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SURVEY OF INVESTIGATIONS INTO
THE MISSILE ALLOCATION PROBLEM

by

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June 1985

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Survey of Investigations into the Missile Allocation Problem

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ABSTRACT

This thesis gives a description of the physical and tactical parameters pertaining to missile defense and offense, and then proceeds with an overview of the mathematical investigations done on the missile allocation problem up to the 1972 publication of the survey monograph on this subject by Eckler and Burr. Finally, it presents the results of a similar survey done by the author of later unclassified studies on the missile allocation problem.

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I. INTRODUCTION

A. BASIC PROBLEM DEFINITION

In the past 35 years, a number of papers and reports, both classified and unclassified, have been published on the missile allocation problem. This problem can be stated very simply as follows:

Given an existing weapon force and a set of targets, what is the 'optimal' allocation of weapons to targets?

The problem can be analyzed from two perspectives:

- that of the defender, in which case the problem concerns the optimal allocation of defense missiles for the defense of a single target or a group of targets, or
- that of the attacker, in which case the problem concerns the optimal allocation of weapons to attack the targets and possibly the defense systems.

There are many elements that comprise the missile allocation problem. These elements can be broadly divided into six groups:

- the attacking force,
- the defending force,
- the target complex,
- intelligence available to both forces,
- scenario of the battle, and
- the criterion upon which the effectiveness of the weapon allocation strategy is based.

The specification of the parameters of these six elements determine the complexity, nature and scope of the particular allocation problem. The solution is a weapon allocation strategy that optimizes the objectives set forth by the force seeking the allocation strategy.

B. PURPOSE

This thesis is motivated by the author's interest in the field of air defense and missile defense systems. Air defense is of particular importance to a small country like Singapore, the author's homeland, where vital military installations and industrial centres are located very close to one another geographically. It is thus especially vulnerable to a concentrated attack of enemy aircraft that can fly at low altitude and unmask only at a close proximity to the intended targets before unloading their ordnance. The other motivation is that the analysis of the missile allocation problem from a mathematical viewpoint necessitates the use of many optimization techniques, such as linear and nonlinear programming, stochastic dynamic programming, game theory and Monte Carlo methods, that form the core of a traditional Operations Research study. As such, the missile allocation problem is a good example of the kind of problem that is amenable to analysis by Operations Research techniques.

C. SCOPE AND ORGANIZATION

The scope of the thesis can be delineated as follows:

- the investigations and results presented are all drawn from the unclassified literature, due to a lack of access to the classified papers.
- no detailed mathematical proofs and derivations are given for most results given. However, the interested reader can examine the original references for more details.
- emphasis is given to results obtained from analytical means rather than from computer simulation. In studies which consider realistic situations, the resulting mathematical analyses are usually so complicated that

it is necessary to resort to Monte Carlo simulation in order to obtain numerical results. The restrictions on computer time usually do not permit extensive variations in parameters in order to find an optimal solution or to conduct sensitivity analyses.

- studies pertaining to specific weapon systems are not included here, in line with the general appeal of this subject.

Chapter 2 presents the missile allocation problem in terms of its components. Terms and nomenclature pertaining to this field of study are given as an aid to understanding. The common notations used in later mathematical formulations of the problem are also given.

Chapter 3 gives a general overview of the investigations and results concerning the missile allocation problem prior to the publication of the monograph by Eckler and Burr [Ref. 1] on this subject. This publication can be considered a landmark as it is the first comprehensive survey of the literature on the missile allocation problem and a compilation of the results obtained in a more or less logical fashion. A total of 138 references are cited therein. Matlin [Ref. 2] is the only author prior to that monograph to attempt a general survey of the missile allocation problem. He presented a total of 40 papers and reports in abstract form with no analytical results or mathematical derivations. The source of the material presented in Chapter 3 is the monograph publication, and the analytical results given form a basis for the further results obtained in the survey by the author of the recent (post 1972) literature on the missile allocation problem. These results are presented in Chapter 4, which forms the core of the thesis.

II. THE MISSILE ALLOCATION PROBLEM

A. INTRODUCTION

In this chapter, the basic missile allocation problem is presented in its simplified form as essentially a stochastic duel between an attacker and a defender, each possessing a stockpile of missiles. The defender defends a single target or a group of targets with surface-to-air (SAM) missiles and the attacker uses tactical missiles that may be aimed at the targets, or at the SAM systems, with the basic objective of destroying as many of the targets as possible. The basic missile allocation problem is the determination of an optimal defensive and/or an optimal offensive strategy that can be described by the number and type of missiles to be allocated to each target or groups of targets, and the firing policy for these missiles so as to minimize (for the defender) or maximize (for the attacker) the destruction of the targets.

The richness and complexity of the missile allocation problem is a consequence of the multitude of factors that bear on this problem, and the influence their parameters have on the determination of the allocation strategy. These factors can be broadly categorized into 6 parts:

- attacker characteristics,
- defender characteristics,
- target characteristics,
- intelligence available on the opposing force,
- scenario, and
- measure of effectiveness of the allocation strategy.

The specifications of and assumptions made for each of these elements in a particular study into the missile allocation problem will determine the degree of complexity and realism of the situation it portrays, and ultimately, the optimal allocation strategy that is sought.

B. ELEMENTS OF THE MISSILE ALLOCATION PROBLEM

1. Attacker

The attacking force, which is assumed to be long-range tactical missiles in most of the literature related to this subject, can be specified by three main characteristics:

- weapon types,
- weapon capabilities, and
- attack strategy.

Each of these characteristics is elaborated on in the following three subsections.

a. Weapon Types

The attacking force can be composed of just a single type of weapon or a number of different weapon types. A single weapon type means that each individual missile has the same physical and performance characteristics such as size, weight, range, accuracy, radar signature, payload and yield, reliability and availability, and will be treated as identical entities in the analysis. The attacking force can also comprise of a mix of different weapon types with different payloads, targeting accuracies, yields, etc., or a mix of real missiles and decoys, which are 'dummy' missiles used to deceive the defense and derive benefit through the exhaustion effect or the saturation effect. The decoy is just an example of a penetration aid for the actual weapons that are aimed at the targets. These penetration aids facilitate the penetration of the main weapons through the defensive systems to the intended targets. Other penetration aids include weapons targeted at the defense systems, chaff, precursor, and ECM.

b. Weapon capabilities

The ability of the attacking weapon to destroy a target that it is aimed at depends on its performance characteristics viz.:

- maximum range of the weapon,
- aiming accuracy of the weapon,
- availability of the weapon for launch,
- reliability of the weapon- whether it can reach its target without degradation in payload or accuracy,
- deliverable payload of the weapon- the number of warheads that the delivery platform can carry,
- yield of the weapon- destruction capability of the warheads, and
- survivability of the weapon- can be affected by such factors as its radar signature, flight profile, or speed.

In many analytical studies, these individual factors are lumped together into parameters that reflect their combined effects, e.g. the availability, reliability and accuracy of a weapon may be expressed as a single quantity called probability of reaching the target that it is aimed at, while the payload and yield of the weapon may be combined together with the hardness of the target into a single parameter called the radius of effectiveness of the weapon. These 'convoluted' quantities may simplify subsequent mathematical analyses considerably, but they should be used with caution for two reasons:

- they are not physical quantities that are directly measureable, and to obtain numerical values for them in specific cases may involve tedious experimentation and gathering of data.

- the way that the physically unequal componental factors are combined into a single quantity may also be subject to debate as to their relative weights.

c. Attack Strategy

The attack strategy can be seen in terms of three dichotomies. The first is concerned with whether a single simultaneous attack of all weapons is utilized or whether the attack is sequenced in several waves, which may be equally or unequally spaced in time. The successive wave attack is normally accompanied by assessments of the attack. The attacker may observe the impact points of his weapons and adjust the aim-points of subsequent weapons accordingly to compensate for aiming errors or dispersion effects. He may also perform a damage assessment at the end of each wave, and aim his weapons only at surviving targets in subsequent attacks. The former assessment is termed a 'shoot-adjust-shoot' strategy, while the latter is termed a 'shoot-look-shoot' strategy.

The second dichotomy is whether the attacker fires at all available targets or just a subset of the target group. Different targets may have different values, and may have associated with each a different kill probability depending on characteristics of the target such as its hardness, location, existence and type of terminal air defenses, etc. If the objective of the attacker is to maximize target value destroyed with a limited stockpile of weapons, he may consider firing at only that subset of targets which have the highest values and kill probabilities.

Another consideration for the attacker is the allocation of his weapons to targets and defense systems, which may include defense radars, command-and-control centres or missile silos. The attacker may choose to fire part of his weapons at the defense systems in an effort to destroy them and thus increase the probability of subsequent weapons penetrating the defenses and reaching their targets. The optimal allocation of weapons to value targets and defense system targets under different assumptions and conditions comprise one class of the missile allocation problem.

For an attacker who is concerned with maximizing target destruction at minimum economic cost, a possible attack strategy is to use a mixture of real missiles and cheaper decoys, or to substitute better (in terms of performance) but more expensive missiles with a numerically greater force of cheaper missiles of relatively inferior performance. By using this strategy, The attacker hopes to bring into play two effects that degrade the capability of the defense to counter the attack. These two effects are:

- exhaustion effect: by firing a larger number of weapons against a fixed stockpile of defensive missiles, the attacker tempts the defender to use up all of his missiles, i.e. to exhaust his stockpile before the attacker exhausts his supply of weapons. At that point, the targets become undefended and would be more vulnerable.
- saturation effect: a defense system is said to be in a state of saturation if the number of attacking weapons arriving simultaneously within its coverage envelope is greater than the number which it is capable of engaging. Thus the defense has to select a limited number of of attackers to engage while the rest of the intruders are allowed to penetrate the defense unhindered (leakage). By having a numerically larger force of weapons, the attacker hopes to induce this condition during his attack on the targets.

2. Defender Characteristics

The defending force, assumed to be surface-to-air (SAM) missile systems in general, can be specified in terms of two major characteristics:

- missile types and capabilities, and
- defense strategy.

Each of these characteristics is elaborated on in the next two subsections.

a. Missile Types and Capabilities

The defending force may be comprised of just a single type of defense missile or of different types of missiles of different ranges, coverages and reliabilities. Here, reliability of a missile means the probability of destroying an attacking weapon it is assigned to, and takes into account such parameters as the probability of successful launch, probability of successful intercept, and probability of kill given intercept, which depend on missile performance specifications. Many defense studies postulate the availability of two types of defense missiles of substantially different coverages:

- a local missile, which can defend against weapons directed at a single target (terminal defense), and
- an area missile, which has a bigger coverage and can defend against weapons directed against one of a group of targets in an extended region (area defense).

b. Defense Strategy

The appropriate (or optimal) defense strategy depends greatly on what the defender knows about the offense's plans, capabilities and resources. Given the extent of intelligence about the attacker, and the resources he possesses, the defensive strategy can be dichotomized in several ways.

The first dichotomy is preallocation strategy vs. non-preallocation strategy. In the former case, a specified number of missiles is assigned to the defense of each target, depending on its value. Preallocation defenses require that the defense keep track of exactly how many attacking weapons have been directed at each target in order to decide whether or not to allocate a missile against the next weapon approaching the target. When this is not possible, a non-preallocation strategy (or group preferential strategy) may nonetheless be possible, wherein the target group is divided into disjoint subsets, and a fraction of the defense stockpile is allocated to each of the target subsets.

The second dichotomy concerns the allocation between local (terminal) and area missiles. Each target can be defended by a mix of local missiles which are allocated to it prior to the attack, and area missiles which can cover any target within some region of protection. The defense strategy in this case is concerned with the relative numbers of each type to be allocated to the target and the firing policy.

The third dichotomy is concerned with whether the defensive strategy is target-oriented or attacker-oriented. In the former case, the defender allocates missiles to specific targets. In some cases, the defender may not be able to determine which target a weapon is directed against in time to make an intercept if desired (attack evaluation). In such a situation, the defender must use an attacker-oriented strategy instead, whereby missiles are assigned to each incoming weapon.

3. Target Characteristics

The target can be characterized by:

- type of target,

- value assigned to the target, and
- defenses associated with the target.

a. Type of Target

A simplifying assumption made in most analyses of the missile allocation problem is that targets are either classified as point targets or area targets. A target is considered a point target if the lethal radius of the attacking weapon is large enough relative to the size of the target so that a single weapon can destroy the target entirely. If more than one weapon is required to cover the target, it is considered an area target. Examples of area targets are a large airbase, a city, or a harbour. However, an area target might be considered to be a collection of point targets if it can be broken down into individual aim-points with values associated with each point rather than with the target as a whole.

A target is considered to be independent of other targets if no single attacking weapon can destroy more than one target at a time, whereas collateral targets can be killed by a single weapon.

b. Value of a Target

The value of a target is an important consideration in the missile allocation problem because the usual objective or measure of effectiveness used in the comparision of alternative allocation strategies is the expected target value destroyed. In most cases, it is assumed that the value or military worth associated with a target is the same as perceived by the offense as for the defense, although in reality it probably is not. Again in most analyses, a single parameter is used to determine the value of the target, e.g. the population of a city target, although in reality, several factors may be of important strategic value, e.g. the industrial capacity and military installations, in addition to the population.

The value of a target may be constant with time, e.g. an ammunition production plant, or it may vary with time, e.g. an airbase from which aircraft are taking off, or a city whose population is being rapidly evacuated. The value scales are usually assumed to be linear, implying for example that a city with two million people is twice as valuable as a city with one million people, all other things being equal, an assumption that is generally inappropriate.

A target may have an indirect value in the sense that no value is assigned for destroying it, but if it is eliminated, it becomes easier to accumulate direct values from other targets. Indirect valued targets are sometimes called secondary targets, whereas direct valued targets are called primary or value targets. Examples of secondary targets are defensive missile silos, air defense radars, and command-and-control centers.

c. Defenses Associated with the Target

A target may either have no defenses at all, or terminal defenses only, area defenses only, or a mixture of both types of defenses. In models which treat the defenses implicitly, the defensive capability of a target is given by that target's penetration probability (or probabilities for combined area and terminal defenses). Where a number of separate defense regions are considered, a region consisting of a subset of targets defended by a single area defense, the defenses in one area cannot be used in another area, and probabilities of penetrating each region are specified separately. Defense regions may overlap to some extent so that some targets are contained in more than one defense region.

4. Intelligence Available on the Opposing Force

The knowledge that each side has regarding the opposing force, its size, capabilities and intentions determines to a large extent the optimal strategy to employ against this force. In all studies on the missile allocation problem, assumptions are made as to the extent of information the attacker and defender has on each other's stockpile size and weapon composition, allocation strategies, and the results of such strategies. Specifically, the intelligence that each side has of its opponent can be delineated as follows:

- the total numbers and types of weapons that the opponent possesses, or if the exact numbers are not known, the probability distribution of the force size;
- the reliabilities of the missiles, given generally as the probability that a defensive missile will intercept and destroy an attacking weapon, or the probability that an attacking weapon will reach and destroy an undefended point target;
- the impact points or probability distribution of impact points of the attacking weapons, and their lethal radii, and
- target damage evaluation, if the attack occurs in successive waves i.e. determine which targets have already been destroyed and allocate missiles or weapons only to surviving ones (shoot-look-shoot strategies).

The most complete intelligence is obtained when one side can see the entire allocation of the opposing force's stockpile to targets before making its own allocation, i.e. the opposing force's strategy (allocation and firing policy) is known beforehand. If the offense has this knowledge, an offense-last-move situation exists, and similiarly, if the defense possesses this intelligence, a defense-last-move situation is present. On the other end of the scale, the offense and defense may allocate their resources each in ignorance of the other's allocations. This allocation problem can in general be formulated as a two-person-zero-sum game.

5. Scenario

A major portion of the studies on the missile allocation problem have been devoted to strategic weapons exchanges between two superpowers. Since strategic nuclear warfare remains outside the realm of military experience, models are proposed and analyzed to provide decision makers with information on possible consequences of policy decisions on the deployment and employment of strategic nuclear weapons.

The scenario usually considered is based on the precept of mutual deterence, i.e. the threat of massive nuclear retaliation to deter aggression. To achieve this, each side maintains a massive and secure strategic force that is expected to retain its capability of delivering a devastating retaliatory strike despite an all-out enemy first strike intended to reduce the retaliatory force (assured destruction policy).

Most studies assume a two-strike nuclear exchange, in which each side possesses two kinds of assets:

- several types of strategic weapons with which each side can strike at the other, e.g. land-based ICBM's, submarine-based SLBM's, or long-range nuclear bombers, and
- value assets, consisting of industrial, economic and governmental facilities and population that contribute to a society's economic viability. By attacking these targets, each side aims to destroy the other as a social and economic entity.

The first striker can allocate his strategic weapons against his opponent's strategic arsenal in a counterforce attack in order to reduce the expected retaliatory damage to himself, or he may target his opponent's value targets, thereby fulfilling the goal of damaging his economic viability in a countervalue attack, or he could mix counterforce and countervalue options to obtain an optimal targeting strategy based on some objective function. Because a two-strike exchange is assumed, there will be no further strikes after the other side retaliates. Therefore the first striker allocates all his weapons in a first strike, and his opponent retaliates with all his weapons against value targets only.

This basic scenario can be enriched by considering reserve forces, or more than two sequential strikes. Selective threat targeting and progressive confrontation targeting may also be considered as alternative scenarios of the real world situation.

6. Measures of Effectiveness of the Allocation Strategy

The criterion of effectiveness used to compare alternative strategies or to find an 'optimal' strategy in a given situation is determined by the decision maker faced with the problem. The choice of an appropriate measure of effectiveness (MOE) is determined largely by the physical parameters of the problem, such as relative stockpile sizes, nature of the targets, degree of knowledge about the opponent's weapons and allocation strategy, as well as political objectives and subjective perceptions. Such a choice may depend largely on intuition, and hence be somewhat arbitrary. In many studies, a particular MOE is chosen for its mathematical tractability rather than its closeness to real political objectives.

The MOE's normally used in missile allocation problems are:

- probability of target destruction- this MOE is appropriate if the target consists of a single point;
- expected target value destroyed- this MOE is suitable if the target is an area target or a composition of many point targets, and both sides know the size of the opponent's stockpile;
- expected number of attacking weapons not intercepted by the defense- this MOE is used because the expected target value destroyed is directly related to the number of penetrating weapons;
- expected target value surviving a certain percentage of all attacks of a given size- in some instances, this MOE is used. It is more difficult to deal with analytically; however it is easily evaluated using Monte Carlo methods;
- probability that no target value is destroyed- this MOE is appropriate if the number of defensive missiles available to a target is greater than the number of attacking weapons directed at the target, and the nature of the target is such that even a relatively small amount of damage inflicted would be as catastrophic as a large amount of damage;
- expected cost of achieving destruction of the target- this MOE is used in a situation where the offense is not restricted to a number of attack waves, but could continue with the attacks until the target is destroyed. This attack strategy is used in the case where the operational value or worth of the target is very high and the number of weapons that the attacker can expend on its destruction is practically unlimited;
- expected number of weapons expended until the first penetrator- this MOE is suitable if the attacker fires

one weapon at a time against the target and the defense has no information about the size of the offense stockpile. In this situation, it is not possible for the defense to design a strategy which minimizes the expected fraction of targets destroyed or to maximize the probability that no target value is destroyed.

