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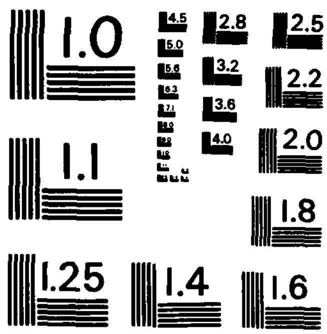
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REPORT WU/CCM-83/3

# COMPUTATION OF STRESS INTENSITY FACTORS

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OCTOBER, 1983



Prepared for:
Office of Naval Research
Arlington, VA 22217

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## ABSTRACT

The performance characteristics of the generalized influence function method for the approximate computation of the amplitudes of the eigenfunctions of the equations of plane elasticity in the vicinity of sharp reentrant corners were evaluated. The eigenfunctions satisfy the equations of equilibrium, compatibility and stress-strain laws and the free-free boundary conditions at reentrant corners. The amplitudes of the eigenfunctions are called the *generalized stress intensity factors*.

It is concluded that the generalized stress intensity factors can be computed to within one percent relative error with small computational effort. Therefore the essential characteristics of the elastic stress field in the neighborhood of reentrant corners can be determined with great precision. This computational technology is essential for the development of theories of crack initiation in metals and composites.

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## TREATMENT OF GEOMETRIC SINGULARITIES WITH THE p-VERSION OF THE FINITE ELEMENT METHOD

### 1. INTRODUCTION

In the displacement formulation of the finite element method the solution minimizes the strain energy of the error for the given finite element mesh and polynomial degree of elements. It has been shown that this is closely related to minimizing the root-mean-square error in stress [1]. In engineering computations, however, the strain energy and the root-mean-square error in stress are not the quantities of primary interest. The goal of computation is usually to estimate within a reasonably small margin of error, some functional of the displacement field, such as stress at a point, stress intensity factors etc. These quantities are usually computed from the finite element solution directly, without taking into account that the essential character of the solution is known a priori. For example, stresses at a point within an element are computed as some linear

*The numbers in brackets in the text indicate references in the Bibliography.

combination of the derivatives of the shape functions at the point. Stress intensity factors for crack problems are computed from the strain energy release rate, which is obtained by computing the strain energy for two bodies with slightly different crack sizes. Although the method performs well when used in conjunction with the p-version of the finite element method [2], the stress intensity factors for mixed mode problems cannot be obtained separately by this method.

In three related papers Babuska and Miller introduced new techniques that permit the extraction of various functionals of the displacements, such as pointwise displacements, stresses and stress intensity factors, with greater reliability and accuracy than was previously possible [3, 4, 5]. The main idea of the new extraction techniques is that not only the finite element solution but also the essential characteristics of the exact solution are utilized in making the computations. This requires a modest amount of extra effort but a great deal is gained in accuracy and reliability. One of the extraction techniques is the generalized influence function method for the computation of stress intensity factors. By stress intensity factors we mean not only the stress intensity factors defined in linear elastic fracture mechanics but also the analogous quantities associated with any sharp reentrant corner.

Stress singularities in the solution of elasticity problems can be caused by loading, sudden changes in the boundary conditions or material properties and reentrant corners. A typical corner detail is shown in figure 1.1. This investigation is concerned with stress singularities at reentrant corners with free-free boundary conditions on the sides of the angle.

In the neighborhood of the corner the solution vector can be written in terms of polar coordinates centered on the corner in the following form:

$$\underline{u} = \sum_i K_i \, r^{\kappa_i} \, \underline{F}_i(\theta) + \underline{G}(r, \theta) \tag{1.1}$$

where $\underline{u} = (u_1, u_2)$, $\underline{F}$ and $\underline{G}$ are functions that are smoother than $r^{\kappa_i} F_i(\theta)$ and the $\kappa_i$'s are positive numbers. The amplitudes $K_i$ are the generalized stress intensity factors. The eigenvalues $\kappa_i$ depend on the angle $\beta$, the boundary conditions imposed on the two sides of the angle, and Poisson's ratio. The eigenfunction $F_i(\theta)$ depends on the angle $\beta$, the corresponding eigenvalue $\kappa_i$ and the elastic constants. A typical stress component is of the form:

$$\sigma = \sum_i K_i \, r^{\kappa_i - 1} \, f(\theta) + g(r, \theta) \tag{1.2}$$

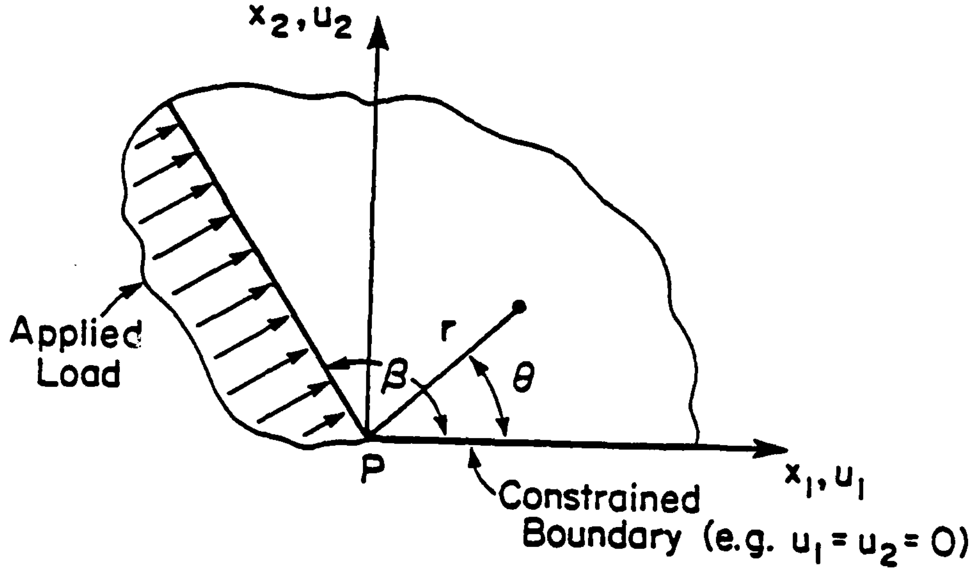


Figure 1.1

Typical corner detail in plane elastic problems

It can be seen that for values of $\kappa_i < 1$ the stresses become infinite at the corner.

A special case is the case of a crack (angle $\beta$ equal to 360 degrees). The solution of crack problems and the study of the conditions under which cracks propogate is the subject of linear elastic fracture mechanics. Its applicability rests on the satisfaction of a criterion known as small scale yielding. This means that any nonlinearities are confined to a region which is small in comparison with the size of the body and is completely surrounded by a region in which the solution of the elasticity problem is an adequate representation of the real response of the material. The elastic stress field and strain field in the neighborhood of the singularity can be written respectively as:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}(\theta) + \text{higher order terms} \qquad (1.3)$$

$$\varepsilon_{ij} = \frac{K}{\sqrt{2\pi r}} \tilde{\varepsilon}_{ij}(\theta) + \text{higher order terms} \qquad (1.4)$$

where K is the stress intensity factor and $\tilde{\sigma}_{ij}(\theta)$, $\tilde{\varepsilon}_{ij}(\theta)$ are smooth functions of $\theta$. When the restrictions stated in the preceding paragraph are met, these fields are the same for any crack, regardless of the overall geometry and applied loading. The stress and deformation fields ahead of the crack tip for a purely symmetric or antisymmetric crack configuration are then characterized by a single parameter, the stress intensity factor K. Linear elastic fracture mechanics is based on the observation that crack growth is controlled by the parameter K. A pre-cracked specimen in which the elastic field is known is tested under monotonic loading conditions until the first occurence of crack propagation is observed. At this point the stress intensity factor is said to have a critical value which is usually denoted by $K_c$. This phenomenological approach ignores the micromechanisms of void nucleation ahead of the crack tip and the way in which these voids are joined to increase the crack length, yet it has been highly successful in giving reliable answers to important practical questions such as the expected life of structures under existing flaws, the maximum flaw size that quality control is allowed to miss and maximum allowable time intervals between inspections. The critical stress intensity factor $K_{IC}$ in mode I under plane strain conditions is termed fracture toughness and is accepted as the material constant that characterizes material resistance to fracture.

This approach has been extended to the nonlinear regime. The J-integral is the intensity of the singular nonlinear elastic strain field in the neighborhood of the of the crack tip (HRR field). In this sense $J_{IC}$ is analogous to $K_{IC}$ but of course the small scale yielding condition is now replaced by the "J-dominance" condition, which means that the area ahead of the tip, in which intense nonlinearities deviating from deformation theory of plasticity take place, is relatively small and is completely surrounded by a region in which the singular nonlinear elastic field represents adequately the real material response [6, 7, 8, 9, 10].

Despite its great success, linear elastic fracture mechanics is difficult to apply to structures of complicated geometry. The presence of a large number of points where cracks are apt to occur would require a large number of analyses to be performed with a hypothetical crack at each of these points. On the other hand, the designer's intent is that the structure should spend most of its life in the crack initiation stage rather than in the crack propagation stage. The ability to formulate crack initiation criteria based on linear elasticity could open the way to more rational design procedures.

Although the singular nature of elasticity solutions in the neighborhood of reentrant corners was reported in the engineering literature as early as 1933 [11], and the 1952 paper by M. L. Williams [12] gave the strength of these singularities for the various angles, no attempt was previously made to assign any physical significance to the amplitudes of the stress singular terms (except in linear elastic fracture mechanics) and no numerical method existed for the computation of these quantities. Based on the work presented herein, these amplitudes can now be computed with levels of precision normally expected in engineering computations at the expenditure of a relatively modest computational effort.

Our ability to compute the amplitude of all terms of the asympotic expansion in the neighborhood of a reentrant corner of any size offers new possibilities in the area of failure initiation. When sufficient number of terms are used in the expansion and the amplitudes are accurately computed, then all stress field parameters are known in the neighborhood of corner points, therefore various hypotheses concerning relationships between elastic stress field parameters and failure initiation can be tested. In the absence of proper extracton methods, uncertainties in numerically computed stress field parameters render the formulation, testing and applications of such hypotheses very tenuous.

A possible hypothesis for example is that the generalized stress intensity factors are responsible for crack initiation in reentrant corners in the same way that

Let us consider the case of mode I and rewrite equations (2.16) and (2.17) in the following form:

$$u = K_I \, r^{1/2} \, G(\theta) \tag{2.20}$$

$$v = K_I \, r^{1/2} \, H(\theta) \tag{2.21}$$

Their derivatives can be expressed in the form:

$$\frac{\partial u}{\partial v} = K_I \, r^{-1/2} \, G_1(\theta) ; \; \frac{\partial u}{\partial y} = K_I \, r^{-1/2} \, G_2(\theta) \tag{2.22}$$

$$\frac{\partial v}{\partial x} = K_I \, r^{-1/2} \, H_1(\theta) ; \; \frac{\partial v}{\partial y} = K_I \, r^{-1/2} \, H_2(\theta) . \tag{2.23}$$

We choose the auxiliary functions to be of the following form:

$$\phi = r^{-1/2} \, \Phi(\theta) \tag{2.25}$$

$$\psi = r^{-1/2} \, \Psi(\theta) . \tag{2.26}$$

Their derivatives can be expressed as:

$$\frac{\partial \phi}{\partial x} = r^{-3/2} \, \Phi_1(\theta) , \; \frac{\partial \phi}{\partial y} = r^{-3/2} \, \Phi_2(\theta) \tag{2.27}$$

$$\frac{\partial \psi}{\partial x} = r^{-3/2} \, \Psi_1(\theta) , \; \frac{\partial \psi}{\partial y} = r^{-3/2} \, \Psi_2(\theta) . \tag{2.28}$$

the displacements u and v is known [13, 14] and is given in the case of plane strain by:

$$Gu = K_I \ (2\pi)^{-1/2} \ r^{1/2} \cos \frac{\theta}{2} \ (2-2\nu-\cos^2 \frac{\theta}{2}) \qquad (2.16)$$

$$Gv = K_I \ (2\pi)^{-1/2} \ r^{1/2} \sin \frac{\theta}{2} \ (2-2\nu-\cos^2 \frac{\theta}{2}) \qquad (2.17)$$

for mode I (opening mode), and by:

$$Gu = K_{II} \ (2\pi)^{-1/2} \ r^{1/2} \sin \frac{\theta}{2} \ (2-2\nu+\cos^2 \frac{\theta}{2}) \qquad (2.18)$$

$$Gv = K_{II} \ (2\pi)^{-1/2} \ r^{1/2} \cos \frac{\theta}{2} \ (2\nu-\cos^2 \frac{\theta}{2}) \qquad (2.19)$$

for mode II (sliding mode).

These are only the first terms in the asymptotic expansions. The second terms are of the order r. The third terms are of the order $r^{3/2}$ and so on. The asymptotic expansions also contain the corresponding negative powers of r (-1/2, -1, -3/2 etc.), but these terms make no sense physically since they would imply infinite displacements at the crack tip, so they are not considered. Nevertheless, these eigenfunctions corresponding to the negative eigenvalues will be very useful for our formulation: the auxiliary functions $\phi$ and $\psi$ are chosen to be precisely these eigenfunctions.

In view of (2.3) and (2.4) we can write (2.12) and (2.13) as:

$$I_5 = \overline{X}\ \phi \tag{2.14}$$

$$I_6 = \overline{Y}\ \psi. \tag{2.15}$$

On that part of the boundary where surface tractions are specified, the quantities $I_5$ and $I_6$ are known and their integrals are computed numerically. On that part of the boundary where displacements are specified, the $\phi$ and $\psi$ functions vanish. The reason that the functions $\phi$ and $\psi$ are chosen to vanish there is that we do not wish the extraction expression to contain a contour integral of the derivatives of displacement components. The extraction method has a rate of convergence equal to the rate of convergence in energy if the integrals used in the extraction represent only energy expressions. This means that area integrals may contain up to first order derivatives of the displacements, whereas contour integrals may contain only the displacements themselves. It is observed that expressions $I_3$ and $I_4$ fall into this category and so they do not reduce the rate of convergence.

In order to evaluate the contour integrals on the circular arc $\Gamma_\varepsilon$ we proceed as follows: in the neighborhood of the crack where the circular arc is located, the form of

$$I_4 = [\lambda(\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y})n_y + G(\frac{\partial\psi}{\partial x} n_x + \frac{\partial\psi}{\partial y} n_y) +$$

$$+ G(\frac{\partial\phi}{\partial y} n_x + \frac{\partial\psi}{\partial y} n_y)]v \qquad (2.11)$$

$$I_5 = [\lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})n_x + G(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y) +$$

$$+ G(\frac{\partial u}{\partial x} n_x + \frac{\partial v}{\partial x} n_y)]\phi \qquad (2.12)$$

$$I_6 = [\lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + G(\frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y) +$$

$$+ G(.\frac{\partial u}{\partial y} n_y + \frac{\partial v}{\partial y} n_y)]\psi. \qquad (2.13)$$

In (2.8) we recognize the expression in the bracket as the force in the x direction in terms of displacements $(\phi, \psi)$. Similarly in (2.9) the bracket represents the force in the y direction corresponding to those displacements $(\phi, \psi)$. If the auxiliary functions $\phi$ and $\psi$ are chosen to satisfy the equations of equilibrium, then $I_1$ and $I_2$ vanish.

$$- \int_{\Gamma+\Gamma_\varepsilon} [(\lambda+G) \frac{\partial\psi}{\partial y} vn_y + \lambda \frac{\partial\psi}{\partial y} un_x + G \frac{\partial\psi}{\partial x} n_x v + + G \frac{\partial\psi}{\partial y} n_y v + G \frac{\partial\psi}{\partial x} un_y]ds$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [(\lambda+G) \frac{\partial v}{\partial y} \psi n_y + \lambda \frac{\partial u}{\partial x} \psi n_y + G \frac{\partial v}{\partial x} n_x\psi + + G \frac{\partial v}{\partial y} n_y\psi + G \frac{\partial u}{\partial y} n_x\psi]ds = 0 \tag{2.6}$$

Adding (2.5) and (2.6) we obtain:

$$\int_A \int (I_2+I_2)dA - \int_{\Gamma+\Gamma_\varepsilon} (I_3+I_4)ds + \int_{\Gamma+\Gamma_\varepsilon} (I_5+I_6)ds=0 \tag{2.7}$$

where:

$$I_1 = [(\lambda+G) \frac{\partial}{\partial x} ( \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y} ) + G( \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} )]u \tag{2.8}$$

$$I_2 = [(\lambda+G) \frac{\partial}{\partial y} ( \frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial x} ) + G( \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2} )]v \tag{2.9}$$

$$I_3 = [\lambda( \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y} ) n_x + G( \frac{\partial\phi}{\partial x} n_x + \frac{\partial\phi}{\partial y} n_y) + G( \frac{\partial\phi}{\partial x} n_x + \frac{\partial\psi}{\partial x} n_y]u \tag{2.10}$$

and, finally:

$$\int_A\int [(\lambda+G)\frac{\partial^2\phi}{\partial x^2}u + (\lambda+G)\frac{\partial^2\phi}{\partial x\partial y}v + G(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2})u]dA$$

$$- \int_{\Gamma+\Gamma_\varepsilon} [(\lambda+G)\frac{\partial\phi}{\partial x}un_x + \lambda\frac{\partial\phi}{\partial x}vn_y + G\frac{\partial\phi}{\partial x}n_x u + G\frac{\partial\phi}{\partial y}n_y u + G\frac{\partial\phi}{\partial y}vn_x]ds$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [(\lambda+G)\frac{\partial u}{\partial x}\phi n_x + \lambda\frac{\partial v}{\partial y}\phi n_x + G\frac{\partial u}{\partial x}n_x\phi + G\frac{\partial u}{\partial y}n_y\phi + G\frac{\partial v}{\partial x}\phi n_y]ds = 0 \qquad (2.5)$$

Similarly, multiplying (2.2) by $\psi$ and integrating, after identical operations, we obtain:

$$\int_A\int [(\lambda+G)\frac{\partial^2\psi}{\partial x^2}v + (\lambda+G)\frac{\partial^2\psi}{\partial x\partial y}u + G(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2})v]dA$$

$$= - \int_A \int [\lambda ( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} ) \frac{\partial \phi}{\partial x} + G \frac{\partial u}{\partial x} \frac{\partial \phi}{\partial x} + G \frac{\partial u}{\partial y} \frac{\partial \phi}{\partial y} +$$

$$+ G \frac{\partial u}{\partial x} \frac{\partial \phi}{\partial x} + G \frac{\partial v}{\partial x} \frac{\partial \phi}{\partial y} ] dA$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [\lambda ( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} ) \phi n_x + G \frac{\partial u}{\partial x} \phi n_x + G \frac{\partial u}{\partial y} \phi n_y +$$

$$+ G \frac{\partial u}{\partial x} \phi n_x + G \frac{\partial v}{\partial x} \phi n_y] ds$$

$$= + \int_A \int [\lambda \frac{\partial^2 \phi}{\partial x^2} u + \frac{\partial^2 \phi}{\partial x \partial y} v + G \frac{\partial^2 \phi}{\partial y^2} u + G \frac{\partial^2 \phi}{\partial y^2} u +$$

$$+ G \frac{\partial^2 \phi}{\partial x^2} u + G \frac{\partial^2 \phi}{\partial x \partial y} v] dA$$

$$- \int_{\Gamma+\Gamma_\varepsilon} [\lambda \frac{\partial \phi}{\partial x} u\ n_x + \lambda \frac{\partial \phi}{\partial x} v\ n_y + G \frac{\partial \phi}{\partial x} u\ n_x +$$

$$+ G \frac{\partial \phi}{\partial y} u\ n_y + G \frac{\partial \phi}{\partial x} u\ n_x + G \frac{\partial \phi}{\partial y} v\ n_x] ds$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [\lambda \frac{\partial u}{\partial x} \phi\ n_x + \lambda \frac{\partial v}{\partial y} \phi\ n_x + G \frac{\partial u}{\partial x} \phi\ n_x +$$

$$+ G \frac{\partial u}{\partial y} \phi\ n_y + G \frac{\partial u}{\partial x} \phi\ n_x + G \frac{\partial v}{\partial x} \phi\ n_y] ds$$

$$(\lambda+G)\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+G\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0. \qquad (2.2)$$

