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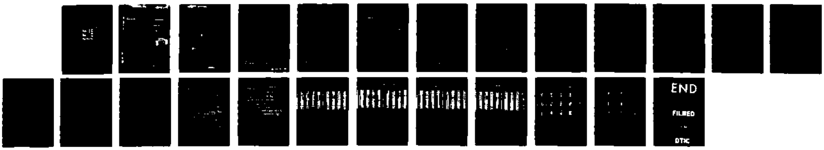

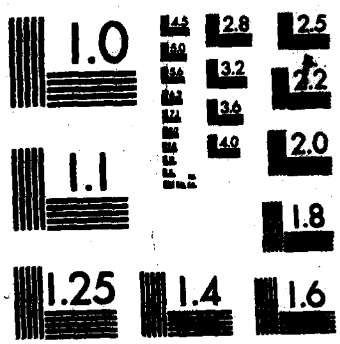

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Summary: Some power series approximations to the exact null distribution of the Chi-bar-square statistic for several testing situations are developed using the first four cumulants of the null distributions, and their performance is investigated numerically. The series expansions use Laguerre polynomials and the associated gamma densities. Chi-bar-square statistics arise when testing the homogeneity of normal means with the alternative restricted by a partial ordering on the means and when testing the ordering against all alternatives. Approximations are provided for the case of a total order and a simple tree with equal, or nearly equal, sample sizes. The numerical investigations indicate the accuracy and usefulness of these approximations.

1. INTRODUCTION We consider situations in which one wishes to test hypotheses about normal means which involve order restrictions. For instance, one may wish to test homogeneity, $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$, with the alternative restricted by the total ordering $H_1: \mu_1 \leq \ldots \leq \mu_k$. On the other hand, one may wish to test H_1 versus $H_2: \mu_i > \mu_{i+1}$ for some i. In comparing several treatments with a control, a test of H_0 with the alternative restricted by the simple tree ordering $H_1': \mu_1 \leq \mu_i$ for $i = 2,3,\ldots,k$ and of H_1' versus $H_2': \mu_1 > \mu_i$ for some $i = 2,3,\ldots,k$ are of interest. If the common variance of these normal populations is known, then the likelihood ratio test statistics have null distributions which are mixtures of chi-square distributions, which Bartholomew (1959) called chi-bar-square statistics. They also provide approximations for large degrees of freedom.

The chi-bar-square distributions also arise as approximations when considering multinomial parameters (Robertson, 1978) one-parameter exponential families (Robertson and Wegman, 1978), Poisson intensities (Magel and Wright, 1984) and nonparametric tests (Shirley, 1977 and Robertson and Wright, 1985).

A great deal of information (e.g. the location, variability about the mean, skewness and kurtosis of a distribution) is contained in the first four moments of a distribution, and so we consider four-moment approximations for these chi-bar-square distributions. A natural choice is to use the first four terms of a series expansion involving Laquerre polynomials and the associated gamma distributions. It should be noted that numerical investigations show that using higher moments, such as the fifth and sixth, does not seem to improve the approximation enough to warrant the extra effort. Sasabuchi and Kulatunga (1985) provide similar approximations using the first three moments for the test of H_0 versus $H_1 - H_0$ with unknown variance and they are based on expansions using Jacobi polynomials and the associated beta distributions.

The approximations presented here are based on the first four moments, or equivalently on the first four cumulants, of the chi-bar-square distributions. Because the mixing coefficients for these distributions are intractable for unequal sample sizes and even moderate k, we restrict attention to the case of equal sample sizes. However, Robertson and Wright (1983) and Wright and Tran (1985) have shown that the chi-bar-square distributions are robust to moderate changes in the sample sizes for both the total order and the simple tree. Hence, the approximations would be reasonable if there is not too much variation in the sample sizes.

Approximations for the totally ordered case are presented in Section 2. The simple tree ordering is considered in Section 3 and the results of our numerical investigation are summarized in Section 4. Bartholomew (1959, p. 330) proposed a two-moment approximation which is equivalent to using the first term, ie the zero-th order term, in the Laguerre expansion. The chi-bar-square distributions may assign positive probability to {0} and so we show how the two and four-moment approximations can be corrected for the discrete part. This type of correction was employed by Sasabuchi and Kulatunga (1985). We found that, independent of the value of k, the corrected two-moment approximation is adequate except in the far right tail of the chi-bar-square distributions, but to the right of the 99th percentile the increase in accuracy warrants the use of the corrected four-moment approximation.