- expected target value extracted per offensive weapon fired- this MOE is appropriate if the defense designs a strategy such that the expected fraction of targets destroyed is proportional to the attack size. These are also known as 'Prim Read' deployments.

The selection of an appropriate MOE is important in the missile allocation problem because the optimal allocation strategy in most cases depends critically on this choice. In some situations however, different criteria of effectiveness lead to the same allocation strategy or lead to similiar results.

C. TERMINOLOGY AND NOTATION

The terminology and notation used throughout this thesis will be consistent in the most part with those used in the monograph of Eckler and Burr. This will provide a sense of continuity in going from Chapter 3, in which an overview of the pre-1972 investigations into the missile allocation problem is presented, with material largely extracted from Eckler and Burr's publication, to Chapter 4, which gives the results of studies done subsequent to the publication of the monograph.

Most of the terminology related to the missile allocation problem has been articulated and explained in the previous sections of this chapter, when the elements of the missile allocation problem are described. Nevertheless, it is worthwhile to summarize the salient terms here to avoid any confusion.

In a typical situation, the offense has a stockpile of weapons which are used to attack a target or targets belonging to the defense. The defense has a stockpile of missiles which can be used to intercept the attacking weapons. The targets may be either point targets, any of which can be destroyed by a single weapon, or area targets, which require several weapons to destroy. A missile has an inherent reliability or probability that it will destroy the weapon it is assigned to. A weapon in turn has a weapon kill probability, which is the probability that it will destroy the target it is aimed at if it is not intercepted by a defense missile. The value of a target is the military worth assigned to it and is assumed to be the same from both the offense's and the defense's point of view. The attack can occur simultaneously in a salvo, or it can occur in several successive waves separated in time. Sequential attacks on a target makes possible damage assessment and leads to shoot-look-shoot strategies.

The most common symbols used in the analyses presented later are given below. Other notation peculiar to a particular analysis will be given as required.

A = total number of weapons in the offense stockpile,

D = total number of missiles in the defense stockpile,

T = total number of targets,

$a = A/T$ = normalized offense stockpile on a per target basis,

$d = D/T$ = normalized defense stockpile on a per target basis,

p = weapon kill probability,

ρ = missile reliability,

$q = 1-p$ = probability an unintercepted weapon fails to kill its target,

$q_1 = 1-p(1-\rho)$ = probability that a weapon to which a missile has been assigned fails to destroy its target,

v_i = value of the ith target,

R = lethal radius of a weapon,

E(X) = expected value of quantity X,

[x] = greatest integer less than or equal to x, and

Pr(X) = probability of event X occuring.

III. PRE-1972 INVESTIGATIONS INTO THE MISSILE ALLOCATION PROBLEM

A. INTRODUCTION

In this chapter, an overview of the investigations into the missile allocation problem from the unclassified literature prior to the 1972 survey monograph by Eckler and Burr is given. The order of presentation follows that of this monograph; however only a summary of the major results of interest are given, since this is aimed at giving a general idea of the state of research on the missile allocation problem up to 1972 rather than a lengthy exposition of all these studies. No references to the original publications are given for the results quoted, since the monograph by Eckler and Burr provides a comprehensive list of the original papers in its bibliography.

The purpose of this chapter is to give an overview of the state of research into the missile allocation problem up to 1972, so that the results of subsequent analyses presented in chapter 4 could be better appreciated and the development of certain key ideas and applications could be more easily traced. The key results are organized in the following manner:

- defense strategies for a single point target,
- offense and defense strategies for a group of identical point targets,
- offense and defense strategies for a group of non-identical targets with different values, and
- offense and defense strategies in special situations.

Each of these classes of problems will be addressed in the following sections.

B. DEFENSE STRATEGIES FOR A SINGLE POINT TARGET

In this section, the defense of a single point target or a single area target with uniform value against a single salvo of weapons or sequential waves of weapons are considered. The MOE in the case of a single target is Pr(the target survives), and in the case of an area target is E(number of penetrators).

The standard defense problem assumes that the damage function is a 'cookie-cutter' function in the case of a point target, i.e. a weapon destroys the target if and only if it lands within a distance R of the target, R being its lethal radius. It is also assumed that individual missiles and weapons operate independently of each other.

If the defense knows the lethal radius and also that a weapons out of a salvo of A weapons will land within distance R before making his allocation, the optimal defense strategy is to salvo his D missiles as uniformly as possible against each of the a weapons, and the probability of target destruction is

$$P_a = 1-\{1-(1-\rho)^k\}^{a-r}\{1-(1-\rho)^{k+1}\}^r$$

where $k = [D/A]$ and r is the remainder when D is divided by a, i.e. $D = ka + r$. P can be approximated by permitting non-integer allocations of missiles to each weapon:

$$P \approx 1 - \{1-(1-\rho)^{\frac{D}{a}}\}^a \text{ .}$$

The approximation will, in all cases, be at most as large as the actual value.

The unconditional probability of target destruction is

$$\sum_{a=0}^{A} \binom{A}{a} p^a (1-p)^{A-a} P_a \text{ ,}$$

where p = Pr(a weapon lands within distance R of the target).

Various modifications to this standard defense problem can be considered viz.

- defense does not know lethal radius R in a salvo attack,
- defense does not know attack size A in a sequential attack,
- defense does not know lethal radius R in a sequential attack,
- defense knows that the sequential attack contains one weapon mixed with decoys, and
- defense can do damage assessment on attacking weapons.

The following subsections describe each of these five cases in turn.

1. Defense Strategies when Lethal Radius is Unknown

It is assumed that the defense knows the attack size a and the impact points r_i of each of these a weapons prior to allocation of his missiles. However the lethal radius R of the weapons is not known. An appropriate MOE to use is to maximize E(distance of the target to the impact point of the nearest penetrator) = E.

The probability that the ith closest weapon will penetrate is

$$A_i = (1-\rho)^{m_i} ,$$

where m_i = no. of missiles allocated to the ith weapon.

Then $E = r_1 A_1 + r_2 A_2 (1-A_1) + \ldots + r_j A_j \prod_{i=1}^{j-1} (1-A_i) + r_{j+1} \prod_{i=1}^{j} (1-A_i) ,$

if the nearest j weapons are assigned missiles. To find the optimum allocations m^* , dynamic programming could be used to maximize E such that

$$\prod_{i=1}^{j} A_i = (1-\rho)^m ,$$

where $m = \sum_{i=1}^{j} m_i$.

An approximate solution can be derived by allowing the unknowns to be continuous, and differentiating E with respect to m_i. In this case a set of recursive equations is obtained:

$$P_k = P_{k+1}(1-A^*_{k+1}),$$

$$Q_k = Q_{k+1}(1-A^*_{k+1}) + r_{k+1} - r_k,$$

$$A^*_k = P_k/Q_k \text{ for } k = j-2, j-3, \ldots, 1,$$

where $P_1 = r_{j+1} - r_j$, $Q_1 = r_{j+1} - r_{j-1}$, $A^*_1 = P_1/Q_1$.

This recurrence enables one to get successively A^*_{j-1}, A^*_{j-2}, ..., A^*_1. Then

$$m^*_i = \log A^*_i / \log(1-\rho).$$

2. Strategies for a Sequential Attack of Unknown Size

It is assumed here that the weapons arrive one at a time and the defense knows the lethal radius R of the weapon but not the size of the attack. The objective of the defense is to maximize E(number of weapons to the 1st penetrator). This problem is very nearly identical to that of the preceding subsection, and one can similiarly derive an approximate solution by means of a set of recursive formulae. A very nearly optimal defense strategy can be stated simply as follows:

For a stockpile of D missiles, allocate approximately D/h of them to the 1st h weapons, and none to the (h+1)th weapon.

Numerical calculations indicate that the choice of h for this near optimal defense strategy is about 90% of the first unengaged weapon under the optimal allocation. The loss in the expected number to the 1st penetrator is only about 6% compared with the continuous optimum.

An exact procedure which provides integer allocations can be derived directly if it is assumed that D is not too large, or that no more than 2 missiles may be assigned to each weapon. By comparing the MOE if 1 missile is assigned to each of the first (m+1) weapons to the MOE if 2 missiles are allocated to the first weapon and 1 missile to each of the next (m-1) weapons, the following defense allocation strategy is derived: let

$$m_0 = \{-\log(1-\rho-p+p\rho)\}/(\log q_1) + 2$$

where p = weapon kill probability, and

$$q_1 = 1-p(1-\rho) .$$

Then if $D \le m_0$, assign 1 missile each to the first D weapons, and if $D > m_0$, assign 2 missiles each to the first $(D-m_0+1)/2$ weapons and 1 missile each to the next $(D+m_0-1)/2$ weapons.

Using the same methodology, one could also derive a procedure to obtain the optimal defense allocation strategy given that no more than 3 missiles may be assigned to each weapon. However, no solution has been given which permits more than 3 missiles to be assigned to a weapon; it is then neccessary to resort to dynamic programming to obtain a solution.

An alternative defense strategy can be obtained if, instead of maximizing E(number of weapons to the 1st penetrator), the defense chooses to make Pr(target destruction) proportional to the attack size up to the point of missile exhaustion. In this case, the marginal increase in the target destruction probability achieved by allocating 1 more weapon to the target is constant. This doctrine of 'constant value decrement' yields the following near optimum allocation strategy:

$$m_i = -\log\{(1-i+n)p\}/\log(1-\rho) , \quad i = 1,\ldots,n ,$$

where m_i is the number of missiles assigned to the ith weapon, and n is the number of weapons needed to exhaust the missile stockpile.

If the defense knows the probability distribution of the attack size, and its objective is to minimize E = E(number of penetrators), then

$$E = \sum_{i=1}^{n} p_i (1-\rho)^{m_i} ,$$

where p_i = Pr(offense will attack with i or more weapons inside the lethal radius). p_i may be assumed to be either binomial or geometric. The reduction in E resulting from adding the jth missile to the ith weapon is

$$R(i,j) = p_i \{(1-\rho)^{j-1} - (1-\rho)^j\} .$$

To obtain the optimum allocation, the missiles are assigned one at a time to that weapon which gives the greatest value of R(i,j).

3. Strategies against a Sequential Attack of Unknown Lethal Radius

It is assumed here that the attack occurs in waves of one weapon at a time. The defense does not know the lethal radius of the weapons, but knows the attack size and can predict the impact point of each weapon relative to the target. Two simplifying extreme cases can be considered, according to whether the defense has no knowledge or complete knowledge of the impact point distribution. Two MOE's are possible:

- MOE 1: max. Pr(the offensive weapon landing nearest the target is assigned a missile), or
- MOE 2: max E(total score of weapons destroyed), where score of a weapon is the probability that a random weapon will land further from the target than it did.

In the case where the impact point distribution is unknown, and MOE 1 is used, the optimum defense strategy is as follows:

observe the smallest miss-distance in a fraction α_i, i = 1,...,D of the attack, and assign a missile to the 1st weapon appearing with a smaller miss-distance.

This observation is done D times, where D is the total number of missiles available. The optimum fractions α_i have been computed and are tabulated. An alternative near optimum strategy for large attack sizes which is simpler to compute is as follows:

observe the smallest miss-distance in a fraction $\alpha = \exp\{-(D!)^{\frac{1}{D}}\}$ of the attack and assign m missiles to the 1st m weapons whose miss-distances are smaller.

If the impact point distribution is known, a near optimal defense strategy for large attack sizes A can be given as follows:

observe the miss-distance r of the ith weapon and assign a missile to it if $r_i \leq r^*$, where $r^* = k/A = \int_{-\infty}^{r^*} p(r)\,dr$.

Optimum values of k for different values of D have been determined.

In the case where the impact point distribution is known and MOE 2 is used, the optimum defense strategy has the following form.

Suppose there are $t \leq D$ missiles remaining, and $k \leq A$ weapons yet appear in the attack. When the first of the k weapons appears with miss-distance r, allocate a missile if $r \leq r(k,t)$, where r(k,t) is defined implicitly by:

$$U(k,t) = \int_{-\infty}^{r(k,t)} p(r)\,dr$$

If E(k,t) is the average value of the t probabilities that a random weapon exceeds the observed miss-distances of the weapons destroyed by the final t missiles, E(total score of the t weapons) is given by the iterative equation:

$$tE(k,t) = \{1+U(k,t)\}\{0.5\{1+U(k,t)\}+(t-1)E(k-1,t-1)\} + U(k,t)tE(k-i,t) .$$

This yields:

$$U^*(k,t)+(t-1)E^*(k-1,t-1) = tE^*(k-1,t) ,$$

and the optimum values $E^*(k,t)$ and $U^*(k,t)$ can be found recursively using the initial conditions $U^*(k,k) = 0$, $E^*(k,k) = 0.5$, $1 \le k \le n$.

4. Strategies against a Sequential Attack of One Weapon with Decoys

The assumptions made are the defense knows that the sequential attack of size A contains one weapon mixed with (A-1) decoys, and the missile reliability $\rho < 1$ while the weapon kill probability p = 1. In this situation, an appropriate MOE for the defense would be to minimize the probability that a weapon is not intercepted. The weapon is characterized by a single observation (real number) drawn from a probability distribution with pdf $f_w(x)$, and the decoy is also characterized by an observation drawn from a pdf $f_d(x)$, both of which are known to the defense.

The optimum strategy can be specified as follows:

Suppose there are $t \le D$ missiles remaining, and $k \le A$ attacking objects yet to appear. When the first of the objects appear, note the value c of its observation, and allocate i missiles to it if $c_i \le c \le c_{i+1}$.

Optimal values of c_i, $i = 1,\ldots,t+1$ can be derived from the analytical expression of Pr(the weapon penetrates) = p, which is a complicated function of the c_i values. Tabulated values of p associated with optimum c_i values are available for $f_w(x)$ and $f_d(x)$ being Normal distributions with unit variance.

A more general model with more than one weapon among the A attacking objects has been postulated. The optimal strategies for two different criteria of effectiveness viz. min. Pr(1 or more weapons penetrate) and min. E(number of weapons penetrating) have been determined, for $f_w(x)$ being a Normal density function with unit mean and variance and $f_d(x)$ being a Normal density function with zero mean and unit variance.

5. Defense Strategies with Damage Assessment

When the defense is able to perform damage assessment on the attacking weapons, he can use a k-stage shoot-look-strategy, whereby m_1 missiles are allocated to A weapons in the first stage, then m_2 missiles are allocated to $A-n_1$ surviving weapons in the second stage after observing which n_1 weapons have been destroyed in the first stage, and so on, and finally $D-(m_1+m_2+...+m_{k-1})$ missiles are allocated to the $A-(n_1+n_2+...+n_{k-1})$ surviving weapons. The MOE used in this case is max. Pr(no weapons survive).

An algorithm for determining the optimal shoot-look-shoot strategy for any number of stages can be devised by using a set of recursive equations, whereby the optimum k-stage strategy is determined from the (k-1), (k-2),..., 1st stage strategies.

For a 2-stage shoot-look-shoot strategy with A = 2, the optimal allocation is D/2 missiles to the 1st stage and D/2 missiles to the 2nd stage; in both cases, the missiles are assigned uniformly to all weapons. When A = 3 or more, analytical results are difficult to obtain, and a computer must be used to obtain the optimum allocation for each A and D.

For a D-stage shoot-look-shoot strategy, a missile is assigned at each stage, and in this situation, Pr(no weapons survive) is

$$p = \sum_{i=A}^{D} \binom{D}{i} (1-\rho)^{D-i} \rho^{i} \ .$$

For high values of ρ, providing a single 'look' in a 2-stage shoot-look-shoot strategy is quite worthwhile in terms of the gain in Pr(no weapons survive) over a 1-stage strategy (no damage assessment), but providing more than one look is much less so unless the missile reliability is low.

A special consideration for k-stage shoot-look-shoot strategies is when a single missile is allocated to each weapon and time is limited. This gives rise to what is known as a fire-power limited shoot-look-shoot defense. If T is the time interval between the 1st possible assignment of a missile to a weapon and the destruction of the target by that weapon, and τ is the time required for a missile to attack the weapon and evaluate the outcome, then a k-stage shoot-look-shoot strategy can be used against each weapon, where $k = [T/\tau]$. It is assumed that the offense attacks with A weapons arriving at equally spaced intervals of length $s\tau$. Four cases can be considered depending on the value of s.

When $s \geq k$, the successive weapon engagements are independent of each other, and the probability that a weapon will destroy the target is

$$P = 1-(1-(1-\rho)^{k})^{A} \ .$$

When $s < k$, successive weapon engagements are not independent of each other and delays in engagements of successive weapons can occur. The evaluation of P is consequently much more involved, and it becomes neccessary to use a computer to evaluate P.

When s = 1, the time interval between successive weapon arrivals is equal to the time required to engage a weapon with a missile, and P can be given by the cumulative negative binomial distribution

$$P = \sum_{i=0}^{A-1} \binom{k}{i} (1-\rho)^{k-i} \rho^{i} .$$

In the previous analyses, it was assumed that the arrival times of weapons are equally spaced. In an attempt to be more realistic, it is sometimes assumed that the arrival times consist of order statistics obtained from a Normal or an Exponential distribution. In these cases, it may occur that certain weapons cannot be engaged at the time of their arrival because the defense is still occupied with earlier weapons, if the arrival time of a weapon is less than the time T required for a missile to engege a weapon. The probability of no delay of the weapons can be given, in the case where $A = 2$, $\rho = 1$, and the arrival time distribution is a Normal distribution with standard deviation , as

$$\int_{-\infty}^{0} \frac{1}{\sigma\sqrt{\pi}} \exp\left\{ \frac{-(t-T)^2}{4\sigma^2} \right\} dt .$$

For values of A greater than 2, it is necessary to resort to Monte Carlo simulation to obtain the values of the maximum delay times.

If the weapon arrival times are assumed to be exponential with parameter a, the probability of no delay can be given in closed form as

$$a^{A-1}(A-1)!\exp(-aTA(A-1)/2) .$$

A more general result assumes that T is not constant, but a random variable from a Gamma distribution with parameters n, λ. In this case, the probability of no delay is

$$\prod_{i=1}^{A-1} (1 - ia/\lambda)^{-n} .$$

C. STRATEGIES FOR A GROUP OF IDENTICAL TARGETS

In the previous sections, optimum defense strategies were presented in the case of a single point target. In contrast this section considers offense and defense strategies for a group of independent identical point targets with identical values under different degrees of knowledge each side has of the other's stockpile size and allocations to individual targets.

The offense and defense strategies that are considered here are organized in the following manner:

- preallocation strategies - offense-last-move
  - defense-last-move
  - neither side knows the other's allocation
- non-preallocation strategies - varying attack size
  - fixed attack size
- mixed non-preallocation and preallocation strategies
- damage assessment strategies - defense damage assessment
  - offense damage assessment
- attacker-oriented strategies - neither side knows the other's allocation
  - offense knows defense allocation

Each of these topics will be dealt with in the following subsections.

1. Preallocation strategies

Strategies allocating weapons and missiles to individual targets rather than to subgroups of targets are called preallocation strategies, and are based on the assumption that attack evaluation by the defense is possible. Two advantages of preallocation strategies are that they represent effectively computable exact solutions of fairly realistic problems, and that they are more effective for the defense than other strategies if the offense outnumbers the defense and missile reliability is not very high.