The stress boundary conditions are:

$$\bar{X}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)n_x+G\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y\right)+$$

$$+\ G\left(\frac{\partial u}{\partial x}n_x+\frac{\partial v}{\partial x}n_y\right) \qquad (2.3)$$

$$\bar{Y}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)n_y+G\left(\frac{\partial v}{\partial x}n_x+\frac{\partial v}{\partial y}n_y\right)+$$

$$+\ G\left(\frac{\partial u}{\partial y}n_x+\frac{\partial v}{\partial y}n_y\right) \qquad (2.4)$$

where $\lambda$, $G$ are Lame's constants and $\bar{n}=(n_x,\ n_y)$ is the outward normal to the surface. $\bar{X}$ and $\bar{Y}$ are the vector components of specified tractions in the x and y directions. Let us now choose two functions $\phi$ and $\psi$, the properties of which will be discussed later and let us multiply equation (2.1) by $\phi$ and integrate over the domain. Applying the Gauss theorem twice we obtain:

$$0=\int_A\int\Big[\lambda\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+$$

$$+\ G\frac{\partial^2 u}{\partial x^2}+G\frac{\partial^2 v}{\partial x\partial y}\Big]\phi dA$$

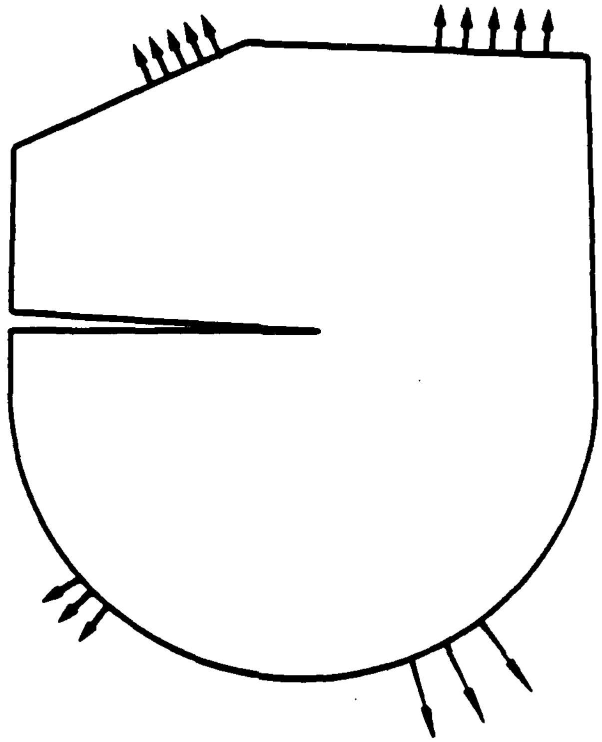

Figure 2.1

Two dimensional elastic body with a crack

## 2. THE GENERALIZED INFLUENCE FUNCTION METHOD IN LINEAR ELASTIC FRACTURE MECHANICS

We consider a two dimensional body containing a crack as shown in figure (2.1). The boundary consists of piece-wise smooth curves and there is no re-entrant corner other than the one at the crack tip. Extension to cases where the domain contains more than one geometric singularity will not introduce any additional difficulties. This will be shown later.

We now remove from the domain a small disk of radius $r=\varepsilon$ centered on the crack tip as shown in figure (2.2). The boundary now consists of two parts: the circular arc $\Gamma_\varepsilon$ and the rest of the original boundary $\Gamma$. The removal of the singularity enables us to perform integration by parts. The limit is then taken as r tends to zero.

Let us choose a Cartesian coordinate system (x,y) centered on the crack tip with the x-axis in the direction of the crack, and the corresponding polar system $(r,\theta)$ as shown in figure (2.3), and write the equations of equilibrium in terms of displacements (u,v) in the Cartesian system:

$$(\lambda+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 \qquad (2.1)$$

In chapter 7 it is concluded that the method is feasible and reliable, its potential value in engineering design is discussed and suggestions are made for further research.

separate the stress intensity factors for modes I and II is demonstrated.

In chapter 3 explicit expressions are derived for the asymptotic expansions of the displacements in the neighborhood of reentrant corners of any size. The real eigenvalues corresponding to modes I and II for various angles are tabulated. The displacement eigenfunctions are also derived for the case of complex eigenvalues.

In chapter 4 the generalized influence function method is implemented for the case of a reentrant corner of arbitrary size. The method for obtaining the amplitude of any term in the expansion is demonstrated. Special consideration is given to the case of complex eigenvalues.

In chapter 5 the implementation is tested against a model problem for which the exact solution is known. This allows rigorous convergence study to be performed. The theoretically predicted rates of convergence for various reentrant corners are verified numerically.

In chapter 6 tests performed on double edge notched epoxy specimens are discussed. This test data shows that a monotonic relationship exists between the stress intensity factor and failure initiation for a wide range of solid angles.

function method for the extraction of stress intensity factors in plane elasticity.

In the case of linear elastic fracture mechanics the generalized influence function method yields separately the stress intensity factors corresponding to the symmetric mode of deformation (mode I) and to the antisymmetric mode (mode II). Our ability to separate of the two modes may prove to be important in linear elastic fracture mechanics, where only the combination of the two modes could be computed previously with reasonable accuracy and the contribution of mode II has not been well understood.

The method has a rate of convergence equal to the rate of convergence in energy, which is twice the rate of convergence in energy norm [4]. Our ability to achieve convergence in practical computations can be used as a tool for error estimation. Knowing the theoretical rate of convergence, we can obtain an estimate of the exact value of the stress intensity factor (also strain energy, root-mean square stress measure and other functionals) accurate to within one percent relative error.

In chapter 2 the method is implemented for crack problems. The rationale of the method is explained and the extraction functions are derived. The ability to the stress intensity factor in linear elastic fracture mechanics is responsible for crack propagation. Again, some small scale yielding criterion must be satisfied. If experiments verify this hypothesis and establish its limits of applicability, then we have at our disposal a very simple and powerful method for the design of structures with geometric singularities. The methodology of linear elastic fracture mechanics can be generalized in a straightforward manner. For a given angle and mode of loading the singular elastic field in the neighborhood of a reentrant corner is of the same form regardless of the overall configuration and load distribution. Any change in the loading or the boundary conditions affects material behavior at the tip of the notch only through the generalized stress intensity factors which are the amplitudes of the terms in the asymptotic expansion of the linear elastic solution in the neighborhood of reentrant corners of arbitrary size. For large notch angles the higher order terms may also be important as crack initiation parameters. We can then argue that under conditions of small scale yielding the generalized stress intensity factors can be used for predicting failure initiation events.

The scope of this report is the implementation, application and evaluation of the generalized influence

The integrals of $I_3$, $I_4$, $I_5$ and $I_6$ along the circular arc can now be computed. A typical term for the integral of $I_3$ will be:

$$\int_{\Gamma_\varepsilon} \frac{\partial \phi}{\partial x} n_x \, u \, ds$$

substituting

$$\frac{\partial \phi}{\partial x} = r^{-3/2} \, \Phi_1(\theta), \; n_\eta = -\cos\theta, \; u = \kappa_I \, r^{1/2} \, G(\theta),$$

$$ds = r d\theta$$

we obtain

$$\int_{\Gamma_\varepsilon} \frac{\partial \phi}{\partial x} n_\eta \, u \, ds = - K_I \int_{-\pi}^{\pi} \Phi_1(\theta) \, G(\theta) \cos\theta \, d\theta \quad (2.29)$$

This last integral is independent of the radius r, it contains only known functions of θ and can be easily computed either analytically or numerically. It also contains as a multiplicative constant the stress intensity factor $K_I$. This is fundamental to the extraction technique. By choosing the auxiliary functions $\phi$ and $\psi$ to be of the proper asymptotic behavior in the neighborhood of the crack tip, expressions containing the radius r disappear. By using the known asymptotic expansions of the displacements in the neighborhood of the crack tip, the stress intensity factor $K_I$ appears as a multiplicative constant. All the terms in the integrals of $I_3$ and $I_4$ behave exactly in the same way. Each one will give an integral of a function of $\theta$ only, containing $K_I$ as a multiplicative constant.

Let us now examine the integrals of $I_5$ and $I_6$ on the arc $\Gamma_\varepsilon$. A typical term will be:

$$\int_{\Gamma_\varepsilon} \frac{\partial u}{\partial x} n_x \phi \, ds.$$

Substituting:

$$\frac{\partial u}{\partial x} = K_I \, r^{-1/2} \, G_1(\theta), \quad n_x = -\cos\theta, \quad \phi = r^{-1/2} \, \Phi(\theta),$$

$$ds = r d\theta$$

we obtain:

$$\int_{\Gamma_\varepsilon} \frac{\partial u}{\partial x} n_x \phi \, ds = -K_I \int_{-\pi}^{\pi} G_1(\theta) \; \Phi(\theta) \; \cos\theta \; d\theta. \quad (2.30)$$

This is again independent of the radius r and the same arguments apply. It is then seen that contour integration around the circular arc $\Gamma_\varepsilon$ will yield the stress intensity factor $K_I$ multiplied by a constant.

In view of the singularity of the auxiliary functions $\phi$ and $\psi$ and their derivatives at the crack tip, the existence of the area integral over the domain and the contour integrals over $\Gamma$ in (2.7) must now be examined. Since the integrands $I_1$ and $I_2$ are highly singular (they are of order $r^{-5/2}$ in the neighborhood of the crack tip), the auxiliary functions $\phi$ and $\psi$ are chosen to satisfy the equations of equilibrium in that region, so that $I_1$ and $I_2$ vanish there. As far as the contour integration is concerned, only the upper and lower faces of the crack are of interest. We normally assume no tractions there and it can be seen from (2.14) and (2.15) that $I_5$ and $I_6$ vanish there. In the case where the crack faces have applied tractions on them, these integrands are of order $r^{-1/2}$. The integrals of $I_5$ and $I_6$ exist, but care must be exercised in their numerical evaluation.

The integral of $I_3$ and $I_4$ on the crack faces requires further consideration. By comparing with (2.3) and (2.4) it can be seen that (2.10) and (2.11) can be rewritten as

$$I_3 = \overline{X}(\phi, \psi)u \tag{2.31}$$

$$I_4 = \overline{Y}(\phi, \psi)v \tag{2.32}$$

where $\overline{X}(\phi,\psi)$ and $\overline{Y}(\phi,\psi)$ are the tractions corresponding to displacements $\phi$ and $\psi$. We have already seen that the auxiliary functions $\phi$ and $\psi$ satisfy the equilibrium equations in the neighborhood of the crack tip. By considering their form in (2.25) and (2.26) and recalling that the displacement expansions in the neighborhood the crack tip contain terms of the order $r^{-1/2}$ we conclude that the auxiliary functions $\phi$ and $\psi$ can be chosen to be the eigenfunctions corresponding to the eigenvalue -1/2, therefore they satisfy the traction free boundary conditions on the crack surfaces. In other words $\overline{X}(\phi, \psi)$ and $\overline{Y}(\phi, \psi)$ and consequently $I_3$ and $I_4$ vanish there.

Let us now summarize the conditions that the auxiliary functions $\phi$ and $\psi$ must fulfill and then proceed to construct them:

i) $\phi$ and $\psi$ must satisfy the equations of equilibrium in the neighborhood of the crack tip;

ii) they must satisfy the traction free boundary conditions on the faces of the crack near the crack tip;

iii) they must have a singularity of the order $r^{-1/2}$ in the neighborhood of the crack tip;

iv) they must vanish on the part of the boundary where displacements are prescribed.

Conditions i), ii) and iii) together mean that $\phi$ and $\psi$ are eigenfunctions for the problem of an infinite body containing a crack, and correspond to the eigenvalue equal to -1/2.

We now observe that in the absence of surface tractions both the equilibrium equations (2.1), (2.2) and the stress boundary conditions (2.3) and (2.4) are homogeneous and contains only material constants and the derivatives of the displacement vector components. By differentiating them with respect to x it is seen that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ also satisfy the same equations. Therefore they are possible candidates for $\phi$ and $\psi$ satisfying requirements i) and ii) above. By comparison of (2.20), (2.21) with (2.25), (2.26) it is seen that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ also satisfy requirement iii) above. If we ncw consider the case of a body where only tractions are specified on the contour, then requirement iv) does not have to be satisfied and we have arrived at an explicit form for $\phi$ and $\psi$ satisfying all the requirements. By differentiating (2.16) and (2.17) with respect to x and setting the constant multipliers equal to one, we obtain for mode I:

$$\phi = r^{-1/2} \cos\frac{\theta}{2} \left(2-2\nu-5\cos^2\frac{\theta}{2} +4\cos^4\frac{\theta}{2}\right) \tag{2.33}$$

$$\psi = r^{-1/2} \sin\frac{\theta}{2} \left(-2+2\nu-3\cos^2\frac{\theta}{2} +4\cos^4\frac{\theta}{2}\right) \tag{2.34}$$

and for mode II:

$$\phi = r^{-1/2} \sin\frac{\theta}{2} \left(-2+2\nu+3\cos^2\frac{\theta}{2} - 4\cos^4\frac{\theta}{2}\right) \quad (2.35)$$

$$\psi = r^{-1/2} \cos\frac{\theta}{2} \left(2\nu-5\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right). \quad (2.36)$$

The contour integrals on the circular arc that appear in (2.7) can now be computed explicitly. Omitting intermediate results, we obtain:

$$-\int_{\Gamma_\varepsilon} (I_3+I_4)ds + \int_{\Gamma_\varepsilon} (I_5+I_6)ds = 4(1-\nu)\pi(2\pi)^{1/2}K_I. \quad (2.37)$$

Taking into account (2.14) and (2.15) we obtain from (2.7) the extraction formula:

$$4(1-\nu)\pi(2\pi)^{1/2} K_I = -\int_A\int (I_1+I_2)dA + \int_\Gamma (I_3+I_4)ds - \int_\Gamma \bar{X}\phi ds - \int_\Gamma \bar{Y}\psi ds. \quad (2.38)$$

In the case where only tractions are specified on the boundary of the body this simplifies to:

$$4(1-\nu)\pi(2\pi)^{1/2} K_I = \int_\Gamma (I_3+I_4)ds - \int_\Gamma \bar{X}\phi ds - \int_\Gamma \bar{Y}\psi ds \quad (2.39)$$

In the case of mode II, the integrals on the circular arc $\Gamma_\varepsilon$ yield the same constant and we obtain:

$$4(1-\nu)\pi(2\pi)^{1/2} K_{II} = -\int_A\int (I_1+I_2)dA + \int_\Gamma (I_3+I_4)ds - \int_\Gamma \bar{X}\phi ds - \int_\Gamma \bar{Y}\psi ds \quad (2.40)$$

and in the case of only applied tractions specified:

$$4(1-\nu)\pi(2\pi)^{1/2}K_{I} = \int_\Gamma (I_3+I_4)ds - \int_\Gamma \bar{X}\phi ds - \int_\Gamma \bar{Y}\psi ds. \quad (2.41)$$

In the case where the domain contains more than one geometric singularity, we simply remove a small disk of radius $r = \varepsilon$ from every singular point. This is done in order to be able to perform integration by parts. The asymptotic expansions (2.16) and (2.17) or (2.18) and (2.19) are considered only in the neighborhood of the point where extration of the stress intensity factor is desired. The auxiliary functions $\phi$ and $\psi$ are also referenced to that point. The other singular points are thus treated as the rest of the boundary and expressions (2.38), (2.39), (2.40) and (2.41) are still valid.

In the formulation described above only one mode was considered, that is, the vicinity of the crack tip was assumed to experience either mode I or mode II deformation, and the corresponding stress intensity factor was extracted. Of course, in practical situations both modes will be present and the two intensity factors must be computed separately. One of the major advantages of the generalized influence function method is the ability to obtain the intensity factors for the two modes separately. It will now be shown that the extraction formulae (2.38), (2.39), (2.40) and (2.41) are still valid when both modes are present simultaneously. Let us first formally introduce the following operators that give the tractions in the x and y directions corresponding to displacements u and v through the strain-displacement, stress-strain and Cauchy boundary relations:

$$\bar{X}(u,v) = \lambda\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_x + G\left( \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) + G\left( \frac{\partial u}{\partial x} n_x + \frac{\partial v}{\partial x} n_y \right) \qquad (2.42)$$

$$\overline{Y}(u,v) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_y + G\left( \frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y \right) +$$

$$+ G\left( \frac{\partial u}{\partial y} n_x + \frac{\partial v}{\partial y} n_y \right) \qquad (2.43)$$

Equation (2.7) can now be written as:

$$\int_A \int (I_1+I_2)dA - \int_{\Gamma+\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u + \overline{Y}(\phi,\psi)v]ds +$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [\overline{X}(u,v)\phi + \overline{Y}(u,v)\psi]ds = 0. \qquad (2.44)$$

The only difference now is the asymptotic expansion for the displacements which is valid along the circular arc $\Gamma_\varepsilon$. Let us use the notation

$$u = u_I + u_{II} \qquad (2.45)$$

$$v = v_I + v_{II} \qquad (2.46)$$

where subscripts I and II refer to the corresponding modes and $u_I$, $v_I$, $u_{II}$ and $v_{II}$ are given by (2.16), (2.17), (2.18) and (2.19) respectively. They are listed below together with their derivatives as well as the auxiliary functions and their derivatives.