Roy and Tiku (1962), Tiku (1964, 1965, 1971, 1975), Tan and Wong (1977, 1978, 1980), and Hirotsu (1979) have used Laguerre series approximations to approximate the sampling distributions of F-ratios in the analysis of variance problems and related topics.

2. Series Approximations: The Totally Ordered Case. In this section, we consider approximations to the null distributions of the likelihood ratio test of H_0 versus H_1-H_0 and of H_1 versus H_2 based on Laguerre polynomial expansions.

Assume that $\{y_{ij};\ j = 1,\ldots,n\}$ for $i = 1,\ldots,k$ are independent random samples from k normally distributed populations with mean μ_i and common variance σ^2. Consider the hypotheses H_0, H_1 and H_2 as defined in introduction, ie.

$$H_0: \mu_1 = \mu_2 = \ldots = \mu_k$$

$$H_1: \mu_1 \leq \mu_2 \leq \quad \leq \mu_k$$

$$H_2: \mu_i > \mu_{i+1} \quad i = 1,2,\ldots,k-1.$$

When σ^2 is known, the likelihood ratio test of H_0 versus H_1-H_0 rejects H_0 for large values of

$$T_{01} = n \sum_{i=1}^{k} (\mu_i^* - \hat{\mu})^2/\sigma^2$$

where $\mu^* = (\mu_1^*,\ldots,\mu_k^*)$ is the maximum likelihood estimate of $\mu = (\mu_1,\ldots,\mu_k)$ under H_1 and $\hat{\mu} = \sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}/nk$; and under H_0,

$$\mathrm{pr}(T_{01} \geq t) = \sum_{\ell=2}^{k} P(\ell,k)\ \mathrm{pr}(\chi^2_{\ell-1} \geq t), \quad t > 0$$

$$\mathrm{pr}(T_{01} = 0) = 1/k$$

where $P(\ell,k)$ denotes the probability, under H_0, that the coordinates of μ^* have exactly ℓ distinct values, and $\chi^2_{\ell-1}$ denotes a standard chi-squared variable having $\ell-1$ degrees of freedom, with $\chi^2_0 \equiv 0$, cf. Barlow et al. (1972). The likelihood ratio test of H_1 versus H_2 rejects H_1 for large values of

$$T_{12} = n \sum_{i=1}^{k} (\mu_i^* - \bar{y}_i)^2/\sigma^2, \quad \bar{y}_i = \sum_{j=1}^{n} y_{ij}/n,$$

H_0 is least favorable within H_1, and under H_0

$$\mathrm{pr}(T_{12} \geq t) = \sum_{\ell=1}^{k-1} P(\ell,k)\mathrm{pr}(\chi^2_{k-\ell} \geq t) \ , \ t > 0$$

$$\mathrm{pr}(T_{12} = 0) = 1/k!,$$

cf. Robertson and Wegman (1978).

To compute a p-value for either T_{01} or T_{12}, one needs to obtain the $P(\ell,k)$ either from Table A.5 of Barlow et al. (1972) if $k \leq 12$ or from their recursive relation, p. 145, for $k > 12$, and then compute the $k - 1$ chi-square tail probabilities. Hence, approximations are of interest for large k.

In the following paragraphs, four approximations to the null distribution of the statistic T_{01} are presented in detail, and the corresponding approximations to the null distribution of the statistic T_{12} are described very briefly.

2.1 Approximations to the Null Distribution of T_{01}.

Now, four series approximations to the null distribution of T_{01} are discussed.

(i) Four-Moment Approximation of T_{01}

First, the null distribution of T_{01} is approximated by a scaled gamma density. That is, $T_{01} = \rho X_b$ where $\rho > 0$ and X_b has density

$$g_b(x) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x}, \quad x > 0$$

ie., the gamma density with parameters (b,1). Equating the first two cumulants of T_{01} with those of ρX_b, one obtains

$$b = k_1/\rho, \quad \rho = k_2/k_1 \tag{2.1}$$

where k_1 amd k_2 are the first two cumulants as in equation (3.47) of Barlow et. al. (1972, p. 151). Then, following Davis (1976), Gideon and Gurland (1977), and Kotz, Johnson and Boyd (1967 a,b), it can be shown that the probability density function of $X = T_{01}/\rho$ can be expanded in a convergent infinite series involving Laguerre polynomials and the associated gamma densities as