It is generally assumed here that missile engagements are one-on-one, and that both sides know the other's stockpile size and weapon kill probability p and missile reliability ρ. The MOE is E(fraction of targets saved), and can be given by

$$E(f) = \sum_{i=1}^{T} P_i ,$$

where P_i is Pr(the ith target survives), and T is the total number of targets.

a. Offense-Last-Move

The offense-last-move situation represents a lower bound for E(fraction of targets saved), since it implies that the offense can see the entire defense allocation of missiles to individual targets before making his own allocation. In this case, the best possible defense strategy is to allocate an equal number of missiles to each target.

The optimum offense strategy against this defense can be derived as follows: Let the offense attack a fraction of the targets $y_k = a/k$ with k weapons per target, $k \geq a$. If P(k) is the probability that the target is destroyed if attacked by k weapons and defended by d missiles, then E(fraction of targets saved) $E(f) = 1-y_k P(k)$ Assuming that P(k) is a function for which a unique value of k, denoted k* maximizes P(k)/k (the average return per weapon at an attacked target), the offense allocation that maximizes P(k)/k also minimizes E(f) if k* > a. Hence

$E(f) = 1 - \{aP(k^*)\}/k^*$ if $0 < a \leq k^*$, and

$E(f) = 1 - P(a)$ if $k^* \leq a$.

For the one-on-one defense that is assumed,

$$P(k) = q_1^{\min(d,k)} q_0^{\max(0,k-d)} \quad .$$

b. Defense-Last-Move

The defense-last-move situation represents an upper bound for the expected fraction of targets saved, since it implies that the defense can see the entire offense allocation of weapons to targets before making his own allocation of missiles. The best possible offense strategy in this case is to allocate an equal number of weapons to each target. For the defense, if $d \geq a$, the maximizing defense strategy is to attack each weapon with a single missile (since engagements are assumed to be one-on-one only). If $d < a$, the optimal defense strategy is assign 1 missile each to a fraction d/a of the targets, and no missiles to the rest. The corresponding value of $E(f)$ is

$$E(f) = (a-d)q_0^a/a + dq_1^a/a \quad .$$

c. Neither Side Knows the Other's Allocation

In the situation where neither side knows the other's allocation to targets, the problem can be formulated in terms of a two-person-zero-sum game, with the payoff being the fraction of targets saved. A generalization of the fundamental theorem of games states that there exists optimum pdf's of offense and defense strategies, and the game has a value V given by $\max_{\underline{x}} \min_{\underline{y}} E(f)$, where $\underline{x}$ and $\underline{y}$ represent the different defense and offense levels respectively. The solution to the allocation problem consists of finding these vectors $\underline{x} = (x_0, x_1, x_2, \ldots)$ and $\underline{y} = (y_0, y_1, y_2, \ldots)$ such that a fraction x_0 of the targets are selected at random for no defense, a fraction x_1 are selected for defense by 1 missile, etc., and similiarly for the vector $\underline{y}$. Then E(fraction of targets saved) is

$$v = \sum_{i,j} x_i \, y_j \, q_1^{\min(i,j)} \, q_0^{\max(0,j-i)} \, .$$

This problem can be expressed as a constrained game which are usually solved by linear programming. However, Matheson [Ref. 3] has found a solution to the preallocation problem without using linear programming explicitly. The results of Matheson's work are rather difficult to describe concisely; the reader is urged to refer to the original paper for details. The problem can be simplified by setting $p = \rho = 1$ (perfect weapons and missiles). In this situation, the optimum offense and defense strategies can be given in terms of a and d in each of two cases:

- defense dominant, i.e. $[2d+1] \ge [2a]$. In this case, the defense strategy is

  $x_i = 2([2d+1]-d)/[2d+2][2d+1]$ for $i = 0, 1, \ldots, [2d]$,

  and $x_{[2d+1]} = (2d-[2d])/[2d+2]$,

  and the offense strategy is

  $y_i = 2a/[2d+1][2d+2]$ for $i = 1, 2, \ldots, [2d+1]$, and

  $y_0 = 1 - 2a/[2d+2]$.

- offense dominant, i.e. $[2d+1] < [a]$. In this case, the defense strategy is

  $x_i = 2d/[2a][2a-1]$ for $i = 1, 2, \ldots, [2a-1]$,

  $x_0 = 1 - 2d/[2a]$,

  and the offense strategy is

  $y_i = 2([2a]-a)/[2a][2a-1]$ for $i = 1, \ldots, [2a-1]$,

  $y_{[2a]} = (2a-[2a])/[2a]$.

In order to get integer allocations which may not be possible using the previous analyses, an integer strategy game analogous to the Matheson game can be defined, whereby the mixed strategy used is a probability distribution function $(p_1, p_2, \ldots, p_N)$ taken over N different pure strategies (the actual allocations of an integer number of missiles or weapons to each of the T targets). This integer allocation game is impossible to solve in closed form except for very small numbers of weapons, missiles, and targets, because the number of pure strategies becomes very large quickly. For $q_0 = 0$, and $q_1 = 1$ however, the value of the Matheson game is the same as that of the integer strategy game. If D (or A) and T are not too large, it is possible to find the optimum strategies by using linear programming, which can also be used to solve various generalizations to the Matheson game such as:

- upper limits on the number of missiles or weapons that can be allocated to a target,
- allocation doctrines besides one-on-one,
- several different types of missiles or weapons,
- independent defense regions, and
- generalized shoot-look-shoot strategies.

The variance in the total number of targets saved if both sides use pure strategies can be given by the upper bound $Var(Z) \le T^2 V(1-V)/(T-1)$, which shows that the variance bound depends only on E(fraction of targets saved), and not on the missile and weapon allocations.

It can also be argued that if both offense and defense use pure strategies, then as $T \to \infty$, the distribution of the number of targets saved converges to a Normal distribution with mean 0 and variance less than 1. This limiting Normal distribution can be used to make estimates of the probabilities that the number of surviving targets is less than, or greater than, a specified value.

Another model for the preallocation offense and defense when neither side knows the other's allocation is known frequently as a 'Blotto game', whereby the defense has a single real target mixed with (T-1) dummy targets, and the offense, not knowing which is the real target, allocates weapons among the targets. The Blotto game can be formulated either as a discrete game or as a continuous game. When $\rho = p = 1$, and $a \geq d$ (offense dominant), the optimum offense strategy is to attack a typical target with a_i weapons, where a_i is a random variable drawn from a Uniform distribution $U(0,2a)$; the optimum defense strategy is to defend a typical target with probability d/a, using d_i missiles, where d_i is distributed according to the same Uniform distribution.

If $a \leq d$ (defense dominant), the optimum offense strategy is to attack a typical target with probability a/d using a_i weapons, where a_i is a random drawing from a Uniform distribution $U(0,2d)$, and the corresponding optimum defense strategy is to defend a typical target with d_i missiles, where d_i is drawn from the same probability distribution.

The general form of the optimum offense and defense strategies for a continuous Blotto game with one-on-one engagements was derived assuming that the probability that a target survives when attacked by y weapons and defended by x missiles is of the form

$$P(x,y) = s(y) \;,\; 0 \leq y \leq x$$
$$= s(x)\,t(y-x) \;,\; x \leq y \;,$$

where s(x) and t(y) are convex functions with continuous derivatives, and $s(0) = t(0) = 1$. If $f(y)dy$ and $g(x)dx$ are the fractions of targets attacked by y weapons and defended by x missiles respectively, the optimum defense strategy is given by g(x) satisfying the equation:

$$\int g(x)P(x,y)dx = m-h(y) \text{ in some interval } U < y < V\text{, and}$$

$$\int g(x)P(x,y)dx > m-h(y) \text{ outside this interval.}$$

The quantities m,h,U,V are determined so that m-ha is maximized. The corresponding offense strategy is given by f(y) satisfying the equation:

$$\int_0^\infty f(y)P(x,y)dy = n+kx \text{ in some interval } U < x < V \text{ and}$$

$$\int_0^\infty f(y)P(x,y)dy < n+kx \text{ outside this interval.}$$

Again, the quantities n,k,U,V are determined so that n+kd is minimized.

2. Non-preallocation Strategies

When the defense is not able to perform attack evaluation for each target, a group preferential strategy would need to be adopted by the defense instead of a preallocation strategy. In this case, the defense allocates all of its missiles to defend only a subgroup of the targets. In this subsection, group preferential strategies are considered in two situations:

- varying attack size
- fixed attack size

When the attack size is varying, one possible defense strategy is to defend a random subset d/k of the targets with the entire stockpile, where k is an integer value. When any target within the subset is attacked, a missile is allocated to it. It is assumed that the offense knows the value of the fraction d/k, but not the actual defended subset, and attacks the targets in waves of one weapon against each target with a total of i waves, where i is a random variable from a probability distribution with a mean of a. In this situation, the optimum offense strategy is a strategy containing a lower and upper attack level denoted by i and (a+j) respectively.

If k = a, the defense stockpile will be equal to the expected attack size on the defended subset. As d→a, the advantages of randomization are lost. If $d \geq a$, the best defense would be to engage each weapon, abondoning the group preferential strategy.

If the value of k is not known to the offense, he could tailor his attack such that E(fraction of targets saved) is the same no matter what value k is selected, $d \leq k \leq t$, by selecting the Matheson strategy corresponding to $d = h$, $t = m$.

When the attack size is fixed, two extreme cases can be considered:

- weapons arrive at random, and
- weapons arrive in an order controlled by the offense.

Each side knows the other's stockpile but not the specific allocation of weapons to targets, or which subset of targets have been selected for defense. It is assumed that $p = \rho = 1$. When the weapon arrival order is controlled, the decision to use a group preferential or a preallocation strategy depends on what the defense thinks the offense knows about his plans. If the weapon arrivals are random, it is likely to be profitable for the defense to shift from a preallocation to a group preferential strategy.

3. Mixed Preallocation and Non-preallocation Strategies

A mixture of preallocation and non-preallocation strategies can be selected by the defense as follows. The target set is divided randomly into disjoint groups of various sizes, and a fraction of the total stockpile of missiles is allocated to each group for defense. It appears quite difficult to determine the optimal offense and defense strategies as a function of A, D, and T if $D < A$ and $T < A$.

For defense-last-move, the determination of an optimal offense strategy is equivalent to solving a set of nonlinear equations, and becomes computationally formidable as the complexity of the problem increases. If neither side knows the other's strategy before choosing his allocation, the problem becomes a game-theoretic one. The expected number of targets saved will lie between the offense-last-move and defense-last-move values, and both sides must use mixtures of strategies. In general, these game-theoretic problems are even more difficult to solve. One can use a linear program to determine approximate optimum non-preallocation defense strategies when both p and are less than 1. However, since the expected fraction of targets saved is not linear in the offensive allocations y_i , where y_i is the fraction of targets attacked by i weapons, an exact linear programming solution to the allocation problem must consider as many linear constraints as there are pure offense strategies (since a mixed offense strategy is a linear combination of pure offense strategies), which is a very large number.

4. Damage Assessment Strategies

Damage assessment by the defense enables him to increase the expected fraction of targets saved by evaluating target damage during the course of the engagement and subsequently defending only undestroyed targets. On the other hand, the offense can also damage assessment by attacking in waves and obtaining information about the effectiveness of earlier waves before deciding on the targets for the next wave. The potential gains in using damage assessment strategies are analysed in the following two subsections.

a. Defense Damage Assessment

A general defense-last-move damage assessment model can be developed assuming that the parameters A,D,T,p and are known to both sides, and that the defense knows that the offense will attack in waves of one weapon per target in each wave. To simplify the analysis, it is further assumed that the number of targets surviving after each wave is given deterministically by its expected value.

When $p < 1$, the optimal defense has the following form, with wave a arriving first, and wave 1 arriving last:

wave a through n+1: defend no targets,
wave n: defend a fraction of the surviving targets, and
wave n-1 through 1: defend all surviving targets.

When $p = 1$ (perfect weapons), this strategy must be modified so that a fraction of the targets is defended starting at wave a. The value n is equal to the smallest value of i for which $Q_i \geq d$ where

$$Q_i = \frac{1 - \{q_0 + (1-q_0)q_1\}^i}{(1-q_0)(1-q_1)} q_1^{a-i}, \quad i = a, a-1, \ldots, 1 .$$

The expected number of missiles to allocate to targets on the ith wave d_i^* can be given by a set of recursive equations, and E(f) is given by

$$E(f) = \{d_i^*/T\}\{q_0 + (1-q_0)q_1\} .$$

The maximum value of D required if all targets are defended at all waves is

$$D_{max} = T \frac{1 - \{q_0 + (1-q_0)q_1\}^a}{(1-q_0)(1-q_1)} .$$

Comparisons of the expected fraction of targets saved in the case of preallocation strategy and damage assessment strategy show that there is not much improvement made by damage assessment. Thus these strategies gain little for the defense in the case of defense-last-move.

b. Offense Damage Assessment

Offense damage assessment strategies have been considered in the cases where both missiles and weapons are perfect, only the defensive missiles are perfect, and only the attacking weapons are perfect.

In the case of perfectly reliable missiles and weapons, the defense can maximize E(fraction of targets saved) in a k-wave attack by observing the number a_i of weapons per surviving target allocated by the offense at the ith attack wave, and then selecting d_i, the corresponding number of missiles allocated per surviving target at the ith wave such that $a_i/d_i = a/d$. In this case,

$$E(f) = (d/a)^k .$$

A better strategy for the defense would be to select $d = dT/k$ missiles to be used in each wave. The fraction of targets saved using this strategy is greater than E(f) with equality occuring when aT/k weapons are allocated to each wave.

In the case where weapon kill probability is less than 1, the problem becomes more complex. To simplify the analysis, it is assumed that the offense does not reattack a target if a weapon assigned to that target was not intercepted by the defense, even though the target may survive. Then

$$E(f) = 1 - \sum_{i=1}^{k} f_i \{1-(d_i/a_i)\}\{1-(1-p)^{\frac{a_i}{f_i}}\}$$

where $f_i = \prod_{j=1}^{i-1} d_i/a_i$.

The optimal strategies satisfying $\max_a \min_d E(f)$ appears unsolvable in closed form. An upper bound can however be easily obtained if an infinite number of waves is assumed. Let the 1st wave attack be $a_1 = a-d$ weapons per target. In subsequent waves, if the defense allocates d_i missiles per target in the ith wave, the attacker allocates $a_{i+1} = d_i$ weapons per target in the (i+1)th wave. The expected fraction of targets killed is $1-(1-p)^{a-d}$.

A somewhat different offensive damage assessment problem can be considered assuming that the defense does not know the weapon stockpile size A. The offense is assumed to allocate one weapon at a time to a target, and continue firing at undestroyed targets until all T targets are destroyed. In this situation, an appropriate MOE for the defense would be to maximize E(number of weapons required to destroy T targets). The optimum missile allocation can then be found by dynamic programming using the recursion:

$$f(i,j) = \max_{0 \le m \le j} \{1+f(i,j-m)(1-p(1-\rho)^m)+f(i-1,j-m)p(1-\rho)^m\},$$

where $f(i,j)$ = E(number of weapons required to destroy i targets given j missiles are available).

5. Attacker-Oriented Defense Strategies

The preceding section considered the gain in effectiveness if the defense could assess damage to its targets. In contrast, there may arise a situation where the defense is not able to predict which target a weapon is aimed at before allocating a missile to engage it. The best that the defense can do in such a situation would be to use an attacker-oriented strategy and assign missiles at random to the weapons on a one-to-one basis, and knowing this strategy, the offense would attack each target with a weapons.

If $d \ge a$, every weapon will be allocated 1 missile. If $d < a$, the number of weapons which are actually intercepted would be a random variable from a binomial distribution with parameter d/a.

Two distinct cases can be considered for attacker-oriented defense: when neither side knows the other's allocation, and when the attacker knows the defender's allocation. In both cases, it is assumed that both sides know the value of A, D, T, ρ, and p.

When both sides must make their allocations in ignorance of the other's allocation, the optimal strategies for both are to allocate missiles and weapons randomly and as uniformly as possible. In the case where the attacker has the last move, the optimal defense strategy is to allocate missiles as uniformly as possible to the targets. If D/A is an integer and the defense uses his optimal strategy, the optimal offense strategy would be to assign [A/T] weapons to T-(A-T[A/T]) targets and [A/T]+1 weapons to A-T[A/T] targets.

D. STRATEGIES FOR A GROUP OF NON-IDENTICAL TARGETS

In this section, offense and defense strategies for a group of targets with unequal values v_i are considered. The value of a target may be related to some physical parameter of the target such as the human population for a city target. It is assumed that the target values and stockpile sizes are known to both offense and defense. An appropriate MOE in this case would be the expected value of targets saved, E(V). Since the targets have different values, it is reasonable to assume that they would have different vulnerabilities; hence the value of p, the weapon kill probability, will not be constant, but will vary with the target with which it is associated. In general, the approaches that have been developed to find optimum offense and defense strategies for targets of unequal values lead to approximate solutions rather than exact ones. The following situations have been analyzed by researchers:

- one-sided allocation problems,
- offense-last-move strategies,
- strategies when neither side knows the other's allocation,
- strategies when offense stockpile size is unknown, and

- attacker-oriented defense strategies.

Each of these situations are presented in the following subsections.

1. One-Sided Allocation Problem

A one-sided allocation problem exists when the allocation strategy of one side has been specified and is known to the other side who then designs his optimal allocation to counter that specific enemy strategy. Two mathematical techniques available for this type of problem are dynamic programming and Lagrange Multipliers. To utilize these methods for finding the maximum value of E(V) and the optimal defense allocation for a specified offense allocation, the problem can be formulated as

$$\max \sum_{i=1}^{K} E(i,d_i) \quad \text{subject to} \quad \sum_{i=1}^{T} c_i d_i \leq C \quad ,$$

where $E(i,d_i)$ is a general function denoting the expected value saved at the ith target if d_i missiles each of cost c_i are allocated to it, and C is the total available defense budget for missiles.

The dynamic programming approach solves successive maximization problems using a recursion equation, whereas the Lagrange Multiplier method finds the unconstrained maximum of the Lagrangian function either by direct differentiation of the Lagrangian, or by direct search methods.

2. Strategies with Offense-Last-Move

Various methods for determining offense and defense strategies when the offense has the last move have been proposed. The approaches to this problem can be divided into two categories. The first category uses an arbitrary payoff function $E(i,a_i,d_i)$, while the other category assumes specific payoff functions. In general, specialized payoff functions simplify the analysis considerably.

Using the Lagrange Multiplier approach, approximate upper and lower bounds for $E^*(V)$ can be obtained, if optimal strategies are used by both sides, by introducing the Lagrangian function:

$$L(\lambda, w) = \max_{d} \min_{a} \{\sum_{i=1}^{T} E(i, a_i, d_i) - \lambda \sum_{i=1}^{T} d_i + w \sum_{i=1}^{T} a_i\} .$$

A lower bound to $E^*(V)$ is given by

$$E(i, a_i^*, d_i^*) - \lambda_o d_i^* + w_o \sum_i a_i^* ,$$

where $(\lambda_o, w_o, a_i^*, d_i^*)$ is the maximum solution to the Lagrangian. An upper bound to $E^*(V)$ can be obtained by finding maximum solutions to the Lagrangian for other values of λ and w, e.g. λ_1, w_1 with corresponding values a_i', d_i', A' and D'. Then if $E(i, a_i', d_i') - \lambda_1(A-A') + w_1 (D-D') \le \sum_i E(i, a_i^*, d_i^*)$, a range of λ, $\lambda_1 \le \lambda \le \lambda_1'$ can be eliminated, where λ_1' in the above equation changes the inequality. Using this elimination procedure successively for different Lagrangian solutions, only a small region of in the vicinity of λ_o will not be eliminated, e.g. $\lambda_L \le \lambda_o \le \lambda_H$. An upper bound for $E^*(V)$ is then the maximum value of $L(\lambda, w)$ in the region $\lambda_L \le \lambda \le \lambda_H$, $w = w_o$. If the difference between these bounds is small, the use of the Lagrangian strategies a_i^* and d_i^* is practicable.