$$u_I = a_I \, r^{1/2} \cos\frac{\theta}{2} \left(2-2\nu-\cos^2\frac{\theta}{2}\right) \tag{2.47}$$

$$v_I = a_I \, r^{1/2} \sin\frac{\theta}{2} \left(2-2\nu-\cos^2\frac{\theta}{2}\right) \tag{2.48}$$

$$\frac{\partial u_I}{\partial x} = \frac{1}{2} a_I \, r^{-1/2} \cos\frac{\theta}{2} \left(2-2\nu-5\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right) \tag{2.49}$$

$$\frac{\partial u_I}{\partial y} = \frac{1}{2} a_I \, r^{-1/2} \sin\frac{\theta}{2} \left(2-2\nu-3\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right) \tag{2.50}$$

$$\frac{\partial v_I}{\partial x} = \frac{1}{2} a_I \, r^{-1/2} \sin\frac{\theta}{2} \left(-2+2\nu-3\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right) \tag{2.51}$$

$$\frac{\partial v_I}{\partial y} = \frac{1}{2} a_I \, r^{-1/2} \cos\frac{\theta}{2} \left(-2\nu+5\cos^2\frac{\theta}{2} - 4\cos^4\frac{\theta}{2}\right) \tag{2.52}$$

$$\phi_I = r^{-1/2} \cos\frac{\theta}{2} \left(2-2\nu-5\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right) \tag{2.53}$$

$$\psi_I = r^{-1/2} \sin\frac{\theta}{2} \left(-2+2\nu-3\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right) \tag{2.54}$$

$$\frac{\partial \phi_I}{\partial x} = \frac{1}{2} r^{-3/2} \cos\frac{\theta}{2} \left[6-6\nu+(-43+8\nu)\cos^2\frac{\theta}{2} + 84\cos^4\frac{\theta}{2} - 48\cos^6\frac{\theta}{2}\right] \tag{2.55}$$

$$\frac{\partial \phi_I}{\partial y} = \frac{1}{2} r^{-3/2} \sin \frac{\theta}{2} [2-2\nu+(-23+8\nu)\cos^2 \frac{\theta}{2} + + 60\cos^4 \frac{\theta}{2} -48\cos^6 \frac{\theta}{2} ] \quad (2.56)$$

$$\frac{\partial \psi_I}{\partial x} = \frac{1}{2} r^{-3/2} \sin \frac{\theta}{2} [-2+2\nu-(7+8\nu)\cos^2 \frac{\theta}{2} + + 60\cos^4 \frac{\theta}{2} -48\cos^6 \frac{\theta}{2} ] \quad (2.57)$$

$$\frac{\partial \psi_{II}}{\partial y} = \frac{1}{2} r^{-3/2} \cos \frac{\theta}{2} [-6\nu+(35+8\nu)\cos^2 \frac{\theta}{2} - - 84\cos^4 \frac{\theta}{2} + 48\cos^6 \frac{\theta}{2}] \quad (2.58)$$

$$u_{II} = a_{II} r^{1/2} \sin \frac{\theta}{2} (2-2\nu+\cos^2 \frac{\theta}{2} ) \quad (2.59)$$

$$v_{II} = a_{II} r^{1/2} \cos \frac{\theta}{2} (2\nu-\cos^2 \frac{\theta}{2} ) \quad (2.60)$$

$$\frac{\partial u_{II}}{\partial x} = \frac{1}{2} a_{II} r^{-1/2} \sin \frac{\theta}{2} [-2+2\nu+3\cos^2 \frac{\theta}{2} -4\cos^4 \frac{\theta}{2} ] \quad (2.61)$$

$$\frac{\partial u_{II}}{\partial y} = \frac{1}{2} a_{II} r^{-1/2} \cos \frac{\theta}{2} [4-2\nu-5\cos^2 \frac{\theta}{2} +4\cos^4 \frac{\theta}{2} ] \quad (2.62)$$

$$\frac{\partial v_{II}}{\partial x} = \frac{1}{2} a_{II} r^{-1/2} \cos\frac{\theta}{2} [2\nu - 5\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}] \quad (2.63)$$

$$\frac{\partial v_{II}}{\partial y} = \frac{1}{2} a_{II} r^{-1/2} \sin\frac{\theta}{2} [2\nu - 3\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}] \quad (2.64)$$

$$\phi_{II} = r^{-1/2} \sin\frac{\theta}{2} [-2 + 2\nu + 3\cos^2\frac{\theta}{2} - 4\cos^4\frac{\theta}{2}] \quad (2.65)$$

$$\psi_{II} = r^{-1/2} \cos\frac{\theta}{2} [2\nu - 5\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}] \quad (2.66)$$

$$\frac{\partial \phi_{II}}{\partial x} = \frac{1}{2} r^{-3/2} \sin\frac{\theta}{2} [-2 + 2\nu + (23 - 8\nu)\cos^2\frac{\theta}{2} - 60\cos^4\frac{\theta}{2} + 48\cos^6\frac{\theta}{2}] \quad (2.67)$$

$$\frac{\partial \phi_{II}}{\partial y} = \frac{1}{2} r^{-3/2} \cos\frac{\theta}{2} [12 - 6\nu + (-51 + 8\nu)\cos^2\frac{\theta}{2} + 84\cos^4\frac{\theta}{2} - 48\cos^6\frac{\theta}{2}] \quad (2.68)$$

$$\frac{\partial \psi_{II}}{\partial x} = \frac{1}{2} r^{-3/2} \cos\frac{\theta}{2} [6\nu - (35 + 8\nu)\cos^2\frac{\theta}{2} + 84\cos^4\frac{\theta}{2} - 48\cos^6\frac{\theta}{2}] \quad (2.69)$$

$$\frac{\partial \psi_{II}}{\partial y} = \frac{1}{2} r^{-3/2} \sin \frac{\theta}{2} [2\nu-(15+8\nu)\cos^2 \frac{\theta}{2} + $$

$$+ 60\cos^4 \frac{\theta}{2} -48\cos^6 \frac{\theta}{2} ] \qquad (2.70)$$

$$a_I = \frac{K_I}{G\sqrt{2\pi}} , \quad a_{II} = \frac{K_{II}}{G\sqrt{2\pi}} \qquad (2.71)$$

The contour integral on the circular arc $\Gamma_\varepsilon$ in (2.44) can now be written as:

$$- \int_{\Gamma_\varepsilon} [\bar{X}(\phi,\psi)u_I + \bar{Y}(\phi,\psi)v_I]ds -$$

$$- \int_{\Gamma_\varepsilon} [\bar{X}(\phi,\psi)u_{II} + \bar{Y}(\phi,\psi)v_{II}]ds$$

$$+ \int_{\Gamma_\varepsilon} [\bar{X}(u_I,v_I)\phi + \bar{Y}(u_I,v_I)\psi]ds +$$

$$+ \int_{\Gamma_\varepsilon} [\bar{X}(u_{II}, v_{II})\phi + \bar{Y}(u_{II},v_{II})\psi]ds. \qquad (2.72)$$

In order to extract the mode I stress intensity factor we substitute $\phi = \phi_I$ and $\psi = \psi_I$. It can now be seen that $\phi$, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \psi}{\partial y}$, $v_{II}$, $\frac{\partial u_{II}}{\partial y}$, $\frac{\partial v_{II}}{\partial x}$ and $n_x = -\cos\theta$ are symmetric with respect to the x axis whereas $\psi$, $\frac{\partial\phi}{\partial y}$ , $\frac{\partial\phi}{\partial x}$ , $u_{II}$, $\frac{\partial u_{II}}{\partial y}$ , $\frac{\partial v_{II}}{\partial x}$ and $n_y = -\sin\theta$ are antisymmetric with respect to the x axis. By virtue of (3.42) and (3.43) we conclude that $\overline{X}(\phi, \psi)$, $\overline{X}(u_I, v_I)$ and $\overline{Y}(u_{II}, v_{II})$ are symmetric whereas $\overline{Y}(\phi, \psi)$, $\overline{Y}(u_I, v_I)$ and $\overline{X}(u_{II}, v_{II})$ are antisymmetric with respect to the x axis. Since the expressions in the radius r disappear when integrating on the circular arc $\Gamma_\varepsilon$ and the domain of integration $(-\pi, \pi)$ is symmetric with respect to the x axis it follows that the integrals of $\overline{X}(\phi, \psi)u_{II}$, $\overline{Y}(\phi, \psi)v_{II}$, $\overline{X}(u_{II}, v_{II})\phi$, $\overline{Y}(u_{II}, v_{II})\psi$ vanish, these products being antisymmetric with respect to the x axis.

The remaining integrals are:

$$- \int_{\Gamma_\varepsilon} [\overline{X}(\phi, \psi)u_I + \overline{Y}(\phi, \psi)v_I]ds + \\ + \int_{\Gamma_\varepsilon} [\overline{X}(u_I, v_I)\phi + \overline{Y}(u_I, v_I)\psi]ds \qquad (2.73)$$

and these are the same as those considered in the derivation of the extraction formula for $K_I$.

In order to extract $K_{II}$ we substitute $\phi = \phi_{II}$, $\psi = \psi_{II}$. In this case $\psi$, $\frac{\partial\phi}{\partial y}$ , $\frac{\partial\psi}{\partial x}$ are the symmetric terms

# 3. EIGENVALUES AND EIGENFUNCTIONS FOR AN INFINITE NOTCH OF ARBITRARY SOLID ANGLE

Let us consider a two dimensional elastic body of infinite dimensions with a notch, the solid angle of which is equal to $2\alpha$ as shown in figure (3.1). We choose the vertex of the angle to be the center of the coordinate system. The bisector line of the solid angle is selected as the x axis of a Cartesian system (x, y) and the reference ($\theta = 0$) axis of the corresponding polar system (r, $\theta$) as shown in the figure. The faces of the angle are assumed to be free of tractions. We shall investigate the state of stress and deformation in the neighborhood of the apex. In the interest of completeness, all required equations are derived from first principles [15, 16].

Let us write the equations of equilibrium in the case where body forces are absent in the form:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad (3.1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \qquad (3.2)$$

In the usual way, let the stresses be defined in terms of the Airy stress function $\chi$ as:

$$\overline{X}(u_1,v_1) = r^{-1/2} X_1(\theta) \tag{2.127}$$

$$\overline{Y}(u_1,v_1) = r^{-1/2} Y_1(\theta). \tag{2.128}$$

this integral is equal to:

$$-r^{1/2} \int_{-\pi}^{\pi} [X_{\phi\psi}(\theta)G_1(\theta) + Y_{\phi\psi}H_1(\theta)]d\theta +$$

$$+ r^{1/2} \int_{-\pi}^{\pi} [X_1(\theta)\Phi(\theta) + Y_1(\theta)\Psi(\theta)]d\theta$$

$$= r^{-1/2} C_5 + r^{-1/2} C_6 = r^{-1/2} C_7 \tag{2.129}$$

The necessary and sufficient condition for this to have the same value for all values of r is that $C_7 = 0$, that is $L_1 = 0$.

$$- \int_{\Gamma_1+\Gamma_3^*} [\bar{X}(\phi,\psi)u_1 + \bar{Y}(\phi,\psi)v_1]ds +$$

$$+ \int_{\Gamma_1+\Gamma_3^*} [\bar{X}(u_1,v_1)\phi + \bar{Y}(u_1,v_1)\psi]ds = 0 \qquad (2.125)$$

therefore:

$$- \int_{\Gamma_3} [\bar{X}(\phi,\psi)u_1 + \bar{Y}(\phi,\psi)v1]ds +$$

$$+ \int_{\Gamma_3} [\bar{X}(u_1,v_1)\phi + \bar{Y}(u_1,v_1)\psi]ds$$

$$- \int_{\Gamma_3^*} [\bar{X}(\phi,\psi)u_1 + \bar{Y}(\phi,\psi)v_1]ds +$$

$$+ \int_{\Gamma_3^*} [\bar{X}(u_1,v_1)\phi + \bar{Y}(u_1,v_1)\psi]ds \qquad (2.126)$$

which means that the contour integral $L_1$ has the same value on all circular arcs $\Gamma_\varepsilon$. (In fact $L_1$ is path independent but it is convenient to consider only circular arcs here). By using (2.112), (2.113), (2.77), (2.80), (2.99), (2.100) and noting that

$$\int_A \int [L_x(\phi,\psi)u_1 + L_y(\phi,\psi)v_1]dA -$$

$$- \int_\Gamma [\overline{X}(\phi,\psi)u_1 + \overline{Y}(\phi,\psi)v_1]ds$$

$$+ \int_\Gamma [\overline{X}(u_1,v_1)\phi + \overline{Y}(u_1,v_1)\psi]ds = 0. \qquad (2.123)$$

The auxiliary functions $\phi$ and $\psi$ satisfy the equilibrium equations in the locality of the crack tip, therefore the area integral vanishes. Both the auxiliary functions and the eigenfunctions satisfy the traction free conditions on the crack surfaces, therefore the contour integrals vanish on the segments $\Gamma_2$ and $\Gamma_4$ of the boundary. Equation (2.123) then reduces to:

$$- \int_{\Gamma_1+\Gamma_3} [\overline{X}(\phi,\psi)u_1 + \overline{Y}(\phi,\psi)v_1]ds +$$

$$+ \int_{\Gamma_1+\Gamma_3} [\overline{X}(u_1,v_1)\phi + \overline{Y}(u_1,v_1)\psi]ds = 0. \qquad (2.124)$$

We now choose a larger annulus bounded by the circular arcs $\Gamma_1$ and $\Gamma_3^*$ and apply the same arguments. Then:

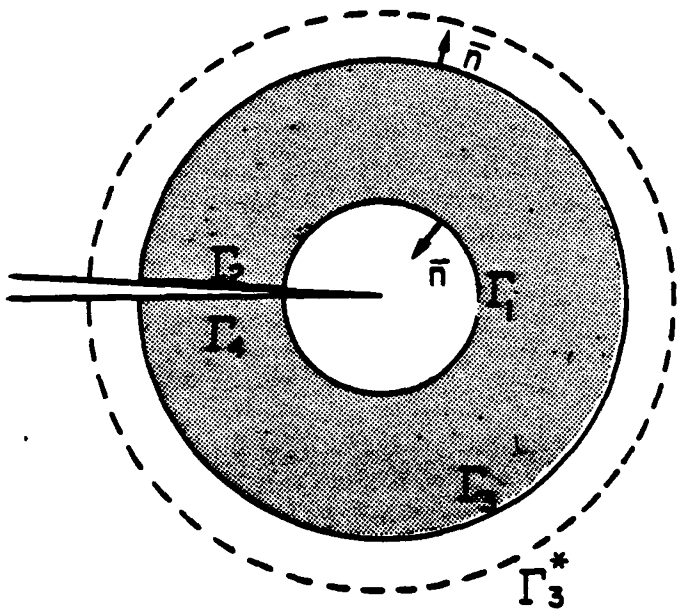


Figure 2.4

Annular ring with a slit

$$\int_{\Gamma_\varepsilon} [\bar{X}(u_3,v_3)\phi + \bar{Y}(u_3,v_3)\psi]ds =$$

$$= r^{1/2} \int_{-\pi}^{\pi} [X_3(\theta)\Phi(\theta) + Y_3(\theta)\Psi_3(\theta)]d\theta = r^{1/2} C_4. \tag{2.121}$$

It can be easily seen that in the limit as $r \to 0$ both expressions vanish.

The integral $L_1$ appears to be more troublesome as it will be of the form $r^{-1/2}$ C and this is unbounded as $r \to 0$. We shall now prove that this integral is zero.

Let us consider a two dimensional body in the form of an annulus with a slit, centered at the crack tip as shown in figure (2.4). The boundary of this body consists of the two circular arcs $\Gamma_1$ and $\Gamma_3$ and the straight segments, $\Gamma_2$ and $\Gamma_4$ on the crack surfaces. Since the eigenfunctions satisfy the equilibrium equations we can write

$$L_x(u_1,v_1)\phi + L_y(u_1,v_1)\psi = 0 \tag{2.122}$$

integrating over the domain and applying the Gauss theorem twice we obtain:

$$\int_{\Gamma_\varepsilon} [\overline{X}(u_2,v_2)\phi + \overline{Y}(u_2,v_2)\psi]ds =$$

$$= \int_{-\pi}^{\pi} [X_2(\theta)\Phi(\theta) + Y_2(\theta) \quad \Psi(\theta)]d\theta = C_2 \qquad (2.117)$$

which is again independent of r and can be readily computed.

In order to evaluate the integral $L_3$ on $\Gamma_\varepsilon$ we use (2.79), (2.82) and observe that:

$$\overline{X}(u_3,\ v_3) = r^{1/2}\ X_3(\theta) \qquad (2.118)$$

$$\overline{Y}(u_3,\ v_3) = r^{1/2}\ Y_3(\theta) \qquad (2.119)$$

therefore:

$$\int_{\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u_3 + \overline{Y}(\phi,\psi)v_3]ds =$$

$$= r^{1/2} \int_{-\pi}^{\pi} [X_{\phi\psi}(\theta)G_3(\theta) + Y_{\phi\psi}(\theta)H_3(\theta)]d\theta = r^{1/2}C_3 \qquad (2.120)$$

and:

Let us now examine each of these expressions separately. The integral $L_2$ will provide the required $K_2$. In view of (2.42), (2.43), (2.78), (2.81), (2.99), (2.100) we can write

$$\overline{X}(\phi,\psi) = r^{-2} X_{\phi\psi}(\theta) \tag{2.112}$$

$$\overline{Y}(\phi,\psi) = r^{-2} Y_{\phi\psi}(\theta) \tag{2.113}$$

$$\overline{X}(u_2,v_2) = r^{-1} X_2(\theta) \tag{2.114}$$

$$\overline{Y}(u_2,v_2) = r^{-1} Y_2(\theta) \tag{2.115}$$

therefore, since $ds = r\, d\theta$:

$$\int_{\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u_2 + \overline{Y}(\phi,\psi)v_2]ds =$$

$$= \int_{-\pi}^{\pi} [X_{\phi\psi}(\theta)\; G_2(\theta) + Y_{\phi\psi}(\theta)\; H_2(\theta)]d\theta = C_1. \tag{2.116}$$

This last integral is independent of r and can be computed either analytically or numerically. Similarly:

$$- \int_{\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u + \overline{Y}(\phi,\psi)v]ds +$$

$$+ \int_{\Gamma_\varepsilon} [\overline{X}(u,v)\phi + \overline{Y}(u,v)\psi]ds = L_1 + L_2 + L_3 \quad (2.108)$$

where:

$$L_1 = -K_1 \int_{\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u_1 + \overline{Y}(\phi,\psi)v_1]ds +$$

$$+ K_1 \int_{\Gamma_\varepsilon} [\overline{X}(u_1,v_1)\phi + \overline{Y}(u_1,v_1)\psi]ds \quad (2.109)$$

$$L_2 = -K_2 \int_{\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u_2 + \overline{Y}(\phi,\psi)v_2]ds +$$

$$+ K_2 \int_{\Gamma_\varepsilon} [\overline{X}(u_2,v_2)\phi + \overline{Y}(u_2,v_2)\psi]ds \quad (2.110)$$

$$L_3 = -K_3 \int_{\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u_3 + \overline{Y}(\phi,\psi)v_3]ds +$$

$$+ K_3 \int_{\Gamma_\varepsilon} [\overline{X}(u_3,v_3)\phi + \overline{Y}(u_3,v_3)\psi]ds \quad (2.111)$$

Let us further introduce the notation $L_x(u, v)$ and $L_y(u, v)$ for the differential operators of equilibrium in the x and y directions:

$$L_x(u,v) = (\lambda+G) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + G\left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.105)$$

$$L_y(u,v) = (\lambda+G) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + G\left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \quad (2.106)$$

Equation (2.44) can now be written:

$$\int_A \int [L_x(\phi,\psi)u + L_y(\phi,v)v]dA -$$

$$- \int_{\Gamma+\Gamma_\varepsilon} [\overline{X}(\phi,\psi)u + \overline{Y}(\phi,\psi)v]ds \quad (2.107)$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [\overline{X}(u,v)\phi + \overline{Y}(u,v)\psi]ds = 0.$$