$$\begin{aligned} f(x) &= \{1 + \sum_{j=3}^{\infty} c_j L_j^b(x)\} g_b(x) \\ &= g_b(x) + \sum_{j=3}^{\infty} d_j \sum_{s=0}^{j} \binom{j}{s} (-1)^s g_{b+s}(x) \end{aligned}$$

where

$$L_j^b(x) = \frac{1}{j!} \sum_{s=0}^{j} \binom{j}{s} (-x)^s \frac{\Gamma(b+j)}{\Gamma(b+s)}$$

is the Laguerre polynomial of degree j, and

$$d_j = c_j \binom{b+j-1}{j} = E\{L_j^b(X)\}.$$

To approximate the distribution only the terms up to and including $j = 4$ are retained. That is, with $f(x)$ the density of T_{01}/ρ,

$$f(x) = g_b(x) + \sum_{j=3}^{4} d_j \sum_{s=0}^{j} \binom{j}{s} (-1)^s g_{b+s}(x) \tag{2.2}$$

where

$$d_3 = \frac{1}{3!}(-k_3^* + 2b)$$

$$d_4 = \frac{1}{4!}(k_4^* - 12\ k_3^* + 18b) \qquad (2.3)$$

$$k_3^* = k_3/\rho^3,\ k_4^* = k_4/\rho^4$$

and k_3, k_4 are the third and fourth cumulants given by equation (3.47) in Barlow et. al. ((1972, p. 151).

(ii) Four-Moment Approximation of T_{01} With Correction

Note that $pr(T_{01} = 0) = 1/k$. Therefore, the characteristic function of the conditional distribution of T_{01} given that $T_{01} > 0$, is given by

$$\phi^*(t) = (\phi(t) - k^{-1})/(1-k^{-1})$$

where

$$\phi(t) = (z + 1)(z + 2)\ldots(z + k-1)/k!$$

and $z = (1-2it)^{-1/2}$.

The first four cumulants of the conditional distribution of T_{01} given that $T_{01} > 0$, are given by

$$k_1^{**} = \frac{k}{(k-1)}\ k_1$$

$$k_2^{**} = \frac{k}{(k-1)}\ (k_2 + k_1^2) - \frac{k^2}{(k-1)^2}\ k_1^2$$

$$k_3^{**} = \frac{k}{(k-1)}(k_3 + 3k_2k_1 + k_1^3) - 3\ \frac{k^2}{(k-1)^2}\ k_1(k_2 + k_1^2)$$

$$+ 2\ \frac{k^3}{(k-1)^3}\ k_1^3$$

$$k_4^{**} = \frac{k}{(k-1)}(k_4 + 3k_2^2 + 4\ k_1k_3 + 6\ k_1^2k_2 + k_1^4)$$

$$- 4 \frac{k^2}{(k-1)^2} k_1(k_3 + 3k_2k_1 + k_1^3)$$

$$- 3 \frac{k^2}{(k-1)^2}(k_2 + k_1^2) + 12 \frac{k^3}{(k-1)^3} k_1^2(k_2 + k_1^2)$$

$$- 6 \frac{k^4}{(k-1)^4} k_1^4$$

where $k_1,\ldots,k_4$ are the cumulants given by equation (3.47) in Barlow et. al. (1972, p. 151).

The corrected four-moment approximation is obtained by taking

$$b = k_1^{**}/\rho \ , \ \rho = k_2^{**}/k_1^{**}$$

$$d_3 = \frac{1}{3!}(-k_3^* + 2b)$$

$$d_4 = \frac{1}{4!}(k_4^* - 12\ k_3^* + 18b) \tag{2.4}$$

$$k_3^* = k_3^{**}/\rho^3 \text{ and } k_4^* = k_4^{**}/\rho^4$$

in the series expansion for $f(x)$ in(2.2.)

In particular, let

$$\bar{G}_b(x) = \int_x^\infty g_b(x)dx.$$

For $t > 0$, under H_0, $pr(T_{01} \geq t)$ is approximated by

$$(1-k^{-1})\sum_{j=0}^{4} a_j\ \bar{G}_{b+j}(t/\rho) \tag{2.5}$$

with

$$a_0 = 1 + d_3 + d_4,\ a_1 = -(3d_3 + 4d_4),\ a_2 = (3d_3 + d_4),\ a_3 = -(d_3+4d_4) \tag{2.6}$$

$$\text{and} \quad a_4 = d_4.$$

For $5 \le k \le 40$, the values of b, ρ, d_3 and d_4 are given in Table 1.