Another approach to the same problem is by using the dynamic programming relation

$$c_i(A, D) = \max_{0 \le a_i \le A} \min_{0 \le d_i \le D} \{E(i, a_i, d_i) + c_{i-1}(A-a_i, D-d_i)\} ,$$

starting with $i = 1$ and solving iteratively for a_i^*, d_i^*, $i = 1, \ldots, T$. The final $c_T(A, D)$ will however only be an upper bound to $E^*(V)$, and the allocation found will be non-optimal. A lower bound can be found by adopting the weapon allocation a_i^*, and using dynamic programming to determine the corresponding defensive allocations.

Three explicit payoff functions with increasing degrees of simplicity are considered.

In an idealized defense in which each weapon is intercepted by d /a missiles, the payoff function E(i) can be given by

$$E(i) = V_i \{1 - q_{ai} \exp(-t_i d_i / a_i)\} ,$$

where $t_i = -\ln(1-\rho_i)$.

The expected value of targets saved with optimal strategies is then

$$E^*(V) = \max_d \min_a \sum_{i=1}^{T} E(i) .$$

The problem of finding the optimal strategies a_i^* and d_i^* is a very difficult analytical problem. An approximation to the optimal strategies can be derived in the case when the total attack size is very large compared with the defense stockpile and the number of targets. Then

$$a_i^* = \{\ln c - \ln(-v_i \ln u_i)\}/\ln u_i ,$$

where $u_i = 1 - q_{ai} \exp(-t_i D/A)$,

$$\ln c = \{A + \sum_{i=1}^{T} \ln(-v_i \ln u_i)/\ln u_i\}/\sum_{i=1}^{T} 1/\ln u_i ,$$

$$d_i^* = Da_i^*/A \quad \text{and} \quad E^*(V) = \sum_{i=1}^{T} v_i u_i^{a_i^*} .$$

A valid solution is obtained when any negative a_i^* are eliminated (target i left undefended), the closed form solution derived for the remaining targets, and the positive a_i^* satisfy the inequality

$$a_i^* \ln u_i + u_i^{-a_i^*} < 1 + t_i d_i (1-u_i)/u_i .$$

If the missiles are assumed to be reliable, i.e. ρ = 1, the payoff function is given by

$$E(i) = v_i (1-p)^{\max(0, a_i - d_i)},$$

and $E^*(V) = \max_{d_i} \min_{a_i} (E(i) - \lambda d_i + w a_i)$.

The optimizing values of d_i and a_i can be found for any λ and w in three cases:

$\lambda < w$: then $d_i^* = (v_i - t_i)/w$ for $w < v_i x$, and

$a_i^* = 0$ or $v_i/w - 1/x$ at will,

where $x = -\ln(1-p)$ and $t_i = (w/x)\{1 - \ln(w/v_i x)\}$;

$\lambda = w$: then d_i^* = any value in the range $\{0, (v_i - t_i)/w\}$ for $w < v_i x$,

$a_i^* = 0$ or $v_i/w - 1/x$ if $d_i^* = (v_i - t_i)/w$ and

$a_i^* = d_i^* - \ln(w/v_i x)/x$ if $d_i^* < (v_i - t_i)/w$;

$\lambda > w$: then $d_i^* = 0$ and $a_i^* = \max\{0, -(\ln(w/v_i x))/x\}$;

λ and w are selected by trial and error so that $\sum_{i=1}^{T} a_i^* = A$, and $\sum_{i=1}^{T} d_i^* = D$. In the first two cases, $d_i^* = a_i^* = 0$ if $w \geq v_i x$.

In the case where $p = \rho = 1$ (perfect missiles and weapons),

$E(i) = v_i$ if $a_i \leq d_i$, and 0 if $a_i > d_i$.

One treatment of this problem assumes a weapon stockpile size normalized to 1 and a missile stockpile size of $H = D/A$. Using techniques from the theory of linear equations and number theory, it can be shown that there exists certain canonical defense strategies corresponding to defense stockpiles $H, \ldots, H_M$, such that the defense can achieve the same $E^*(V)$ by using only H_i missiles, where $H_i \leq H \leq H_{i+1}$, i.e. one lists the complete set of offense and defense strategies for $1 \leq D/A \leq T$, and the optimal offense/defense strategies are those that maximizes/minimizes the expected value destroyed. This method is however only feasible for small numbers of targets, as the combinatorial possibilities go up rapidly with increases in the number of targets.

3. Strategies when Neither Side Knows the Other's Allocation

The game-theoretic situation where each side knows the other's stockpile size but not his allocation to targets is a very difficult problem mathematically. In order to obtain optimum strategies, it is necessary to make a number of simplifying assumptions to make the problem more tractable analytically.

If it is assumed that $A \geq D$ and $p = \rho = 1$ with the payoff function being the expected value of targets destroyed, the optimal offense strategy is to attack the a single target with the entire stockpile A, and the defense allocates its missiles among the more valuable targets, leaving the less valuable targets undefended.

If there are only two targets with values v_1 and v_2, the optimum defense and offense strategies can be obtained in 5 cases:

- $D = A-1$ (neither side dominant): the unique optimal offense strategy is to allocate all weapons to the more valuable target, while all defense strategies are equivalent. The value of the game V is max (V_1, V_2);
- $D \geq 2A$ (defense overwhelming): any defense strategy is optimum as long as at least A missiles are allocated to each target, while all offense strategies are equivalent with $V = 0$;
- $2D+2 \leq A$ (offense overwhelming): any offense strategy is optimal as long as at least D+1 weapons are allocated to each target. The defense strategy has no effect and $V = v_1 + v_2$.

In the remaining two cases where $2A-1 \geq D \geq A$ (defense dominant) and $D+2 \leq A \leq 2D+1$ (offense dominant), the optimal strategies can be written as a convex linear combination of extremal strategies of the general form:

allocate i missiles (or weapons) to the target of value v_1 and the remaining missiles (or weapons) to the other target of value v_2 with probability x_i, where $\sum x_i = 1$. When defense is dominant, each extremal optimal defense strategy corresponds to a sequence $M = (m_1, \ldots, m_k)$ of integers such that $1 \le m_1 \le m_2 \le \ldots \le m_k \le R$ where k is the smallest integer $\ge (A+1)/(D-A+1)$, and $R = k(D-A+1)-A$, and allocates $i(D-A+1)-m_i$ missiles to the target with value v_1 (and the remaining missiles to the target with value v_2) with probability

$$(v_1^{i-1} v_2^{k-i}) / (v_2^{k-1} + v_1 v_2^{k-2} + \ldots + v_1^{k-2} v_2 + v_1^{k-1}) \;, \; i = 1, \ldots, k \;.$$

Similiarly each extremal optimal offense strategy corresponds to a sequence $(n_1, n_2, \ldots)$ of integers such that $(D-A+1) \ge n_1 \ge n_2 \ge \ldots \ge R$, and allocates $i(D-A+1)-n_i$ weapons to the target of value v_1 (and the remainder to target v_2) with probability

$$(v_1^{k-i} v^{i-1}) / (v_2^{k-1} + v_1 v_2^{k-2} + \ldots + v_1^{k-2} v_2 + v_1^{k-1}) \;, \; 1 = 1, \ldots, k \;.$$

When the offense is dominant, the extremal optimal defense strategies are obtained by substituting $\bar{D} = A-2$ and $\bar{A} = D$ into the extremal optimal offense strategy given above, and the extremal optimal offense strategies are obtained by substituting $\bar{D}$ and $\bar{A}$ into the extremal optimum defense strategy formula.

4. Strategies for Unknown Weapon Stockpile Size

If the defense has no knowledge of the offensive stockpile size, it is reasonable to design a strategy such that the expected value of targets destroyed is approximately proportional to the attack size (robust strategy). If the offense has the last move, the objective of the defense would be to minimize the maximum (over all possible attack strategies) expected value destroyed per weapon expended at the ith target, i.e. $\min S_i$ where $S_i = \max_{a_i} \{(v_i - E(i,a_i,d_i))/a_i\}$. This is achieved by selecting an optimal defense strategy $(d_1^*,\ldots,d_T^*)$ such that $S_i = k$ at all defended targets and $S_i \le k$ at all undefended targets, where k is found by trial and error satisfying $\Sigma d_i^* = D$.

When neither side knows the other's allocation, a near optimal defense strategy can be constructed if the missile reliability is assumed to be 1, and an uninterrupted weapon damages exactly one unit of target value. If v_i is an integer and $\Sigma v_i = D$, then $0, 1, \ldots, 2v_i$ missilies are assigned to the defense of a target of value v_i , each with probability $1/(2v_i+1)$. If the stockpile size is $D = k\Sigma v_i$, the corresponding defense allocation would be scaled up to be $0, 1, 2, \ldots, 2kv_i$ missiles assigned with probabilities $1/(2kv_i+1)$.

5. Attacker-Oriented Defense Strategies

Attacker-oriented defense strategies are used when the defense is ignorant of which targets the incoming weapons are attacking. If the offense has the last move, the uniform attacker-oriented strategy described earlier for identical targets is also optimum in the case of unequal-valued targets.

If both sides are ignorant of the other's allocation, the optimal defense is a uniform random attacker-oriented strategy similiar to the case where targets are identical, i.e. allocate $[D/A]$ missiles randomly to $A-D+A[D/A]$ incoming weapons and $[D/A]+1$ missiles to the remainder. The optimal offense strategy can be approximated to be as follows: allocate weapons to the T_0 targets of greatest value, where T_0 is the maximum value of i satisfying the inequality

$$v_i \ge \left(\prod_{j=1}^{i} v_j\right)^{\frac{1}{i-1}} C^{\frac{1-i}{i-1}}, \quad 1 \le i \le T ,$$

and $Q = 1-p(1-p_e)$,

with

$$p_e = (1+D/A+[D/A])(1-(1-\rho)^{[\frac{D}{A}]}) + (D/A-[D/A])(1-(1-\rho)^{[\frac{D}{A}]+1}).$$

The number of weapons assigned to v_j , $1 \le j \le T$ is

$$a_j = (\log c - \log v_j)/\log Q ,$$

where

$$c = (\prod_{i=1}^{T_0} v_i)^{\frac{1}{T_0}} Q^{\frac{A}{T_0}} .$$

E. STRATEGIES IN SPECIAL SITUATIONS

In this section, the problem of allocating offensive weapons and defensive missiles in three special situations are presented:

- attacks on the defense system,
- defense using local and area missiles, and
- budget constrained defense using local and area missiles.

These represent more realistic scenarios than the previously idealized cases of offense and defense strategies. The mathematical models are consequently more difficult to solve analytically, and it is necessary in most cases to resort to iterative search procedures or Monte Carlo simulations on a computer in order to find the optimal allocation strategies.

1. Attacks on the Defense System

It was mentioned earlier that an alternative feasible strategy for the offense would be to allocate some of his weapons to attack the defense system itself on the premise that undefended targets would be more vulnerable than defended ones. The offense would normally attack a critical component of the defense system such that when it is destroyed, the entire defense system would be rendered either inoperative or its operation would be seriously degraded. Examples of such critical components are radars, command-and-control centres or tactical communication links. It is assumed that there are R such identical components e.g. R radars all of which must be destroyed before the entire defense system is considered destroyed. It is also assumed that there are T identical point targets, the defense can carry out attack evaluation, and both sides know the other's stockpile size.

The MOE is the expected fraction of targets saved, and can be given generically by

$$E(f) = pE_u(f) + (1-p)E_d(f) ,$$

where p is Pr(all radars are destroyed), $E_u(f)$ and $E_d(f)$ are the expected fractions of targets saved if undefended and defended, respectively.

If the offense has the last move and if missiles are completely reliable, but the radars are completely vulnerable to attack, i.e. Pr(an undefended radar is destroyed by a weapon) is 1, then the optimal defense strategy would be to divide the missile stockpile into two equal parts, and allocate each part evenly to the radars and targets respectively, if the offense allocates his weapons evenly among the defended targets. In some circumstances when the attack is not uniform, a better defense strategy would be to shift some missiles from radars to targets, since only one radar is required for the defense system to be operative.

If the defense has the last move and has a central stockpile from which missiles are drawn either to defend a radar or a value target, he will defend a randomly selected radar against attack as long as missiles remain in the stockpile, and then use an attacker-oriented strategy to assign missiles to incoming weapons starting with the most lightly attacked targets. The offense will attack all radars with the same number of weapons in order to reduce or exhaust the defense stockpile. Above a certain number, the radars will no longer be a soft spot in the defense, and a better offense strategy would be to attack the targets directly rather than attack the radars. In the defense-last-move model, the defense must make allocation decisions in the course of the attack, based on up-to-date information. An alternative defense strategy analogous to the Matheson strategy could be devised, that do not depend on the capability to make 'on-the-spot' decisions. However, this strategy is inferior to the defense-last-move strategy.

In the case where the defense is restricted to a one-on-one defense for both radars and targets, and the defense intercepts each attacker as long as there are still missiles available, the problem of determining the minimum necessary number of radars so that the offense attacks targets only can be solved. In a target-only attack, the expected fraction of targets saved is given by

$$E_t(f) = \{q_o + \rho(1-q_o)\}^{\frac{A}{T}} ,$$

and in a mixed target-radar attack, the expected fraction of targets saved is given by

$$E_m(f) = k q_o^{\frac{A-Ra_r}{T}} + (1-k)\{q_o + \rho(1-q_o)\}^{\frac{A-Ra_r}{T}} ,$$

where k = Pr(all radars are destroyed) = $\{1-(q_{or}+\rho(1-q_{or}))^{a_r}\}^R$, a_r is the number of weapons allocated to radars, and q_{or} is Pr(an undefended radar survives an attack by a weapon). The minimum neccessary value of R is the smallest R for which $E_m(f) \geq E_t(f)$ for all a_r in the interval $(0, A/R)$.

In a model with offensive damage assessment, it is assumed that the offense knows the defensive stockpile size but not vice versa, the attack is sequential with 1 weapon at a time allocated to either a radar or a target, and the offense can carry out damage assessment between firings. The

MOE used is the expected number of weapons required to kill the T targets. Dynamic programming can be used to obtain the optimal defense allocation to each incoming weapon, and the offense allocation to either target or radar in each successive wave. If the expected number of weapons required to destroy i targets given j radars and k missiles and the next attack is on a target is denoted by $f_t(i,j,k)$, and the analogous expected number of weapons, given the next attack is on a radar, is denoted by $f_r(i,j,k)$, then the recursive equations are

$$f_t(i,j,k)=\max_{0\le m\le k}\{1+f(i,j,k-m)\,(1-\rho(1-p)^m)+f(i-1,j,k-m)\,p(1-\rho)^m\} ,$$

$$f_r(i,j,k)=\max\{1+f(i,j,k-m)\,(1-p_r(1-\rho)^m)+f(i,j-1,k-m)\,p_r(1-\rho)^m\} ,$$

$$f(i,j,k) = \min\{f_t(i,j,k),\ f_r(i,j,k)\} \text{ , where } p_r = 1-q_{or} .$$

If an offense strategy that includes attacks on missile silos is considered, the problem becomes more complex. In order to evaluate this situation, the following assumptions are made: the offense can attack missile silos, radars and value targets in waves of one weapon directed at each of the D missile silos, or at each of the R radars, or at each of the T targets, and continues with the attacks until I or fewer targets survive, the value of I being known to the defense. All engagements are one-on-one given that $p = \rho = 1$, and there is no offense damage assessment, although the offense has the last move. The MOE used is the expected number of weapons required to destroy I or more targets.

The defense strategy is as follows: if the offense attacks the radars, allocate 1 missile to defend a specific (unknown to the offense) radar; if the offense attacks the targets, then allocate I missiles to defend a specific subset (unknown to the offense) containing I targets; and if the missile silos are attacked, allocate half of the unused and undamaged missile stockpile to defend the silos of the other half of the stockpile. If R and T/I are both integer powers of 2, the number of weapons required to ensure I or more targets are destroyed is

$$A = T-IR+D(1+\log_2 R) \quad \text{for } R \le T/I$$
$$= D\{1+\log_2(T/I)\} \quad \text{for } R > T/I .$$

The offense strategy is as follows: if $R \le T/I$, attack missile silos in $\log_2 R$ waves of D weapons each, then attack radars in (D/R - 1) waves of R weapons each, and finally attack (T-I) targets in a single wave of T weapons. If $R > T/I$, attack missile silos in $\log_2(T/I)$ waves of D weapons each, then attack targets in D/T waves of T weapons each.

2. Defense Using Local and Area Missiles

In the preceding discussion, it was assumed that there is only one type of defensive missile. A more realistic situation would be to allow two types of missiles: a short-range local missile which defends single targets (terminal defense), and a longer range area missile which can defend against weapons directed at one of a group of targets in an extended region (area missiles). Various possibilities for defense using both local and area missiles are considered here.

The simplest model involves the defense of a set of targets of different values using D_A area missiles which can cover any target in the set and for which the defense has the last move, and D local missiles which are allocated to single targets prior to the attack. It is assumed that both sides know the other's stockpile size, and both weapons and missiles are perfectly reliable. The offense is assumed to attack a subset of the targets, each one with a number of weapons proportional to its value, while the defense allocates local missiles in numbers also proportional to target value. The area missiles are allocated to targets such that they destroy just enough of the weapons directed at each target to let the remainder be destroyed by the local missiles defending that target. The optimum fraction of total target value to be attacked is given by A_L^*/D_L if $A_L^* < D_L$, where $A_L^* = A-(D_A A)^{\frac{1}{2}}$; otherwise the offense attacks the entire set of targets. If the local defense covers only a fraction h of target value instead of the entire set of targets, $A_L^* = A - D_A\{A+D_L(1/h - 1)\}^{\frac{1}{2}}$.

If instead of the missile reliability being equal to 1, it is assumed that t local missiles or s area missiles are required to kill a weapon, $A_L^* = A - (D_A A/s)^{\frac{1}{2}}$, and the optimum fraction of total target value attacked is tA_L^*/D.

If the defense uses a preallocation strategy for area missiles, and it is assumed that the targets have identical values, the missiles and weapons have perfect reliability, and both sides know the other's stockpile size, the problem can be formulated as a continuous Blotto game by allowing the offense and defense allocations to vary continuously. The local missiles are allocated evenly among the targets. The allocation of area missiles and weapons depends, however, on whether the offense or defense is dominant.

If $d_a(a - d_l/2) \leq (a - d_l)^2$ where d_a and d_l are the number of area and local missiles available per target, and the offense is dominant and he attacks a typical target with a_i weapons where a is a_i random variable drawn from the Uniform distribution $U(d_l, 2a-d_l)$. If, however, $d_a(a - 0.5d_l) \geq (a-d_l)^2$, the defense becomes dominant, and in this case, the offense should attack a target with probability

$$2a/\{d_a+2d_l+ (d_a^2+ 2d_a d_l)^{\frac{1}{2}}\}$$

using a_i weapons, where a_i is a random variable from the Uniform distribution $U(d_l,\ d_l + d_a + \sqrt{d_a^2 + d_a d_l})$. The defense defends a target with probability

$$(1/d_l)\sqrt{d_a^2 + 2d_a d_l} - (d_a/d_l)$$

using d_i area missiles, where d_i is a random variable from the Uniform distribution $(0,\ d_a + \sqrt{d_a^2 + d_a d_l})$.

