2 appearing rather than m, allows

$$\bar{u} = \exp[+\frac{i\,mc^2}{\hbar}t] \tag{18}$$

to be a solution as well as u from (2). The same derivation of the nonrelativistic equation goes through, and as long as $e\varphi \ll mc^2$, the two states will not mix. As the energy $e\varphi$ gets larger, the coupling between the two states becomes more important, leading in lowest order to vacuum polarization, Zitterbewegung, etc., and eventually to antimatter production and the necessity for second quantization and quantum electrodynamics.

It was the unwelcome existence of negative-energy solutions to the Klein-Gordon equation that led Dirac to search for an equation in first derivatives. His search led eventually to four component spinors, and thus he was forced back to the situation that is evident in the Klein-Gordon equation at the start—the simultaneous existence of positive and negative-energy solutions.

The discrepancy in the hydrogen fine structure cannot be dismissed as an effect of relativity. The fact is that the spin of the electron is 1/2, not 0. Some people feel that starting with the Klein-Gordon equation to describe the electron is therefore too misleading. It should be pointed out that learning the Klein-Gordon route is quite valuable, since, as Case[7] has shown, the Foldy-Wouthuysen transformation makes the resulting forms for operators and the nonrelativistic limits quite similar for integer and half-integer spins.

VI. Relation to Hamilton's Optical-Mechanical Analogy

It is of the greatest importance to realize that (17) is a *low-frequency* approximation to the relativistic equation. It was Schrödinger's great desire to make the connection between quantum mechanics and classical mechanics by the Hamiltonian method that Sommerfeld and Runge used to connect wave optics to ray optics.[8] That connection requires taking the *high-frequency* limit of the wave theory. Schrödinger was fully aware that the frequency was associated with the energy, and that the logical total energy would involve a much larger value, mc^2. It was no problem for him to realize that the requirement for the classical equations of motion to be valid would be a requirement that the wavelength is small compared with some characteristic length L for the potential to change:

$$\lambda \ll L \ . \tag{19}$$

which combines with (10) to give

$$\frac{\hbar}{2m} L^2 \ll \omega \ . \tag{20}$$

Thus Schrödinger took the point of view that classical mechanics should be included in his equation, and took the high-frequency, small-wavelength limit to satisfy that requirement, but he ignored the fact that the nonrelativistic requirement at the same time restricted the frequency to be *small* compared with a fixed quantity $(mc^2/\hbar)$. He did realize implicitly that a restricted range of frequency does exist that satisfies both the nonrelativistic and the classical mechanics requirements:

$$\frac{\hbar}{2mL^2} \ll \omega \ll \frac{mc^2}{\hbar} \tag{21}$$

Note that as $\hbar \to 0$ the range of validity of nonrelativistic classical mechanics gets larger at *both* ends. That is why the $\hbar \to 0$ limit is often used as the classical limit.

It is also useful to put the requirement (21) in terms of a spatial wavelength requirement $(\lambda = 2\pi/k)$;

$$\frac{\hbar}{mc} \ll \lambda \ll L \tag{22}$$

The left-hand inequality expresses the nonrelativistic requirement and the validity of the Schrödinger wave equation, while the right-hand inequality expresses the classical-mechanics requirement.

Schrödinger's novel use of Hamilton's methods, wherein he used de Broglie's relations to define ω and k in terms of E and p and then created a wave equation by identifying ω and k with operators on a wavefunction, allowed him to bypass the requirements of relativity and allowed him to ignore the left-hand inequality in (22). By this method of starting with classical nonrelativistic equations he avoided both the pitfalls and opportunities associated with antimatter and electron spin.

VII. The Classical Schrödinger Equation: The Parabolic Wave Equation

The Schrödinger equation comes out of the Klein-Gordon equation solely because of the m^2 term. If the m^2 term were not there, all waves would move at speed c, and there would be no hope for solutions with small velocity. Therefore the classical wave equation (1) can never yield the Schrödinger equation with a first derivative with respect to time.

However, an analog of the Schrödinger equation in which the first derivative is with respect to a spatial coordinate can be obtained in the following way. Suppose the speed of the waves, c, is a function of position, and we look for time-harmonic solutions

$$F(x,t) = u(x)e^{-i\omega t} \tag{23}$$

Then our equation becomes the Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{24}$$

where $k = \omega / c$ is the wavenumber, which is a function of position because of c.

The general solution to this equation involves waves in all directions. However, let us single out the z direction as of special interest, and write

$$[\nabla_T^2 + \partial_{zz} + k^2]u = 0 \tag{25}$$

where $\nabla_T^2 = \partial_{xx} + \partial_{yy}$. Equation (25) has some of the characteristics of the Klein-Gordon equation (3). Let us try a solution of a form which reminds us of waves travelling in the positive z direction:

$$u = \psi(\underset{\sim}{x})e^{ik_0z} \tag{26}$$

The resulting equation for $\psi(x, y, z)$ is

$$\nabla_T^2\psi + [\partial_{zz} + 2ik_0\partial_z + k^2]\psi - k_0^2\psi = 0 \tag{27}$$

which is remarkably analogous to (5). We see that the k_0^2 term is playing the role of the m^2 term in the Klein-Gordon equation. If k^2 is slowly varying in space, then we can pick k_0 such that

$$|k_0^2 - k^2| \ll k_0^2 \tag{28}$$

In other words, the wave speed $c(\underset{\sim}{x})$ varies by a small fraction of itself, and does so slowly with respect to $\underset{\sim}{x}$. In that case, ψ will have only small-wavenumber components

$$\psi(\underset{\sim}{x}) \sim e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} \;\; ; \;\; |\underset{\sim}{q}| \ll k_0 \tag{29}$$

and the ∂_{zz} term in (27) is negligible. The resulting equation is

$$2ik_0\partial_z\psi = -\nabla_T^2\psi + (k_0^2 - k^2)\psi \; , \tag{30}$$

which is exactly analogous to the Schrödinger equation (17), except that ∇_T^2 is a two-dimensional Laplacian rather than a three-dimensional one. The constants in (30) are a bit different from (17). The analogy more direct. Let

$$k(x) = k_0[1 + \mu(\underset{\sim}{x})] \tag{31}$$

where $\mu(\underset{\sim}{x})$ is the variation from unity of the index of refraction. Now write (30) as

$$2ik_0\partial_z\psi = -\nabla_T^2\psi + 2k_0^2\mu\psi \tag{32}$$

and write (17) as

$$2i(\frac{mc}{\hbar})\partial_{ct}\psi = -\nabla^2\psi + 2(\frac{mc}{\hbar})^2(\frac{V}{mc^2})\psi \quad (33)$$

The analogy is complete, and we see that the appropriate k_0 is the inverse of the Compton wavelength of the electron, and the variable index of refraction is analogous to the potential V as a fraction of mc^2.

There is an analogy to antimatter in (32). We can find solutions to (25) which are travelling in the *negative* z direction. These are analogous to antiparticle solutions. As long as μ is slowly varying and small, the waves going in the positive and negative directions are decoupled. However, if μ is strong, or varies over a small distance, then coupling will occur in the form of *backscattering*. This is a classical realization of Feynman's picture of positrons being electrons moving backwards in time.[9]

Equation (32) is called the parabolic wave equation in the classical context in which it is used.[10] It is most used in solving problems in wave propagation through continuously variable media; for example light through the atmosphere, radio waves through the ionosphere or interplanetary plasma, or sound through the ocean.[11]

Let us discuss the validity requirements for the parabolic wave equation. The main ingredient is (28) which states that the variations in the wave speed must be small compared with unity, in analogy with the nonrelativistic requirement (18).

In the quantum-mechanical context the nonrelativistic requirement can be expressed in a variety of ways; for example, that the wavelength be large compared to some quantity. The analog of this requirement in the classical context is quite different, because the analog of the wavelength of the quantum-mechanical ψ is *not* the classical wavelength of the propagating wave.

In the classical context we must distinguish between the longitudinal and transverse components of $\underset{\sim}{q}$, the wavevector of ψ, and furthermore we have the wavelength λ, defined from k_0.

Let $\underset{\sim}{q} = (k_T, k_L)$ where k_T has components in the x-y plane, and k_L is along the z direction. The first term of (32) must be larger than $\partial_{zz}\psi$. This results in the requirement

$$k_T \ll k_0 \tag{34}$$

which means that the total wavevector $\underset{\sim}{k}$ is directed at a small angle to the z axis. This latter requirement is the analog of the nonrelativistic requirement, $v \ll c$ in the Schrödinger equation case.