(iii) Two-Moment Approximation of T_{01}

In the two-moment approximation, the first two cumulants of the exact null distribution of T_{01} are made equal to those of a scaled gamma distribution, and it can be obtained as a special case of the four-moment series approximation by taking $d_3 = d_4 = 0$ in (2.2). That is under H_0

$$pr(T_{01} \ge t) \approx \bar{G}_b(t/\rho) \quad \text{for } t > 0 \tag{2.7}$$

where b and ρ are given by (2.1). Note that this approximation is due to Bartholomew (1959, p. 330).

(iv) Two-Moment Approximation of T_{01} With Correction

The two-moment approximation to the null distribution of T_{01} with correction is obtained by using (2.7) where now b and ρ are given by (2.4). Hence, under H_0,

$$pr(T_{01} \ge t) \approx (1-k^{-1})\, \bar{G}_b(t/\rho) \quad \text{for } t > 0, \tag{2.8}$$

and the values of b and ρ are given in Table 1. This kind of approximation with correciton is suggested by Sasabuchi and Kulatunga (1985) in approximating the null distribution of the E-bar-square statistic.

2.2 Approximations to the Null Distribution of T_{12}. Note that $pr(T_{12} = 0) = 1/k!$, which is small even for moderately large value of k, and so, correcting for the discrete part may not improve the approximation significantly. Therefore, only two approximations to the null distribution of T_{12} are given.

The characteristic function of the null distribution of T_{12} is

$$\phi(t) = E(e^{itT_{12}}) = \frac{(z+1)(z+2)\dots(z+k-1)}{z^{k-1}\,(k!)}$$

where $z = (1-2it)^{\frac{1}{2}}$. The cumulant generating function is thus

$$\psi(t) = \ln \phi(t) = \sum_{j=1}^{k-1} \ln(z+j) - (k-1)\ln z - \ln k!$$

The first four cumulants of T_{12} are given by

$$k_1 = (k-1) - \sum_{j=2}^{k} j^{-1}$$

$$k = 2(k-1) - \sum_{j=2}^{k} j^{-1} - \sum_{j=2}^{k} j^{-2}$$

$$k_3 = 8(k-1) - \sum_{j=2}^{k} 3j^{-1} - \sum_{j=2}^{k} 3j^{-2} - \sum_{j=2}^{k} 2j^{-3} \tag{2.9}$$

$$k_4 = 48(k-1) - \sum_{j=2}^{k} 15\, j^{-1} - \sum_{j=2}^{k} 15\, j^{-2}$$

$$- \sum_{j=2}^{k} 12 j^{-3} - \sum_{j=2}^{k} 6\, j^{-4}$$

(i) Four-Moment Approximation of T_{12}

Again let

$$b = k_1/\rho \text{ and } \rho = k_2/k_1 \tag{2.10}$$

where k_1 and k_2 are the first two cumulants of T_{12} given by (2.9). Then, the four-moment approximation to the null distribution of T_{12}/ρ is given by (2.2) and (2.3) where now k_3 and k_4 are the third and fourth cumulants of T_{12} given by (2.9). In particular, for $t > 0$, under H_0, $pr(T_{12} \geq t)$ is approximated by

$$\sum_{j=0}^{q} a_j\, \bar{G}_{b+j}(t/\rho) \tag{2.11}$$

with a_j given by (2.6) and b, ,d_3 and d_4 are given in Table 2 for $5 \le k \le 40$.

(ii) Two-Moment Approximation of T_{12}

The two-moment approximation to the distribution of T_{12}/ρ, under H_0, is

$$pr(T_{12} \geq t) \approx \bar{G}_b(t/\rho) \quad \text{for } t > 0$$

where b and ρ are given by (2.10) or maybe found in Table 2.