At this point we are interested only in the contour integrals around the circular arc $\Gamma_\varepsilon$. The other integrals will not be affected as the asymptotic expansions are not valid there. The integrals on $\Gamma_\varepsilon$ can be written as:

$$v_{31}(r,\theta) = r^{1/2} H_{31}(\theta) \tag{2.97}$$

$$v_{32}(r,\theta) = r^{1/2} H_{32}(\theta). \tag{2.98}$$

The intensity factors $K_i$ and the eigenfunctions $u_i$, $v_i$ may correspond to either mode I or mode II. The intensity factor $K_2$ corresponding to the second term in the expansion will now be extracted. To this end the auxiliary functions are chosen to be of the form:

$$\phi = r^{-1} \Phi(\theta) \tag{2.99}$$

$$\psi = r^{-1} \Psi(\theta). \tag{2.100}$$

Their derivatives can be written in the form:

$$\frac{\partial\phi}{\partial x} = r^{-2} \Phi_1(\theta) \tag{2.101}$$

$$\frac{\partial\phi}{\partial y} = r^{-2} \Phi_2(\theta) \tag{2.102}$$

$$\frac{\partial\psi}{\partial x} = r^{-2} \Psi_1(\theta) \tag{2.103}$$

$$\frac{\partial\psi}{\partial y} = r^{-2} \Psi_2(\theta). \tag{2.104}$$

$$\frac{\partial v}{\partial y} = K_1\ v_{21}(r,\theta)\ +\ K_2\ v_{22}(r,\theta)\ +\ K_3\ v_{32}(r,\theta)\ +\ \ldots \tag{2.86}$$

where:

$$u_{11}(r,\theta) = r^{-1/2}\ G_{11}(\theta) \tag{2.87}$$

$$u_{12}(r,\theta) = r^{-1/2}\ G_{12}(\theta) \tag{2.88}$$

$$u_{21}(r,\theta) = u_{21}(\theta) = G_{21}(\theta) \tag{2.89}$$

$$u_{22}(r,\theta) = u_{22}(\theta) = G_{22}(\theta) \tag{2.90}$$

$$u_{31}(r,\theta) = r^{1/2}\ G_{31}(\theta) \tag{2.91}$$

$$u_{32}(r,\theta) = r^{1/2}\ G_{32}(\theta) \tag{2.92}$$

$$v_{11}(r,\theta) = r^{-1/2}\ H_{11}(\theta) \tag{2.93}$$

$$v_{12}(r,\theta) = r^{-1/2}\ H_{12}(\theta) \tag{2.94}$$

$$v_{21}(r,\theta) = v_{21}(\theta) = H_{21}(\theta) \tag{2.95}$$

$$v_{22}(r,\theta) = v_{22}(\theta) = H_{22}(\theta) \tag{2.96}$$

$$u_1(r,\theta) = r^{1/2}\, G_1(\theta) \tag{2.77}$$

$$u_2(r,\theta) = r\, G_2(\theta) \tag{2.78}$$

$$u_3(r,\theta) = r^{3/2}\, G_3(\theta) \tag{2.79}$$

$$v_1(r,\theta) = r^{1/2}\, H_1(\theta) \tag{2.80}$$

$$v_2(r,\theta) = r\, H_2(\theta) \tag{2.81}$$

$$v_3(r,\theta) = r^{3/2}\, H_3(\theta). \tag{2.82}$$

Their derivatives can be written in the form:

$$\frac{\partial u}{\partial x} = K_1\, u_{11}(r,\theta) + K_2\, u_{21}(r,\theta) + K_3\, u_{31}(r,\theta) + \ldots \tag{2.83}$$

$$\frac{\partial u}{\partial y} = K_1\, u_{12}(r,\theta) + K_2\, u_{22}(r,\theta) + K_3\, u_{32}(r,\theta) + \ldots \tag{2.84}$$

$$\frac{\partial v}{\partial x} = K_1\, v_{11}(r,\theta) + K_2\, v_{21}(r,\theta) + K_3\, v_{31}(r,\theta) + \ldots \tag{2.85}$$

and $\phi$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\psi}{\partial y}$ are the antisymmetric terms. It follows that $\overline{X}(\phi, \psi)$ is now antisymmetric and $\overline{Y}(\phi, \psi)$ is symmetric. The integrands that now cancel are: $\overline{X}(\phi, \psi)u_I$, $\overline{Y}(\phi, \psi)v_I$, $\overline{X}(u_I, v_I)\phi$ and $\overline{Y}(u_I, v_I)\psi$. The remaining integrals are:

$$- \int_{\Gamma_\varepsilon} [\overline{X}(\phi, \psi)u_{II} + \overline{Y}(\phi, \psi)v_{II}]ds + $$
$$+ \int_{\Gamma_\varepsilon} \overline{X}(u_{II}, v_{II})\phi + \overline{Y}(u_{II}, v_{II})\psi]ds \qquad (2.74)$$

which are precisely those considered in deriving the formula for $K_{II}$.

Another advantage of the extraction technique is the ability to extract the intensity factors corresponding to higher order terms in the asymptotic expansions. Let us write the asymptotic expansions for the displacements in the form:

$$u = K_1 u_1(r, \theta) + K_2 u_2(r, \theta) + K_3 u_3(r, \theta) + \ldots \qquad (2.75)$$

$$v = K_1 v_1(r, \theta) + K_2 v_2(r, \theta) + K_3 v_3(r, \theta) + \ldots \qquad (2.76)$$

where:

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2} \tag{3.3}$$

$$\sigma_y = \frac{\partial^2 \chi}{\partial x^2} \tag{3.4}$$

$$\tau_{xy} = - \frac{\partial^2 \chi}{\partial x \partial y} \tag{3.5}$$

It can be easily verified that the equilibrium equations are identically satisfied. The stress-strain relations in the case of plane strain can be written in terms of the displacements as:

$$\sigma_x = \lambda\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2G \frac{\partial u}{\partial x} \tag{3.6}$$

$$\sigma_y = \lambda\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2G \frac{\partial v}{\partial y} \tag{3.7}$$

$$T_{xy} = G\left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{3.8}$$

Adding (3.6) and (3.7) yields:

$$\sigma_x + \sigma_y = 2(\lambda + G) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \tag{3.9}$$

Combining (3.6) and (3.9) and putting $\frac{\lambda}{2(\lambda+G)} = \nu$ we obtain:

$$2G \frac{\partial u}{\partial x} = \sigma_x - \nu(\sigma_x + \sigma_y) \tag{3.10}$$

Analogously:

$$2G \frac{\partial v}{\partial y} = \sigma_y - \nu(\sigma_x + \sigma_y) \tag{3.11}$$

In the case of plane stress we can write:

$$\sigma_x = \lambda\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial u}{\partial x} \tag{3.12}$$

$$\sigma_y = \lambda\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial v}{\partial y} \tag{3.13}$$

$$0 = \lambda\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial w}{\partial z} \tag{3.14}$$

from which by summation:

$$\sigma_x + \sigma_y = (3\lambda + 2G)\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \tag{3.15}$$

and by substituting back for $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ and putting $\frac{\lambda}{3\lambda+2G} = \frac{\nu}{1+\nu}$ we obtain:

$$2G \frac{\partial u}{\partial x} = \sigma_x - \frac{\nu}{1+\nu} (\sigma_x + \sigma_y) \tag{3.16}$$

$$2G \frac{\partial v}{\partial y} = \sigma_x - \frac{\nu}{1+\nu} (\sigma_x + \sigma_y). \tag{3.17}$$

We define σ as:

$$\sigma = \begin{cases} \nu & \text{for plane strain} \\ \dfrac{\nu}{1+\nu} & \text{for plane stress} \end{cases} \tag{3.18}$$

and combine (3.10), (3.11), (3.16) and (3.17) to obtain:

$$2G\,\frac{\partial u}{\partial x} = \sigma_x - \sigma(\sigma_x + \sigma_y) \tag{3.19}$$

$$2G\,\frac{\partial v}{\partial y} = \sigma_y - \sigma(\sigma_x + \sigma_y). \tag{3.20}$$

We now differentiate (3.19) twice with respect to y, differentiate (3.20) twice with respect to x, add them and use (3.8) to obtain:

$$2\,\frac{\partial^2\tau_{xy}}{\partial x\partial y} = \frac{\partial^2\sigma x}{\partial y^2} + \frac{\partial^2\sigma y}{\partial x^2} - \sigma\nabla^2(\sigma_x + \sigma_y). \tag{3.21}$$

Finally, substituting (3.3), (3.4), (3.5) into (3.21) and dividing by $(1 - \sigma)$ we obtain:

$$\nabla^4\chi = 0. \tag{3.22}$$

We now write:

$$\sigma_x + \sigma_y = \nabla^2\chi = \frac{\partial^2\psi}{\partial x\partial y}\,. \tag{3.23}$$

This equation defines the function $\psi$ in terms of the stress function $\chi$ up to the functions of integration. Equations (3.19) and (3.20) can be written in the form:

$$2G \frac{\partial u}{\partial x} = -\sigma_y + (1 - \sigma)(\sigma_x + \sigma_y) \tag{3.24}$$

$$2G \frac{\partial v}{\partial y} = -\sigma_x + (1 - \sigma)(\sigma_x + \sigma_y) \tag{3.25}$$

and by use of (3.23) they become:

$$2G \frac{\partial u}{\partial x} = - \frac{\partial^2 \chi}{\partial x^2} + (1 - \sigma) \frac{\partial^2 \psi}{\partial x \partial y} \tag{3.26}$$

$$2G \frac{\partial v}{\partial y} = - \frac{\partial^2 \chi}{\partial y^2} + (1 - \sigma) \frac{\partial^2 \psi}{\partial x \partial y} \tag{3.27}$$

These can now be integrated to give:

$$2G\ u = - \frac{\partial \chi}{\partial x} + (1 - \sigma) \frac{\partial \psi}{\partial y} \tag{3.28}$$

$$2G\ v = - \frac{\partial \chi}{\partial y} + (1 - \sigma) \frac{\partial \psi}{\partial x} \tag{3.29}$$

where u and v are known up to possible rigid body motion.

These displacements u and v were derived on the basis of equations (3.6) and (3.7). The third equation of equilibrium (3.8) also has to be satisfied which imposes an one more condition that the function $\psi$ has to satisfy. From (3.8) and (3.5):

$$G\left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau_{xy} = - \frac{\partial^2 \chi}{\partial x \partial y} \tag{3.30}$$

Differentiating (3.28) and (3.29) and adding:

$$2\, \tau_{xy} = -2 \frac{\partial^2 \chi}{\partial x \partial y} + (1 - \sigma)\, \nabla^2 \psi \tag{3.31}$$

and in view of (3.30):

$$\nabla^2 \psi = 0. \tag{3.32}$$

The question now arises whether it is always possible to find a function $\psi$ which is harmonic and related to the stress function by (3.23). By taking Laplacians of both sides of (3.23) we obtain:

$$\nabla^4 \chi = \nabla^2 (\nabla^2 \chi) = \nabla^2 \left( \frac{\partial^2 \chi}{\partial x \partial y} \right) = \frac{\partial^2}{\partial x \partial y} (\nabla^2 \psi) = 0. \tag{3.33}$$

We now separate the terms that are independent of x and the terms that are independent of y by writing $\psi$ as:

$$\psi = f_1(x) + f_2(y) + w(x, y) \tag{3.34}$$

from which:

$$\nabla^2 \psi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = f_1''(x) + f_2''(y) + \nabla^2 w(x, y) \tag{3.35}$$

and by (3.33):

$$\frac{\partial^2}{\partial x \partial y} (\nabla^2 \psi) = \frac{\partial^2}{\partial x \partial y} (\nabla^2 w) = 0. \tag{3.36}$$

From which:

$$\frac{\partial}{\partial x} [ \frac{\partial}{\partial y} \nabla^2 w(u, y)] = 0 \tag{3.37}$$

consequently, the expression in the bracket is independent of x:

$$\frac{\partial}{\partial y} \nabla^2 w(x, y) = g_1(y) \tag{3.38}$$

Therefore, $\nabla^2 w(x, y)$ is itself independent of x:

$$\nabla^2 w(x, y) = \nabla^2 w(y) = g_2(y). \tag{3.39}$$

Interpreting now the mixed derivative in the reverse order:

$$\frac{\partial}{\partial y} [\frac{\partial}{\partial x} \nabla^2 w(x, y)] = 0 \tag{3.40}$$

Therefore, the expression in the bracket in independent of y:

$$\frac{\partial}{\partial x} \nabla^2 w(x, y) = g_3(x) \tag{3.41}$$

from which $\nabla^2 w(x, y)$ is itself independent of y:

$$\nabla^2 w(x, y) = \nabla^2 \quad (x) = g_4(x) \tag{3.42}$$

From (3.39), (3.42) and noting that constant terms have been incorporated in $f_1(x)$ and $f_2(y)$ it follows that:

$$\nabla^2 w(x, y) = 0 \tag{3.43}$$

or

$$\nabla^2 [\psi(x, y) - f_1(x) - f_2(y)] = 0 \tag{3.44}$$

which means that the function $\psi$ can always be adjusted by means of functions of x only and functions of y only to become harmonic. These functions are the integration functions needed to obtain $\psi$ from $\chi$. There are no other requirements that $\psi$ has to satisfy, therefore, it is always possible to find such a function.

Stress functions in polar coordinates

Expressions for the stress components in polar coordinates in terms of the stress function $\chi$ can be derived as follows:

Noting (3.3) and using the stress transformation law we obtain:

$$\sigma_x = \sigma_r \cos^2\theta - 2T_{r\theta}\cos\theta\sin\theta + \sigma_\theta \sin^2\theta = \frac{\partial^2 \chi}{\partial y^2} \tag{3.45}$$

transforming $\frac{\partial^2 \chi}{\partial y^2}$ into polar coordinates:

$$\frac{\partial \chi}{\partial y} = \sin\theta \frac{\partial \chi}{\partial r} + \frac{\cos\theta}{r} \frac{\partial \chi}{\partial \theta} \tag{3.46}$$

$$\frac{\partial^2 \chi}{\partial y^2} = (\sin\theta \frac{\partial}{\partial r} + \frac{\cos}{r} \frac{\partial}{\partial \theta})(\sin\theta \frac{\partial \chi}{\partial r} + \frac{\cos}{r} \frac{\partial \chi}{\partial \theta})$$

$$= \cos^2\theta(\frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \chi}{\partial r}) + 2\sin\theta\cos\theta \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \chi}{\partial \theta})$$

$$+ \sin^2\theta \frac{\partial^2 \chi}{\partial r^2} \tag{3.47}$$

comparing the coefficients of like trigonometric terms we obtain:

$$\sigma_r = \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} \tag{3.48}$$

$$\tau_{r\theta} = - \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \chi}{\partial \theta}) = - \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta \partial r} + \frac{1}{r^2} \frac{\partial \chi}{\partial \theta} \tag{3.49}$$

$$\sigma_\theta = \frac{\partial^2 \chi}{\partial r^2} \tag{3.50}$$

and it can be easily seen that:

$$\sigma_r + \sigma_\theta = \nabla^2 \chi \tag{3.51}$$

The stress-strain relations of two-dimensional elasticity in polar coordinates can be written concisely as:

$$\varepsilon_r = \frac{1}{2G}\left[-\sigma_\theta + (1 - \sigma)(\sigma_r + \sigma_\theta)\right] \tag{3.52}$$

$$\varepsilon_\theta = \frac{1}{2G}\left[-\sigma_r + (1 - \sigma)(\sigma_r + \sigma_\theta)\right] \tag{3.53}$$

with $\sigma$ given by (3.18).

We now introduce the function $\psi_1$ which is defined (up to the functions of integration) as:

$$\nabla^2\chi = \frac{\partial}{\partial r}\left(r\,\frac{\partial\psi_1}{\partial\theta}\right) \tag{3.54}$$

Taking into account (3.50) and (3.52) the expression for $\varepsilon_r$ can be written as:

$$\frac{\partial u_r}{\partial r} = \frac{1}{2G}\left[-\frac{\partial^2\chi}{\partial r^2} + (1 - \sigma)\,\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi_1}{\partial\theta}\right)\right] \tag{3.55}$$

integrating with respect to r:

$$u_r = \frac{1}{2G}\left[-\frac{\partial\chi}{\partial r} + (1 - \sigma)\, r\,\frac{\partial\psi_1}{\partial\theta}\right] \tag{3.56}$$

where $\psi_1$ now incorporates the integration function.

In view of (3.54) and (3.51) we can write:

$$\sigma_r + \sigma_\theta = \frac{\partial}{\partial r} (r \frac{\partial \psi_1}{\partial \theta}) = \frac{\partial \psi_1}{\partial \theta} + \frac{\partial^2 \psi}{\partial r \partial \theta} = \frac{\partial}{\partial \theta} (\psi_1 + r \frac{\partial \psi_1}{\partial r}) \tag{3.57}$$

and expression (3.53) for $\varepsilon_\theta$ can be written as:

$$\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{2G} [- \frac{1}{r}\frac{\partial \chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2} + $$

$$+ (1 - \sigma) \frac{\partial}{\partial \theta} (\psi_1 + r \frac{\partial \psi_1}{\partial r})] \tag{3.58}$$

Multiplying through by 2Gr and substituting for $u_r$ from (3.56) gives after some cancellation of some terms:

$$2G \frac{\partial u_\theta}{\partial \theta} = - \frac{1}{r}\frac{\partial^2 \chi}{\partial \theta^2} + (1 - \sigma) r^2 \frac{\partial^2 \psi_1}{\partial r \partial \theta} \tag{3.59}$$

integrating with respect to $\theta$:

$$u_\theta = \frac{1}{2G} [- \frac{1}{r}\frac{\partial \chi}{\partial \theta} + (1 - \sigma) r^2 \frac{\partial \psi_1}{\partial r}] \tag{3.60}$$

where once again the function of integration has been incorporated in $\psi_1$.

So far only the radial and circumferential components of stress and strain have been considered. The consideration of the stress-strain and strain-displacement relations for shear stress and strain will lead to an extra condition that the function $\psi_1$ must satisfy. From:

$$\tau_{r\theta} = G\gamma_{r\theta} \tag{3.61}$$

and:

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{3.62}$$

Substituting for $u_r$ and $u_\theta$ from (3.56) and (3.60) after some manipulation we obtain:

$$\gamma_{r\theta} = \frac{1}{G}\left[-\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial\theta} + \frac{1}{2}r^2(1-\sigma)\left(\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} + \right.\right.$$

$$\left.\left. + \frac{1}{r}\frac{\partial\psi_1}{\partial r} + \frac{\partial^2\psi_1}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial\chi}{\partial\theta}\right]$$

or

$$\gamma_{r\theta} = \frac{1}{G}\left[-\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial\theta} + \frac{r^2}{2}(1-\sigma)\nabla^2\psi_1 + \frac{1}{r^2}\frac{\partial\chi}{\partial\theta}\right] \tag{3.63}$$

Comparing with (3.56) and (3.61) we deduce that:

$$\nabla^2\psi_1 = 0 \tag{3.64}$$

The solution of the two-dimensional elasticity problem then reduces to the determination of a biharmonic function $\chi$ and a harmonic function $\psi_1$ related to it through (3.54).

Eigenvalues and eigenfunctions for the notch problem

We now seek the general solution in the neighborhood of an angular point. We shall investigate solutions of the form

$$\chi = r^{\kappa+1} F(\theta) \tag{3.65}$$

Then since

$$\nabla^4\chi = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \right.$$

$$\left. + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\chi = 0$$

we obtain:

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) r^{\kappa+1} F(\theta) =$$

$$= r^{\kappa-1}[(\kappa+1)^2 F(\theta) + F''(\theta)] \tag{3.66}$$

and:

$$\nabla^4\chi = r^{\kappa-3}[F^{iv}(\theta)+2(\kappa^2+1)F''(\theta)+(\kappa-1)^2(\kappa+1)^2 F(\theta)] \tag{3.67}$$

from which:

$$F^{iv}(\theta) + 2(\kappa^2+1)F''(\theta) + (\kappa-1)^2(\kappa+1)^2F(\theta) = 0 \quad (3.68)$$

Let

$$F(\theta) = a\ e^{m\theta} \quad (3.69)$$

The characteristic equation becomes:

$$m^4 + 2(\kappa^2+1)m^2 + (\kappa-1)^2(\kappa+1)^2 = 0 \quad (3.70)$$

which has the roots:

$$m = \pm (\kappa + 1)i$$

$$m = \pm (\kappa - 1)i \quad (3.71)$$

these roots leading to solutions of the form:

$$\sin(\kappa+1)\theta,\ \cos(\kappa+1)\theta,\ \sin(\kappa-1)\theta,\ \cos(\kappa-1)\theta$$

and the general solution can then be written in the form:

$$\chi = r^{\kappa+1}[C_1\sin(\kappa+1)\theta+C_2\cos(\kappa+1)\theta+$$

$$+ C_3\sin(\kappa-1)\theta+C_4\cos(\kappa-1)\theta] \quad (3.72)$$

At this point it should be noted that the above form for $\chi$ satisfies the biharmonic equation for any value of $\kappa$, real or complex.