The relation of the various wavenumbers to the characteristics of the medium index of refraction is quite involved, and lies beyond the scope of this paper, because it brings in the strength of the fluctuations (μ) as well as the scale L.

The analogy between the parabolic wave equation and the Schrödinger equation is expressed pictorially in Figures 1 and 2, and is summarized in Table I.

VIII. History of the Parabolic Wave Equation

The parabolic wave equation could be called the classical Schrödinger equation. The strongest difference is just one of variables, since the parabolic equation is first order in a space derivative rather than the time derivative. A parabolic wave equation was written down for the first time in 1946 by Leontovich and Fock,[12] but they did not really write down (32). They wrote down a parabolic equation for radio wave propagation over the surface of the earth, with no potential term. Their effect was controlled by diffraction due to the curved boundary condition over the surface of the spherical earth.

Fock was one of the pioneers in searching for wave equations for quantum mechanics, so it is not surprising that he was the first to point out the relationship between the parabolic wave equation and the Schrödinger equation.[13] He also pointed out that the dependence of the index of refraction for radio waves on height, which occurs because the atmosphere is stratified, is an analog to the quantum mechanical potential. It is perhaps surprising that he did not make the connection until 1950; he also did not discuss the requirements for validity as is done in this paper.

Thanks to Fock, the parabolic wave equation with external potential as in (32) was known to Soviet workers in the 1960's. It was common knowledge by the time of Tatarskii's classic monograph on light through the turbulent atmosphere.[14] (Tatarskii gives neither references nor a very straightforward derivation.) A further interesting case in which the potential term is replaced by a nonlinear term was written down by Kelley,[15] who was dealing with self-trapped laser beams.

One may also ask why the parabolic wave equation was not written down one hundred years earlier, since all the mathematical tools were available once Hamilton had used complex numbers to represent oscillatory phenomena. Why did not those mathematical physicists interested in the wave theory of light write down a parabolic equation for the propagation of light?

The parabolic wave equation is fundamentally a small-wavenumber approximation. Most physicists were working with the large-wavenumber limit of wave propagation, namely, geometrical optics. This was especially true of Hamilton, whose work had tremendous influence over those who followed.

Probably more importantly, experiments were not done in continuous media. Typical problems involved diffraction around obstacles, and for these problems, solutions in the form of integrals were directly written down.

Rayleigh typifies the attitude of several generations of physicists by referring to the problem of continuous media in his Theory of Sound;[14]

> The variation is supposed to be so slow that no sensible reflection occurs, and this is not inconsistent with decided refraction of the rays in travelling distances which include a very great number of wavelengths... The further development of this part of the subject would lead us too far into the domain of geometrical optics. The fundamental assumption of the smallness of the wavelength..., having a far wider application to the phenomena of light than to those of sound, the task of developing its consequences may properly be left to the cultivators of the sister science.

In one paragraph he washes his hands of the question and tells his optical colleagues that it is a geometrical optics problem! It was not until 1948, when Feynman[17] wrote down his path integral, that the relation between the various integral solutions to the differential equations describing wave propagation were made clear.

IX. Huygens-Fresnel-Feynman Theory

Huygens attempted to describe light as a wave, in analogy to sound. Ironically, he used his ideas about secondary wavelets to prove that light travelled in straight lines; that is, he dealt only with geometrical optics.[18] In fact, Huygens' construction is mathematically equivalent to the small-wavelength limit of the full wave equation, namely the unfolding of a contact transformation (the Hamilton-Jacobi equation for the phase).[19]

Over one hundred years passed before attempts were made to generalize Huygens construction to include diffraction. Fresnel's first attempt to do so uncovered disturbing difficulties such as the backward wave and the requirement that the secondary wavelets had to be emitted one-quarter cycle out of phase with the incident wave.[20] Helmholtz and Kirchhoff attempted to improve on Fresnel by introducing the so-called obliquity factor (which, among other things eliminated the backward wave), but neither they nor anyone else have been able to create a Huygens-like construction that provides a solution to the full wave equation.[21]

After Fresnel, over one hundred more years passed before Huygens' construction was generalized in a way that exactly solved a wave equation. Feynman's path integral is an exact solution of the Schrödinger equation.[22] Therefore, when translated into classical terms, it is an exact solution of the parabolic wave equation. No obliquity factor appears, and the quarter-cycle phase shift is relegated to the status of a mathematical normalization factor. No backward wave appears, since no coupling to a backward wave is allowed in the parabolic wave equation, as discussed above. Thus the Feynman path integral can be regarded as the simplest Huygens-Fresnel theory repeatedly applied over infinitesimal steps in range.

Feynman was concerned with quantum mechanics, not classical wave propagation; hence he worked with the Schrödinger equation. The analog of generalizing to the full classical wave equation would have been to try to solve all the problems associated with relativity, antimatter, and particle production. Thus the road of Helmholtz and Kirchhoff was not particularly tempting in the quantum-mechanical context. One could say that Fresnel's difficulties were solved not by constructing a solution to the full wave equation, but by going back to Fresnel's simple construction and realizing that it was the solution to an approximate wave equation, one that is valid for waves propagating in a narrow angular cone.

Geometrical optics is valid in the limit of infinitesimal wavelength. The parabolic wave equation and the Feynman path integral are valid for waves propagating in a narrow angular cone.

X. Conclusion

In 1926, the Schrödinger equation was a wave equation that had *never* been seen before. Yet it does appear in classical contexts, a fact only realized well after Schrödinger's work. In classical form it is called the parabolic wave equation, and its derivation from the full classical wave equation involves a restriction to waves travelling within a narrow cone of angles in a particular direction. An analogous derivation of Schrödinger's equation begins with the Klein-Gordon equation, and need only require that the potential energy is small compared with mc^2. Undergraduates might well be introduced to Schrödinger's equation by this route rather than the standard *ad hoc* approaches.

The development of theories of wave propagation through continuous media, sometimes with a random component, owes much to the use of the parabolic wave equation. The analogy between that equation and the Schrödinger equation has brought the full power of much that was learned in nonrelativistic quantum mechanics to bear on classical wave propagation problems.[11] Most notably, Feynman's path-integral technique has been important in solving problems in wave propagation in random media (WPRM). Perhaps in the future, quantum-mechanical problems in condensed matter may be helped by recent progress in

WPRM, where the medium is considered as having a statistically fluctuating wave speed. That would have been appreciated by Schrödinger.

XI. Acknowledgements

I am grateful for useful conversations with Frank Henyey. I thank the Aspen Center for Physics for providing facilities in which part of this work was done. This work was supported by DARPA.

References

1. L. Wessels, "Schrödinger's Route to Wave Mechanics," *Studies in History and Philosophy of Science*, 10, 311-340 (1979) p. 328.

2. O. Klein, "Quantentheorie und Fünfdimensionale Relativitätstheorie," *Zeitschrift fur Physik*, 37, 895-906 (1926).

3. W. Gordon, "Der Comptoneffekt nach der Schrödingerschen Theorie," *Zeitschrift fur Physik*, 40, 117-133 (1926).

4. L. de Broglie, "Sur la fréquence propre de l'électron," Comptes Rendues *180*, 498-500 (1925).

5. L.I. Schiff, *Quantum Mechanics, Third Edition*, McGraw-Hill, New York, 1968, pp. 466-486.

6. E. Schrödinger, *Wave Mechanics*, translated by J.F. Shearer and W.M. Deans, Chelsea, New York, 1982, pp. 118-19.

7. K.M. Case, Some Generalizations of the Foldy-Wouthuysen Transformation," *Phys. Rev.*, 95, 1323-1328 (1954).

8. Reference 5, page 13.

9. R.P. Feynman, *The Theory of Fundamental Processes*, Benjamin, New York, 1962, p. 25.

10. F.D. Tappert, "The Parabolic Approximation Method," in *Wave Propagation and Underwater Acoustics*, J.B. Keller and J.S. Papadakis, eds., Lecture Notes in Physics Vol. 70, Springer-Verlag, Berlin, 1977.

11. See, for example, S.M. Flatté, "Wave Propagation through Random Media: Contributions from Ocean Acoustics," *Proc. IEEE*, 71, 1267-94 (1983), and S.M. Flatté, R. Dashen, W.H. Munk, K.M. Watson, and F. Zachariasen, *Sound Transmission Through a Fluctuating Ocean*, Cambridge University Press, New York, 1979.