3. Series Approximations: The Simple Tree Ordering. In this section, we consider approximations to the null distributions of the likelihood ratio test of H_0 versus $H_1'-H_0$ and of H_1' versus H_2' based on Laguerre polynomial expansions. Recall, $H_1':\mu_1 \le \mu_i$ for $i = 2,3,\ldots,k$ and $H_2':\mu_1 > \mu_i$ for some $i = 2,3,\ldots,k$. As in Section 2, we let y_{ij}, $1 \le j \le n$ and $1 \le i \le k$, denote the observations with $y_{ij} \sim N(\mu_i,\sigma^2)$ and consider the case of known variances. If $\tilde{\mu} = (\tilde{\mu}_1,\tilde{\mu}_2,\ldots\tilde{\mu}_k)$ denotes the maximum likelihood estimate of μ subject to the restriction $\tilde{\mu}\in H_1'$, then the likelihood ratio test rejects H_0 for large values of

$$T_{01}' = n\,\Sigma_{i=1}^{k}(\tilde{\mu}_i-\hat{\mu})^2/\sigma^2$$

and under H_0,

$$pr(T_{01}' \geq t) = \Sigma_{\ell=2}^{k} Q(\ell,k)\,pr(\chi^2_{\ell-1} \geq t)\ ,\ t > 0$$

$$pr(T_{01}' = 0) = Q(1,k)$$

where $Q(\ell,k)$ is the probability, under H_0, that the coordinates of $\tilde{\mu}$ have exactly ℓ distinct values, cf. Barlow et. al. (1972). The likelihood ratio test of H_1' versus H_2' rejects H_1' for large values of

$$T_{12}' = n\ \Sigma_{i=1}^{k}(\tilde{\mu}_i - \bar{y}_i)^2/\sigma^2,$$

H_0 is least favorable within H_1' and under H_0,

$$pr(T_{12}' \geq t) = \Sigma_{\ell=1}^{k-1} Q(\ell,k)\ pr(\chi^2_{k-\ell} \geq t),\ t > 0$$

$$pr(T_{12}' = 0) = Q(k,k) = 1/k,$$

cf. Robertson and Wegman (1978). For $k \leq 12$, the $Q(\ell,k)$ are given in Table A.6 of Barlow et. al. (1972) and for $k > 12$, they may be obtained from their (3.38) and (3.39). However, a numerical integration is needed to obtain $Q(\ell,k)$ for $2 \leq \ell \leq k$.

The characteristic functions of T_{01}' and T_{12}' are given by

$$\phi_1(t) = \Sigma_{\ell=1}^{k} Q(\ell,k)\ (1-2it)^{-\frac{\ell-1}{2}} \text{ and } \phi_2(t) = \Sigma_{\ell=1}^{k} Q(\ell,k)(1-2it)^{-\frac{k-\ell}{2}},$$

respectively, Carrying out the numerical integrations needed to compute $Q(\ell,k)$, one can obtain the first four cumulants of T_{01}' and T_{12}'. We see from Table A.6 of Barlow et.al. (1972) that $Q(1,k)$ is converging to zero fairly rapidly, i.e. $Q(1,5) < .01$ and hence we need not correct for the discrete part of T_{01}'. As is the case for the approximations without correction, $pr(T_{01}' \geq t)$ is approximated by (2.1) with $b = k_1/\rho$, $\rho = k_2/k_1$ and d_3 and d_4 are given by (2.3). For T_{01}' with $5 \leq k \leq 40$, the values of b, ρ, d_3 and d_4 are given in Table 3. Furthermore, the two-moment approximation gives

$pr(T_{01}' \geq t) = \bar{G}_b(t/\rho)$ for $t > 0$ with b and ρ taken from Table 3.

3.2 Approximations to the Null Distribution of T'_{12}

In this case, $pr(T'_{12} = 0) = 1/k$ and so we consider approximations corrected for the discrete part of T'_{12} under H_0, the four-moment approximation for $pr(T'_{12} \geq t)$ with $t > 0$ is given by (2.5) with b, ρ, d_3 and d_4 given in Table 4. Of course, the two-moment approximation under H_0 is given by (2.8) for $t > 0$.

4. Numerical Comparisons. For $k = 5$, 10, 15 and 20 and t successive integers the exact value of $pr(T_{01} \geq t)$ under H_0, the two-moment, the corrected two-moment, the four-moment and the corrected four-moment approximations were computed. Table 5 gives these values to four decimal places along with the percentage errors to the nearest 1/10 of a percent for $k = 5$, 10 and 20 and those t which make the exact values closest to 0.2, 0.1, 0.05, 0.01 and 0.005.

Examining Table 5, one sees that the correction for the discrete part is worthwhile. Even for k as large as 20 this is true in the right tail. For practical purposes the corrected two-moment approximation could be used except possibly for the far right tail, say at the 99th percentile and beyond. There was considerable improvement obtained by using the corrected four-moment approximation for such values for all k studied.