A further relaxation of the assumptions would be to allow several non-overlapping area defense regions, each containing several point targets of different values v_i which are protected by local defenses as well. One-on-one missile engagements are assumed, together with weapon kill probability being equal to 1. An approximate solution to this nested allocation problem can be found if the offense stockpile is assumed to be of infinite size, and weapons are allocated to minimize the cost in terms of weapons destroyed per unit target value destroyed. The defense strategy is to allocate area missiles among regions so that the offense minimum cost per unit value destroyed is the same for every sector. The local missiles are allocated among targets within a region such that the minimum cost per unit value killed is the same for every target in the region, ignoring the contribution of the area missiles.

3. Budget Constrained Defense Using Local and Area Missiles

The models considered here differ from the previous models in that the defense is given a fixed budget to divide among local and area missiles. The optimization therefore involves this division as well as the allocation of the two types of missiles to the defense of targets. It is assumed that the defense can purchase d area missiles per target with his budget, and that the ratio of the cost of an area missile to that of a local missile is k, both values being known to the offense. Both sides know the other's stockpile size, and a one-on-one defense is used. Furthermore, it is assumed that the weapons and missiles are perfectly reliable. Since the defense can use an attacker-oriented strategy and save all targets with area missiles if $d \geq a$, the analyses that follow assume $d < a$, and consider three cases based on specific assumptions about the area defense:

- defense-last-move strategy for area missiles,
- area defense strategy for area missiles, random weapon arrivals, and
- area defense strategy for area missiles, controlled weapon arrivals.

In the first case of defense-last-move for area missiles, the defense has $d-j$ area missiles per target and jk local missiles per target, $j = 0,1,\ldots,d$. If the offense attacks a fraction $i/(jk)$ of the targets with ajk/i weapons apiece, then jk of these weapons will be destroyed by local missiles at each target attacked, leaving $(a-i)jk/i$ weapons to which area missiles are assigned. The offense can choose i after observing the defense's choice of j. The optimal strategies are found by differential calculus to be as follows:

if $1 \leq a/d \leq k$, $j = d(1 - d/a)$, i.e. allocate d^2/a area missiles per target, and $i = a(1 - d/a)$, i.e. a fraction $a/(dk)$ of the targets are attacked.

In the case of random weapon arrivals, as each weapon arrives, the defense assigns an area missile to it without knowing which target is being attacked, until the area missile stockpile is exhausted; then local missiles are used. Weapon arrivals are random with respect to the targets the weapons are directed against. Assuming independent engagements of weapons by local missiles at different targets, the probability that a weapon is intercepted by an area missile is $(d-j)/a$. If there are $a+i$ weapons,

$i = 0, 1, 2, \ldots$ allocated to a target, the probability that exactly m of them are intercepted by area missiles is given by the approximation

$$p_m = \binom{a+i}{m}\left(\frac{d-j}{a}\right)^m \left(1-\frac{d-j}{a}\right)^{a+i-m} .$$

No simple analytical solution to this problem can be found; however the offense strategy i can be approximated by $dk-a+1$ if d and k are small and $d \ll a$. If $a > 3d/2$, the defense strategy j is approximated by $d-1$.

The model with controlled weapon arrivals is similiar to the one analyzed previously except that in this case, the offense can control the order of arrival of his weapons on targets. The offense exhausts the area missile stockpile with $(d-j)T$ weapons, then attacks as many targets as possible with $(jk+1)$ weapons per target. The fraction of targets to be attacked is determined by

$$i = \max \{0, a(jk+1)/(a-d+j) - a\} .$$

The optimal defense strategy is one of two extremes: all local or all area missiles, according to whether $1/k$ is greater than or less than $a-d$.

IV. LATER INVESTIGATIONS INTO THE MISSILE ALLOCATION PROBLEM

A. INTRODUCTION

In Chapter 3, an overview of the studies done on the missile allocation problem that are mentioned in the monograph by Eckler and Burr was presented. This chapter gives a survey of the investigations in this field conducted after the monograph's publication, with material drawn from papers published in scientific journals and postgraduate theses. A list of these publications is given in the Reference section of this thesis.

It is generally observed that the later investigations into the missile allocation problem tend to model more realistic and hence more complex scenarios of the battle, in contrast to the situations presented in Chapter 3, which are fairly simple models with a number of simplifying assumptions made to make the problem solvable. As a result, the mathematical formulations of the problem are not generally amenable to solution in closed form, and various solution techniques such as implicit enumeration algorithms, dynamic programming techniques, linear and nonlinear programming algorithms and other constrained optimization procedures were utilized to obtain numerical results.

This survey of the recent literature on the missile allocation problem is by no means comprehensive due to the restrictions on the scope of the thesis given in Chapter 1. However, the literature that was reviewed revealed a number of interesting analytical approaches to the missile allocation problem in specific, and sometimes novel, situations.

The papers that were surveyed analyzed the missile allocation problem from a number of different perspectives and used various analytical techniques. They can however be loosely grouped for exposition purposes here according to the specific scenario the model seeks to represent, or to the objectives that the defender or attacker seeks to achieve, as follows:

- strategies involving decoys,
- strategies involving attacks on the defense system,
- strategies involving specific types of defensive allocations,
- strategies involving targets of opportunity,
- strategies with specific target assumptions,
- strategic nuclear exchange situations,
- strategies involving proportional defense, and
- strategies in a game theoretic situation.

The studies will be presented in the following sections under these scenarios.

B. STRATEGIES INVOLVING DECOYS

In Chapter 3, Section B.4, the problem of allocating defensive missiles to a mixture of attacking weapons and decoys was considered in the case where a limited capability of the defense to distinguish between actual weapons and decoys exists, expressed in terms of his knowledege of the probability distributions $f_w(x)$ and $f_d(x)$ of some arbitrary physical characteristic.

1. Defense Strategy

Layno [Ref. 4] also considered the defense allocation against a mixture of weapons and decoys when the defense is assumed to possess a limited capability of distinguishing between a weapon and a decoy, this capability being quantified by the probabilities of mistaking a decoy for a weapon p_1 and mistaking a weapon for a decoy p_2. The defense is assumed to know the total number of threat objects, the number of defensive missiles available and their kill probabilities, and the values of p_1 and p_2. The objective of the defense is to minimize the expected total number of real weapons penetrating the defense, by finding an optimal allocation of missiles against an incoming object diagnosed as being a weapon and an object diagnosed as being a decoy.

In the case where the defense has no discrimination capability, the expected number of penetrating weapons can be given by

$$L_0 = A_r (1-\rho)^i (1-f\rho) ,$$

where A_r is the total number of attacking weapons, i is the integer part of d, the average number of missiles allocated per attacking object, and f is the fractional part of d, i.e. $d = i+f$. If the approximation $1-f\rho \approx (1-\rho)^f$ is used, then

$$L_0 = A_r (1-\rho)^{i+f} = A_r (1-\rho)^d .$$

In the case where the defense possesses a limited discrimination capability, the average number of attacking objects which are diagnosed as being weapons is $A_r' = A_r - p_2 A_r + p_1 A_d$, where A_d is the total number of incoming decoys. Similarly, the number of objects diagnosed as being decoys can be given by

$A_d' = A_d - p_1 A_d + p_2 A_r$. The expected number of penetrating weapons in the limited discrimination case can be given by

$$L = A_r \{(1-p_2)(1-\rho)^{d_r} + p_2 (1-\rho)^{d_d}\} ,$$

where d_r and d_d are the numbers of missiles allocated to each incoming object diagnosed as a weapon and a decoy respectively.

The problem of finding the values of d_r and d_d to minimize L reduces to being a nonlinear program with a linear constraint:

$$\min A_r\{(1-p_2)(1-\rho)^{d_r}+p_2(1-\rho)^{d_d}\}$$

subject to $A'_r d_r + A'_d d_d = D$,

where D is the total number of missiles. The optimal solution given by Layno is:

$d_d^* = (b-B)/(m'+1)$ if $(b-B)/(m'+1) > 0$, and 0 otherwise; and

$d_r^* = (b+m'B)/(m'+1)$ if $d_r^* > 0$, and b otherwise,

where $b = D/A'_r$, $B = \log\{p_2/m'(1-p_2)\}/\log(1-\rho)$ and $m' = A'_d/A'_r$.

The solution is, however, not correct since B can become a large negative number if p_2 is close to 1, in which case $d_d^* > 0$, and d_r^* could be negative if $-m'B > b$. For example, if $A'_r = 1$, $A'_d = 2$, $p_2 = 0.95$, $\rho = 0.6$, and $D = 4$, then $d_d^* = 2.15$ and $d_r^* = -0.31$ using the above two equations.

The correct solution is as follows: letting $1-\rho = e^{-\alpha}$, the objective function becomes:

$$\min A_r\{(1-p_2)e^{-\alpha d_r}+ p_2 e^{-\alpha d_d}\} ,$$

and using the Lagrange Multiplier technique, the optimal solutions are found to be:

$$d_r^* = (1/\alpha)\ln\{\alpha(1-p_2)/(\lambda A'_r)\} , \quad d_d^* = (1/\alpha)\ln\{p_2\alpha/(\lambda A'_d)\} ,$$

where λ is the Lagrange Multiplier.

If $\alpha(1-p_2)/(\hat{\lambda}A'_r) > 1$ and $\alpha p_2/(\hat{\lambda}A'_d) > 1$, where

$$\hat{\lambda} = \exp\{(A'_d\ln(\alpha(1-p_2)/A'_r) + A'_r\ln(\alpha p_2/A'_d) - \alpha D)/(A'_r+A'_d)\} ,$$

then $d_r^* = (1/\alpha)\ln\{\alpha(1-p_2)/(\hat{\lambda}A'_r)\}$, $d_d^* = (1/\alpha)\ln\{p_2\alpha/(\hat{\lambda}A'_d)\}$.

Otherwise, suppose $(1-p_1)/\lambda_r' > p_1/\lambda_d'$, then $d_r^* = D/\lambda_r'$, and $d_d^* = 0$, and if $(1-p_1)/\lambda_r' < p_1/\lambda_d'$, then $d_r^* = 0$, and $d_d^* = D/\lambda_d'$.

2. <u>Offense Strategy</u>

Sverdlov [Ref. 5: pp. 183-264] considered this subject within a different context. Whereas Layno analyzed the problem from the defense's viewpoint, Sverdlov considered the problem of deploying weapons and decoys in an attack on targets utilizing the two effects that were mentioned in Chapter 2, namely the defense exhaustion effect, and the saturation effect. In both cases, it is assumed that the defender does not possess any weapon-decoy discrimination capability, and that the engagements are one-on-one. The MOE used is the expected cost of killing the value target, and the offense strategy consists of deciding whether to fire a weapon or a decoy at each stage of the game while the defense strategy consists of either intercepting the incoming object with a missile or not. It is assumed that there is perfect information to both sides about the state of the process.

When the exhaustion effect is utilized, the offense launches wave after wave until the single target is destroyed. It is assumed that N missiles are available. If the value of the game is V_N, the cost of destruction $\bar{V}_N$, measured in terms of the cost of destruction incurred if the attacker uses real weapons only, is given by

$$\tilde{V}_N = V_N/(c_R/(pq)),$$

where c_R is the cost of a real weapon, p is the probability that the weapon destroys the target given that it survived interception by the defense, and q is the probability that the weapon survives the intercept. $\bar{V}_N$ can be written in recursive form as

$$0.5\left(B + \bar{V}_{N-1} + \sqrt{B+\bar{V}_{N-1} - 4C\bar{V}_{N-1}}\right), \quad \bar{V}_0 = q ,$$

with $B = q-qr_c(1-p)$, $c = q-qr_c(1-pq)$, $r_c = c_D/c_R$, c_D being the cost of a decoy.

The sclution to the problem can be stated as follows: if $N \le N^*$, where $N^* = \min (N: \bar{V}_N > 1-r_c)$, the optimal offense strategy is randomized, characterized by the probability that the attacker launches a real weapon,

$$p_c^* = (\bar{V}_N - \bar{V}_{N-1})/\{p(\bar{V}_N - q\bar{V}_{N-1})\} .$$

The corresponding optimal defense strategy is also randomized and is characterized by the probability that the defense fires at the incoming object,

$$p_d^* = (\bar{V}_N + qr_c - q)/(\bar{V}_N - q\bar{V}_{N-1}) .$$

However for $N > N^*$, the optimal strategies are pure: the attacker always uses weapons and the defense always fires at them.

When the saturation effect is utilized to overcome the defense, the offense strategy consists of finding the optimum number of decoys tc accompany the real weapons in each attack wave. Two cases are considered, firstly when the attacker can launch only a single weapon mixed with decoys in each wave, and secondly when the number of weapons is not restricted to one.

In the first case, the expected cost of destruction when m decoys acccmpany the single weapon is given by

$$c(m_D) = (m_D c_D + c_R)/\{p(1- ((1-q)/(m_D+1))^{N_s})\}$$

where N_s is the number of defense systems protecting the target, and each is assumed to act indepently of the others. To minimize the expected cost this expression is differentiated with respect to m_D to obtain the following optimal offense strategy:

if $q > N_s/(N_s+r_c)$, the optimal value $m_D^* = 0$, i.e. there is no need to have decoys;.

if $q \le N_s/(N_s+r_c)$, m_D^* is either $[m_D']$ or $[m_D']+1$, depending on whether $c([m_D'])$ is less than or greater than $c([m_D']+1)$, where m_D' is the positive root of the quadratic equation

$$c_D m_D^2 + c_D\{q+1-N_s(1-q)\}m_D + c_D q - c_R N_s(1-q) = 0 .$$

In the other case, where no restriction on the number of real weapons m_R per wave is imposed, but assuming only one defense system is available, i.e. $N_s = 1$, the minimum expected cost of destruction when the attacker is constrained to launch a total of m objects at a time is

$$c^*(m) = c_D m + \min_{1 \le m_R \le m}\{(c_R - c_D)m_R + (1-p)^{m_R - 1}(1-p+m_R p(1-q)/m)c^*(m)\}.$$

Numerical procedures must be employed to solve this equation.

C. STRATEGIES INVOLVING ATTACKS ON THE DEFENSE SYSTEM

In the previous chapter, the problem of attacks on the defense system itself was analysed essentially from the defender's viewpoint under a variety of assumptions. In contrast, Sverdlov [Ref. 5: pp. 31-182] considers the problem from the point of view of the attacker, who seeks to allocate his weapons in a successive wave attack between defense systems and a single value target such that various objectives are achieved, e.g. maximizing the probability of hitting the target, or maximizing the expected number of penetrators. In solving for the optimal strategies under different sets of assumptions, various applications of stochastic dynamic programming and game theory are employed.

In general, the sequential optimal attack on the defense starts with attacks on the defense system (if the offense stockpile is large enough) until the weapon stockpile is reduced to N^*, then the offense switches over to attack on the value target which is assumed to suffer 0-1 damage, and continues until the weapon stockpile is depleted. It is not feasible for the offense to switch back to attacking the defense system, hence only one switchover at M^* is optimal and no switch is possible from an attack on a target to the defense system in an optimal policy.

In the case where the defense system comprises a single point target (a defense target), and the MOE used is the probability of hitting the value target, the optimal policy can be obtained using dynamic programming, and is given by:

$$M^* = 1 + \left[\frac{\ln\left(1 - \frac{P_p}{P_s}\right)}{\ln\left(\frac{1-P_p}{1-P_p \cdot q}\right)} \right] .$$

P_p and P_s are the probabilities that an unintercepted weapon destroys the value target and the defense target respectively, and q is the probability a weapon will survive an intercept by the defense system.

If the MOE is the expected number of penetrators, the optimal policy is

$$M^* = 1 + [\, 1/P_s (1-q) \,] .$$

When the problem is generalised to include N_s defense systems (and hence N_s defense targets), and the assumption is made that there is no collateral damage among targets and the operation of defense targets is independent, the optimal attack strategy, using the maximum probability of hit criterion, is

$$M^*(N_s) = 1 + \left[\frac{\ln\left(1 - \frac{P_p}{P_s}\right)}{\ln\left(\frac{1 - P_p q(N_s - 1)}{1 - P_p q(N_s)}\right)} \right] .$$

$M^*(n)$ is non-increasing if the miss probability ratio $f(n)$, defined as

$$f(n) = \{1 - P_p q(n)\} / \{1 - P_p q(n-1)\} ,$$

is monotone increasing. Most weapon survival functions q do not have the monotone miss probability ratio property, and $M^*(N)$ is actually strictly monotone increasing, i.e. $M^*(n+1) > M^*(n)$. In this case, an algorithm based on the maximizing probability of hit criterion was derived for solving $M^*(N_s)$.

If the MOE is to maximize the expected number of penetrators, the optimal policy is

$$M^*(n+1) = 1 + \left[\frac{1}{P_s\{q(n)-q(n+1)\}}\right] .$$

If q(n) is strictly concave, M* is non-increasing.

A special situation arises when the defense is assumed to have the capability of switching to a cautious mode of operation, in which the defense system becomes much less vulnerable to attack, but at the same time is also much less effective in intercepting attacking weapons. The defense is assumed to consist of a single system, and possesses a limited capability for discerning whether an incoming weapon is aimed at a target or at the defense system itself. The defense thus has four choices of action denoted as follows:

- P1S1: employ ordinary mode of operation (Mode 1) regardless of the classification of an incoming weapon;
- P2S2: employ the cautious mode of operation (Mode 2) regardless of the classification of the weapon;
- P1S2: employ Mode 1 if the weapon is discerned to be aimed at a value target ('anti-primary' weapon), and Mode 2 if it is discerned to be aimed at the defense system ('anti-secondary' weapon);
- P2S1: employ Mode 1 if the weapon is classified as anti-secondary and Mode 2 if it is classified as anti-primary.

In any case, the probabilities that a weapon aimed at a target and at the defense system is correctly classified by the defense are α_p and α_s respectively, and the probabilities of survival of the weapon when the defense uses the normal mode and the secure mode are q_1 and q_2 respectively.

This problem can be formulated as a sequential game. Since one player (the attacker) has only two pure actions available to him, optimal randomized defense strategies exist which mix at most two of the four alternatives mentioned above.

If $\alpha_p \neq 1-\alpha_s$ no optimal defense strategy exists in which P1S1 and P2S2 are the only 'active' actions. If $\alpha_p > 1-\alpha_s$, P1S1 is active in all optimal mixed strategies, and conversely, if $\alpha_p < 1-\alpha_s$, P2S1 will be present in all optimal mixed strategies.

The first value of N in which both players resort to randomized strategies instead of pure strategies (offense attacks value target, defense uses P1S1 strategy) is the N^* of the one-sided dynamic programming model given above.

The general structure of the optimal defense and offense strategies is as follows:

- the number of weapons $N \le N^*$: the optimal defense strategy uses purely the normal mode of operation, and the optimal offense attacks value targets only;
- $N^* < N \le N^{**}$: the optimal defense randomizes over P1S1 and P1S2, and the optimal offense randomizes over attack on the value target and attack on the defense system;
- $N > N^{**}$: the defense randomizes over P2S2, and the offense randomizes over attack on value targets and defense systems.