12. M. Leontovich and V. Fock, "Solution of the Problem of Propagation of Electromagnetic waves along the Earth's Surface by the Method of Parabolic Equation," *J. Phys.*, 10, 13-24 (1946).

13. V.A. Fock, "Theory of radio-wave propagation in an inhomogeneous atmosphere for a raised source," Bulletin de l'Académie des Sciences de l'URSS, sér. phys. 14, p.70 (1950) (in Russian). Translated as Chapter 14 of *Electromagnetic Diffraction and Propagation Problems*, by V.A. Fock, Pergamon Press, Oxford, 1965.

14. V.I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation*. Jerusalem: Israel program for scientific translation, 1971, p. 223.

15. P.L. Kelley, *Phys. Rev. Lett.*, 15, 1005 (1965).

16. J.W. Strutt, Baron Rayleigh, *The Theory of Sound, Vol. II*, Dover, New York, 1945, p. 126.

17. R.P. Feynman, "The Space-Time Approach to Non-relativistic Quantum Mechanics," *Rev. Mod. Phys.*, 20, 367 (1948).

18. C. Huygens, *Treatise on Light*, translated by S.P. Thompson, Dover, New York, 1962, pp. 19-21.

19. B.B. Baker and E.T. Copson, *The Mathematical Theory of Huygens' Principle*, Second Edition, Clarendon Press, Oxford, 1950, pp. 4-5.

20. Ibid., pp. 20-22.

21. Ibid., pp. 109-114.

22. R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

Table 1

ANALOGIES BETWEEN QUANTUM MECHANICS AND CLASSICAL WAVE PROPAGATION

Quantum Mechanics	Classical Wave Propagation
Klein Gordon equation $\nabla^2 u - \frac{1}{c^2}\partial_{tt} u - \frac{m^2c^2}{\hbar^2}u = 0$	Helmholtz equation $\nabla^2 u + k^2 u = 0$
Schrödinger equation $2i\left(\frac{mc}{\hbar}\right)\partial_{ct}\psi = -\nabla^2\psi + 2\left(\frac{mc}{\hbar}\right)^2\left(\frac{V}{mc^2}\right)\psi$	Parabolic wave equation $2ik_0\partial_z\psi = -\nabla_T^2\psi + 2k_0^2\mu\psi$
Frequency of ψ ω	Longitudinal wavenumber of ψ k_L
Wavenumber of ψ k	Transverse wavenumber of ψ k_T
Wavenumber of u $\frac{mc}{\hbar}$	Longitudinal wavenumber of u k_0
Frequency of u $\frac{mc^2}{\hbar}$	Frequency of u $k_0 c$
Spatial scale of V L	Transverse scale of μ L
Validity of S.E. $\lambda \gg \frac{\hbar}{mc}$	Validity of P.W.E. $k_T \ll k_0$
Validity of Classical Mechanics $\lambda \ll L$	Validity of Geometrical Optics $\frac{1}{L} \ll k_0$

Figure Captions

1. Typical solutions to classical wave equations.

   (a) The full wave equation allows waves to travel in all directions. For an incident plane wave, such solutions will be generated by medium variations with scales comparable with a wavelength. Waves at large angles to the forward or backward directions can be thought of as linear combinations (couplings) of nearly forward and backward waves.

   (b) The parabolic wave equation describes only waves travelling in the nearly forward or backward directions, and allows no coupling between them.

2. Typical solutions to quantum-mechanical wave equations.

   (a) The Klein-Gordon equation allows particle and antiparticle waves to travel at speeds up to the speed of light. For an incident plane wave, such solutions will be generated by external potentials that have strengths comparable with mc^2. Waves at high speeds can be thought of as linear combinations of particle and antiparticle waves, representing the possibility of (real or virtual) particle production.

   (b) The Schrödinger equation describes only slowly moving particles or antiparticles, and allows no particle production.

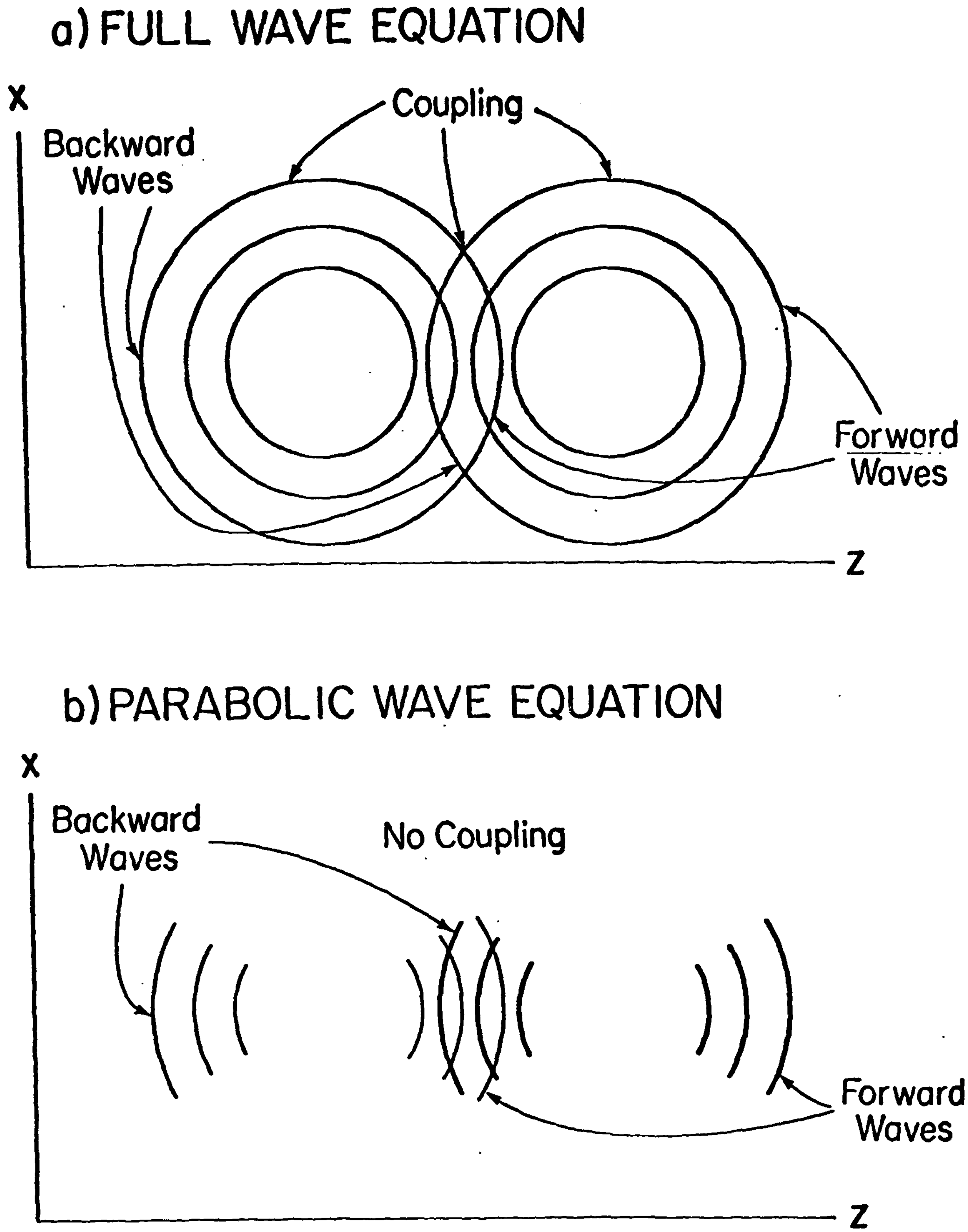

a) FULL WAVE EQUATION
X
Coupling
Backward Waves
Forward Waves
Z
b) PARABOLIC WAVE EQUATION
X
Backward Waves
No Coupling
Forward Waves
Z