In order to determine $\psi_1$ we investigate solutions of the form:

$$\psi_1 = r^m G(\theta) \tag{3.73}$$

Since $\psi_1$ is harmonic, from

$$\left( \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \psi_1 = 0$$

we obtain:

$$G''(\theta) + m^2 G(\theta) = 0 \tag{3.74}$$

Let

$$G = be^{n\theta} \tag{3.75}$$

then:

$$n^2 + m^2 = 0 \tag{3.76}$$

from which:

$$n = \pm mi \tag{3.77}$$

$G_2(r, \theta)$. Thus a complex root of the eigenequation gives rise to two distinct eigenfunctions. This can also be seen if we write the stress intensity factor in complex form and write the term corresponding to (3.140) as:

$$u = (K_1 + iK_2)\ r^{\kappa_1}[G_1(r,\theta) + iG_2(r,\theta)] \qquad (3.141)$$

separating real and imaginary terms:

$$u = K_1\ r^{\kappa_1}\ G_1(r,\theta) - K_2\ r^{\kappa_1}\ G_2(r,\theta) +$$

$$+\ i\ r^{\kappa_1}[K_1\ G_2(r,\theta) + K_2\ G_1(r,\theta)] \qquad (3.142)$$

Both the real and the imaginary parts of this expression are acceptable solutions. However, we observe that the imaginary part contains the same eigenfunctions $G_1(r, \theta)$ $G_2(r, \theta)$ as the real part. Therefore, to a complex eigenvalue there correspond two distinct eigenfunctions, each with its own stress intensity factor. They have a singularity of the same strength $r^{\kappa_1}$. If the complex number in (3.136) is an eigenvalue, its complex conjugate will also be an eigenvalue. To this eigenvalue there correspond two distinct eigenfunctions $G_3(r, \theta)$, $G_4(r, \theta)$, both having a singularity of the same strength $r^{\kappa_1}$. We then conclude that to a pair of complex conjugate

It is seen that only the real part of the complex root determines the strength of the singularity, the imaginary part being incorporated in the eigenequation. The expressions in the bracket in the eigenequations (3.130), (3.121) of mode I and (3.130), (3.131) of mode II will also be complex. Let us write (3.120) for example in the form:

$$u = r^{\kappa}[F_1(\theta) + i\ F_2(\theta)] \tag{3.138}$$

In view of (3.148) this can be written as:

$$u = r^{\kappa_1}[F_1(\theta)\cos(\kappa_2 \ell nr) - F_2(\theta)\sin(\kappa_2 \ell nr)$$

$$+ i(F_2(\theta)\cos(\kappa_2 \ell nr) + F_1(\theta)\sin(\kappa_2 \ell nr))] \tag{3.139}$$

The terms $\cos(\kappa_2 \ell nr)$ and $\sin(\kappa_2 \ell nr)$ are highly oscillatory in the neighborhood of the notch tip, the frequency of oscillations tending to infinity as the radius r approaches zero. Let us now write (3.139) in the form:

$$u = r^{\kappa_1}[G_1(r,\theta) + i\ G_2(r,\theta)] \tag{3.140}$$

There will be a stress intensity factor corresponding to the real part $G_1(r, \theta)$ of the eigenfunction and a different one corresponding to the imaginary part

$$2GC(2\pi)^{1/2}v = K_I^* \, r^{\kappa}[(CB+\kappa+1)\sin\kappa\theta-2\kappa C\sin\theta\cos(\kappa-1)\theta] \quad (3.133)$$

and for mode II:

$$2GC(2\pi)^{1/2}u = K_{II}^* \, r^{\kappa}[(CD-\kappa-1)\cos\kappa\theta+2\kappa C\sin\theta\cos(\kappa-1)\theta] \quad (3.134)$$

$$2GC(2\pi)^{1/2}v = K_{II}^* \, r^{\kappa}[-(CD+\kappa+1)\cos\kappa\theta-2\kappa C\cos\theta\cos(\kappa-1)\theta] \quad (3.135)$$

It is reminded that the eigenvalue $\kappa$ and the constant C are different for each mode.

Behavior of the roots of the eigenequations. The case of complex eigenvalues.

The above derivation is valid for complex values of the eigenvalue $\kappa$ as well. Let us write the complex roots in the form:

$$\kappa = \kappa_1 + i\,\kappa_2 \quad (3.136)$$

With this notation, the singular term can be written as:

$$r^{\kappa} = r^{\kappa_1} \cos(\kappa_2 \ell n r) + i \sin(\kappa_2 \ell n r) \quad (3.137)$$

These are transformed into Cartesian coordinates and after trigonometric transformations they are written in the form:

$$2Gu = r^{\kappa}[(CD-\kappa-1)\sin\kappa\theta+2\kappa C\sin\theta\cos(\kappa-1)\theta] \qquad (3.130)$$

$$2Gv = r^{\kappa}[-(CD+\kappa+1)\cos\kappa\theta-2\kappa C\cos\theta\cos(\kappa-1)\theta] \qquad (3.131)$$

where D is given by (3.128) and C is the ratio in (3.100).

The eigenfunctions (3.120), (3.121) of mode I and (3.130), (3.131) of mode II are determined within a multiplicative constant: any multiple of these eigenfunctions will also be an eigenfunction. This multiplicative constant is chosen to agree with the one adopted in engineering literature on Linear Elastic Fracture Mechanics. The generalized stress intensity factors, which are the amplitudes of the singular terms in the displacement and stress expansions in the neighborhood of the reentrant corner, are then defined to coincide with Irwin's definition [13] of $K_1$ and $K_{II}$ by writing for mode I:

$$2GC(2\pi)^{1/2}u = K_I^* \, r^{\kappa}[(CB-\kappa-1)\cos\kappa\theta-2\kappa C\cos\theta\cos(\kappa-1)\theta] \qquad (3.132)$$

$$F(\theta) = \sin(\kappa+1)\theta + C\sin(\kappa-1)\theta \tag{3.122}$$

$$G(\theta) = -\frac{4}{\kappa-1} C\cos(\kappa-1)\theta \tag{3.123}$$

from which:

$$F'(\theta) = (\kappa+1)\cos(\kappa+1)\theta + C(\kappa-1)\cos(\kappa-1)\theta \tag{3.124}$$

$$G'(\theta) = 4C\sin(\kappa+1)\theta \tag{3.125}$$

and the displacements for mode II can be written as:

$$2Gu_r = r^\kappa[-(\kappa+1)\sin(\kappa+1)\theta+CD\sin(\kappa-1)\theta] \tag{3.126}$$

$$2Gu_\theta = r^\kappa[-(\kappa+1)\cos(\kappa+1)\theta-CE\cos(\kappa-1)\theta] \tag{3.127}$$

where:

$$D = 4(1-\sigma) - \kappa - 1 \tag{3.128}$$

$$E = 4(1-\sigma) + \kappa - 1 \tag{3.129}$$

$$u = \cos\theta \; u_r - \sin\theta \; u_\theta \tag{3.116}$$

$$v = \sin\theta \; u_r + \cos\theta \; u_\theta \tag{3.117}$$

to be:

$$2Gu = r^\kappa[-(\kappa+1)\cos\kappa\theta + CA\cos\theta\cos(\kappa-1)\theta$$

$$- CB\sin\theta\sin(\kappa-1)\theta] \tag{3.118}$$

$$2Gv = r^\kappa[(\kappa+1)\sin\kappa\theta + CA\sin\theta\cos(\kappa-1)\theta + CB\cos\theta\sin(\kappa-1)\theta] \tag{3.119}$$

Through trigonometric transformations these can be written in the form:

$$2Gu = r^\kappa[(CB-\kappa-1)\cos\kappa\theta - 2\kappa C\cos\theta\cos(\kappa-1)\theta] \tag{3.120}$$

$$2Gv = r^\kappa[(CB+\kappa+1)\sin\theta - 2\kappa C\sin\theta\cos(\kappa-1)\theta] \tag{3.121}$$

where B is given by (3.115) and C is the ratio in (3.102).

Mode II

In a similar way by taking $C_1 = 1$ and calling the ratio $C_3/C_1 = C$ in (3.100) the stress functions can be written as:

and:

$$G(\theta) = \frac{4}{\kappa-1} C \sin(\kappa-1)\theta \tag{3.109}$$

from which:

$$F'(\theta) = -(\kappa+1)\sin(\kappa+1)\theta - C(\kappa-1)\sin(\kappa-1)\theta \tag{3.110}$$

$$G'(\theta) = 4C \cos(\kappa-1)\theta \tag{3.111}$$

using (3.82) and (3.83) we can write the displacements for mode I as:

$$2G\, u_r = r^{\kappa}[-(\kappa+1)\cos(\kappa+1)\theta + CA\cos(\kappa-1)\theta] \tag{3.112}$$

$$2G\, u_{\theta} = r^{\kappa}[(\kappa+1)\sin(\kappa+1)\theta + CB\sin(\kappa-1)\theta] \tag{3.113}$$

where:

$$A = 4(1-\sigma) - (\kappa+1) \tag{3.114}$$

$$B = 4(1-\sigma) + \kappa-1 \tag{3.115}$$

The displacements u and v in Cartesian coordinates can now be found through the transformation law:

In the case of a crack ($\alpha = 180°$), the first positive eigenvalue $\kappa = 1/2$ is of great significance in linear elastic fracture mechanics, where the symmetric mode is termed mode I and the antisymmetric mode is termed mode II. In the sequel this terminology will be extended to the case of a notch of an arbitrary angle. In the crack case the two modes have the same eigenvalue but the eigenfunctions corresponding to them are different. In all other cases (except the half plane case as we saw) the two modes have different eigenvalues. This means that whenever $C_1$ and $C_3$ are nonzero then $C_2$ and $C_4$ must be zero and vice-versa.

It is important to note that in either mode I or mode II if $\kappa$ is an eigenvalue, so is $-\kappa$. This is very important for the formulation of the generalized influence function method, where use is made of the eigenfunctions corresponding to these negative eigenvalues. The negative roots yield unbounded displacements at the vertex of the angle, so they do not have physical significance.

Expressions for the displacements

Mode I

By taking $C_2 = 1$ and calling the ratio $C_4/C_2 = C$ in (3.102) we can write:

$$F(\theta) = \cos(\kappa+1)\theta + C\cos(\kappa-1)\theta \qquad (3.108)$$

hold simultaneously. Excluding the meaningless case of $\alpha = 0$, these are satisfied either when:

$$\alpha = 90° \text{ and } \kappa = n \quad \text{(half plane)} \tag{3.105}$$

or:

$$\alpha = 180° \text{ and } \kappa = \frac{n}{2} \quad \text{(crack case)} \tag{3.106}$$

with n = integer.

In the case of the half plane $C_1 = 0$ and $C_3$ is undetermined. The triviality of the case $\kappa = 0$ can be seen by expressing $\sigma_r$ in terms of $\kappa$. Making use of (3.50) and (3.72) we obtain:

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} = r^{\kappa-1}[F''(\theta)+(\kappa+1)F(\theta)] \tag{3.107}$$

and for $\kappa = 0$ it can be verified that the expression in the bracket becomes zero. From (3.84) and (3.85) it is obvious that $\sigma_\theta$ and $\tau_{r\theta}$ are also zero, and from (3.82) and (3.83) we can see that the displacements do not depend on the radius r, therefore, we have a state of rigid body motion. It is also of interest to note that for the antisymmetric case (equation 3.99), the value $\kappa = 1$ is a solution for any notch angle $\alpha$.

$$\frac{C_3}{C_1} = - \frac{\sin(\kappa+1)\alpha}{\sin(\kappa-1)\alpha} = - \frac{(\kappa+1)\cos(\kappa+1)\alpha}{(\kappa-1)\cos(\kappa-1)\alpha} \tag{3.100}$$

but one of them, say $C_1$, can be assigned arbitrarily. Conveniently we choose $C_1 = 1$.

Similarly the other two equations (3.96) and (3.97) yield the eigenequation:

$$\sin 2\alpha + \sin 2\kappa\alpha = 0. \tag{3.101}$$

Again for every root of this equation at least one of $C_2$, $C_4$ is nonzero. One of them can be given an arbitrary value but their ratio is determined from either (3.96) or (3.97) as:

$$\frac{C_4}{C_2} = - \frac{\cos(\kappa+1)\alpha}{\cos(\kappa-1)\alpha} = - \frac{(\kappa+1)\sin(\kappa+1)\alpha}{(\kappa-1)\sin(\kappa-1)\alpha} \tag{3.102}$$

Examining equations (3.99) and (3.101) together it is seen that they are satisfied simultaneously either in the trivial case $\kappa = 0$, or when:

$$\sin 2\kappa\alpha = 0 \tag{3.103}$$

and:

$$\sin 2\alpha = 0 \tag{3.104}$$

$$(\kappa+1)C_2\sin(\kappa+1)\alpha + (\kappa-1)C_1\sin(\kappa-1)\alpha = 0 \qquad (3.97)$$

It is then seen that the original system of four equations in four unknowns is separated into two independent systems, each of two equations in two unknowns. It will be seen that (3.94) and (3.95) lead to a solution which is antisymmetric about the x axis and (3.96), (3.97) lead to a solution which is symmetric about the x axis. This separation into the symmetric and the antisymmetric terms could have been anticipated considering the symmetry of the domain about the x axis. Each system is homogeneous and for non-zero solutions to exist the determinant of each one should vanish. The determinant of the system (3.94), (3.95) is zero if:

$$(\kappa-1)\sin(\kappa+1)\alpha\cos(\kappa-1)\alpha - (\kappa+1)\sin(\kappa-1)\alpha\cos(\kappa+1)\alpha = 0 \qquad (3.98)$$

this simplifies to:

$$\kappa\sin 2\alpha - \sin 2\kappa\alpha = 0. \qquad (3.99)$$

For every root $\kappa$ of this eigenequation at least one of $C_1$ and $C_3$ is nonzero. The ratio $C_3/C_1$ can be determined from either (3.94) or (3.95) as:

substituting for $F(\theta)$ from (3.72) yields:

$$C_1\sin(\kappa+1)\alpha+C_2\cos(\kappa+1)\alpha+C_3\sin(\kappa-1)\alpha+C_4\cos(\kappa-1)\alpha = 0 \tag{3.90}$$

$$-C_1\sin(\kappa+1)\alpha+C_2\cos(\kappa+1)\alpha-C_3\sin(\kappa-1)\alpha+C_4\cos(\kappa-1)\alpha = 0 \tag{3.91}$$

$$(\kappa+1)C_1\cos(\kappa+1)\alpha-(\kappa+1)C_2\sin(\kappa+1)\alpha+(\kappa-1)C_3\cos(\kappa-1)\alpha -(\kappa-1)C_4\sin(\kappa-1)\alpha = 0 \tag{3.92}$$

$$(\kappa+1)C_1\cos(\kappa+1)\alpha+(\kappa+1)C_2\sin(\kappa+1)\alpha+(\kappa-1)C_3\cos(\kappa-1)\alpha +(\kappa-1)C_4\cos(\kappa-1)\alpha = 0 \tag{3.93}$$

By simple additions and subtractions these are easily transformed into:

$$C_1\sin(\kappa+1)\alpha + C_3\sin(\kappa-1)\alpha = 0 \tag{3.94}$$

$$(\kappa+1)C_1\cos(\kappa+1)\alpha+(\kappa-1)C_3\cos(\kappa-1)\alpha = 0 \tag{3.95}$$

and:

$$C_2\cos(\kappa+1)\alpha + C_4\cos(\kappa-1)\alpha = 0 \tag{3.96}$$

The characteristic values of $\kappa$ for which the stress function in (3.72) and consequently the displacements in (3.82) and (3.83) provide a solution for the notch problem will be determined by the requirement that the traction-free boundary conditions be satisfied on the edges of the angle. Let us express $\sigma_\theta$ and $\tau_{r\theta}$ in terms of $\kappa$. Using (3.49), (3.50) and (3.72) we obtain:

$$\sigma_\theta = \frac{\partial^2 \chi}{\partial r^2} = \kappa(\kappa+1) r^{\kappa-1} F(\theta) \tag{3.84}$$

$$\tau_{r\theta} = -\frac{1}{r}\frac{\partial^2 \chi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \chi}{\partial \theta} = -\kappa r^{\kappa-1} F'(\theta) \tag{3.85}$$

In order for tractions to be zero on the two faces of the angle, both $\sigma_\theta$ and $\tau_{r\theta}$ must vanish there:

$$\sigma_\theta(\theta = \pm\, \alpha) = 0 \tag{3.86}$$

$$\tau_{r\theta}(\theta = \pm\, \alpha) = 0 \tag{3.87}$$

from which:

$$F(\theta = \pm\, \alpha) = 0 \tag{3.88}$$

$$F'(\theta = \pm\, \alpha) = 0 \tag{3.89}$$

which leads to solutions of the form:

$$\sin m\theta, \ \cos m\theta$$

The general solution for $G(\theta)$ will then be:

$$G = a_1 \cos m\theta + a_2 \sin m\theta \tag{3.78}$$

and this is valid for complex values of m as well. From the relationship (3.54) between $\chi$ and $\psi_1$ by equating the powers of r we obtain:

$$\kappa - 1 = m \tag{3.79}$$

and by equating subsequently coefficients of similar trigonometric terms we obtain:

$$a_1 = -\frac{4}{\kappa-1} C_3 \tag{3.80}$$

$$a_2 = \frac{4}{\kappa-1} C_4. \tag{3.81}$$

The expressions for the displacements now become:

$$2Gu_r = r^{\kappa}[-(\kappa+1)F(\theta)+(1-\sigma)G'(\theta)] \tag{3.82}$$

$$2Gu_\theta = r^{\kappa}[-F'(\theta)+(1-\sigma)(\lambda-1)G(\theta)]. \tag{3.83}$$

eigenvalues there correspond four independent eigenfunctions with a singularity of the same strength $r^{\kappa_1}$ but each having its own stress intensity factor. By using the complex form of trigonometric functions

$$\sin\kappa = \sin\kappa_1 \cosh\kappa_2 + i\cos\kappa_1 \sinh\kappa_2 \qquad (3.143)$$

$$\cos\kappa = \cos\kappa_1 \cosh\kappa_2 - i\sin\kappa_1 \sinh\kappa_2 \qquad (3.144)$$

and substituting in (3.120), (3.121), (3.130), (3.131) after rather lengthy but otherwise straightforward trigonometric and algebraic transformations we obtain the following explicit expressions for the eigenfunctions:

<u>mode I</u>:

$$4G(2\pi)^{1/2} u_1 = K_I^{(1)} r^{\kappa_1} \Big\{ e^{\kappa_2\theta}[(\kappa_1+3-4\sigma-L_1)c_1 + (\kappa_2-L_2)s_1 - 2(\kappa_1 c_3+\kappa_2 s_3)\cos\theta] + e^{-\kappa_2\theta}[(\kappa_1+3-4\sigma-L_1)c_2 - (\kappa_2-L_2)s_2 - 2(\kappa_1 c_4+\kappa_2 s_4)\cos\theta] \Big\} \qquad (3.145)$$

$$4G(2\pi)^{1/2}\, v_1 = K_I^{(1)}\, r^{\kappa_1} \Big\{ e^{\kappa_2\theta}[(\kappa_1+3-4\sigma+L_1)s_1 - - (\kappa_2+L_2)c_1 - 2(\kappa_1 c_3+\kappa_2 s_3)\sin\theta] + e^{-\kappa_2\theta}[(\kappa_1+3-4\sigma+L_1)s_2 + (\kappa_2+L_2)c_2 + + 2(-\kappa_1 c_4+\kappa_2 s_4)\sin\theta] \Big\} \quad (3.146)$$

$$4G(2\pi)^{1/2}\, u_2 = K_I^{(2)}\, r^{\kappa_1} \Big\{ e^{\kappa_2\theta}[-(\kappa_1+3-4\sigma-L_1)s_1 + + (\kappa_2-L_2)c_1 + 2(\kappa_1 s_3-\kappa_2 c_3)\cos\theta] + e^{-\kappa_2\theta}[(\kappa_1+3-4\sigma-L_1)s_2 + (\kappa_2-L_2)c_2 - - 2(\kappa_1 s_4+\kappa_2 c_4)\cos\theta] \Big\} \quad (3.147)$$

$$4G(2\pi)^{1/2}\, v_2 = K_I^{(2)}\, r^{\kappa_1} \Big\{ e^{\kappa_2\theta}[(\kappa_1+3-4\sigma+L_1)c_1 + + (\kappa_2+L_2)s_1 + 2(\kappa_1 s_3-\kappa_2 c_3)\sin\theta] + e^{-\kappa_2\theta}[-(\kappa_1+3-4\sigma+L_1)c_2 + (\kappa_2+L_2)s_2 - - 2(\kappa_1 s_4-\kappa_2 c_4)\sin\theta] \Big\} \quad (3.148)$$

where

$$c_1 = \cos(\kappa_1\theta-\kappa_2\ell nr) \tag{3.149}$$

$$s_1 = \sin(\kappa_1\theta-\kappa_2\ell nr) \tag{3.150}$$

$$c_2 = \cos(\kappa_1\theta+\kappa_2\ell nr) \tag{3.151}$$

$$s_2 = \sin(\kappa_1\theta+\kappa_2\ell nr) \tag{3.152}$$

$$c_3 = \cos(\kappa_1\theta-\theta-\kappa_2\ell nr) \tag{3.153}$$

$$s_3 = \sin(\kappa_1\theta-\theta-\kappa_2\ell nr) \tag{3.154}$$

$$c_4 = \cos(\kappa_1\theta-\theta+\kappa_2\ell nr) \tag{3.155}$$

$$s_4 = \sin(\kappa_1\theta-\theta+\kappa_2\ell nr) \tag{3.156}$$

$$L_1 = \frac{C_1(\kappa_1+1)+\kappa_2 C_2}{C_1^2 + C_2^2} \tag{3.157}$$

$$L_2 = \frac{C_1\kappa_2-C_2(\kappa_1+1)}{C_1^2 + C_2^2} \tag{3.158}$$

$$C_1 = -\frac{\cosh 2\kappa_2\alpha \cos 2\alpha + \cos 2\kappa_1\alpha}{\cosh 2\kappa_2\alpha + \cos 2(\kappa_1-1)\alpha} \qquad (3.159)$$

$$C_2 = \frac{\sinh 2\kappa_2\alpha \ \sin 2\kappa_1\alpha}{\cosh 2\kappa_2\alpha + \cos 2(\kappa_1-1)\alpha} \qquad (3.160)$$

and $\sigma$ is given by (3.18)

<u>mode II</u>

$$4G(2\pi)^{1/2}\ u_1 = K_{II}^{(1)}\ r^{\kappa_1}\Big\{e^{\kappa_2\theta}[(3-\kappa_1-4\sigma-L_1)s_1 + (\kappa_2+L_2)c_1 + 2(\kappa_1 c_3+\kappa_2 s_3)\sin\theta] + e^{-\kappa_2\theta}[(3-\kappa_1-4\sigma-L_1)s_2 - (\kappa_2+L_2)c_2 + 2(\kappa_1 c_4-\kappa_2 s_4)\sin\theta]\Big\} \qquad (3.161)$$

$$4G(2\pi)^{1/2}\ v_1 = K_{II}^{(1)}\ r^{\kappa_1}\Big\{e^{\kappa_2\theta}[-(3-\kappa_1-4\sigma+L_1)c_1 + (\kappa_2-L_2)s_1 + 2(-\kappa_1 c_3+\kappa_2 s_3)\cos\theta] + e^{-\kappa_2\theta}[-(3-\kappa_1+4\sigma+L_1)c_2 - (\kappa_2-L_2)s_2 - 2(\kappa_1 c_4+\kappa_2 s_4)\cos\theta]\Big\} \qquad (3.162)$$

$$4G(2\pi)^{1/2}\,u_2 = K_{II}^{(2)}\, r^{\kappa_1}\{e^{\kappa_2\theta}[(3-\kappa_1-4\sigma+L_1)c_1 +$$

$$+ (\kappa_2-L_2)s_1 + 2(-\kappa_1 s_3+\kappa_2 c_3)\sin\theta]$$

$$+ e^{-\kappa_2\theta}[-(3-\kappa_1-4\sigma-L_1)c_2 - (\kappa_2-L_2)s_2 +$$

$$+ 2(\kappa_1 s_4+\kappa_2 c_4)\sin\theta]\} \qquad (3.163)$$

$$4G(2\pi)^{1/2}\,v_2 = K_{II}^{(2)}\, r^{\kappa_1}\{e^{\kappa_2\theta}[(3-\kappa_1-4\sigma+L_1)s_1 +$$

$$+ (\kappa_2-L_2)c_1 + 2(\kappa_1 s_3-\kappa_2 c_3)\cos\theta]$$

$$+ e^{-\kappa_2\theta}[-(3-\kappa_1-4\sigma+L_1)s_2 + (\kappa_2-L_2)c_2 -$$

$$- 2(\kappa_1 s_4+\kappa_2 c_4)\cos\theta]\} \qquad (3.164)$$

where

$c_i$ and $s_i$ are given by (3.149) through (3.155)

$$L_1 = \frac{C_1(\kappa_1+1)+\kappa_2 C_2}{C_1^2 + C_2^2} \qquad (3.165)$$

$$L_2 = \frac{C_1\kappa_2-C_2(\kappa_1+1)}{C_1^2 + C_2^2} \qquad (3.166)$$

$$C_1 = \frac{-\cosh 2\kappa_2\alpha \cos 2\alpha + \cos 2\kappa_1\alpha}{\cosh 2\kappa_2\alpha - \cos 2(\kappa_1 - 1)\alpha} \tag{3.167}$$

$$C_2 = \frac{\sinh 2\kappa_2\alpha \ \sin 2\kappa_1\alpha}{\cosh 2\kappa_2\alpha - \cos 2(\kappa_1 - 1)\alpha} \tag{3.168}$$

The heuristic approach that was followed in this chapter can be justified in the framework of a very general theory given by Kondratev [17]. He showed that the solution of an elliptic boundary value problem in the neighborhood of an angle can be written in the form:

$$u = \sum_i a_i \, r^{\kappa_i} \, \ell n^q \, r \, \phi_i(\theta) + w(r, \theta) \tag{3.169}$$

where

$q = 0, \ldots, m_i - 1$

$m_i$ = the multiplicity of the eigenvalue $\kappa_i$

$\phi_i(\theta)$ is a smooth function

and $w(r, \theta)$ is smoother than the $r^{i} \, \ell n^q \, r$ terms

It is now of interest to explore further the nature of the roots of the eigenequations. We are only interested in roots with a positive real part. The roots with a negative real part are readily obtained from the former through a simple sign reversal, due to the symmetry of the eigenequations with respect to the x axis. They are disregarded in the displacement expansions as they would imply unbounded displacements at the tip. Karp and Karal [18] have shown that there exists a nontrivial real root which is always smaller than the positive real part of any of the complex roots.

The smallest positive eigenvalue for mode I is given in table 3.1 for various values of the half angle $\alpha$. Table 3.2 displays the same information for mode II. We can see that except for the crack case ($\alpha$ = 180 degrees), where the eigenvalues for the two modes are the same, the eigenvalue for mode II is always larger than the eigenvalue for mode I. When the half solid angle $\alpha$ is equal to 128.7268 degrees the eigenvalue for mode II is equal to 1, therefore the stresses corresponding to mode II are bounded at the notch tip. Under mode I conditions any reentrant corner will give rise to singular stresses at the tip, the critical half angle being 90 degrees (half plane). The second and higher eigenvalues are complex for angles $\alpha$

equal to or smaller than 157 degrees in the case of mode I, and for angles $\alpha$ equal to or smaller than 164 degrees in the case of mode II. In the case of a crack ($\alpha$ = 180 degrees) all the eigenvalues are real. For smaller angles there is a finite number of real eigenvalues and an infinite number of complex ones. Table 3.1 refers to mode I and displays the first nine real eigenvalues for $\alpha$ equal to 175 degrees together with all the real eigenvalues for $\alpha$ equal to 170, 165, 160 and 158 degrees. Table 3.4 displays the corresponding results for mode II, with the first nine real eigenvalues for $\alpha$ equal to 175 degrees and all the real eigenvalues for $\alpha$ equal to 170 and 165 degrees.

Table 3.1

Smallest positive eigenvalue

for various solid angles, mode I

(roots of the equation $\kappa \sin 2\alpha + \sin 2\kappa\alpha = 0$)

| solid angle | $\alpha$ | first eigenvalue, mode I |
|---|---|---|
| 360° | 180° | 0.5 |
| 350° | 175° | 0.500052987126443 |
| 340° | 170° | 0.500426375426056 |
| 330° | 165° | 0.501453008713551 |
| 320° | 160° | 0.503490483184783 |
| 310° | 155° | 0.506932842286465 |
| 300° | 150° | 0.512221361160512 |
| 290° | 145° | 0.519854303113919 |
| 280° | 140° | 0.530395719129773 |
| 270° | 135° | 0.544483736782464 |
| 260° | 130° | 0.562839480481682 |
| 250° | 125° | 0.586278864957285 |
| 240° | 120° | 0.615731059490783 |
| 230° | 115° | 0.652269555181627 |
| 225° | | 0.67358343 |
| 220° | 110° | 0.697164972097201 |
| 210° | 105° | 0.751974545407642 |
| 200° | 100° | 0.818695851323838 |
| 190° | 95° | 0.900043811488137 |
| 180° | 90° | 1. |

Table 3.2

Smallest positive eigenvalue
for various solid angles, mode II

(roots of the equation $\kappa \sin 2\alpha - \sin 2\kappa\alpha = 0$)

| solid angle | α | first eigenvalue, mode II |
|---|---|---|
| 360° | 180° | 0.5 |
| 350° | 175° | 0.529354738341384 |
| 340° | 170° | 0.562006549619481 |
| 330° | 165° | 0.598191849614085 |
| 320° | 160° | 0.638182471293363 |
| 310° | 155° | 0.682294830307061 |
| 300° | 150° | 0.730900741512950 |
| 290° | 145° | 0.784440552974094 |
| 280° | 140° | 0.843439568929300 |
| 270° | 135° | 0.908529189846099 |
| 260° | 130° | 0.980474925453105 |
| 257.4536° | 128.7268° | 1. |
| 250° | 125° | 1.060214662528446 |
| 240° | 120° | 1.148912751316944 |
| 230° | 115° | 1.248039607030766 |
| 225 | | 1,302086 |
| 220° | 110° | 1.359494953661662 |
| 210° | 105° | 1.485811706900859 |
| 200° | 100° | 1.630525086564494 |
| 190° | 95° | 1.798932622346293 |
| 180° | 90° | 2. |

Table 3.3

Real eigenvalues for various solid angles, mode I

(Roots of the equation $\kappa \sin 2\alpha + \sin 2\kappa\alpha = 0$)

| Solid angle | α | real eigenvalues, mode I |
|---|---|---|
| 350° | 175° | 0.500052987126443<br>1.058842953176205<br>1.499727767815282<br>2.118822841754057<br>2.497979910848996<br>3.181532712089234<br>3.493301550223319<br>4.250184120916684<br>4.482534709491662<br>etc. |
| 340° | 170° | 0.500426375426056<br>1.125406650991640<br>1.497613486365886<br>2.267186596933758<br>2.476769998913093 |
| 330° | 165° | 0.501453008713551<br>1.202957173241424<br>1.490377798463559 |

Table 3.3

(continued)

| Solid angle | α | real eigenvalues, mode I |
|---|---|---|
| 320° | 160° | 0.503490483184783<br>1.302693359118874<br>1.467008439164243 |
| 316° | 158° | 0.504675297031383<br>1.365731350131872<br>1.436282183283593 |

Table 3.4

Real eigenvalues for various solid angles, mode II
(Roots of the equation $\kappa\sin 2\alpha - \sin 2\kappa\alpha = 0$)

| Solid angle | α | real eigenvalues, mode II |
|---|---|---|
| 350° | 175° | 0.529354738341384 |
| | | 1. |
| | | 1.588609191187519 |
| | | 1.999106964391621 |
| | | 2.649698718055999 |
| | | 2.996140969135392 |
| | | 3.714772677395547 |
| | | 3.989019694329133 |
| | | 4.789341976925038 |
| | | 4.972275638161229 |
| | | etc. |
| 340° | 170° | 0.562006549619481 |
| | | 1. |
| | | 1.692250101505664 |
| | | 1.991384797275372 |
| | | 2.883886605832655 |
| | | 2.920169039793589 |
| 330° | 165° | 0.598191849614085 |
| | | 1. |
| | | 1.838934252571961 |
| | | 1.948555887250050 |

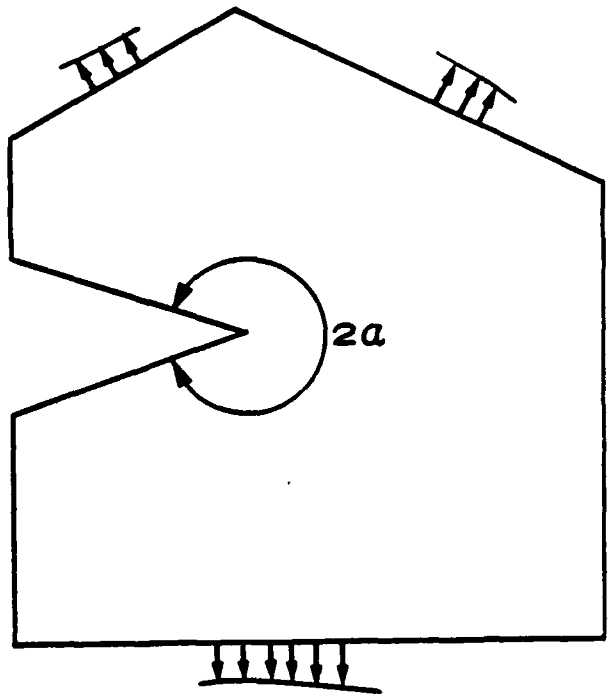


Figure 4.1

Two dimensional body with a notch

(solid angle = 2α)

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MICROCOPY RESOLUTION TEST CHART
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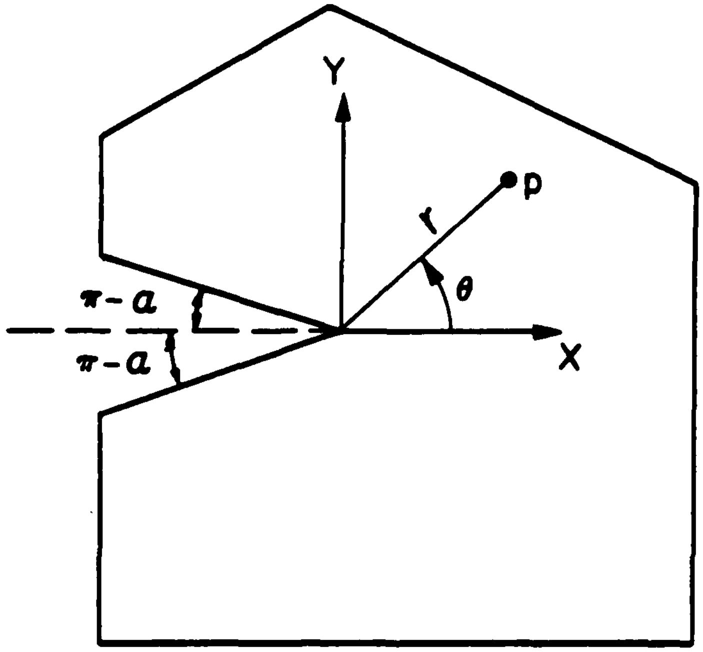


Figure 4.2

Cartesian and polar coordinate systems

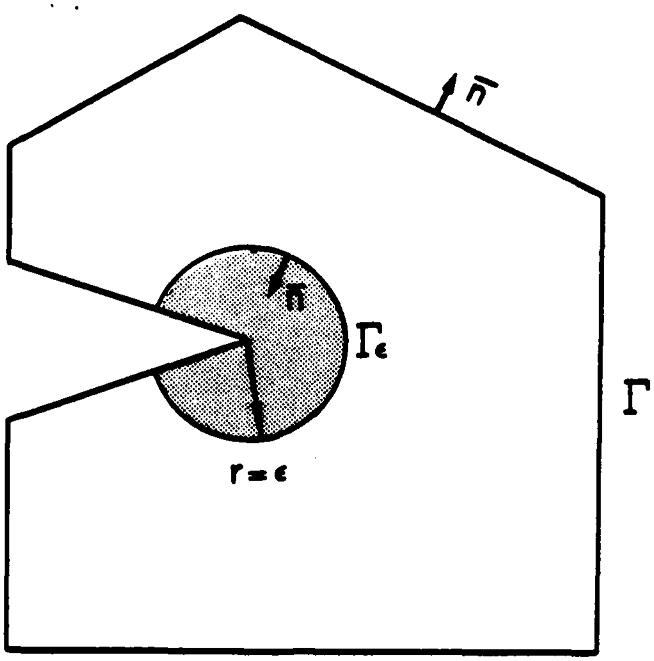


Figure 4.3

Removal of a small disk of radius r = ε

$$u = K_1 r^{\kappa_1} U_1(\theta) + K_2 r^{\kappa_2} U_2(\theta) + \ldots \tag{4.1}$$

$$v = K_1 r^{\kappa_1} V_1(\theta) + K r^{\kappa_2} V_2(\theta) + \ldots \tag{4.2}$$

where $K_1$, $K_2$ are the generalized stress intensity factors and $\kappa_1$, $\kappa_2$ are the eigenvalues. The precise form of functions $U_i(\theta)$ and $V_i(\theta)$ was derived in the previous chapter. We are interested in compute so the generalized stress intensity factors $K_1$, $K_2$, etc. The eigenfunctions $U_i(\theta)$, $V_i(\theta)$ also depend on the elastic constants, the half solid angle $\alpha$ and the corresponding eigenvalue $\kappa_i$.

We choose as extraction functions $\phi$ and $\psi$ the eigenfunctions corresponding to the negative eigenvalue of the term, whose amplitude is being extracted.

$$\phi = r^{-\kappa_2} \Phi(\theta) + \phi_b \tag{4.3}$$

$$\psi = r^{-\kappa_1} \Psi(\theta) + \psi_b \tag{4.4}$$

where $\Phi(\theta)$ and $\Psi(\theta)$ are obtained from $U_1(\theta)$ and $V_1(\theta)$ respectively if we substitute $-\kappa_1$ for $\kappa_1$. The terms $\phi_b$ and $\psi_b$ are smooth functions that vanish in the neighborhood of the notch tip and arc so chosen that the extraction functions satisfy the following requirements:

i) They vanish on the part of the boundary where displacements are specified. This is accomplished by the addition of the smooth functions $\phi_b$ and $\psi_b$, which are called blending functions. Another way is to multiply the singular part in $\phi$ and $\psi$ by a cut-off function, which takes the value of unity in the neighborhood of the notch tip and the value of zero away from it with a smooth transition in-between. Further discussion on the cut-off and blending functions can be found in [3].

ii) They satisfy the equilibrium equations in the neighborhood of the notch tip. If no displacement boundary conditions are specified on any part of the boundary, then they satisfy equilibrium everywhere in the domain.

iii) They satisfy the traction-free boundary conditions on the faces of the notch, that is tractions corresponding to displacements $\phi$ and $\psi$ vanish for $\theta=\alpha$ and $\theta=-\alpha$.

iv) They have a singularity of the order $r^{-\kappa_1}$.