The value of N^{**} can be calculated by the following set of equations:

$$N^{**} = \min\{N: N \ge N^*, q_N > (a'b''-a''b')/((a''-a')-(b''-b'))\} ,$$

where $b' = (q-q_1)/\alpha_s q_1 P_s$,

$$a' = (g-g_1)(1-g_1) \quad ,$$

$$b'' = (g_2-g)/\{g_1 P_s (1-\alpha_s)\} \quad ,$$

$$a'' = \{g_2-g-(1-\alpha_s) g_1 g_2 P_s\}/\{g_1 P_s (1-\alpha_s)\} \quad ,$$

$$g_M = M-V_M' \quad \text{and} \quad g = \alpha_p g_1 - (1-\alpha_p) g_2 \quad .$$

D. STRATEGIES INVOLVING SPECIFIC TYPES OF DEFENSE ALLOCATIONS

In this section, situations involving models of particular defense systems are analyzed. The first concerns the problem of overlapping area defense regions and the optimal allocation of defensive missiles to protect targets within these regions. In Chapter 3, the defense of targets with local and area missiles was also considered, but only in the case where the area defense regions are non-overlapping.

Another interesting problem that is considered here concerns optimal defense and offense strategies when the defense has a choice of allocating defense resources in procuring 'numerically vulnerable' defense systems which are easy to locate but difficult to destroy, or 'percentage vulnerable' systems whichare relatively difficult to locate, but once located can be easily destroyed.

The last model assumes that the defense consists of several 'layers' of defense systems, and that the attacker has to survive all of these layers in order to reach the target. The probability of an attacker penetrating all the layers is analyzed.

1. Overlapping Area Defense Regions

Swinson, et. al. [Ref. 6] consider the problem of overlapping area defense regions containing a number of point targets of different values, and developed a procedure that applies a dynamic programming algorithm within a general framework of successive approximations that allow area missile allocations to target 'sectors' to be optimized sequentially within the constraint of the missile stockpile size.

In the model, several area defense systems are distributed throughout an area containing point targets of different values. Each area defense covers a certain region within which a subset of the targets are located. These regions may intersect, and when they do, the union of these regions may be decomposed by these intersections into non-overlapping areas called 'sectors'. Targets in the sectors are defended by either a single area defense or several area defenses. Associated with a given attack of a weapons against target t is a function $r_t(d_t)$ denoting the expected value saved at the target if d missiles are allocated to intercept the attacking weapons. The function may be given by

$$r_t(d_t) = v_t(1-pq^{c_t}) \quad (1-pq^{c_t+1})^{a_t-f_t} ,$$

where v_t is the value of target t, p is the weapon kill probability, q is the probability that the weapon survives an engagement by a missile, and c_t and f_t are given by

$$c_t = [d_t/a_t] , \quad f_t = a_t(c_t+1) - d_t .$$

The objective of the defense is to maximize the expected total target value saved over all targets. If an optimal within-sector missile allocation policy is employed to allocate a total of D_j missiles to the defense of the T targets in sector j, then the total expected value saved for sector j is given by:

$$f_j(D_j) = \max \sum_{t=1}^{T} r_t(d_t), \quad \text{where} \quad \sum_{t=1}^{T} d_t = D_j .$$

This can be written in the standard functional equation of dynamic programming as:

$$f_{j,T}(D_j) = \max_{0 \le d_T \le D_j} r_T(d_T) + f_{j,T-1}(D_j - d_T)$$

which can be used recursively to find optimal missile allocations within each sector given a total sector allocation of D_j missiles. In order to find a set of optimal missile allocations to each sector, the expected total value saved over all sectors $F(x) = \sum_{j=1}^{n} f_j(\sum_{i \in I_j} x_{ij})$ is to be maximized subject to $\sum_{j \in J_i} x_{ij} = b_i$, $i = 1,\ldots,m$, where n is the total number of sectors, x_{ij} is the number of missiles allocated from area defense i to sector j, I_j is the set of indices of the area defenses that cover sector j, J_i is the set of indices of the sectors that lie within the region of area defense i, b_i is the size of the missile stockpile at area defense i, and m is the total number of area defenses. The sequential optimization procedure developed runs as follows. The missile stockpile of each area defense is first randomly allocated among the sectors which it covers. The expected total value saved as a result of this initial allocation x^0 is then $F(x^0) = \sum_{j=1}^{n} f_j(\sum_{i \in I_j} x^0_{ij})$. The allocations of all area defenses other than a particular area defense k is then held fixed, and the allocations x_{kj}, $j \in J_k$ of area defense k can be determined using the standard dynamic programming technique to maximize the payoff

$$F(x) = \sum_{j \in J_k} f_j(x_{kj} + \sum_{\substack{i \in I_j \\ i \neq k}} x^0_{ij}) + \sum_{j \notin J_k} f_j(\sum_{i \in I_j} x^0_{ij}), \text{ subject to } \sum_{j \in J_k} x_{kj} = b_k.$$

Starting with the matrix of missile allocations x^k resulting from optimizing the allocations for area defense k, the next area defense is optimized in the same way with the other area defense allocations held fixed. This sequential optimization procedure is repeated for all the area defenses cyclically until an entire cycle passes within which no sector payoff changes from that of the previous cycle, thus indicating that a local maximum solution to the problem has been found. A set of local maximum solutions can be generated by either varying the initial random allocation x^0, or by varying the order for optimizing the area defenses.

Furman and Greenberg [Ref. 7] also analyzed the attacker's problem of allocating a fixed stockpile of weapons of different types against targets of different values that are protected by a number of overlapping area defenses. It is assumed that only one weapon type can be allocated to a particular target or area defense, and that a target must first be rendered defenseless, i.e. the area defenses that are protecting the target must first be exhausted before a target can be attacked. The decisions that the offense must make that constitute his allocation strategy can be represented by the following decision variables: the exhaustion strategy E, where $k \in E$ means area defense k is to be exhausted; the binary variable b_{kj} which indicates which weapon type j is used to exhaust area defense k; the binary variable t_{ij} which indicates which weapon type j is allocated to target i; and a_{ij} giving the number of weapons of type j that are allocated to target i. The total payoff to the attacker can be defined as

$$f(\underline{a}) = \sum_{i=1}^{T} \sum_{j=1}^{N} D_{ij}(a_{ij})\, t_{ij} \quad ,$$

where T is the number of targets, N is the number of weapon types, and $D_{ij}(a_{ij})$ is the collection of damage functions representing the expected damage to target i when a weapons of type j are allocated to it. $D_{ij}(a_{ij})$ can, for example, be specifically a square root law damage function

$$D_{ij}(a_{ij}) = v_i\,(1-(1+c_{ij}\sqrt{a_{ij}})\exp(-c_{ij}\sqrt{a_{ij}}))$$

or a power law damage function

$$D_{ij}(a_{ij}) = v_i\,(1-(1-c_{ij})^{a_{ij}}) \quad ,$$

where v_i is the value of target i, and c_{ij} is the damage constant, a value between 0 and 1, depending on the warhead characteristics and certain measures of uncertainty.

The complete mathematical programming formulation for the offense weapon allocation problem is

$$\max f(\underline{a})$$

subject to $E \subset (1,2,\ldots,D)$, where D is the total number of area defenses;

$\sum_{i=1}^{T} a_{ij} \le w_j$, $j = 1,\ldots,N$, where W_j is the number of weapons of type j available;

$\sum_{j=1}^{N} t_{ij} \le 1$, $i = 1,\ldots,T$;

$\sum_{j=1}^{N} t_{kj} = 1$, $k \in E$;

$a_{ij} = 0$ if $j \notin J_i^E$, where J_i^E is the index set of weapon types to which the target i is exposed when using exhaustion strategy E, and $\sum_{i=1}^{T} t_{ij} a_{ij} + \sum_{k \in E} b_{kj} x_{kj} \le w_j$, $j = 1,\ldots,N$, where x_{kj} is the number of weapons of type j required to exhaust area defense k.

This problem can be partitioned and written as: $\max_E \{\max_{s.t.} f(\underline{x})\}$ subject to the above constraints, where E is chosen over all exhaustion strategies. The Lagrangian with respect to the last constraint about available weapon resources can then be formulated, and the generalised Lagrange Multiplier method used to solve the resulting problem:

$$\min_{\lambda} \max_{E} \{\max_{s.t.} f(\underline{a}) - \sum_{i,j} \lambda_j a_{ij} t_{ij} - \sum_{j,k \in E} \lambda_j b_{kj} x_{kj} + \underline{\lambda}\,\underline{w}\} ,$$

where the multiplier λ_j represents the price of a unit of weapon type j.

For given $\underline{\lambda}$ and E, the optimal values $\underline{b}^*$, $\underline{a}^*$ and $\underline{t}^*$ can be found by simple enumeration, and when the coverage of each area defense is the same for each weapon type, the optimal exhaustion strategy E^* can be found (for a given price vector $\underline{\lambda}$) by finding the minimum-cut of a capacitated network with vertices representing targets and area defenses, and arcs representing the area defense coverages. Details are given in the original paper of Furman and Greenberg.

In an earlier paper by Miercourt and Soland [Ref. 8], an offensive optimization model is analyzed given specific defense levels. In a later paper by Soland [Ref. 9], the optimization of the defensive allocations given an offense-last-move situation and optimal offense allocation is considered. The scenario calls for a mixture of overlapping area defenses as well as terminal defenses with perfectly reliable missiles, and an upper limit on the defensive stockpile sizes due to a budget constraint B. The offense is assumed to possess a stockpile of size A of a single type of weapon that exacts a level of damage on an undefended target (after its area and terminal defenses have been exhausted) according to the discrete concave and non-decreasing damage function $f_j(a_j)$, where a_j is the number of weapons directed against target j. The defense's allocation problem consists of finding the optimum number of missiles d_i^A to allocate to area defense region i, $i = 1,\dots,m$, and optimum number of point defense missiles d_j^P to assign to target j, $j = 1,\dots,T$ so as to minimize the maximum damage the offense can inflict. If the number of weapons required to exhaust the area defense region i and point defense of target j are given by e_i^A and e_j^P respectively, then a damage function g_j can be defined such that $g_j(a_j, d_j^P) = 0$ if $a_j < e_j^P$, and $g_j(a_j, d_j^P) = f_j(a_j - e_j^P)$ otherwise. The joint optimization problem can then be formulated as follows:

$$\min_{\underline{d}^A,\underline{d}^P} \left(\max_{a,\delta} \sum_{j=1}^{T} g_j(a_j, d_j^P)\right)$$

subject to
$$0 \le \underline{d}^A \le \underline{r}^A ,$$
$$0 \le \underline{d}^P \le \underline{r}^P ,$$
$$C(\underline{d}^A, \underline{d}^P) \le B ,$$
$$\sum_{j=1}^{T} a_j + \sum_{i=1}^{m} \delta_i e_i^A \le A ,$$
and
$$\sum_{j=1}^{T} d_{ij} a_j \le A\delta_i , \quad i = 1,\dots,m ,$$

where δ_i is an indicator variable such that $\delta_i = 1$ if defense of area i is to be exhausted and $\delta_i = 0$ otherwise. $\underline{D}^A$ and $\underline{D}^P$ are the upper bounds on the number of area and point defense missiles to be allocated, $C(\underline{d}^A, \underline{d}^P)$ is the total cost function associated with the defense allocation $\underline{d}^A$ and $\underline{d}^P$, and d_{ij} is another indicator variable that equals 1 if the defense of area i covers target j, and equals 0 otherwise. The last constraint ensures that no target is attacked unless all area defenses covering it is to be exhausted.

This problem can be reformulated into a simpler form by defining a function $\phi_A(\underline{d}^A, \underline{d}^P)$ such that

$$\phi_A = \max_{\underline{\delta}, \underline{a}} \sum_{j=1}^{T} g_j(a_j, d_j^P)$$

$$\text{subject to} \quad \sum_{j=1}^{T} a_j + \sum_{i=1}^{m} \delta_i e_i^A \le A$$

$$\text{and} \quad \sum_{j=1}^{T} d_{ij} \le A\delta_i, \; i = 1, \ldots, m .$$

ϕ_A can be calculated for given values of $\underline{d}^A$ and $\underline{d}^P$ by a branch-and-bound algorithm. The defender's problem can thus be formulated as

$$\min \; \phi_A(\underline{d}^A, \underline{d}^P)$$

$$\text{subject to} \quad C(\underline{d}^A, \underline{d}^P) \le B ,$$

$$0 \le \underline{d}^A \le \underline{D}^A , \text{ and}$$

$$0 \le \underline{d}^P \le \underline{D}^P .$$

As a final step in the simplification process, the upper bounds on the defense allocations are denoted by

$D_i^A = 2^{p_i} - 1$, and $D_j^P = 2^{q_j} - 1$,

where p_i and q_j are nonnegative integers. This involves no loss of generality because $C(\underline{d}^A, \underline{d}^P)$ for $d_i^A > D_i^A$ for example can be defined as being equal to infinity. New indicator 0-1 variables y_{ik}, $i = 1, \ldots, m$, $k = 1, \ldots, p_i$ and z_{jl}, $j = 1, \ldots, T$, $l = 1, \ldots, q_j$ are defined as follows:

$$d_i^A = (2^{p_i} - 1) - \sum_{k=1}^{p_i} 2^{p_i - k} y_{ik} , \; i = 1, \ldots, m ,$$

$$d_j^P = (2^{q_j} - 1) - \sum_{l=1}^{q_j} 2^{q_j - l} z_{jl} , \; j = 1, \ldots, T .$$

Letting $(\underline{y},\underline{z}) = (y_{11}, y_{12}, \ldots, y_{m\ell_m}, \; z_{11}, z_{12}, \ldots, z_{nq_n})$ to be the new decision vector with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{I} q_j$ binary components, the defense allocation problem can be formulated as

$$\min \; \phi_M(\underline{y},\underline{z})$$

subject to $C(\underline{y},\underline{z}) \le B$.

Since it is assumed that $\phi_M(\underline{y},\underline{z})$ is non-increasing and $C(\underline{y},\underline{z})$ is non-decreasing, this problem can be solved by the Lawler-Bell enumeration algorithm [Ref. 10].

2. <u>Percentage and Numerically Vulnerable Defenses</u>

Shere and Cohen [Ref. 11] analyzed the problem of offense and defense resource allocations involving weapon system development costs from a game theoretic viewpoint. Two classes of defense systems are considered in the model:

- percentage vulnerable (PV) systems, e.g. Polaris submarines, a fixed percentage of which comes under attack for a fixed search effort by the attacker. Using random search theory, the fraction of weapons surviving in the ith PV system can be given by $\exp\{-a_i(y_i - r_i)\}$ and its value after an attack is

  $$f_i(x_i, y_i) = v_i(x_i - q_i)\exp\{-a_i(y_i - r_i)\} ,$$

  where v_i represents the value of the system (in terms of destructive capability), x_i and y_i are the total amount of funds allocated by the defense to the setting up, and by the offense to the destruction of the ith PV system, q_i and r_i are their required development costs associated with the aforementioned purposes, and a_i represents the vulnerability of the ith system.
- numerically vulnerable (NV) systems, comprising of essentially static weapon systems such as the Minuteman ICBM system. The attacker's effort is distributed among all the weapons of the system. In this case, the residual value of the jth NV system is

$$f_j'(x_j',y_j') = v_j'(x_j'-q_j')\exp\{-a_j'(y_j'-r_j')/(x_j'-q_j')\} .$$

The model assumes an offense-last-move situation with counterforce targeting only. The objective of the offense is to minimize the retaliatory capability of the defense. Consequently, the defense allocates his financial resources in a manner which maximizes this minimum. The problem can thus be formulated as:

$$\max_x \min_Y \{\sum_i f_i(x_i,y_i) + \sum_j f_j'(x_j',y_j')\}$$

subject to $\sum_i x_i + \sum_j x_j' = X$ (defense's total resources),

$\sum_i y_i + \sum_j y_j' = Y$ (offense's total resources).

The authors developed an iterative algorithm to solve the allocation problem for a mix of PV systems only by extending the max-min theory, and hypothesizing that if the offense considers attacking the ith PV system, it will allocate resources y_i in excess of q_i, its 'cost of admission' for this system; the defense, if it decides to set up the ith PV system, will similarly allocate funds x_i in excess of the system's development cost r_i, so that it can procure at least one weapon. If the choice of A is unique for some optimal allocation $x = x^*$, then the optimal allocation x^* and y^* is also a solution to the game

$$\max_\beta \min_\alpha\{\sum_A v_i(x_i-q_i)\exp(a_i(y_i-r_i)) + \sum_{B-A} v_i(x_i-q_i)\}$$

subject to $\sum_B x_i = X$, $\sum_A y_i = Y$,

$x_i \geq q_i$, $i \in B$, $y_i \geq r_i$, $i \in A$,

where $A = (i: y_i^* > r_i)$, $B = (i: x_i > q_i)$,

$\beta = (x: x_i = 0 \text{ for } i \notin B)$, $\alpha = (y: y_i = 0 \text{ for } i \notin A)$,

and y_i^* and x_i^* are the optimal offense and defense allocations respectively.

It can be proven that A = B if A is assumed unique. Hence the defense should not invest in a new PV system unless it is of sufficient value for the attacker to pay the penalty for at least a limited counter to this new system.

A solution method for the allocation problem in the case of a general mix of PV and NV defense systems was also developed. It was shown that at most one NV system should be developed, and thus the problem reduces to the previous problem concerning a mix of only PV systems with the amount of investment in at most one NV system a parameter used to determine the remaining amount of resources available to allocate among the PV systems.

3. Layered Defense

Nunn, et. al. [Ref. 12] analyzed the missile allocation problem in the situation where the defense is layered, and the attackers try to penetrate the several layers of defense systems. An example of such a scenario may be an ICBM defense system or a high-rate-of-fire air defense system which adopts a shoot-look-shoot strategy against attacking aircraft. The objective of the defense is to minimize the expected number of penetrators.

The analysis uses a Markov chain formulation. No explicit representation of defense force levels is given. Instead, it is assumed that the numbers of attackers penetrating (i.e. surviving) the ith layer is binomially distributed with parameters n_l, q_l, where n_l is the number of attackers approaching the lth layer, and q_l is the probability that an attacker survives the lth layer defense. The passage through the lth layer is viewed as a transition in a Markov chain, with the associated transition matrix A whose elements a are given as:

$$a_{ij} = \binom{i}{j} q_l^j (1-q_l)^{i-j} .$$

A_l is diagonalizable with AS = SD where S is a lower triangular matrix whose non-zero elements are those of Pascal's triangle, and D is a vector of the form $diag(1, q, q^2, \ldots, q^n)$. It is shown as a consequence that if the distribution of the initial number of attackers is T (a row vector whose elements make up the discrete mass function of the initial number of attackers), then the distribution of survivors after penetrating through L layers of defense is given by $T \prod_{i=1}^{L} A_i$. The product $\prod_{i=1}^{L} A_i$ is just another similiar matrix with parameter $\prod_{i=1}^{L} q_i$. In the case where the initial distribution T is binomial, that distribution is maintained throughout the layers of defense. Moreover, the final distribution of attackers is independent of the ordering of the defense layers since the transition matrices commute.