Figure 1

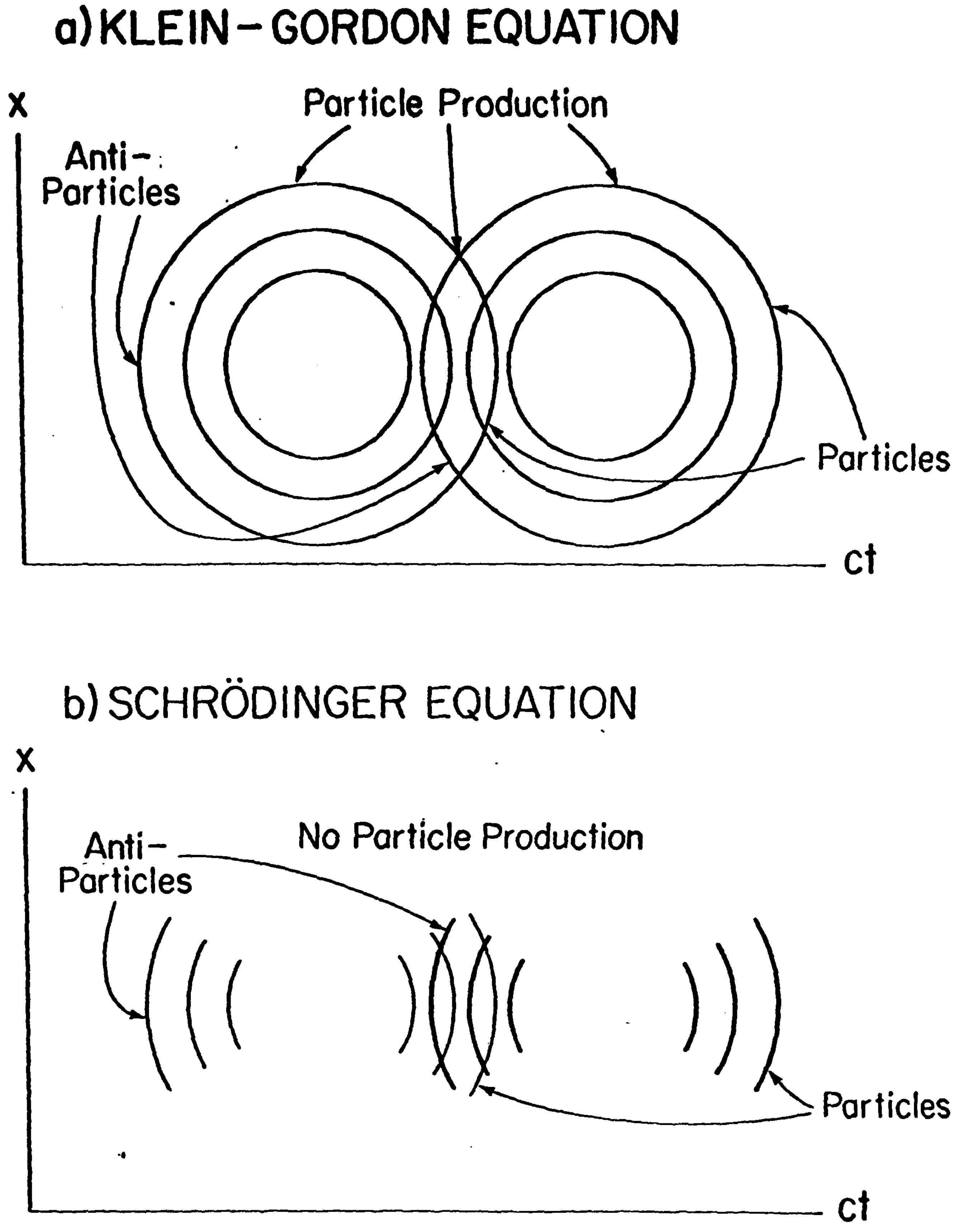

a) KLEIN-GORDON EQUATION
x
Particle Production
Anti-
Particles
Particles
ct
b) SCHRÖDINGER EQUATION
x
Anti-
Particles
No Particle Production
Particles
ct

Figure 2

APPENDIX G

Some Recent Results on Wave Propagation Through Continuous Random Media

Stanley M. Flatté

LJI-P-85-334

Some Recent Results on Wave Propagation Through Continuous Random Media

by

Stanley M. Flatté*

Center for Studies of Nonlinear Dynamics†
La Jolla Institute
3252 Holiday Court, Suite 208
La Jolla, California 92037

July 1985

*On leave from the University of California, Santa Cruz.
†Affiliated with the University of California, San Diego.

7-19-'85

I provide here a guide to a few recent results obtained by the group at the Center for Studies of Nonlinear Dynamics in the area of wave propagation in random media.

I. Average Arrival Time of Wave Pulses Through Continuous Random Media

A random medium consisting of discrete point scatterers in a homogenous background will delay the arrival of a pulse, because all scattered paths are of greater length than the straight-line path from transmitter to reciever. A continuous random medium is different.

Our recent analysis[1] points out that a fluctuating continuous medium can cause an average advance of the pulse arrival time. All previous analyses have dealt with situations in which pulses are delayed on the average.[2,3] By convention, the ensemble average of a random medium is taken as the medium reference state, and the small fluctuations about this reference state are thus by definition a zero-mean random process. The arrival-time advance or delay is relative to the travel time through the reference state. Thus, for example, results through turbulent air or plasma are relative to quiescent air or plasma, not vacuum.

The behavior of a wave propagating through a random medium is controlled by relationships between the wavenumber (k) of the propagating wave, the range (R), and the strength and size of the medium fluctuations.[4] *Unsaturated* behavior corresponds to one stationary-phase path (ray), and occurs if the medium fluctuations are weak enough. In *fully saturated* behavior the original ray breaks up into many new microrays which are statistically independent of each other. Propagation through a medium of discrete scatterers falls in this category. *Partially saturated* behavior occurs in a strongly fluctuating medium with a power-law spectrum, which has enough small-scale fluctuations to cause the breakup into many microrays, and enough large-scale fluctuations to make the microray bundle behave like a single ray in its wandering from the unperturbed ray. Experiments in waves propagating through continuous random media typically fall into this category. We deal only with the important case in which the transverse wandering from the unperturbed ray is small compared with the range of propagation.

Briefly our results are as follows: if the travel time of a pulse is averaged over an ensemble of the random medium, with each pulse weighted by its intensity, then the average pulse is delayed, regardless of the type of propagation behavior, in agreement with previous results.[2,3] However, if the average travel time is obtained without weighting by pulse intensity, then a pulse advance is expected for both unsaturated and partially saturated behavior, while a pulse delay remains for the fully saturated case. The difference between intensity-weighted and unweighted travel time probes the variance of the first derivative of the refractive index, smoothed over a microray bundle.

The effect is illustrated by a simple special case. Consider a point source and point receiver separated by range R, and a homogeneous medium in the absence of fluctuations, so that the unperturbed ray from source to receiver is a straight line. The random medium is concentrated in a "phase screen" at a distance z from the source. This screen has the effect of advancing the time of a wavefront by a random amount $t(x)$ where x is the position on the screen, and $t(x)$ is a stationary Gaussian random process with zero mean. (We take x as one-dimensional for simplicity.)

Weak fluctuations—In the geometrical optics limit only one ray exists from source to receiver. The travel time for a path through point x is:

$$T(x) \approx T_0 + 0.5c_0^{-1} A x^2 - t(x) \tag{1.1}$$

where $A^{-1} \equiv z(R-z)/R$. By Fermat's principle the ray is at the point x_r such that $T(x_r)$ is a minimum. For the case of weak fluctuations we may expand $t(x)$ as

$$t(x) = t_0 + t'x + 0.5t''x^2 \tag{1.2}$$

The position of the ray follows to first order as

$$x_r = A^{-1} c_0 t' \tag{1.3}$$

The travel time of the ray is then

$$T(x_r) = T_0 + 0.5c_0 A^{-1} t'^2 - t_0 - c_0 A^{-1} t'^2 \tag{1.4}$$

This case requires that, typically

$$|c_0 A^{-1} t'^2| \ll |t_0| \tag{1.5}$$

But t (and hence t_0 and t') are (by construction) random variables with zero mean.

Therefore the t_0 term will disappear in the *average* travel time and the only effect of the fluctuations will come from the t' terms. These terms arise because the ray has moved away from its unperturbed position. The first t' term is positive, corresponding to a pulse delay, and represents the effects of geometry; the perturbed path is physically longer than the unperturbed one. The second t' term is negative, corresponding to a pulse advance; we call this the Fermat term; the ray sought out a region of the medium with a pulse advance. The Fermat term is twice as large in magnitude as the geometry term. The average travel time is

$$< T > \; = \; T_0 - 0.5 c_0 A^{-1} < t'^2 > \tag{1.6}$$

so that the pulse on the average arrives early.