The equations of elasticity (plane strain) in the Cartesian system (x,y) can be written as:

$$L_x(u,v) = 0 \tag{4.5}$$

$$L_y(u,v) = 0 \tag{4.6}$$

where

$$L_x(u,v) = (\lambda+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{4.7}$$

$$L_y(u,v) = (\lambda+G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + G\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{4.8}$$

with λ and G being Lame's constants. We now multiply equation (4.5) by $\phi$ and integrate over the domain. Similarly, we multiply equation (4.6) by $\psi$, integrate over the domain and add the resulting equations to obtain:

$$\int_A \int [L_u(u,v)\phi + L_y(u,v)\psi]dA = 0. \tag{4.9}$$

Integrating twice by parts, as explained in Chapter 2, we obtain:

$$\int_A \int [L_x(\phi,\psi)u + L_y(\phi,\psi)v]dA - \int_{\Gamma+\Gamma_\varepsilon} [T_x(\phi,\psi)u + T_y(\phi,\psi)v]ds$$

$$+ \int_{\Gamma+\Gamma_\varepsilon} [T_x(u,v)\phi + T_y(u,v)\psi]ds = 0 \tag{4.10}$$

where:

$$T_x(u,v) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_x + G\left( \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right)$$

$$+ G\left( \frac{\partial u}{\partial x} n_x + \frac{\partial v}{\partial x} n_y \right) \tag{4.11}$$

$$T_y(u,v) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) n_y + G\left( \frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y \right)$$

$$+ G\left( \frac{\partial u}{\partial y} n_x + \frac{\partial v}{\partial y} n_y \right) \tag{4.12}$$

are the tractions corresponding to displacements (u,v) and $\bar{n}=(n_x,n_y)$ is the outward normal to the surface. In the case where only tractions are specified on the boundary the area integral in (4.10) vanishes, since the extraction functions satisfy equilibrium everywhere, and equation (4.10) becomes:

$$- \int_{\Gamma+\Gamma_\varepsilon} [T_x(\phi,\psi)u + T_y(\phi,\psi)v]ds + \int_{\Gamma+\Gamma_\varepsilon} [T_x(u,v)\phi + T_y(u,v)\psi]ds = 0 \qquad (4.13)$$

The integration will be performed separately on the circular arc $\Gamma_\varepsilon$ and the rest of the boundary $\Gamma$. On the part of the boundary, where tractions are specified, $T_x(u,v)$ and $T_y(u,v)$ are known:

$$T_x(u,v) = \overline{X} \qquad (4.14)$$

$$T_y(u,v) = \overline{Y} \qquad (4.15)$$

On the part of the boundary, were displacements are specified, $\phi$ and $\psi$ vanish. The second integral in (4.13) evaluated around the contour $\Gamma$ can then be written:

$$\int_{\Gamma} [T_x(u,v)\phi + T_y(u,v)\psi]ds = \int_{\Gamma} (\overline{X}\phi + \overline{Y}\psi)ds \qquad (4.16)$$

This integral can be evaluated numerically from the data of the problem. The integration around the circular arc yields an expression of the required stress intensity factor. On this arc $\Gamma_\varepsilon$ the displacements u and v are given by equations (4.1) and (4.2). Their derivatives will be of the form:

$$\frac{\partial u}{\partial x} = K_1 r^{\kappa_1 - 1} U_{1x}(\theta) + K_2 r^{\kappa_2 - 1} U_{2x}(\theta) + \ldots \tag{4.17}$$

$$\frac{\partial u}{\partial y} = K_1 r^{\kappa_1 - 1} U_{1y}(\theta) + K_2 r^{\kappa_2 - 1} U_{2y}(\theta) + \ldots \tag{4.18}$$

$$\frac{\partial v}{\partial x} = K_1 r^{\kappa_1 - 1} V_{1x}(\theta) + K_2 r^{\kappa_2 - 1} V_{2x}(\theta) + \ldots \tag{4.19}$$

$$\frac{\partial v}{\partial y} = K_1 r^{\kappa_1 - 1} V_{1y}(\theta) + K_2 r^{\kappa_2 - 1} V_{2y}(\theta) + \ldots \tag{4.20}$$

In view of (4.11) and (4.12) and the fact that on the circular arc $\Gamma_\varepsilon$ the outward normal $\bar{n}$ has components

$$n_x = -\cos\theta \tag{4.21}$$

$$n_y = -\sin\theta \tag{4.22}$$

the tractions $T_x(u,v)$ and $T_y(u,v)$ can be written in the form:

$$T_x(u,v) = K_1 r^{\kappa_1 - 1} T_{x1}(\theta) + K_2 r^{\kappa_2 - 1} T_{x2}(\theta) + \ldots \tag{4.23}$$

$$T_y(u,v) = K_1\, r^{\kappa_1 - 1}\, T_{y1}(\theta) + K_2\, r^{\kappa_2 - 1}\, T_{y2}(\theta) + \ldots \tag{4.24}$$

By noting that on $\Gamma_\varepsilon$:

$$ds = r\, d\theta \tag{4.25}$$

we can write the second integral in (4.13) around $\Gamma_\varepsilon$ in the form:

$$\int_{\Gamma_\varepsilon} [T_x(u,v)\phi + T_y(u,v)\psi] ds = K_1\, M_1 + K_2\, M_2(r) + K_3\, M_3(r) + K_4\, M_4(r) \tag{4.26}$$

where

$$M_1 = \int_{-\alpha}^{\alpha} [T_{x1}(\theta)\ \Phi(\theta) + T_{y1}(\theta)\psi(\theta)] d\theta \tag{4.27}$$

$$M_2(r) = r^{-\kappa_1 + \kappa_2} \int_{-\alpha}^{\alpha} [T_{x2}(\theta)\ \Phi(\theta) + T_{y2}(\theta)\ \Psi(\theta)] d\theta \tag{4.28}$$

$$M_3(r) = r^{\kappa_1} \int_{-\alpha}^{\alpha} [T_{x1}(\theta)\phi_b + T_{y1}(\theta)\psi_b] d\theta \tag{4.29}$$

$$M_4(r) = r^{\kappa_2} \int_{-\alpha}^{\alpha} [T_{x2}(\theta)\phi_b + T_{y2}(\theta)\psi_b] d\theta \tag{4.30}$$

The higher order terms are disregarded in what follows, since two terms are sufficient for the discussion of all the relevant issues. We now observe that $M_1$ is independent of the radius r. This is the reason that the extraction functions are chosen as the eigenfunctions corresponding to the negative eigenvalue $-\kappa_1$. Otherwise $M_1$ would also depend on r. It can be seen that $M_3(r)$ and $M_4(r)$ vanish as r approaches zero.

In view of (4.3) and (4.4) the derivatives of the extraction functions can be written in the form:

$$\frac{\partial\phi}{\partial x} = r^{-\kappa_1 - 1} \Phi_x(\theta) + \phi_{bx} \tag{4.31}$$

$$\frac{\partial\phi}{\partial y} = r^{-\kappa_1 - 1} \Phi_y(\theta) + \phi_{by} \tag{4.32}$$

$$\frac{\partial\psi}{\partial x} = r^{-\kappa_1 - 1} \Psi_x(\theta) + \psi_{bx} \tag{4.33}$$

$$\frac{\partial\psi}{\partial y} = r^{-\kappa_1 - 1} \Psi_y(\theta) + \psi_{by} \tag{4.34}$$

where $\phi_{b2}$, $\phi_{by}$, $\psi_{bx}$, $\psi_{by}$ are again smooth functions. The tractions corresponding to displacements $\phi$ and $\psi$ can be written in the form:

$$T_x(\phi,\psi) = r^{-\kappa_1 - 1} X_{\phi\psi}(\theta) + X_b \tag{4.35}$$

$$T_y(\phi,\psi) = r^{-\kappa_1 - 1} Y_{\phi\psi}(\theta) + Y_b \tag{4.36}$$

with $X_b$ and $Y_b$ being again smooth functions. The first integral around $\Gamma_\varepsilon$ in (4.13) can then be written as:

$$\int_{\Gamma_\varepsilon} [T_x(\phi,\psi)u + T_y(\phi,\psi)v]ds = K_1\ M_5 + K_2\ M_6(r)$$

$$+ K_2\ M_7(r) + K_2\ M_8(r) \qquad (4.37)$$

where:

$$M_5 = \int_{-\alpha}^{\alpha}[X_{\phi\psi}(\theta)\ U_1(\theta) + Y_{\phi\psi}(\theta)\ V_1(\theta)]d\theta \qquad (4.38)$$

$$M_6(r) = r^{-\kappa_1+\kappa_2} \int_{-\alpha}^{\alpha}[X_{\phi\psi}(\theta)\ U_2(\theta) + Y_{\phi\psi}(\theta)\ V_2(\theta)]d\phi \qquad (4.39)$$

$$M_7(r) = r^{\kappa_1+1} \int_{-\alpha}^{\alpha}[X_b\ U_1(\theta) + Y_b\ V_1(\theta)]d\theta \qquad (4.40)$$

$$M_8(r) = r^{\kappa_2+1} \int_{-\alpha}^{\alpha}[X_b\ U_2(\theta) + Y_b\ V_2(\theta)]d\theta \qquad (4.41)$$

We observe that expression $M_5$ is also indpenedent of r. The choice of the extractions functions $\phi$ and $\psi$ as the eigenfunctions corresponding to the negative eigenvalue $-\kappa_1$ makes both $M_5$ and $M_6$ independent of r. Since $X_b$ and $Y_b$ are smooth functions, expressions $M_7(r)$ and $M_8(r)$ vanish in the limit as r approaches zero. Expressions $M_2(r)$ and $M_6(r)$ require further consideration. In the case where $\kappa_1$ is the first eigenvalue, that is $\kappa_1 < \kappa_2$, then the quantity $(-\kappa_1+\kappa_2)$ is positive and both $M_2(r)$ and $M_6(r)$ vanish as r tends to zero. However, if a higher order stress intensity factor is extracted, say $K_2$, the exponent of r in expressions $M_2(r)$ and $M_6(r)$ will be negative. These expressions then become unbounded as r tends to zero, unless the integrals with respect to $\theta$ vanish. It will now be shown that they indeed vanish.

Consider the body shown in figure (4.4), which is bounded by the straight segments $\Gamma_2$, $\Gamma_4$ on the faces of the notch angle and the circular arcs $\Gamma_1$, $\Gamma_3$ of epicentral angle $2\alpha$ centered at the notch tip. Applying equation (4.13) for this body with u, v as the second terms of (4.1) and (4.2) respectively and $\phi, \psi$ as the singular terms of (4.3) and (4.4) we obtain:

$$- \int_{\Gamma_1+\Gamma_3} r^{-\kappa_1+\kappa_2-1} [X_{\phi\psi}(\theta)\ U_2(\theta) + Y_{\phi\psi}(\theta)\ V_2(\theta)]rd\theta$$

$$+ \int_{\Gamma_1+\Gamma_3} r^{-\kappa_1+\kappa_2-1} [T_{x2}(\theta)\Phi(\theta) + T_{y2}(\theta)\Psi(\theta)]rd\theta = 0 \tag{4.42}$$

Both integrals vanish on the straight segments $\Gamma_2$, $\Gamma_4$ because the traction components $T_{x2}$, $T_{y2}$ corresponding to the eigenfunctions, as well as the tractions components $X_{\phi\psi}$, $Y_{\phi\psi}$ corresponding to the extraction functions vanish on the faces of the angle.

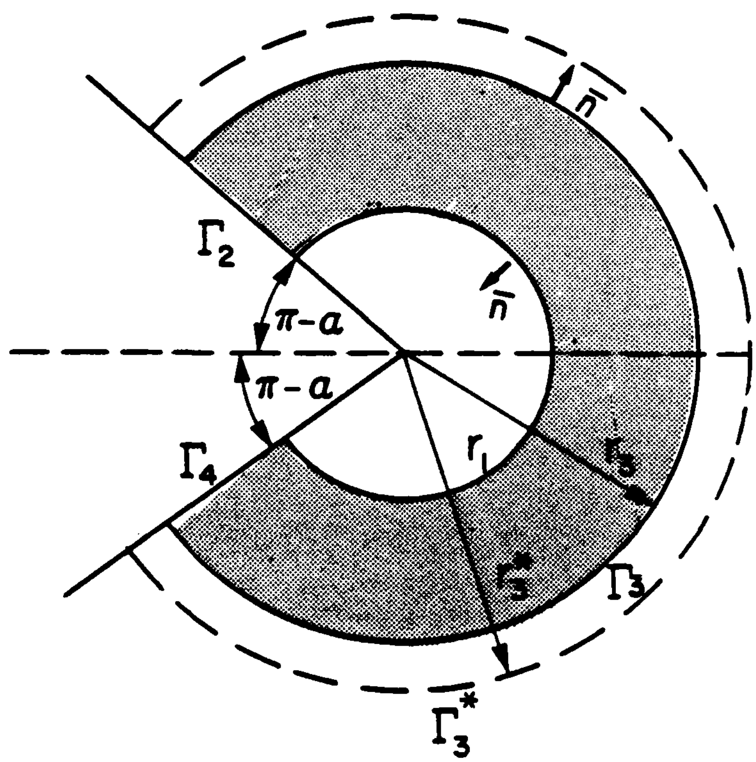


Figure 4.4

Annular ring with a cut

(epicentral angle = $2\alpha$)

We now consider a larger annular sector with the same inside diameter but a larger outside diameter, where the arc $\Gamma_3$ is replaced by $\Gamma_3^*$. We then have:

$$- \int_{\Gamma_1+\Gamma_3^*} r^{-\kappa_1+\kappa_2-1}[X_{\phi\psi}(\theta)U_2(\theta) + Y_{\phi\psi}(\theta)V_2(\theta)]rd\theta$$

$$+ \int_{\Gamma_1+\Gamma_3^*} r^{-\kappa_1+\kappa_2-1}[T_{x2}(\theta)\Phi(\theta) + T_{y2}(\theta)\Psi(\theta)]rd\theta = 0 \quad (4.43)$$

Therefore:

$$- \int_{\Gamma_3} r^{-\kappa_1+\kappa_2-1}[X_{\phi\psi}(\theta)U_2(\theta) + Y_{\phi\psi}(\theta)V_2(\theta)rd\theta$$

$$+ \int_{\Gamma_3} r^{-\kappa_1+\kappa_2-1}[T_{x2}(\theta)\Phi(\theta) + T_{y2}(\theta)\Psi(\theta)]rd\theta =$$

$$= - \int_{\Gamma_3^*} r^{-\kappa_1+\kappa_2-1}[X_{\phi\psi}(\theta)U_2(\theta) + Y_{\phi\psi}(\theta)V_2(\theta)rd\theta$$

$$+ \int_{\Gamma_3^*} r^{-\kappa_1+\kappa_2-1}[T_{x2}(\theta)\Phi(\theta) + T_{y2}(\theta)\Psi(\theta)]rd\theta \quad (4.44)$$

$$2C_I G(2\pi)^{1/2} \frac{\partial u_I}{\partial x} = K_I \; r^{\kappa_I - 1} \; \kappa_I[(C_I B-\kappa_I-1)\cos(\kappa_I-1)\theta \; -$$

$$- \; 2\kappa_I C_I \cos\theta\cos(\kappa_I-2)\theta \; +$$

$$+ \; 2C_I \sin\theta\sin(\kappa_I-2)\theta] \qquad (5.18)$$

$$2C_I G(2\pi)^{1/2} \frac{\partial u_I}{\partial y} = K_I \; r^{\kappa_I - 1} \; \kappa_I[-(C_I B-\kappa_I-1)\sin(\kappa_I-1)\theta \; +$$

$$+ \; 2C_I(\kappa_I-1)\cos\theta\sin(\kappa_I-2)\theta] \qquad (5.19)$$

$$2C_I G(2\pi)^{1/2} \frac{\partial v_I}{\partial x} = K_I \; r^{\kappa_I - 1} \; \kappa_I[(C_I B+\kappa_I+1)\sin(\kappa_I-1)\theta \; -$$

$$- \; 2C_I \sin\theta\cos(\kappa_I-2)\theta]. \qquad (5.20)$$

$$2C_I G(2\pi)^{1/2} \frac{\partial v_I}{\partial y} = K_I \; r^{\kappa_I - 1} \; \kappa_I[(C_I B+\kappa_I+1)\cos(\kappa_I-1)\theta \; +$$

$$+ \; 2\kappa_I C_I \sin\theta\sin(\kappa_I-2)\theta \; -$$

$$- \; 2C_I \cos\theta\cos(\kappa_I-2)\theta] \qquad (5.21)$$

Subscripts I and II refer to modes I and II respectively. The stresses in the Cartesian system (x, y) can be expressed through the transformation formulas:

$$\sigma_x = \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta - 2\,\tau_{r\theta} \sin\theta \cos\theta \tag{5.13}$$

$$\sigma_y = \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta + 2\,\tau_{r\theta} \sin\theta \cos\theta \tag{5.14}$$

$$\tau_{xy} = (\sigma_r-\sigma_\theta)\sin\theta \cos\theta + \tau_{r\theta}(\cos^2\theta-\sin^2\theta) \tag{5.15}$$

The edges of the notch are free from tractions with stresses $\sigma_\theta$ and $\tau_{r\theta}$ being zero there. On any other plane through the body with an outward normal $\overline{n} = (n_x, n_y)$ the tractions are given in terms of stresses through the transformation law:

$$\overline{X} = \sigma_x \cos\alpha + \tau_{xy} \sin\alpha \tag{5.16}$$

$$\overline{Y} = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha \tag{5.17}$$

An alternative way of obtaining expressions for the tractions is to obtain first the derivatives of the displacements and then express the tractions in terms of these derivatives as follows:

$$C_I(2\pi)^{1/2}\,\tau_{r\theta I} = K_I\, r^{\kappa_I - 1}\,\kappa_I[(\kappa_I+1)\sin(\kappa_I+1)\theta + $$

$$+ C_I(\kappa_I-1)\sin(\kappa_I-1)\theta] \qquad (5.7)$$

$$C_{II}(2\pi)^{1/2}\,\sigma_{rII} = -K_{II}\, r^{\kappa_{II}-1}[\kappa_{II}(\kappa_{II}+1)\sin(\kappa_{II}+1)\theta + $$

$$+ C_{II}\kappa_{II}(\kappa_{II}-3)\sin(\kappa_{II}-1)\theta] \qquad (5.8)$$

$$C_{II}(2\pi)^{1/2}\,\sigma_{\theta II} = K_{II}\, r^{\kappa_{II}-1}\,\kappa_{II}(\kappa_{II}+1)[\sin(\kappa_{II}+1)\theta + $$

$$+ C_{II}\sin(\kappa_{II}-1)\theta] \qquad (5.9)$$

$$C_{II}(2\pi)^{1/2}\,\tau_{r\theta II} = -K_{II}\, r^{\kappa_{II}-1}\,\kappa_{II}[(\kappa_{II}+1)\cos(\kappa_{II}+1)\theta + $$

$$+ C_{II}(\kappa_{II}-1)\cos(\kappa_{II}-1)\theta] \qquad (5.10)$$

where

$$C_I = -\frac{\cos(\kappa_I+1)\alpha}{\cos(\kappa_I-1)\alpha} = -\frac{(\kappa_I+1)\sin(\kappa_I+1)\alpha}{(\kappa_I-1)\sin(\kappa_I-1)\alpha} \qquad (5.11)$$

$$C_{II} = -\frac{\sin(\kappa_{II}+1)\alpha}{\sin(\kappa_{II}-1)\alpha} = -\frac{(\kappa_{II}+1)\cos(\kappa_{II}+1)\alpha}{(\kappa_{II}-1)\cos(\kappa_{II}-1)\alpha} \qquad (5.12)$$

$$2GC_I(2\pi)^{1/2}\, u_I = K_I\, r^{\kappa_I}[(C_I B-\kappa_I-1)\cos\kappa_I\theta -$$

$$- 2\kappa_I C_I \cos\theta \cos(\kappa_I-1)\theta] \qquad (5.1)$$

$$2GC_I(2\pi)^{1/2}\, v_I = K_I\, r^{\kappa_I}[(C_I B+\kappa_I+1)\sin\kappa_I\theta -$$

$$- 2\kappa_I C_I \sin\theta \cos(\kappa_I-1)\theta] \qquad (5.2)$$

$$2GC_{II}(2\pi)^{1/2}\, u_{II} = K_{II}\, r^{\kappa_{II}}[(C_{II} D-\kappa_{II}-1)\sin\kappa_{II}\theta +$$

$$+ 2\kappa_{II} C_{II} \sin\theta \cos(\kappa_{II}-1)\theta] \qquad (5.3)$$

$$2GC_{II}(2\pi)^{1/2}\, v_{II} = K_{II}\, r^{\kappa_{II}}[-(C_{II} D+\kappa_{II}+1)\cos\kappa_{II}\theta -$$

$$- 2\kappa_{II} C_{II} \cos\theta \cos(\kappa_{II}-1)\theta] \qquad (5.4)$$

$$C_I(2\pi)^{1/2}\, \sigma_{rI} = -K_I\, r^{\kappa_I-1}[\kappa_I(\kappa_I+1)\cos(\kappa_I+1)\theta +$$

$$+ C_I\kappa_I(\kappa_I-3)\cos(\kappa_I-1)\theta] \qquad (5.5)$$

$$C_I(2\pi)^{1/2}\, \sigma_{\theta I} = K_I\, r^{\kappa_I-1}\, \kappa_I(\kappa_I+1)[\cos(\kappa_I+1)\theta +$$

$$+ C_I\cos(\kappa_I-1)\theta] \qquad (5.6)$$

## 5. THE MODEL PROBLEM

The numerical performance of the extraction method will now be established through numerical experiments based on a model problem. The model problem contains a reentrant corner and a solution is selected which satisfies (1) the Navier equations and (2) the boundary conditions at the reentrant corner. On the other boundaries the tractions corresponding to the exact solution are specified. In this way the exact solution has the main characteristics of typical practical problems (i.e. ones with reentrant corners). It is therefore suitable for benchmark studies. No problem with reentrant corners and possessing an exact solution has been reported in the literature. In all the reported solutions a numerical approximation is made at some stage, in which the error is unknown. Thus precise convergence studies cannot be performed.