E. TARGETS OF OPPORTUNITY

A unique variation of the missile allocation problem concerns so-called 'targets of opportunity', which may be value targets or incoming weapons. These targets of opportunity arrive sequentially within a given time period, each having a random value. In the case of value targets, the problem concerns the allocation of defensive missiles to protect these targets and weapons to destroy these targets. In the case where the targets of opportunity are incoming weapons, the problem consists of allocating defensive missiles to intercept them. This class of problems can be solved by dynamic programming.

1. Sequentially Arriving Targets

Sakaguchi [Ref. 13] formulated a generalised two-person-zero-sum game under the following assumptions: the attacker has A weapons and the defender has D missiles. A total of T targets arrive sequentially, each having a value v_j , $j = 1, \ldots, T$, from a probability distribution $F(v)$. The allocation policy consists of a decision on whether to attack (for the offense) or defend (for the defense) each target as it arrives with a single weapon or missile, and is based on the value of the arriving target, the number of weapons (or missiles for the defender) remaining in the stockpile, and the mission time remaining. The payoff for a target of value v can be given by $p(1-\rho)v$ if the defender decides to defend the target, or pv if the defender decides not to defend this target. The optimal strategies can be characterised by a system of recursive difference equations using a dynamic programming formulation.

If the defense and offense have d missiles and a weapons respectively left in their stockpiles, and there are t targets yet to arrive, the value of the game

$$V_t(a,d) = \int_0^\infty \text{Value}\begin{Bmatrix} p(1-\rho)v+V_{t-1}(a-1,d-1) & pv+V_{t-1}(a-1,d) \\ V_{t-1}(a,d-1) & V_{t-1}(a,d) \end{Bmatrix} dF(v) ,$$

with initial condition $V_0(0,0) = 0$, and boundary conditions

$$V_t(0,d) = 0, \quad V_t(a,0) = p\sum_{i=1}^{k} g_{t,i}, \; 0 \le k \le t ,$$

$$V_t(t,d) = tp\mu - p\rho\sum_{i=1}^{l} g_{t,i}, \; 0 \le l \le t , \text{ and}$$

$$V_t(a,t) = p(1-\rho)\sum_{i=1}^{k} g_{t,i} , \quad 0 \le k \le t ,$$

where $g_{t,i}$, $i = 1,\dots,t$ is a triangular array of positive numbers defined by the recurrence relations

$$g_{t,1} = S_F(g_{t-1,1}) \text{ for } t \ge 2, \; g_{1,1} = \mu , \text{ and}$$

$$g_{t,i} = S_F(g_{t-1,i}) - \sum_{j=1}^{i-1}(g_{t,j} - g_{t-1,j}) \quad \text{for} \quad 2 \le i \le t-1,$$

$$g_{t,t} = t\mu - \sum_{j=1}^{t-1} g_{t,j} .$$

The function $S_F(z)$ is given by $z+T_F(z)$, where $T_F(z)$ is the mean shortage function defined by

$$T_F(z) = \int_z^\infty \{1-F(x)\}dx ,$$

and μ in the above equation is the expected target value, given as

$$\mu = T_F(0) = \int_0^\infty x\,dF(x) .$$

The optimal strategy for the defense and offense is that of the matrix game in the right hand side of the equations for $V_t(a,d)$, if a target of value v arrives in state (t,a,d). The explicit solution of the game is not easily solvable even for the simpliest kind of target value distributions. However, if the simplifying assumption is made that target value is deterministic having a value of 1, the value of the game $V_t(a,d) = pa(1 - \rho d/t)$. The optimal defense strategy is to defend the target with with probability d/t, and siriliarly, the optimal offense strategy is to attack the target with probability a/t. A similiar continuous time solution can be derived if the targets are assumed to arrive according to a Poisson process with rate λ, i.e. the number of targets and their arrival times are assumed to be random. In this case, the value of the game is given by a system of recursive differential equations which characterizes the optimal strategies of the offense and defense.

2. Randomly Arriving Weapons

Kisi [Ref. 14] considered the problem of allocating missiles against attacking weapons (attacker-oriented defense strategy) which arrive randomly according to a Poisson process with rate λ. It is assumed that the defense has a fixed stockpile d of missiles with reliability $\rho < 1$, and adopts a shoot-look-shoot strategy for each incoming weapon. The defense allocation strategy consists of deciding whether or not to engage an incoming weapon, and how many missiles to fire given a limited number of missiles and mission time remaining. It is assumed that the shoot-look-shoot strategy is instantaneous, i.e. no time is wasted between firings within a salvo. Each of the incoming weapons have a random value which is distributed according to a Uniform (0,1) distribution. The objective of the defense is to maximize the expected total value destroyed during a given total mission duration.

The number of weapons that are expected to arrive during a mission time t is λt, and the expected number of weapons destroyed is ρd. Hence, only a fraction $\rho d/(\lambda t)$ of weapons can be destroyed , and the defense should only select targets with high values greater than or equal to a critical threshold value c. The optimal threshold c depends on both the time remaining t and the number of missiles remaining d, and intuitively should increase as t increases, and decrease as d increases. An optimal value function f(t,d) is defined as the expected value destroyed when time t and d missiles are remaining, and the optimal allocation policy is employed by the defense throughout time t. Then the optimal value of c is given by

$$c^*(t,d) = \frac{1}{\rho}\{f(t,d) - f(t,d-1)\}$$

and an incoming weapon is allocated a missile so long as its value $v \geq c^*$. The optimal value function can be derived exactly, and is given by the following recursive relation:

$$\frac{1}{\lambda}\frac{d}{dt}\{f(t,d) - (1-\rho)f(t,d-1)\} = \frac{1}{2\rho}\{f(t,d) - f(t,d-1) - \rho\}^2$$ for $d = 1,2,\ldots$ with initial conditions $f(t,0) = 0$ and $f(0,d) = 0$.

An approximate solution can be given in the form:

$$f(t,d) = \rho\{d - f_d/(\lambda(t-t_0))\} ,$$

where $f_d = f_{d-1} + 1 + \sqrt{2\rho f_{d-1} + 1}$, and $f_0 = 0$, $t_0 = 2/\lambda$.

The difference between the exact optimal and approximate solutions $c^*(t,d)$ is negligible for large t, but increase as t becomes small. However the difference between the values of f(t,d) in the two cases is negligible even for small values of t.

Mastran and Thomas [Ref. 15] analyzed the same problem of attacking targets of opportunity , however under a different set of assumptions. Specifically, it is assumed that the defender can only attack one incoming weapon throughout the mission time available, and that all missiles will be expended in the intercept. A general probability distribution of weapon interarrival times is assumed instead of the exponential interarrival times assumed earlier. The conditional probability D_i that there is an incoming weapon in the next time interval given that the last arrival occured i-1 time intervals ago is given by

$$D_i = T_i/(1-\sum_{j=1}^{i-1} T_j) \quad \text{for } i \geq 2 \text{ and } D_1 = T_1 ,$$

where T_i is the probability that i time intervals separate successive arrivals. The value of the incoming weapon v comes from a general probability density function g(v), instead of a Uniform (0,1) distribution. An optimal value function $f_n(i)$ is defined to be the expected value destroyed when n time intervals remain, and i time intervals have elapsed since the last weapon arrival, and the optimal policy is used. A threshold weapon value K_n that is varying over time can be similiarly defined, such that the defense will attack the incoming weapon when n periods remain, if and only if its value v is greater than K_n. Given that there is an incoming weapon, the expected value destroyed for the case when g(v) is continuous is

$$f_{n-1}(1)\int_{v \leq K_n} g(v)\,dv + \int_{v \geq K_n} vg(v)\,dv .$$

The function $f_n(i)$ is maximized only when a weapon is attacked that has a higher value than would be obtained by waiting another time interval and obtaining $f_{n-1}(1)$; hence $K_n = f_{n-1}(1)$. Thus, the recursive relationship can be written as

$$f_n(i) = D_i\{\int_{v \leq f_{n-1}(1)} f_{n-1}(1)g(v)\,dv + \int_{v \geq f_{n-1}(1)} vg(v)\,dv\} + (1-D_i)\{f_{n-1}(i+1)\},$$

with $f_1(i) = D_i\int_{v \geq 0} vg(v)\,dv$, and $f_0(i) = 0$.

Using this dynamic programming formulation, the value of $f_n(i)$ can be obtained for any n and i.

F. STRATEGIES WITH SPECIFIC TARGET ASSUMPTIONS

Thus far, the values of targets are either assumed to be identical or non-identical from some probability distribution. In this section, the missile allocation problem is analyzed under some special assumptions on the target, viz.,

- deterioration of target value over time,
- complementary targets, and
- collateral damage between strategic and nonstrategic targets.

Each of these situations is discussed in the following three subsections.

1. Deterioration of Target Value over Time

Bracken and McGill [Ref. 16] treats the problem of target value deterioration over time, and seeks an optimal sequential attack strategy to maximize the expected target value destroyed. The model assumes a set of weapon launch centers with different capabilities in terms of the maximum number of weapons that can be launched at time t, where t is discretized into increments of equal length equal to the time between successive weapon launches. The target set consists of a number of point targets with different values which decrease monotonically over time. It is also assumed that the flight times of the weapons from a weapon center to a target is finite, and are different for each weapon center-target pair. No explicit representation of defenses is included; it is only implicitly represented by p_{ij}^t, the probability that a weapon from launch center i hits target j at time t. The conditional probability of destroying target j given that the target has survived until time t is then given by

$$p_j^t = 1 - \prod_{i=1}^{p}(1-p_{ij}^t)^{a_{ij}^t} \quad , \quad j = 1,\ldots T, \; t = 1,\ldots,r \; ,$$

where a_{ij}^t is the number of weapons launched from weapon center i to arrive at target j at time t, p is the total number of weapon centers, T is the total number of targets, and r is a value large enough to allow all weapons to reach their targets by time r.

It is assumed that the offense has a fixed total attack capability in terms of the number of weapons A that can be allocated to the launch centers at each time increment t. Hence, if A_i denotes the number of weapons available at launch center i at each time increment, the constraints $\sum_{i=1}^{p} A_i \le A$ and $\sum_{j=1}^{T} a_{ij}^{t+t_{ij}} \le M$ hold, t_{ij} being the weapon from launch center i to target j. The objective of the offense is then to find optimum values of a_{ij}^t, i = 1,...,p, j = 1,...T, t = 0,1,...,r and M_i, i = 1,...,p to maximize the expected total value destroyed, i.e.

$$\max \sum_{j=1}^{T} \{ \sum_{t=0}^{r} v_j^t p_j^t \prod_{s=0}^{t-1} (1-p_j^s) \}$$

subject to the constraints $\sum_{i=1}^{p} A_i \le A$ and $\sum_{j=1}^{T} a_{ij}^{t+t_{ij}} \le M_i$,
where v_j^t is the value of target j at time t.

If the target values are such that $v_j^t \ge v_j^{t+1}$ for t = 0,1,...,r-1 (value nonincreasing over time), the objective function is concave in the variables x_j^t , and since the constraints are linear, the problem becomes a convex nonlinear program which can be solved to yield a global solution.

2. Complementary Targets

Shubik and Weber [Ref. 17] considered a generalization of the classical Blotto game for allocating forces to independent targets in the case of 'complementary' targets or networks, where the value of a subset of targets v(s) is not equal to the total individual target values, but depends on the target configuration. In this case, the defender's expected payoff can be given by

$$\sum_{s \in N} \{ \prod_{i \in S} f_i(d_i, a_i) \prod_{i \notin S} (1 - f_i(d_i, a_i)) \} v(S) ,$$

where N is the set of all the targets, $f_i(d_i, a_i)$ is the probability that the targets in subset S all survive, given that the defense and offense allocations at target i are d_i and a_i respectively. In the case where homogeneous offense and defense resources are assumed, and the outcome function $f_k(d_k, a_k)$ at the kth target is of the form

$$f_k(d_k, a_k) = 1 - \gamma_k / \{ \gamma_k + (1 - \gamma_k)/k^m \} ,$$

where $k = a_k / d_k$ (attacker to defender force ratio), and γ_k is a target parameter that represents its natural defensibility, and m is a parameter that reflects the importance of the relative difference in size between the attacking and defending forces which have total resources A and D, the force allocations are proportional to the $(f_1, f_2, \ldots f_T)(D, A)$-value of the underlying game if both sides have optimal pure strategies, where T is the total number of targets. Furthermore, for all sufficiently small values of m, these allocations are optimal.

3. Strategic and Nonstrategic Targets

Grotte [Ref. 18] considered a plausible situation where strategic (military) and nonstrategic (nonmilitary) targets are colocated, and the objective is to employ counterforce targeting of weapons such that sufficient damage to strategic targets can be achieved without causing appreciable damage to the surrounding nonstrategic facilities. The problem therefore consists of finding an optimal allocation of weapons to a set of aimpoints such that minimum levels of damage to a set of military targets are achieved while permissable levels of damage to a set of neighbouring nonmilitary targets are not surpassed.

This problem can be formulated as a discrete nonlinear program:

$$\min \sum_{n=1}^{N} g_n(z)\lambda_n$$

$$\text{subject to} \quad f_m(z) \ge c_m, \; m = 1,\ldots,M ,$$

$$g_n(z) \le d_n, \; n = 1,\ldots,N ,$$

$$\sum_{j=1}^{J} z_{ij} \le w_i, \; i = 1,\ldots,I ,$$

$$z_{ij} \in I^+, \; i = 1,\ldots,I, \; j = 1,\ldots,J ,$$

where M and N are the numbers of military and nonmilitary targets, I is the number of weapon types, and J is the number of possible weapon aimpoints, w_i is the number of weapons of type i available, and z_{ij} is the number allocated to aimpoint j, $f_m(z)$ and $g_n(z)$ represent damage functions for the military and nonmilitary targets, c_m and d_n are the minimum acceptable and maximum permissable damage to military target m and nonmilitary target n, and λ_n is a nonnegative weight for nonmilitary target n.

The solution to the problem is by implicit enumeration based on the lexicographic technique of Lawler and Bell.

G. STRATEGIC EXCHANGE MODELS

It was mentioned in Chapter 1 that a number of studies on the missile allocation problem is done in the context of a strategic nuclear exchange between two superpowers. In this section, three such papers are presented which are representative of the studies done in this field. The first paper formulates a general two-strike nuclear exchange as a max-min problem, while the second proposed a model to optimize defense allocations in order to ensure a minimum level of post-attack economic capacity. The last study optimizes the allocation of resources for population defense.

1. General Two-Strike Nuclear Exchange

Bracken, Falk & Miercourt [Ref. 19] present a general formulation of the two-strike strategic nuclear exchange, in which both sides possess multiple weapon systems and value targets. It is assumed that the first striker allocates all his weapons against his opponents value targets and possibly against his strategic weapons in an optimal countervalue-counterforce targeting mix. The second striker then retaliates with all his surviving weapons against the first striker's value targets. This two-strike problem can be formulated in general as:

$$\max_{\underline{x} \in X} \min_{\underline{y} \in Y(\underline{x})} \{D_Y(\underline{x}) - D_X(\underline{y})\} ,$$

where X is the set of allocations x_{ij}, denoting the number of the first striker's type i warheads allocated the second striker's type j resources, Y is the set of allocations y_j denoting the number of the second striker's surviving type j warheads allocated against the opponent's value targets. D_X and D_Y represent maximum value damage to the first and second striker respectively from the opponent's weapon allocation against his resources.

An appropriate function that is convex representing the expected number of surviving second striker's warheads is

$$n_j w_j \prod_{i=1}^{I} q_{ij}^{\frac{x_{ij}}{n_j}} ,$$

where n_j is the total number of the second striker's type j weapons, w_j is the number of warheads per type j weapon of the second striker, and q_{ij} is the single-shot survival probability for the second striker's type j weapon when attacked by a single type i warhead of the first striker.

On the assumption that D_I is a nondecreasing function, the solution to the second striker's allocation problem becomes simply

$$\min_{y \in Y(x)} \{-D_I(y)\} = \max_{y \in Y(x)} D_I(y) = -D_I(z) ,$$

where $z_j = B_j(x)$, i.e. the second striker allocates all his surviving weapons against the first striker's value targets. The two-strike problem can thus be reformulated as:

$$\max_{x \in X} \{D_{\mathbb{I}}(x) - D_I(z)\} \text{ subject to } z \geq B(x) .$$

If the maximum value damage functions are assumed to have the following specific forms

$$D_{\mathbb{I}}(x) = V_{\mathbb{I}}\{1-\exp(-\sum_{i=1}^{I} f_i(x_{i0}))\},\ D_I(y) = V_I\{1-\exp(-\sum_{j=1}^{J} g_j(z_j))\},$$

where the functions f_i and g_j are continuous and assumed to be linear and x_{i0} denotes allocations to value targets, the two-strike exchange problem can be expressed as

$$\max\ (V_{\mathbb{I}} - V_I) - V_{\mathbb{I}}\exp(-t_1) + V_I\exp(-t_2)$$

$$\text{subject to: } \sum_{j=0}^{J} x_{ij} \leq m_i ,\ i = 1,\ldots,I ,$$

$$t_1 \leq \sum_{i=1}^{I} f_i(x_{i0}) ,$$

$$t_2 \geq \sum_{j=1}^{J} g_j(z_j) ,$$

$$\ln z_j \geq \ln n_j w_j + \sum_{i=1}^{I} x_{ij} \ln y_{ij}/n_j ,\ j = 1,\ldots,J ,$$

where m_i is the number of the first striker's type i warheads. This is equivalent to a separable nonconvex program, and an approximate global solution can be found by applying a branch-and-bound algorithm after replacing each nonlinear function by a piecewise linear approximating function.

A later paper by Grotte [Ref. 20] expanded on this model by considering four specific weapon types on each side, namely: ICBM's, submarine launched ballistic missiles

SLBM-at-sea, bombers, and SLBM-in-port, and deriving separate equations for the second striker's surviving force of these four weapon types after a first strike, which have as parameters original force levels, reliabilities of the attacking weapons, penetration and kill probabilities, etc. The maximum value damage functions were specified as

$$D_I(\underline{x}) = V_I \{1-\exp(-\sum_{i=1}^{4} a_i x_{i0}^{u_i})\} \text{ and } D_{II}(\underline{y}) = V_{II} \{1-\exp(-\sum_{j=1}^{4} b_j z_j^{v_j})\},$$

where the parameters a_i, u_i, b_j, v_j are selected to represent the first and second striker's response to allocations $\underline{x}$ and $\underline{y}$. This more detailed problem was solved using the same branch-and-bound algorithm after forming piecewise linear approximations for each function in the separated problem.

2. <u>Ensuring Post-Attack Production Capacities</u>

Bracken & McGill [Ref. 21] propose an economic model of strategic defenses, and formulate a mathematical program for allocating a minimum cost mix of defense resources to geographical regions such that a specified minimum level of economic production capacity will survive after an optimized attack by the offense. It is assumed in the model that the country is divided into geographical regions (defense regions) with different economic sectors, each being characterized by a Cobb-Douglas production function of the form $H_{ij}(K_{ij})^{\alpha_{ij}}(L_{ij})^{\beta_{ij}}$ where H_{ij} represents the technological efficiency of economic sector i in geographical region j, K_{ij} is the corresponding capital base, L_{ij} the labor base, α_{ij} and β_{ij} denote the elasticities of value added with respect to capital and labor respectively.