There is a subtlety to this result. In the weak-fluctuation limit the intensity is controlled by the focussing due to the curvature of the wavefront as it exits the phase screen. It is not difficult to show that the intensity I is, to first order,

$$I \; = \; 1 + A^{-1} c_0 t'' \tag{1.7}$$

Consider the intensity-weighted average travel time:

$$< IT(x_r) > \; = \; T_0 - 0.5 c_0 A^{-1} < t'^2 > - c_0 A^{-1} < t_0 t'' > \tag{1.8}$$

where the last term comes from the correlation between the intensity and the travel time.

For any random function $t(x)$ whose Fourier components are uncorrelated (i.e., the correlation function is translation-invariant):

$$< t_0 t'' > \; = \; - < t'^2 > \tag{1.9}$$

Therefore

$$< IT(x_r) > \; = \; T_0 + 0.5 c_0 A^{-1} < t'^2 > \tag{1.10}$$

In other words, the intensity-weighted average travel-time is *delayed* by fluctuations by exactly the amount that the unweighted average is advanced! The focussing effect exactly canceled the Fermat term leaving a resultant equal to the geometry effect alone. This occurs because a negative fluctuation, which delays the pulse, acts as a converging lens to increase the intensity.

The simple example of a phase screen in the weak-fluctuation geometrical-optics limit has illustrated our point. Other remarks on generalizations to extended media and strong fluctuations, as well as a rigorous extension of these results by means of a path-integral method to include diffractive effects in a power-law medium, are included in Reference 1.

There is no difficulty in extending the above results from a phase screen to extended media in which (1.6) and (1.10) are replaced by

$$< T > - T_0 = -0.5 c_0^{-1} \int dz \, A^{-1}(z) \left[\int dz' \, \rho_{zz}(z, z') \right] \qquad (1.11)$$

$$< IT > - T_0 = +0.5 c_0^{-1} \int dz \, A^{-1}(z) \left[\int dz' \, \rho_{zz}(z, z') \right] \qquad (1.12)$$

$$\rho_{zz}(z, z') = < \partial_z \mu(z) \partial_z \mu(z') > \qquad (1.13)$$

where $\partial_z \mu(z)$ is the transverse gradient of the refractive index due to the fluctuations at location z along the unperturbed ray. These results require the Markov approximation (that is, the quantity in square brackets in (1.11) is a local function of z). If an incident plane wave rather than a point source is used, all three terms (geometry, Fermat, and focussing) are reduced by a factor of three. If the Markov approximation is not made, the ratio between Fermat and geometry remains −2, while all terms are modified by terms of order L_p/R, where L_p is the longitudinal correlation length of the medium fluctuations.

An important modification of the above result occurs if, in the absence of fluctuations, the medium has focussing properties. In ocean sound propagation this is due to the sound channel. In radio wave propagation from pulsars this might be due to very large-scale medium fluctuations that are effectively frozen during the time of observation. The modification can be simply expressed by generalizing $A^{-1}(z)$ in (1.11-1.12) for a curved ray.[5] The key result is that $A^{-1}(z)$ can be negative for various regions along the ray, and hence the *geometry term can be negative for curved rays*. This complication is crucial to the comparison between calculation and experiment in the ocean, though probably not in other media. Note that this effect provides another, different mechanism by which fluctuations in a medium may cause an average pulse advance.

A numerical study of (1.11) for acoustic rays through ocean internal waves has revealed this effect as a significant bias in the measurement of large-scale ocean eddies by acoustic means. Typical examples of experiments would be waves travelling over a range of 1,000 km, whose travel times are modified by about 20 ms due to a single eddy of 100-km size. The effect of random internal waves along this path is calculated to be of the same order. Therefore an average change in the travel time of a long-range ocean ray could be interpreted either as the effect of an eddy wandering across the ray, or as the effect of the entire internal-wave field changing its strength.

II. Comparison between Moment Equations and Path-Integral Expressions for Wave Propagation in Random Media

Many problems in wave propagation through random media concern phenomena in which there is no significant backscatter, so that a parabolic approximation may be made to the wave equation.[6] In these cases a further approximation, called the Markov approximation,[7] leads to relatively tractable mathematical expressions for moments of the field that can be used for practical calculations. Two quite different formalisms have been used in this context: the moment-equation and path-integral techniques.

A path-integral expression for a general moment of the field of a wave propagating through an inhomogeneous, anisotropic medium in the presence of a deterministic background refractive index has been derived,[8] and the expression has been used for specific calculations.[4,5,9]

Moment equations in coordinate representation have been derived for homogeneous isotropic media in the absence of a deterministic background.[7] Treatments of inhomogeneity, anisotropy, and deterministic background by moment-equation techniques have heretofore been confined to special cases involving the first and second moments.[10,11]

We present here general moment equations in coordinate representation that account for inhomogeneity, anisotropy, and deterministic background, but require the Markov approximation. We have derived these equations[12] using the time-ordered-product method of Van Kampen,[13] which also provides a derivation of equations that are valid under conditions more general than the Markov approximation. The modified equations are more complicated than those that require the Markov approximation: a special case was previously derived by Besieris and Tappert.[14]

We have also shown that our new general moment equations derived under the Markov approximation are mathematically equivalent to the path-integral expressions for the moments that have been previously presented. Thus, the two popular formalisms, under the Markov approximation, are not different in content.

Consider waves travelling predominantly in the z direction. Let $\vec{x}$ be a transverse coordinate (e.g. two-dimensional, but in fact general), and k be a reference wave number ($k = 2\pi\omega/C_0$), where ω is the wave frequency and C_0 is a reference wave speed). Express the full wave field as

$$u(\vec{x}, z, t) = \psi(\vec{x}, z) \exp\left[ik(z - C_0 t) \right] \tag{2.1}$$

Let the wave speed (a function of position only) be

$$C(\vec{x}, z) = C_0 \left[1 - 2U_0(\vec{x}) - 2\mu(\vec{x}, z) \right]^{-1/2} \approx C_0 \left[1 + U_0(\vec{x}) + \mu(\vec{x}, z) \right] \tag{2.2}$$

where U_0 represents the deterministic background and μ represents the fluctuating random medium, assumed to be a realization of a zero-mean Gaussian process.

Then, the parabolic equation (in rectangular coordinates) for the reduced wave function ψ is:

$$ik\partial_z \psi = -\frac{1}{2} \nabla^2 \psi + k^2 U_0(\vec{x})\psi + k^2 \mu(\vec{x}, z)\psi \tag{2.3}$$

where ∇^2 is the transverse Laplacian.

A moment Γ is the ensemble expectation value of a product of ψ's and ψ^*'s where each ψ or ψ^* is evaluated at a different position $\vec{x}_j$ and wavenumber k_j. We write, in abbreviated form,

$$\Gamma_{mn} = \langle \psi_1^* \cdots \psi_m^* \psi_{m+1} \cdots \psi_{m+n} \rangle \tag{2.4}$$

Define an operator L_0 such that

$$L_0 = \sum_{j=1}^{m+n} \pm \frac{1}{k_j} \left(-\frac{1}{2} \nabla_j^2 + k_j^2 U_{0j} \right) \tag{2.5}$$

The terms that apply to the ψ's use the plus sign and those that apply to the ψ^*'s use the minus sign. The subscript j requires that ∇_j^2 operate only on $\vec{x}_j$ and $U_{0j} \equiv U_0(\vec{x}_j)$.

Define the important combination of fluctuation quantities as

$$M(z) = \sum_{j=1}^{m+n} \pm k_j \mu(\vec{x}_j, z) \tag{2.6}$$

Our general moment equation under the Markov approximation can be written

$$\partial_z \Gamma_{mn}(z) = -i L_0 \Gamma_{mn}(z) - \frac{1}{2} \int_{-\infty}^{\infty} dz' < M(z) M_{shift}(z') > \Gamma_{mn}(z) \tag{2.7}$$

where $M_{shift}(z')$ is obtained by evaluating $M(z)$ with all the $\vec{x}_j$ at z shifted by the transverse distance that a deterministic ray through $(\vec{x}_j, z)$ moves in travelling from z to z' (see Figure 1). In other words $M_{shift}(z')$ is evaluated at point B: i.e. $\vec{x}_j = \vec{x}_{ray}(z')$ where the ray is forced to go through $\vec{x}_j(z)$. The particular ray is determined not only by the local position $(\vec{x}_j, z)$, but also by the initial conditions on the moment; for example, the location of a point source, or the direction of a plane wave.[15] The unphysical assumption of delta-correlated medium fluctuations along the propagation direction would imply that $M_{shift}(z')$ would be evaluated at point C: i.e. $\vec{x}_j(z)$ (and z'). In the isotropic case (or in the case of propagation along a principal axis of the anisotropy) the difference between evaluating $M_{shift}(z')$ at $\vec{x}_j(z)$ and $\vec{x}_{ray}(z')$ is negligible, and the delta-correlated assumption is adequate. *In the anisotropic case, the necessity of* defining the unperturbed ray makes (2.7) somewhat complicated to apply for general initial conditions. However, since (2.7) is a linear equation, superposition can be used whether the source is a point, an incident plane wave, or an arbitrary coherent or incoherent sum of point sources.