Similar computations were carried out for $pr(T_{12} \geq t)$ and the results are summarized in Table 6. While the trend observed in the approximation of $pr(T_{01} \geq t)$ continues in this case, it seems that for $k \geq 10$ the two-moment approximation would be adequate for practical purposes.

Studying Tables A.5 and A.6 of Barlow et al. (1972) we see $Q(\ell,k)$ behaves somewhat like $P(k- +1,k)$, and so one would expect that the behavior of the approximations for $pr(T'_{01} \geq t)$ would be like those for $pr(T_{12} \geq t)$ and those for $pr(T'_{12} \geq t)$ would behave like those for $pr(T_{01} \geq t)$. For this reason we did not conduct as thorough a study of the approximations for $pr(T'_{01} \geq t)$ and $pr(T'_{12} \geq t)$. However, for k = 10 we did compute $pr(T'_{01} \geq t)$ for t = 15 and 21, as well as the two-moment and four-moment approximations. The error percentages are for t = 15 (21) 0.5% (8.1%) for the four-moment approximation, and 1.9%(10.5%) for the two-moment approximation. These percentages are very similar to those for $pr(T_{12} \geq t)$. For $pr(T'_{12} \geq t)$, we computed the exact value. The corrected two-moment and corrected four-moment approximations for t = 7 and 12. The error percentages for t = 7 (12) are 6.0%(4.5%) for the four-moment approximation, and 0.7%(15.8%) for the two-moment approximation. Again, these percentages are much like those for $pr(T_{01} \geq t)$.

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Table 1. Coefficients for the corrected two-moment and four-moment approximations to the null distribution of T_{01} .

k	ρ	b	d_3	d_4	k	ρ	b	d_3	d_4
5	2.31791	0.69207	0.01352	0.01691	23	2.65541	1.07651	0.04081	0.05420
6	2.37111	0.73383	0.01650	0.02083	24	2.66168	1.08828	0.04163	0.05536
7	2.41322	0.77006	0.01908	0.02428	25	2.66757	1.09961	0.04242	0.05647
8	2.44757	0.80213	0.02137	0.02735	26	2.67310	1.11054	0.04318	0.05755
9	2.47626	0.83093	0.02343	0.03013	27	2.67832	1.12110	0.04392	0.05859
10	2.50066	0.85709	0.02529	0.03267	28	2.68325	1.13131	0.04463	0.05959
11	2.52174	0.88108	0.02700	0.03501	29	2.68791	1.14119	0.04532	0.06056
12	2.54017	0.90325	0.02858	0.03717	30	2.69233	1.15077	0.04598	0.06150
13	2.55646	0.92386	0.03004	0.03919	31	2.69654	1.16006	0.04662	0.06242
14	2.57098	0.94313	0.03141	0.04108	32	2.70053	1.16909	0.04725	0.06330
15	2.58403	0.96122	0.03269	0.04285	33	2.70434	1.17785	0.04786	0.06416
16	2.59584	0.97828	0.03390	0.04453	34	2.70798	1.18638	0.04845	0.06500
17	2.60658	0.99442	0.03504	0.04612	35	2.71145	1.19469	0.04902	0.06581
18	2.61641	1.00973	0.03612	0.04762	36	2.71477	1.20277	0.04958	0.06661
19	2.62544	1.02431	0.03714	0.04906	37	2.71796	1.21066	0.05012	0.06738
20	2.63378	1.03823	0.03812	0.05043	38	2.72101	1.21835	0.05065	0.06813
21	2.64151	1.05153	0.03906	0.05174	39	2.72394	1.22586	0.05116	0.06887
22	2.64870	1.06428	0.03995	0.05299	40	2.72675	1.23319	0.05167	0.06958

Table 2. Coefficients for the two-moment and four-moment approximations to the null distribution of T_{12} .