The post-attack production function (in terms of value added) in sector i of region j can be given by:

$$P = HKL$$

where $H = H_{ij}(1 - (\exp(-\sum_{k=1}^{p} x_{ijk}^{M} d_{jk}))(1 - \exp(-\sum_{l=1}^{q} y_{ijl}^{M} a_{jl})))$,

$$K = K_j^{\alpha_{ij}}\{1 - (\exp(-\sum_{k=1}^{p} x_{ijk}^{K} d_{jk}))(1 - \exp(-\sum_{l=1}^{q} y_{ijl}^{K} a_{jl}^{i}))\}^{\alpha_{ij}} ,$$

$$L = L_j^{\beta_{ij}}\{1 - (\exp(-\sum_{k=1}^{p} x_{ijk}^{L} d_{jk}))(1 - \exp(-\sum_{l=1}^{q} y_{ijl}^{L} a_{jl}^{i}))\}^{\beta_{ij}} ,$$

where the standard Cobb-Douglas form has been modified to make the expression a function of the offense and defense allocations, d_{jk} being the number of defense resources of type k allocated to region j, $k = 1,\ldots,p$, $j = 1,\ldots,n$, and a_{jl}^{i} being the number of weapons of type 1 targeted on region j in an attack on economic sector i, $i = 1,\ldots,m$, $l = 1,\ldots,q$.

The parameters x_{ijk}^{M}, x_{ijk}^{K}, and x_{ijk}^{L}, y_{ijl}^{M}, y_{ijl}^{K}, and y_{ijl}^{L} might be estimated from detailed analyses. Assuming that the unit cost of defense resource k in region j is c_{jk}, and the required minimum level of post-attack production capacity for economic sector i is r_i, the obective of the defense to find an optimal (minimum cost) allocation of defense resources to geographical regions to ensure the survivability of a minimum level of production capability can be given by the mathematical program:

$$\min \sum_{k=1}^{p} \sum_{j=1}^{n} c_{jk} d_{jk}$$

subject to: $\min \sum_{j=1}^{n} p_{ij} \geq r_i$, $i = 1,\ldots,m$,

and $\sum_{j=1}^{n} a_{jl}^{i} \leq A_l$ (the number of offensive weapons of type 1) .

This is a convex mathematical program with nonlinear programs in the constraints, and can be solved by a SUMT computer program.

3. Population Defense in a Nuclear Attack

Kupperman & Smith [Ref. 22] approached the problem of optimal offense and defense strategies in a unique way in their study of the role of population defense in mutual deterence. Their model assumed that 'centres of destruction' are placed at random in a plane, forming a Poisson process of density m points per unit area, with a circle of area A (area of destruction) centered at each of these points. The probability that an arbitrary point in the plane will not be covered by any of these circles would be the probability that none of the points of the Poisson process lies within a circle of area A centered at that point, which is $\exp(-mA)$. If a value density $v(x)$ is associated with points x in the plane, and value is considered to be destroyed in regions covered by the circles centered at the Poisson distributed points, the expected remaining value density is $v\{\exp(-mA)\}$, and conversely, the density of destruction is $v\{1-\exp(-mA)\}$. This formula yields a good approximation in the case where weapons are delivered with random errors which are a substantial fraction of their lethal radius. If this Poisson type model is applied to compute the maximum destruction inflicted on a circularly symmetric Normal value distribution, the maximum damage function can be given in the form

$$f(n) = 1 - (1+B\sqrt{n})\exp(-B\sqrt{n}) ,$$

where n is the number of weapons, and B is a parameter constant. This function gives a reasonably accurate representation of maximum net destruction for urban areas in the US based on census data and weapons of less than one megaton yield.

To compute optimal offense and defense strategies, it is observed that the effect of an antimissile defense would be to reduce the value of m, so that a general destruction density of the form $p = v\{1-\exp(-w.m.\phi)\}$ is obtained, with ϕ being a parameter between zero and one reflecting both the deployment of the defense and its technical characteristics, and w is a parameter. Using a generalization of Gibb's Lemma and the concept of decreasing marginal utility, an optimization of the defense to minimize at fixed total cost the maximum destruction of value caused by an attack of fixed size is found. In essence, each defense force is characterized by a value λ such that the total defense stockpile D is given by

$$D = (1/w) \int \{wv(x)/\lambda - 1 - \log\{wv(x)/\lambda\}\} dA(x) ,$$

and the defense allocation d(x) is given by

$$d(x) = (1/w)\{wv(x)/\lambda - 1 - \log(wv(x)/\lambda)\} , \quad wv(x) > \lambda ,$$
$$= 0 \text{ otherwise} ,$$

where the integral is taken over all points x such that $wv(x) \geq \lambda$. This strategy is optimal whatever the size of the offense stockpile.

Every level of offense marginal utility $\mu < \lambda$ has a unique force level. The total attack size A is given by

$$A = \frac{1}{w} \int \left(\frac{wv(x)}{\lambda} - 1 + \log \frac{\lambda}{\mu} \right) dA(x) + \frac{1}{w} \int \log \left(\frac{w - v(x)}{\mu} \right) dA(x) ,$$

where the first integral is taken over all x such that $wv(x) \geq \lambda$, and the second integral is taken over all x such that $\lambda > wv(x) \geq \mu$. The payoff to the offense is

$$P = \int (v(x) - \mu/w) dA(x) ,$$

with the integral taken over all x such that $v(x) \geq \mu$.

H. PROPORTIONAL DEFENSE STRATEGIES

In the case where the defense is at a disadvantage, e.g. when the offense has prior knowledge of the defense allocation before making his own allocation of weapons to targets (offense-last-move situation), the defense can 'insure' against excessive losses by making the defense proportional, in the sense that the attacker must pay a 'price' that is proportional to the target value extracted.

A class of proportional defense models comprises the so-called 'Prim-Read' missile deployments (named after their developers R.C. Prim and W.T. Read), for defending against an attack by an unknown number of independent and sequentially arriving weapons, with the objective of minimizing the total expected number of defenders subject to an upper bound on the maximum expected target value damaged per attacker, i.e. the maximum possible damage under any attack is bounded by a linear function of the attack size.

Burr, Falk and Karr [Ref. 23] developed a method to produce globally optimum solutions of integer versions of a class of problems whose continuous solutions are of the Prim-Read variety. It is assumed that the offense has the last move, and the target set consists of T point targets with values v_i, i = 1,...,T, each protected by its own independent terminal defense. The defensive missile has a reliability $\rho < 1$, and the attacking weapon kill probability p = 1.

The expected target value destroyed can then be given by

$$V_{da} = \sum_{i=1}^{T} v_i \{1 - \prod_{j=1}^{a_i} (1-(1-\rho)^{d_{ij}})\} \ .$$

where a_i is the number of weapons allocated to target i, and d_{ij} is the number of missiles assigned at target i to be directed at the jth incoming weapon, both numbers assumed to be nonnegative integers. Letting s denote the upper bound on the maximum expected target damage per attacker, this defense problem can thus be formulated as:

$$\min_{d_{ij}} \sum_i \sum_j d_{ij}$$

subject to: $V_{da} \leq s \sum_i a_i$,

$d_{ij} \geq 0$, $a_i \geq 0$, d_{ij}, a_i integers .

For each value of s, the problem has a solution which can be found by solving a collection of single-target problems, one for each target in the target set. The single target problem can be formulated as:

$$\min \sum_j d_j$$

subject to: $v\{1 - \prod_{j=1}^{k}(1-(1-\rho)^{d_j})\} < sk, \quad k = 1,2,\ldots,$

where d_j is the number of missiles assigned to attacking weapon j, and v is the value of the single target. It is shown that for every r = s/v, there exists an optimal solution $\underline{d}^*$ such that $d_1^* \geq d_2^* \geq d_3^* \geq \ldots$.

The solution to this single-target problem can be given in the form of a recursive relation:

$$2 \leq k < 1/r, \quad d_k^* = \left\lceil \frac{\ln\left\{1 - \frac{1-rk}{\prod_{j=1}^{k-1}(1-(1-\rho)^{d_j^*})}\right\}}{\ln(1-\rho)} \right\rceil$$

with initial condition $d_1^* = \lceil\{\ln r\}/\{\ln(1-\rho)\}\rceil$;

$k > 1/r, \quad d_k^* = 0.$

($\lceil x \rceil$ denotes the smallest integer $\geq x$.) The individual optimal solutions to the single-target problems form the set of optimal solutions to the original multi-target problem.

A different algorithm for the all-integer version of the Prim-Read model was derived by Burr, which is similar in nature to the above algorithm, but unlike this method, always produces monotone deployments.

In the case where both defensive missiles and offensive weapons are perfectly reliable, i.e. $p = \rho = 1$, the defender can ensure destruction of the attacker by allocating a single missile to it, and a target will be destroyed the first time it is left undefended; hence the value of d can be expressed as

$$d_{ij} = 1, \; j = 1,\ldots,\delta_i$$
$$= 0, \; j = \delta_i+1, \; \delta_i+2,\ldots,$$

δ_i being a nonnegative integer representing the number of weapons against which target i will be defended and to which it is therefore invulnerable. This simplified problem can thus be reformulated as:

$$\min_{\delta_i} \sum_i d_i$$

subject to: $v_{i} \leq s \sum_i a_i$.

The unique solution to this problem is given by:

$$\delta_i^* = \lceil (v_i/s) - 1 \rceil \ , \ i = 1,2,\ldots,T.$$

Haaland & Wigner [Ref. 24] derived optimal defense and offense strategies using elementary mathematical techniques. The assumptions which they made in their model are that the weapons and missiles are perfectly reliable, known to the offense which, as before, has the last move. The damage function at target i is denoted by $f_i(a_i)$, and represents the maximum damage inflicted on undefended target i (i.e. its missile defenses having been exhausted) by a weapons. This function is assumed to be monotone increasing with decreasing slope.

An example of such a function is the square root law damage function.

The optimal attack strategy is shown to allocate a number of weapons a to each target such that the marginal increase in damage by the last weapon is equal for all targets, i.e.

$$f_i(a_i+1) - f_i(a_i) < c < f_i(a_i) - f_i(a_i-1) \text{ for all } i \ ,$$

or ignoring integer value considerations, $\partial f_i(a_i)/\partial a_i = c$, where c is a constant denoting the marginal increase.

The criterion for not attacking a particular target j is given by the inequality $f_j(a_j)/(d_j+a_j) < c$, where d_j is the number of missiles defending target j. Hence, the optimal attack strategy is obtained as follows: an arbitrary value of c is chosen, and all values of a_i are calculated using the equation $\partial f_i(a_i)/\partial a_i = c$. Those targets for which there is no solution to that equation are disregarded. The criterion for not attacking a target is then applied to the remaining targets, and the sum $\Sigma(a_i + d_i)$ over all those targets to be attacked is compared with the total offense stockpile A. If $\Sigma(a_i + d_i) > A$, the procedure is repeated with a larger value of c, and if $\Sigma(a_i + d_i) < A$, it is repeated with a smaller value of c. The optimal strategy is found when $\Sigma(a_i + d_i) = c$. This tactic has the property that a larger attack size A does not decrease the number of weapons aimed at a particular target, and would not cause a target that is attacked at a smaller attack force level to be bypassed.

The optimal defense strategy is analogous to the offense strategy in that missiles are allocated to each target such that the marginal increase in damage by the last weapon is equal for all targets. This is determined principally by their ability to decrease the effect of an attack in which not all defended targets are attacked, since if A is much greater than the defense stockpile D, the offense would simply send in weapons to exhaust the defense stockpile D, and then would allocate the rest of his weapons over any targets he wishes, resulting in damages independent of the defense allocation. The defense strategy is specified as follows: an arbitrary B is chosen, and the number of weapons allocated to undefended target i, a_i' is determined using the equation

$$f_i(a_i' + 1) - f_i(a_i') < B < f_i(a_i') - f_i(a_i' - 1) .$$

Then the defense allocation d_i for each target is determined from the equation

$$f_i(d_i')/(d_i + a_i') = B.$$

If $f_j(a_j')/a_j' \leq B$, d_j is set to 0, i.e. the target j is left undefended. Then the sum of the defense allocations are compared with the total defense stockpile D. If $\sum d_i < D$, the process is repeated with a smaller B, and if $\sum d_i > D$, it is repeated with a larger B. The optimal defense strategy is found when $\sum d_i = D$ for a certain value of B. This defense strategy is optimal when all targets are attacked. A 'tuned attack' is said to occur if the number of attacking weapons is just equal to $\sum (d_i + a_i^*)$, and in this case, all weapons will be aimed at defended targets, and the total damage will be $B \sum (d_i + a_i^*) = BD$.

I. STRATEGIES IN A GAME-THEORETIC SITUATION

Croucher [Ref. 25] uses game theory to analyse the missile allocation problem. It is assumed that a target i is attacked by a weapon carrying r_i reentry vehicles, and is defended by d_i missiles. Given this situation, the probability that an incoming weapon that is aimed at target i destroys it can be given by a 'natural' payoff function of the following form:

$P(r_i, d_i) = \{1-\exp(-a_i r_i)\}\exp(-b_i d_i)$, where a_i and b_i are constants representing vulnerability factors associated with target i.

If each target i has a value v_i associated with it, then the total expected target value destroyed is given by

$$P(\underline{r}, \underline{d}) = \sum_{i=1}^{T} v_i \{1-\exp(-a_i r_i)\}\exp(-b_i d_i) ,$$

subject to the constraints $\sum_{i=1}^{T} r_i = R$ (the total offensive stockpile of reentry vehicles), and $\sum_{i=1}^{T} d_i = D$ (the total defense stockpile). The vectors $\underline{r}$ and $\underline{d}$ represent the offense and defense strategies respectively. The function $P(r_i, d_i)$ is concave in r_i for fixed d_i and convex in d_i for fixed r_i; consequently, it can be proven using the fundamental theorem of games that there exists a pure strategy solution for the game with the payoff function $P(\underline{r}, \underline{d})$. The optimal max-min strategies $\underline{r}^*$ and $\underline{d}^*$ are derived using Gibbs Lemma to be as follows:

$r_i^* = (1/b_i)\ln\{1 + (\mu b_i)/(\lambda a_i)\}$, $d_i^* = (1/a_i)\ln\{v_i/(\mu/a_i + \lambda/b_i)\}$ if $r_i^* > 0$, $d_i^* > 0$, and

$r_i^* = (1/b_i)\ln(v_i b_i/\lambda)$ if $r_i^* > 0$ and $d_i^* = 0$.

The values of λ and μ are uniquely determined by the equations:

$$\sum_{\frac{\lambda}{b_i} \le v_i \le \frac{\mu}{a_i} + \frac{\lambda}{b_i}} (1/b_i)\ln(v_i b_i/\lambda) + \sum_{v_i > \frac{\mu}{a_i} + \frac{\lambda}{b_i}} \ln\{1 + \mu b_i/(\lambda a_i)\} = R$$

and $$\sum_{v_i > \frac{\mu}{a_i} + \frac{\lambda}{b_i}} (1/a_i)\ln\{v_i/(u/a_i + \lambda/b_i)\} = D .$$

The criterion for attacking or defending a target i is characterized as follows:

no offense and defense at target i: $r_i^* = d_i^* = 0$ if $v < \lambda/b_i$;

no defense: $r_i^* > 0$, $d_i^* = 0$ if $\lambda/b_i < v_i < (\mu/a_i) + (\lambda/b_i)$;

both offense and defense: $r_i^* > 0$, $d_i^* > 0$ if $v_i > (\mu/a_i) + (\lambda/b_i)$.

A defense allocation $d_i^* > 0$ implies $r_i^* > 0$. The total expected value destroyed when offense and defense are both using their optimal strategies is given by the value of the game

$$V = \lambda \sum_{i \in I_1} 1/b_i + \sum_{i \in I_2} (v_i - \lambda/b_i) ,$$

where I_1 is the index set i such that $r_i^* > 0$, $d_i^* > 0$, and I_2 is the index set i such that $r_i^* > 0$, $d_i^* = 0$.

In a later paper, Croucher [Ref. 26] derives corresponding results in the case where the damage function is given by

$$P(r_i, d_i) = 1 - \exp\{-b_i r_i/(1+a_i d_i)\} .$$

On the condition that $R < 2/\max_i b_i$, the optimal offense and defense strategies are given by:

$$r_i^* = \{v_i b_i \mu/(a_i \lambda^2)\}\exp\{b_i \mu/(\lambda a_i)\} ,$$

$$d_i^* = \{v_i b_i/\lambda\}\exp\{(b_i \mu/(a_i \lambda) - 1)/a_i\} \text{ if } r_i^*>0 \text{ and } d_i^*>0 ,$$

and $r_i^* = (1/b_i)\log(v_i b_i/\lambda)$ if $r_i^* > 0$ and $d_i^* = 0$.

Thus the criteria for attacking or defending a particular target i is characterized by the following equations:

$$r_i^* = d_i^* = 0 \text{ if } v_i < \lambda/b_i ;$$

$$r_i^* > 0 , d_i^* = 0 \text{ if } \lambda/b_i < v_i \le (\lambda/b_i)\exp\{b_i \mu/(a_i \lambda)\} ;$$

$$r_i^* > 0 , d_i^* > 0 \text{ if } v_i > (\lambda/b_i)\exp\{b_i \mu/(a_i \lambda)\} .$$

The value of λ and μ is determined by the equations:

$$\sum_{\frac{\lambda}{b_i} < v_i < \frac{\lambda}{b_i}\exp\left(\frac{b_i\mu}{a_i\lambda}\right)} (1/b_i)\log(v_i b_i/\lambda) + \sum_{v_i > \frac{\lambda}{b_i}\exp\left(\frac{b_i\mu}{a_i\lambda}\right)} (b_i v_i \mu/a_i\lambda^2)\exp(b_i \mu/a_i \lambda) = R ,$$

$$\text{and} \quad \sum_{v_i > \frac{\lambda}{b_i}\exp\left(\frac{b_i\mu}{a_i\lambda}\right)} \{(v_i b_i/\lambda)\exp(b_i \mu/a_i \lambda) - 1\}/a_i = D .$$

The value of the game V is given by

$$V = \sum_{i \in I_1} v_i \{1 - \exp(-b_i \mu/(a_i \lambda))\} + \sum_{i \in I_2} (v_i - \lambda/b_i) .$$

V. CONCLUSION

This thesis has attempted to provide a description of the missile allocation problem and a general survey of the mathematical investigations, models, and results related to this problem. The treatments have not included classified studies, weapon specific studies or computer simulation combat models, and are hence by no means comprehensive. It is hoped, however, that the reader who is interested in missile defense and offense either from a practical or mathematical standpoint would, after reading this thesis, gain a better appreciation of the range of problems involved in this field, the successful attempts that have been made in solving this problem, and the areas in which no solutions have yet been found, and which therefore merit the attention of mathematical analysts or operations analysts who are interested in pursuing this field of research.

The general trend has been towards the building of more realistic and aggregated models of missile offense and defense. This is especially so in models which represent national-level strategic exchanges between superpowers (see for example [Ref. 27]). However, as the degree of realism and complexity of these models increase, it is generally more difficult to obtain analytical solutions in closed form or even through the use of iterative search algorithms, and it seems that computer simulation offers the only hope for a solution to the problem. However, simulation studies carry with them the disadvantage that sensitivity analyses and exploration of alternatives are extremely tedious and time-consuming because of the large number of variables or parameters required to characterize the model. This would hopefully motivate researchers to search for more 'elegant' mathematical solution methods for these problems.

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