We now turn to the path integral method. Equation (2.3) has the formal solution

$$\psi = \int D\,\vec{x}(z)\, e^{iS} \tag{2.8}$$

where $\int D\vec{x}(z)$ means integration over paths, $\vec{x}(z)$ is a transverse vector indicating the position of the path at z, and

$$S = k \int_0^R dz \left(\frac{1}{2} \left(\frac{d\vec{x}}{dz} \right)^2 - U_0(\vec{x}) - \mu(\vec{x}, z) \right) \tag{2.9}$$

In order to obtain a given moment, expressions like (2.8) (or its complex conjugate) are multiplied together, and the ensemble average is taken:

$$\Gamma_{mn} = \int \prod_{j=1}^{m+n} D\vec{x}_j(z) < e^{\sum_j \pm i S_j} > \tag{2.10}$$

The Markov approximation yields (See Section V):

$$\Gamma_{mn} = \int \prod_{j=1}^{m+n} D\vec{x}_j(z) \exp \int_0^R dz \left[\sum_j \pm ik_j \left(\frac{1}{2} \left(\frac{d\vec{x}_j}{dz} \right)^2 - U_{0j} \right) - \frac{1}{2} \int_{-\infty}^{\infty} dz' < M(z) M_{shift}(z') > \right] \tag{2.11}$$

We have shown that the moment equations (2.7) and the path integral expressions (2.11) are mathematically equivalent.[12]

Moment equations can be formulated in a variety of coordinate systems, while path integrals require a rectangular coordinate system. There has been a fair amount of effort expended on using polar coordinate systems, especially for point source problems.

The same results (for point sources among others) can be obtained in either polar or rectangular coordinates. Thus, the results of Shishov[16] on the intensity correlation, derived in spherical polar coordinates, can be seen to be identical (after an appropriate transformation) to the results of Codona *et al.*,[17] derived in rectangular coordinates. It was necessary for Shishov to make small angle approximations in addition to the parabolic approximation of dropping the second derivative in the propagation direction, whereas Codona *et al.* only require the single parabolic approximation.

We have derived moment equations in coordinate representation under the Markov approximation that apply in anisotropic, inhomogeneous media with deterministic background. The derivation shows the relationship between these moment equations and modified equations that are valid under approximations weaker than Markov; the second-moment equation of Besieris and Tappert is a special case of these modified equations.

In a hierarchy of approximations we begin with the parabolic wave equation itself. A path integral with non-local exponent can be written as an exact solution, although it is not yet useful in practice. The next level is the approximation that the interaction strength over a correlation length is small—this "first-order perturbation theory" leads to the modified moment equations, and in homogeneous, isotropic media, to the standard moment equations and path-integral expressions. In anisotropic, inhomogeneous media, however, a further approximation is necessary to obtain the moment equations and path integral expressions. This further approximation is that the significant flow of wave energy, or the important paths, are parallel to the unperturbed ray; we call this the Markov approximation because its violation implies the appearance of correlations between successive scatterings. We have shown that the moment equations and the path-integral expressions for the moments are mathematically equivalent under the Markov approximation. Thus the two formalisms have exactly the same physical content. In an anisotropic medium, the moment equation involves a shift operation to calculate the medium correlation function along the unperturbed ray; this form of the moment equation has not been given before.

III. Series Expansion of the Fourth Moment

We have developed a series expression for the fourth moment of a beamed field incident on a random phase screen or an extended medium.[17] The series has a symmetry that allows its first few terms to generate useful approximations at both low and high spatial frequency. The parabolic wave equation, the Markov approximation, and Gaussian refractive index fluctuations are assumed. The result for the phase screen is obtained by Green's-function techniques. The extended-medium result is derived in an analogous manner using path integral methods. The same results can also be derived by moment-equation methods. The behavior of certain leading terms agree with previous results for plane-wave and point-source geometries.

We consider waves propagating from an arbitrary source distribution in a random medium. We assume the statistics of the medium are locally homogeneous, and we make the Markov approximation; i.e. the field fluctuations induced within a correlation length along the propagation direction are weak. For a more complete discussion see another of our papers.[12] The wave propagation is characterized by narrow angular scattering due to the small random fluctuations in refractive index. It is then convenient to write the complex monochromatic scalar field as $E(\vec{x}, z)e^{ikz}$ where z is the propagation direction, $\vec{x}$ is the transverse coordinate and k is the wavenumber of the wave with no refractive index fluctuations.

The random nature of the fields is conveniently described by statistical moments evaluated in the transverse plane located at distance R. Ensemble averages of random variables are denoted by $< >$. The first moment

$$\Gamma_1(\vec{x}, R) = <E(\vec{x}, R)> \quad (3.1)$$

or average of the field and the second moment

$$\Gamma_2(\vec{x}_1, \vec{x}_2, R) = <E(\vec{x}_1, R) E^*(\vec{x}_2, R)> \quad (3.2)$$

or mutual coherence function are well understood.[7] However, there are few analytic results for the fourth moment

$$\Gamma_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R) = <E(\vec{x}_1, R) E^*(\vec{x}_2, R) E(\vec{x}_3, R) E^*(\vec{x}_4, R)> \quad (3.3)$$

Previous theoretical work concentrated on plane-wave and point-source geometry. We present three main results for arbitrary source distribution.

A series expression for the fourth moment is derived as an expansion of the Green's function for the fourth moment, thus avoiding the difficulties associated with the source distribution. For the thin-screen problem, the expansion quantity is a combination of phase structure functions. For the extended random media, the expansion quantity is an analogous combination of phase structure function densities. The Green's function is expressed as a multiple path integral. The resulting series of path integrals is evaluated with a useful identity.

Our second result is the generation of two series for the intensity correlation or intensity spectrum. The fourth moment $\Gamma_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R)$ has the obvious symmetries that it is unchanged by interchanging $\vec{x}_1$ and $\vec{x}_3$ or by interchanging $\vec{x}_2$ and $\vec{x}_4$. Each term of the series expansion does not share the symmetry of the entire expression. Thus two separate series are obtained by invoking symmetry. In principle, either series could be summed to give Γ_4. We demonstrate, however, that it is better to consider both series in order to describe the fourth moment with the fewest number of terms. This assertion is demonstrated for the second moment of intensity or intensity correlation, $C(\vec{x}_1, \vec{x}_2, R)$, which is a special case of the fourth moment, i.e.

$$C(\vec{x}_1, \vec{x}_2, R) = <I(\vec{x}_1, R)\, I(\vec{x}_2, R)> = \Gamma_4(\vec{x}_1, \vec{x}_1, \vec{x}_2, \vec{x}_2) = \Gamma_4(\vec{x}_1, \vec{x}_2, \vec{x}_2, \vec{x}_1) \quad (3.4)$$

Note that the symmetry of the fourth moment has been explicitly indicated. A clear presentation of the behavior of the intensity correlation series obtained from the fourth moment expansion requires the introduction of a spatial spectrum of intensity fluctuations for a spatially nonstationary random process. We adopt the definition

$$\Phi(\vec{\rho},\vec{q},R) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} C(\vec{\rho},\vec{r},R)\, e^{-i\vec{q}\cdot\vec{r}}\, d\vec{r} \tag{3.5}$$

where

$$\vec{\rho} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2) \qquad \vec{r} = \vec{x}_1 - \vec{x}_2 \tag{3.6}$$

(Note the free format of the argument list of functions). The spectrum has the property

$$\int_{-\infty}^{\infty} \Phi(\vec{\rho},\vec{q},R)\, d\vec{q} = C(\vec{\rho},0,R) = \langle I(\vec{\rho},R)^2 \rangle \tag{3.7}$$

It should be noted that the spatial spectrum may depend on the centroid $\vec{\rho}$.