k	ρ	b	d_3	d_4	k	ρ	b	d_3	d_4
5	2.30174	1.18027	0.03277	0.04535	23	2.11066	9.12780	0.10752	0.14656
6	2.27003	1.56386	0.03977	0.05490	24	2.10740	9.59670	0.11004	0.14987
7	2.24530	1.96283	0.04618	0.06363	25	2.10434	10.06682	0.11247	0.15350
8	2.22537	2.37360	0.05207	0.07164	26	2.10148	10.53808	0.11471	0.15623
9	2.20891	2.79370	0.05752	0.07903	27	2.09880	11.01036	0.11708	0.15976
10	2.19505	3.22135	0.06258	0.08588	28	2.09627	11.48364	0.11914	0.16213
11	2.18319	3.65527	0.06729	0.09225	29	2.09389	11.95782	0.12120	0.16499
12	2.17290	4.09443	0.07171	0.09823	30	2.09164	12.43284	0.12329	0.16814
13	2.16388	4.53809	0.07587	0.10384	31	2.08951	12.90865	0.12532	0.17099
14	2.15589	4.98562	0.07978	0.10911	32	2.08749	13.38521	0.12721	0.17388
15	2.14876	5.43651	0.08349	0.11414	33	2.08557	13.86249	0.12902	0.17501
16	2.14235	5.89037	0.08700	0.11888	34	2.08375	14.34039	0.13097	0.17884
17	2.13656	6.34687	0.09035	0.12339	35	2.08201	14.81892	0.13281	0.18221
18	2.13128	6.80571	0.09352	0.12765	36	2.08036	15.29805	0.13444	0.18362
19	2.12646	7.26665	0.09656	0.13181	37	2.07877	15.77775	0.13608	0.18514
20	2.12203	7.72950	0.09947	0.13567	38	2.07727	16.25794	0.13788	0.19053
21	2.11795	8.19407	0.10227	0.13949	39	2.07582	16.73867	0.13936	0.18951
22	2.11417	8.66020	0.10501	0.14326	40	2.07444	17.21983	0.14125	0.19188

Table 3. Coefficients for the two-moment and four-moment approximations to the null distribution of T'_{01} .

k	ρ	b	d_3	d_4	k	ρ	b	d_3	d_4
5	2.31135	1.17029	0.03366	0.04674	23	2.13511	8.93305	0.13041	0.16981
6	2.28329	1.54569	0.04152	0.05762	24	2.13155	9.39352	0.13192	0.16507
7	2.26135	1.93529	0.04899	0.06797	25	2.12814	9.85572	0.13251	0.15653
8	2.24357	2.33592	0.05611	0.07784	26	2.12485	10.31963	0.13188	0.14279
9	2.22877	2.74539	0.06289	0.08726	27	2.12167	10.78520	0.12983	0.12234
10	2.21619	3.16214	0.06939	0.09627	28	2.11859	11.25236	0.12632	0.09456
11	2.20532	3.58499	0.07560	0.10491	29	2.11559	11.72110	0.12103	0.05753
12	2.19580	4.01305	0.08157	0.11322	30	2.11266	12.19135	0.11386	0.01116
13	2.18738	4.44559	0.08732	0.12122	31	2.10982	12.66299	0.10506	-0.04644
14	2.17986	4.88206	0.09285	0.12893	32	2.10705	13.13600	0.09394	-0.11654
15	2.17309	5.32199	0.09818	0.13633	33	2.10437	13.61022	0.08100	-0.19787
16	2.16694	5.76502	0.10330	0.14343	34	2.10178	14.08549	0.06637	-0.29185
17	2.16132	6.21083	0.10821	0.15016	35	2.09929	14.56163	0.05016	-0.39810
18	2.15616	6.65916	0.11291	0.15644	36	2.09694	15.03839	0.03309	-0.51275
19	2.15138	7.10984	0.11729	0.16191	37	2.09475	15.51533	0.01625	-0.63262
20	2.14693	7.56269	0.12131	0.16635	38	2.09273	15.99235	-0.00054	-0.75338
21	2.14277	8.01756	0.12495	0.16958	39	2.09096	16.46856	-0.01429	-0.86484
22	2.13884	8.47437	0.12804	0.17112	40	2.08930	16.94505	-0.02789	-0.97903

Table 4. Coefficients for the corrected two-moment and four-moment approximations to the null distribution of T'_{12} .