Since there are two series for the intensity correlation there are also two series for the intensity spectrum. The leading terms of one series for $\Phi(\vec{\rho},\vec{q},R)$ describe the small $\vec{q}$ behavior while the other series is valid at high $\vec{q}$. The rate of convergence of each series provides a criterion for merging the two results to produce a complete expression for the intensity spectrum. In general, an analogous treatment of the intensity correlation series is not possible since the leading terms of both series do not converge to the variance as the spatial separation approaches zero.

Our third result is the demonstration of the equivalence of path integral and moment-equation methods. Early theoretical work on WPRM concentrated on geometrical optics and the method of small perturbations.[7,18] These two approaches were limited to weak scattering conditions. This restriction was removed with the introduction of differential equations for the moments of the field.[19] Functional techniques of high energy physics (path integrals and operator methods) provided another point of view to WPRM.[20,8] The moment equation method and functional techniques are equivalent[12] and must generate identical results when expansions are performed in the same quantity. This equivalence can be demonstrated by deriving the same fourth moment series expression using moment-equation methods.[17]

An example of the calculation of the first two terms in the expansion for a particular case is shown in Figure 2.

IV. Summary

Descriptions have been given of three recent results in the theory of wave propagation through random media. The first result is that pulse arrivals can be advanced in time by the imposition of a zero-mean random wave-speed fluctuation on the medium. There are many subtleties to the interpretation of this effect; these subtleties involve intensity weighting, curved rays, and strength of fluctuations. The second result is that moment equations can be written that are mathematically equivalent to the path integral expressions for a general moment, but the moment equations so obtained under the Markov approximation are not quite the standard ones used. The equivalent moment equations require knowledge of the unperturbed rays between source and the point of interest. The third result is a series expansion of the general fourth moment with arbitrary source distribution: a series expansion whose first few terms adequately approximate the high-spatial-frequency part of the intensity spectrum. Previous work in the high-spatial-frequency regime had required the summation of a large number of terms.

References

1. J.L. Codona, D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F.S. Henyey, "Average Arrival Time of Wave Pulses Through Continuous Random Media," *Phys. Rev. Lett.*, vol. 55, 9, 1985.
2. I.P. Williamson, *Proc. R. Soc. Lond.*, vol. 342, 131-147, 1975.
3. C.H. Liu and K.C. Yeh, *J. Opt. Soc. Am.*, vol. 70, 168-172, 1980.
4. S.M. Flatté, R. Dashen, W.M. Munk, K.M. Watson, and F. Zachariasen, *Sound Transmission Through a Fluctuating Ocean,* Cambridge University Press, New York, 1979.
5. S.M. Flatté, "Wave Propagation Through Random Media: Contributions from Ocean Acoustics," *Proceedings of the IEEE*, vol. 71, 1267-1294, 1983.
6. V.A. Fock, "Theory of Radio-Wave Propagation in an Inhomogeneous Atmosphere for a Raised Source," *Bulletin de l'Acadeḿie des Sciences de l'URSS*, 70, 1950. Translated as Ch. 14 of *Electromagnetic Diffraction and Propagation Problems*, by V.A. Fock, Permagamon Press, Oxford, 1965.
7. V.I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation,* p. 223, Israel Program For Scientific Translation, Jerusalem, 1971.
8. R. Dashen, "Path Integrals for Waves in Random Media," *J. Math. Phys.*, vol. 20, 894-920, May 1979.
9. S. M. Flatté, D.R. Bernstein, and R. Dashen, "Intensity Moments by Path Integral Techniques for Wave Propagation Through Random Media, with Application to Sound in the Ocean," *Physics of Fluids*, vol. 26, 1701-1713, 1983.
10. M.J. Beran, A.M. Whitman, and S. Frankenthal, "Scattering Calculations Using the Characteristic Rays of the Coherence Function," *J. Acoust. Soc. Am.*, vol. 71, 1124, 1982.
11. R.N. Baer, J.S. Perkins, E.B. Wright, and J.J. McCoy, "Stochastic Propagation of the Mutual Coherence Function in the Deep Ocean," *J. Acoust. Soc. Am.*, vol. 75, 1407, 1984.
12. J.L. Codona, D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F.S. Henyey, "Moment-Equation and Path-Integral Techniques for Wave Propagation in Random Media," *to be published in J. Math Phys.*, 1985.
13. N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry,* North Holland, New York, 1981.
14. I.M. Besieris and F.D. Tappert, "Kinetic Equations for the Quantized Motion of a Particle in a Randomly Perturbed Potential Field," *J. Math. Phys.*, vol. 14, 1829, 1973.
15. In the ocean there may be many deterministic rays connecting the source and receiver. In that case (2.7) is still correct, but an additional index needs to be added to Γ_{mn} to indicate which deterministic ray is connected to each transverse variable $\vec{x}_j$. An entire matrix of moments would be followed, though each element of the matrix can be followed independently. If the deterministic rays are far enough from each other as to be uncorrelated, or if the source or receiver

distinguishes between rays (e.g. by travel time or angle), then each ray can be treated independently. Near a caustic, at least two rays are very close to each other, but this does not cause any difficulty since the rays are also adequately coincident in angle.

16. V.I. Shishov, "Strong Fluctuations of the Intensity of a Plane Wave Propagating in a Random Medium," *Soviet Physics JETP*, vol. 34, 744, 1972.

17. J.L. Codona, D.B. Creamer, S.M. Flatté, R.G. Frehlich, and F.S. Henyey, "Solution for the Fourth Moment of Waves Propagating in Random Media," *submitted for publication*, 1985.

18. Y.N. Barabanenkov, Y.A. Kravtsov, S.M. Rytov, and V.I. Tatarskii, "Status of the Theory of Propagation in Randomly Inhomogenous Media," *Sov. Phys. Usp.*, vol. 13, 551-680, 1971.

19. A.M. Prokhorov, F.V. Bunkin, K.S. Gochelashvily, and V.I. Shishov, "Laser Irradiance Propagation in Turbulent Media," *Proceedings of the IEEE*, vol. 63, 790-811, 1975.

20. V.I. Klyatskin, "Statistical Theory of Light Propagation in a Turbulent Medium (Review)," *Izv. Vysh. Ucheb. Zaved. Radiofiz.*, vol. 15, 1433-1455, English, Radiophysics and Quantum Electronics 15, 1095-1111, 1973.

Figure Captions

Figure 1a. Moment-equation expression of the Markov approximation. The correlation should be taken between a point at z (point P) and an arbitrary point at z' (point A). Instead it is taken with the point B, obtained by extrapolating along the unperturbed ray from P. The assumption of delta-correlated medium fluctuations leads to the incorrect formulation of correlations between points P and C. The dashed lines indicate the idea of a scattering as a function of angle from point P.

Figure 1b. Path-integral expression of the Markov approximation. The general path at z' (point A) is approximated by the path at z extrapolated along the unperturbed ray (point B).

Figure 2. The leading terms of the intensity spectrum versus normalized spatial frequency, qR_f, where $R_f = (R/k)^{1/2}$ is the Fresnel scale. The curves are calculated from expressions given by Gochelashvily and Shishov [1975] for the case of plane waves incident on a random phase screen with a Kolmogorov spectrum of phase fluctuations and $D_\Theta(R_f)=100$. The (−) sign indicates that $\Phi_1^{hf}(q)$ is negative at high frequency.

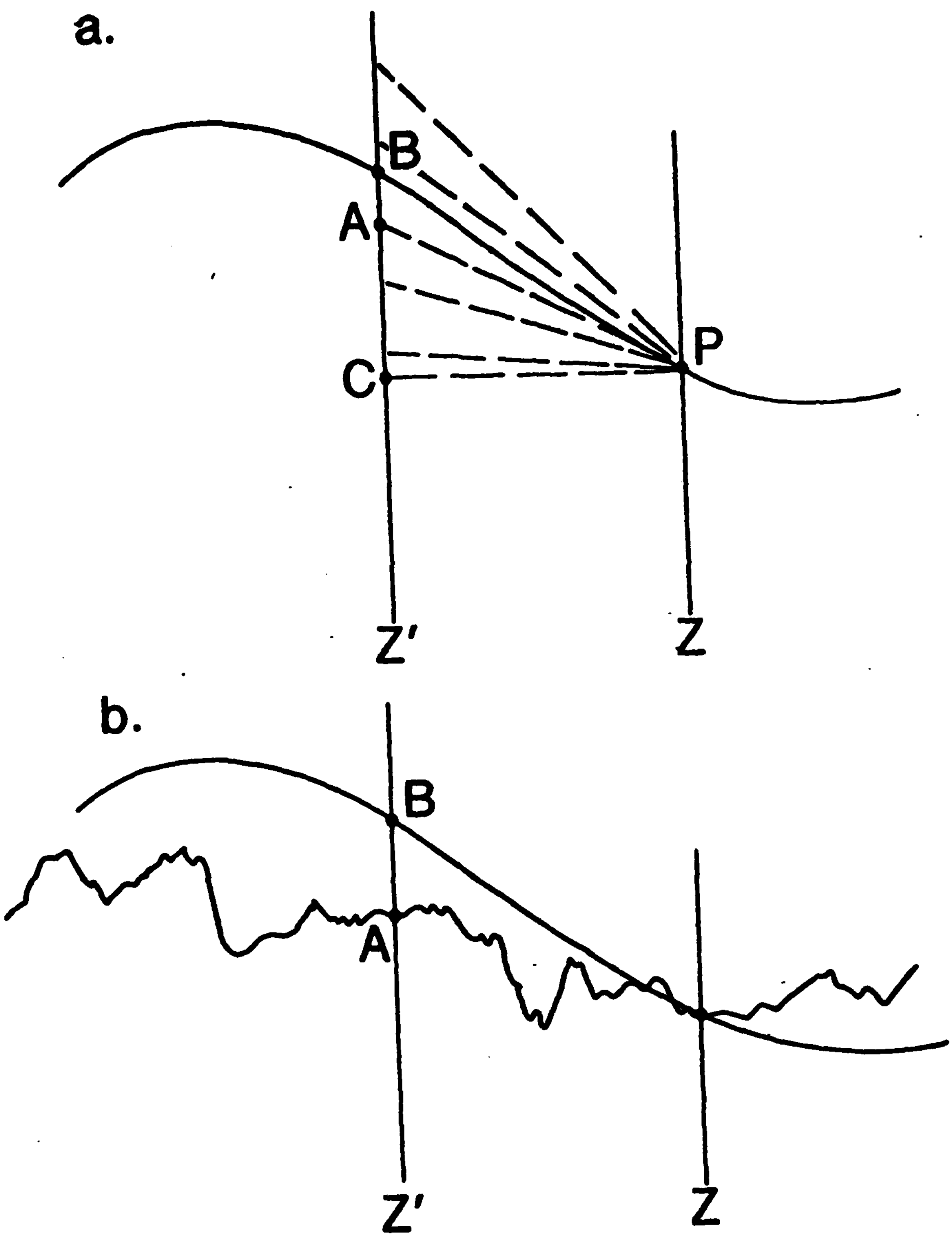
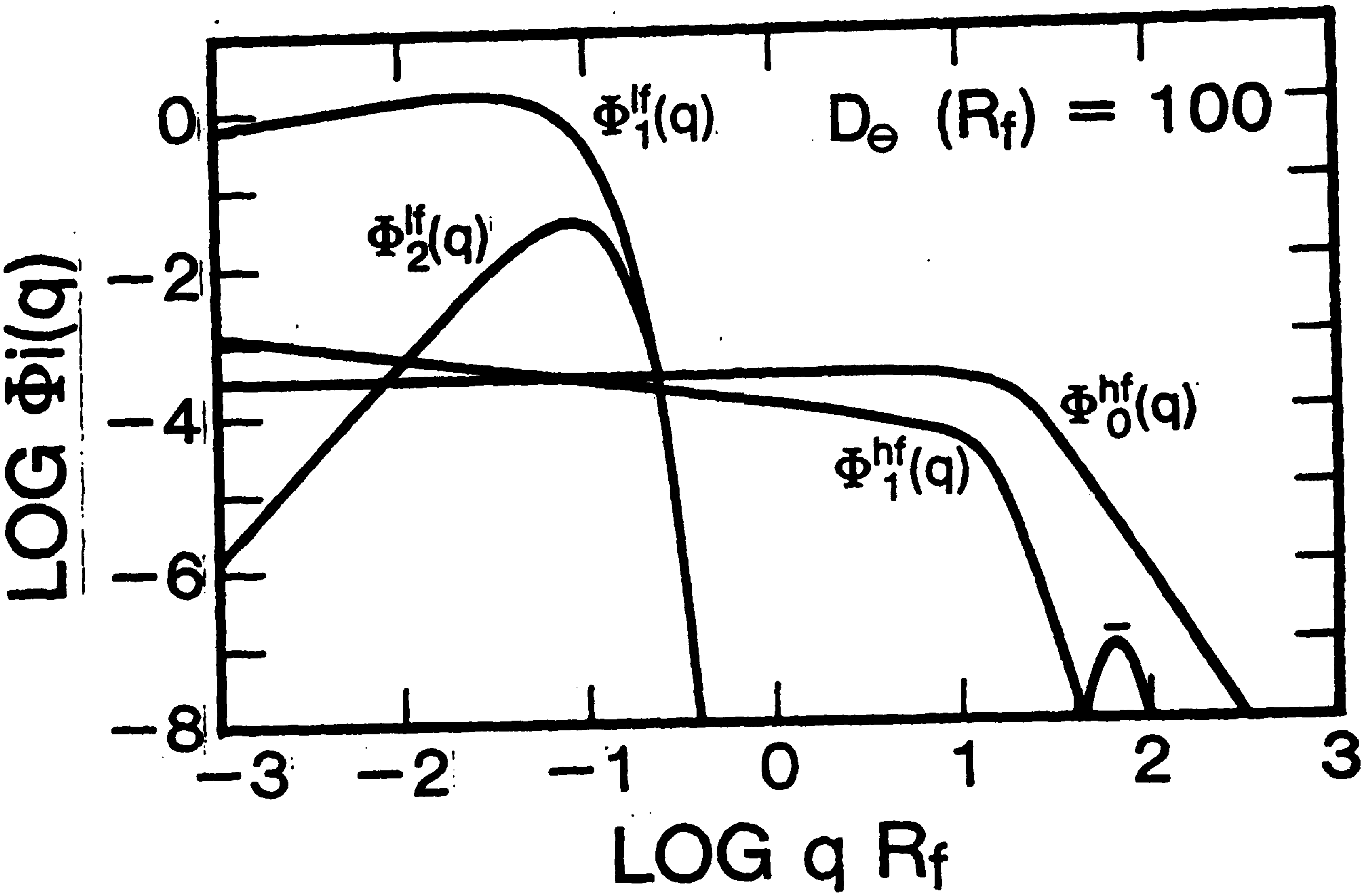


Figure 1



Φ1lf(q)
Dθ (Rf) = 100
Φ2lf(q)
Φ0hf(q)
Φ1hf(q)
0
−2
−4
−6
−8
LOG Φi(q)
−3
−2
−1
0
1
2
3
LOG q Rf

Figure 2

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO. AD-A160 150

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
Theoretical Studies and Data Analysis of Wave Propagation in Random Media

5. TYPE OF REPORT & PERIOD COVERED
Final Technical Report
12-1-83 - 9-30-85

6. PERFORMING ORG. REPORT NUMBER
LJI-R-85-350

7. AUTHOR(s)
Flatte, Stanley M.

8. CONTRACT OR GRANT NUMBER(s)
MDA903-83-C-0515

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Center for Studies of Nonlinear Dynamics
La Jolla Institute
3252 Holiday Court, Suite 208
La Jolla, California 92037

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
4852

11. CONTROLLING OFFICE NAME AND ADDRESS
Director, Defense Advanced Research Projets Agy.
Attn: Program Management Office (PMO)
1400 Wilson Boulevard
Arlington, Virginia 22209

12. REPORT DATE
September 15, 1985

13. NUMBER OF PAGES

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
Defense Support Service - Washington
Rm 1D-245, The Pentagon
Washington, DC 20310
Attn: Mr. Sidney Koss

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release - Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Same

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Wave propatation in random media, path integrals

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is the final technical report of work under this two-year contract on wave propagation in random media with applications to light through the atmosphere, radio waves through the ionosphere, sound through the ocean, and seismic waves through the earth.

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)