k	ρ	b	d_3	d_4	k	ρ	b	d_3	d_4
5	2.32655	0.69580	0.01390	0.01740	23	2.75232	1.11216	0.04553	0.06070
6	2.38563	0.73980	0.01715	0.02166	24	2.76220	1.12520	0.04653	0.06210
7	2.43385	0.77829	0.02001	0.02548	25	2.77160	1.13777	0.04748	0.06345
8	2.47431	0.81255	0.02258	0.02893	26	2.78058	1.14990	0.04839	0.06476
9	2.50898	0.84349	0.02492	0.03209	27	2.78916	1.16163	0.04928	0.06601
10	2.53917	0.87172	0.02706	0.03500	28	2.79739	1.17298	0.05013	0.06722
11	2.56582	0.89770	0.02905	0.03771	29	2.80529	1.18397	0.05095	0.06839
12	2.58961	0.92178	0.03089	0.04025	30	2.81288	1.19463	0.05175	0.06953
13	2.61103	0.94424	0.03262	0.04262	31	2.82020	1.20497	0.05253	0.07062
14	2.63048	0.96528	0.03424	0.04486	32	2.82725	1.21502	0.05328	0.07169
15	2.64827	0.98509	0.03576	0.04698	33	2.83406	1.22479	0.05401	0.07272
16	2.66463	1.00380	0.03721	0.04899	34	2.84064	1.23429	0.05472	0.07373
17	2.67978	1.02153	0.03858	0.05091	35	2.84701	1.24353	0.05541	0.07471
18	2.69386	1.03840	0.03988	0.05273	36	2.85319	1.25254	0.05609	0.07567
19	2.70701	1.05447	0.04112	0.05446	37	2.85918	1.26130	0.05675	0.07661
20	2.71935	1.06982	0.04230	0.05612	38	2.86501	1.26985	0.05740	0.07752
21	2.73096	1.08452	0.04342	0.05771	39	2.87068	1.27817	0.05804	0.07842
22	2.74193	1.09861	0.04450	0.05924	40	2.87618	1.28630	0.05866	0.07930

Table 5. Exact and Approximate Values for $pr(T_{01} \geq t)$ Under H_0

t	two-moment	%-error	corrected two-moment	%-error	four-moment	%-error	corrected four-moment	%-error	exact
				k=5					
2	0.2114	6.7	0.2221	2.0	0.2136	5.8	0.2219	2.1	0.2267
4	0.0786	6.0	0.0815	2.5	0.0853	2.0	0.0842	0.8	0.0836
5	0.0495	3.3	0.0503	1.7	0.0543	6.0	0.0526	2.7	0.0512
8	0.0132	10.9	0.0123	3.6	0.0128	7.3	0.0124	4.5	0.0119
10	0.0056	25.0	0.0049	9.1	0.0044	3.1	0.0045	1.0	0.0045
				k=10					
3	0.2151	6.4	0.2219	3.4	0.2211	3.7	0.2244	2.3	0.2297
5	0.0931	5.1	0.0949	3.3	0.1014	3.4	0.1000	1.9	0.0981
7	0.0414	0.3	0.0411	0.4	0.0449	8.8	0.0438	6.0	0.0413
10	0.0126	14.8	0.0119	8.0	0.0115	4.2	0.0115	4.1	0.0110
12	0.0058	28.5	0.0052	16.1	0.0041	9.7	0.0043	5.7	0.0045
				k=20					
4	0.2156	5.9	0.2196	4.1	0.2243	2.0	0.2251	1.7	0.2290
6	0.1029	4.5	0.1039	3.5	0.1122	4.2	0.1108	2.9	0.1077
8	0.0493	0.3	0.0490	0.3	0.0534	8.6	0.0526	6.9	0.0492
12	0.0114	18.2	0.0109	12.7	0.0096	0.6	0.0097	0.9	0.0097
14	0.0055	31.6	0.0051	22.4	0.0035	17.1	0.0036	12.8	0.0042

Table 6. Exact and Approximate Values for $pr(T_{12} \geq t)$ Under H_0.

t	two-moment	%-error	four-moment	%-error	exact
			k=5		
4	0.2266	2.9	0.2319	0.6	0.2334
6	0.1002	1.8	0.1055	3.3	0.1021
8	0.0438	1.2	0.0456	5.5	0.0432
11	0.0125	8.7	0.0114	0.5	0.0115
13	0.0054	15.3	0.0041	11.9	0.0047
			k=10		
10	0.2000	1.1	0.2039	0.9	0.2022
12	0.1114	0.4	0.1134	1.3	0.1119
15	0.0430	1.6	0.0426	0.5	0.0424
19	0.0110	6.1	0.0099	4.0	0.0104
21	0.0054	9.0	0.0046	7.0	0.0049
			k=20		
21	0.2021	0.4	0.2034	0.3	0.2029
24	0.1064	0.0	0.1066	0.2	0.1063
27	0.0520	0.8	0.0515	0.2	0.0516
33	0.0104	3.4	0.0099	1.4	0.0101
35	0.0058	4.5	0.0055	1.7	0.0056