Let us consider the two-dimensional domain of infinite dimensions with a notch of total solid angle equal to $2\alpha$. As it was explained in Chapter 3 the stress and deformation fields are given by series expansions in which the typical terms are as follows:

The same will happen when more terms are kept in the expansion. Equation (4.76) then reduces to:

$$- \int_{\Gamma_\varepsilon} [T_x(\phi,\psi)u + T_y(\phi,\psi)v]ds + \int_{\Gamma_\varepsilon} [T_x(u,v)\phi +$$

$$+ T_y(u,v)\psi]ds = \sum_{i=1}^{4} K_i \; N_{1i} \tag{4.82}$$

The other terms in equation (4.13) are not affected and the analogous equation of (4.50) is now:

$$\sum_{i=1}^{4} K_i \; N_{1i} = \int_{\Gamma} \overline{X}\phi ds + \int_{\Gamma} \overline{Y}\psi ds - \int_{\Gamma} [T_x(\phi,\psi)u +$$

$$+ T_y(\phi,\psi)v]ds \tag{4.83}$$

By choosing in turn the extraction functions to be the eigenfunctions with the negative eigenvalue $-\kappa_1$ and corresponding to the eigenfunctions that have as stress intensity factors the constants $K_2$, $K_3$, $K_4$ respectively in equations (4.54), (4.55), we obtain three more equations of the same form. We have then a system of four equations in four unknowns to determine $K_1$, $K_2$, $K_3$, and $K_4$.

where:

$$M_i = \sum_{i=1}^{4} K_j N_{ij} \tag{4.77}$$

$$N_{ij} = \int_{-\alpha}^{\alpha} h_{ij}(\theta) d\theta \tag{4.78}$$

We have shown that the integral in (4.78) must be independent of r. Since the functions $c(r)$, $s(r)$, $c_1(r)$, $s_1(r)$ are linearly independent, this can happen only if:

$$M_2 = M_3 = M_4 = M_5 = 0 \tag{4.79}$$

So far only one term in the expansion has been considered. The inclusion of further terms will give rise to an expression of the form:

$$M_1 + c(r)M_2 + s(r)M_3 + c_1(r)M_4 + s_1(r)M_5$$
$$+ r^{\kappa_3}c(r)M_6 + r^{\kappa_3}s(r)M_7 + r^{\kappa_3}c_1(r)M_8 + r^{\kappa_3}s_1(r)M_9 \tag{4.80}$$

as the right-hand-side of (4.76). It can be shown that the new functions of r ($c$, $s$, $c_1$, $s_1$, $r^{\kappa_3} s$, $r^{\kappa_3} c_1$, $r^{\kappa_3} s_1$) are also linearly independent. Thus expression (4.79) can be independent of r only if:

$$M_i = 0 \quad i = 2, \ldots, 9 \tag{4.81}$$

$$F_i(r,\theta) = g_{1i}(\theta) + c(r)\ g_{2i}(\theta) + s(r)\ g_{3i}(\theta)$$

$$+ c_1(r)\ g_{4i}(\theta) + s_1(r)\ g_{5i}(\theta) \qquad (4.73)$$

where now $c(r)$, $s(r)$, $c_1(r)$, $s_1(r)$ are functions of r that are linearly independent. The other three products $T_y(u,v)\psi ds$, $T_x(\phi,\psi)xds$ and $T_y(\phi,\psi)vds$ are of the same form and we can write for the contour $\Gamma_\varepsilon$:

$$- [T_x(\phi,\psi)u + T_y(\phi,\psi,v]ds + [T_x(u,v)\phi + T_y(u,v)\psi]ds =$$

$$= \sum_{i=1}^{4} K_i\ H_i(r,\theta)d\theta \qquad (4.74)$$

where:

$$H_i(r,\theta) = h_{1i}(\theta) + c(r)h_{2i}(\theta) + s(r)h_{3i}(\theta) +$$

$$+ c_1(r)h_{4i}(\theta) + s_1(r)h_{5i}(\theta) \qquad (4.75)$$

Integrating:

$$- \int_{\Gamma_\varepsilon} [T_x(\phi,\psi)u+T_y(\phi,\psi)v]ds + \int_{\Gamma_\varepsilon}[T_x(u,v)\phi+T_y(u,v)\psi]ds =$$

$$M_1 + c(r)M_2 + s(r)M_3 + c_1(r)M_4 + s_1(r)M_5 \qquad (4.76)$$

$$T_x(u,v)\phi ds = \sum_{i=1}^{4} K_i F_i(r,\theta)d\theta \tag{4.66}$$

where

$$F_i(r,\theta) = G_{1i}(\theta) + c(r)G_{2i}(\theta) + s(r)G_{3i}(\theta) + + c^2(r)G_{4i}(\theta) + s^2(r)G_{5i}(\theta) + c(r)s(r)G_{6i}(\theta) \tag{4.67}$$

By denoting:

$$c_1(r) = \cos(2\kappa_2 \ell nr) \tag{4.68}$$

$$s_1(r) = \sin(2\kappa_2 \ell nr) \tag{4.69}$$

we can write:

$$c^2(r) = \cos^2(\kappa_2 \ell nr) = \frac{1}{2}[c_1(r)-1] \tag{4.70}$$

$$s^2(r) = \sin^2(\kappa_2 \ell nr) = \frac{1}{2}[c_1(r)+1] \tag{4.71}$$

$$c(r)s(r) = \cos(\kappa_2 \ell nr)\sin(\kappa_2 \ell nr) = \frac{1}{2} s_1(r) \tag{4.72}$$

and rearrange (4.67) to obtain:

$$T_x(u,v) = r^{\kappa_1 - 1} K_1[T_{x11}(\theta)+c(r)T_{x12}(\theta)+s(r)T_{x13}(\theta)] +$$

$$+ r^{\kappa_1 - 1} K_2[T_{x21}(\theta)+c(r)T_{x22}(\theta)+s(r)T_{x23}(\theta)]$$

$$+ r^{\kappa_1 - 1} K_3[T_{x31}(\theta)+c(r)T_{x32}(\theta)+s(r)T_{x33}(\theta)] +$$

$$+ r^{\kappa_1 - 1} K_4[T_{x41}(\theta)+c(r)T_{x42}(\theta)+s(r)T_{x43}(\theta)] \quad (4.62)$$

$$T_y(u,x) = r^{\kappa_1 - 1} K_1[T_{y4}(\theta)+c(r)T_{y12}(\theta)+s(r)T_{y13}(\theta)] +$$

$$+ r^{\kappa_1 - 1} K_2[T_{y21}(\theta)+c(r)T_{y22}(\theta)+s(r)T_{y23}(\theta)]$$

$$+ r^{\kappa_1 - 1} K_3[T_{y31}(\theta)+c(r)T_{y32}(\theta)+s(r)T_{y33}(\theta)] +$$

$$+ r^{\kappa_1 - 1} K_4[T_{y41}(\theta)+c(r)T_{y42}(\theta)+s(r)T_{y43}(\theta)] \quad (4.63)$$

$$T_x(\phi,\psi) = r^{-\kappa_1 - 1}[X_1(\theta) + c(r)\ X_2(\theta) + s(r)\ X_3(\theta)] \quad (4.64)$$

$$T_y(\phi,\psi) = r^{-\kappa_1 - 1}[Y_1(\theta) + c(r)\ Y_2(\theta) + s(r)\ Y_3(\theta)] \quad (4.65)$$

We are now in a position to evaluate the integrals appearing in (4.13) around the circular arc $\Gamma_\varepsilon$. The product $T_x(u,v)\phi ds$ can be written as:

$$+ r^{\kappa_1} K_3[V_{31}(\theta) + c(r)\ V_{32}(\theta) + s(r)\ V_{33}(\theta)] +$$

$$+ r^{\kappa_1} K_4[V_{41}(\theta) + c(r)\ V_{42}(\theta) + s(r)\ V_{43}(\theta)] \tag{4.55}$$

where:

$$c(r) = \cos\ (\kappa_2 \ell n r) \tag{4.56}$$

$$s(r) = \sin\ (\kappa_2 \ell n r) \tag{4.57}$$

The corresponding extraction functions for extracting $\kappa_1$ are:

$$\phi = r^{-\kappa_1}[\Phi_1(\theta) + c(r)\ \Phi_2(\theta) + s(r)\ \Phi_3(\theta)] \tag{4.58}$$

$$\psi = r^{-\kappa_1}[\Psi_1(\theta) + c(r)\ \Psi_2(\theta) + s(r)\ \Psi_3(\theta)] \tag{4.59}$$

Since

$$\frac{\partial}{\partial r}\ [\cos(\kappa_2 \ell n r)] = - \kappa_2\ r^{-1} \sin(\kappa_2 \ell n r) \tag{4.60}$$

$$\frac{\partial}{\partial r}\ [\sin(\kappa_2 \ell n r)] = \kappa_2\ r^{-1} \cos(\kappa_2 \ell n r) \tag{4.61}$$

the corresponding tractions can be written in the form:

As it was explained in the previous chapter, to the positive real part there correspond four independent eigenfunctions, each with its own stress intensity factor and a singularity of the order $r^{\kappa_1}$. The smooth part also contains terms of the form:

$$\sin(\kappa_2 \ell n r), \ \cos(\kappa_2 \ell n r) \tag{4.53}$$

which means that the eigenfunctions can be written in the form:

$$\begin{aligned} u = {} & r^{\kappa_1} K_1[U_{11}(\theta) + c(r)\, U_{12}(\theta) + s(r)\, U_{13}(\theta)] + \\ & + r^{\kappa_1} K_2[U_{21}(\theta) + c(r)\, U_{22}(\theta) + s(r)\, U_{23}(\theta)] \\ & + r^{\kappa_1} K_3[U_{31}(\theta) + c(r)\, U_{32}(\theta) + s(r)\, U_{33}(\theta) + \\ & + r^{\kappa_1} K_4[U_{41}(\theta) + c(r)\, U_{42}(\theta) + s(r)\, U_{43}(\theta)] \end{aligned} \tag{4.54}$$

$$\begin{aligned} v = {} & r^{\kappa_1} K_1[V_{11}(\theta) + c(r)\, V_{12}(\theta) + s(r)\, V_{13}(\theta)] + \\ & + r^{\kappa_1} K_2[V_{21}(\theta) + c(r)\, V_{22}(\theta) + s(r)\, V_{23}(\theta)] \end{aligned}$$

Equation (4.13) then becomes:

$$K_1(-M_2+M_5) = \int_\Gamma \overline{X}\phi ds + \int_\Gamma \overline{Y}\ ds - \int_\Gamma [T_x(\phi,\psi)u + $$

$$+ T_y(\phi,\psi)v]ds \tag{4.50}$$

When displacements are prescribed on some part of the boundary, the area integral is also present and (4.50) becomes:

$$K_1(-M_1+M_5) = \int_A\int [L_x(\phi,\psi)x + L_y(\phi,\psi)v]dA +$$

$$+ \int_\Gamma \overline{X}\phi ds + \int_\Gamma \overline{Y}\psi ds - \int_\Gamma [T_x(\phi,\psi)x + T_y(\phi,\psi)v]ds \tag{4.51}$$

In order to obtain an approximation to $K_1$ we substitute in (4.50) or (4.51) $u = u_{FE}$ and $v = v_{FE}$.

<u>The case of complex eigenvalues</u>:

Let us now consider the case of a pair of complex conjugate eigenvalues in the form:

$$\kappa = \kappa_1 + i\,\kappa_2 \tag{4.52}$$

or:

$$-M_2(r_3) + M_6(r_3) = -M_2(r_3^*) + M_6(r_3^*) \qquad (4.45)$$

where $r_3$ is the radius of the arc $\Gamma_3$ and $r_3^*$ is the radius of the arc $\Gamma_3^*$. This means that the expression $(-M_2+M_6)$ takes the same value for any value of the radius r. By writing this expression in the form:

$$-M_2 + M_6 = r^{-\kappa_1+\kappa_2} \int_{-\alpha}^{\alpha} F(\theta)d\theta = r^{-\kappa_1+\kappa_2} C \qquad (4.46)$$

it is obvious that this can be independent of r only if the integral is zero (the eigenvalues $\kappa_1$, $\kappa_2$ are distinct), therefore:

$$-M_2 + M_6 = 0 \qquad (4.47)$$

When more terms are kept in the asymptotic expansion the corresponding expression will be:

$$-M_2 + M_6 = r^{-\kappa_1+\kappa_2} C_2 + r^{-\kappa_1+\kappa_3} C_3 + r^{-\kappa_1+\kappa_4} C_4 + \ldots \qquad (4.48)$$

where the eigenvalues $\kappa_2$, $\kappa_3$, $\kappa_4$ are all different from $\kappa_1$. This expression can be independent of r only if:

$$C_2 = C_3 = C_4 = 0 \qquad (4.49)$$

consequently, we obtain again equation (4.47).

$$2C_{II}G(2\pi)^{1/2}\ \frac{\partial u_{II}}{\partial x} = K_{II}\ r^{\kappa_{II}-1}\ \kappa_{II}[(C_{II}-\kappa_{II}-1)$$

$$\sin(\kappa_{II}-1)\theta + 2\ C_{II}(\kappa_{II}-1)$$

$$\sin\theta\cos(\kappa_{II}-2)\theta] \qquad (5.22)$$

$$2C_{II}G(2\pi)^{1/2}\ \frac{\partial u_{II}}{\partial y} = K_{II}\ r^{\kappa_{II}-1}\ \kappa_{II}[(C_{II}D-\kappa_{II}-1)$$

$$\cos(\kappa_{II}-1)\theta - 2\kappa_{II}C_{II}\sin\theta$$

$$\sin(\kappa_{II}-2)\theta + 2C_{II}\cos\theta\cos(\kappa_{II}-2)\theta] \qquad (5.23)$$

$$2C_{II}G(2\pi)^{1/2}\ \frac{\partial v_{II}}{\partial x} = K_{II}\ r^{\kappa_{II}-1}\ \kappa_{II}[-(C_{II}D+\kappa_{II}+1)$$

$$\cos(\kappa_{II}-1)\theta - 2\kappa_{II}C_{II}\cos\theta$$

$$\cos(\kappa_{II}-2)\theta + 2C_{II}\sin\theta\sin(\kappa_{II}-2)\theta] \qquad (5.24)$$

$$2C_{II}G(2\pi)^{1/2}\,\frac{\partial v_{II}}{\partial y} = K_{II}\,r^{\kappa_{II}-1}\,\kappa_{II}[(C_{II}D+\kappa_{II}+1)$$

$$\sin(\kappa_{II}-1)\theta + 2C_{II}(\kappa_{II}-1)\cos\theta$$

$$\sin(\kappa_{II}-2)\theta] \tag{5.25}$$

$$X_I = (\lambda+2G)\,\frac{\partial u_I}{\partial x}\,n_x + \lambda\,\frac{\partial v_I}{\partial y}\,n_x + G\left(\frac{\partial u_I}{\partial y} + \frac{\partial v_I}{\partial x}\right)n_y \tag{5.26}$$

$$Y_I = (\lambda+2G)\,\frac{\partial v_I}{\partial y}\,n_y + \lambda\,\frac{\partial u_I}{\partial x}\,n_y + G\left(\frac{\partial u_I}{\partial y} + \frac{\partial v_I}{\partial x}\right)n_x \tag{5.27}$$

$$X_{II} = (\lambda+2G)\,\frac{\partial u_{II}}{\partial x}\,n_x + \lambda\,\frac{\partial v_{II}}{\partial y}\,n_x + G\left(\frac{\partial u_{II}}{\partial y} + \frac{\partial v_{II}}{\partial x}\right)n_y \tag{5.28}$$

$$Y_{II} = (\lambda+2G)\,\frac{\partial v_{II}}{\partial y}\,n_y + \lambda\,\frac{\partial u_{II}}{\partial x}\,n_y + G\left(\frac{\partial u_{II}}{\partial y} + \frac{\partial v_{II}}{\partial x}\right)n_x \tag{5.29}$$

Let us now cut out from the infinite domain a body of finite dimensions which contains the notch, but is otherwise arbitrary. If we apply to this body the tractions specified by (5.26) to (5.29), then the state of stress and deformation will be given by (5.1) to (5.10) in accordance with the uniqueness theorem of elasticity. Since the body was in equilibrium when embedded in the infinite domain, it must be in equilibrium under the tractions on the cut-out surface. When these tractions are then applied as external loading, this loading is self-equilibrating. The displacement boundary conditions that have to be enforced are then the ones necessary to prevent rigid body motions. In the problems that we analyzed we chose to fix both displacements at the notch tip and the u displacement at the upper right corner.

We can then predetermine the singular elastic field that exists throughout the body by arbitrarily choosing the generalized stress-intensity factors. The applied tractions necessary to induce this field are then given by (5.26) to (5.29). Any number of terms can be retained in the expansion. Thus we can create problems for which the exact values of the generalized stress intensity factors of any order are known. In the problems studied herein only the first terms in the expansions were kept and the stress intensity factors $K_I$ and $K_{II}$ were chosen to have an exact value of unity. The combined loads corresponding to mode I and mode II were applied in each case. Three problems were analyzed with a total solid angle of 360, 330 and 270 degrees respectively as shown in figures 5.1, 5.2, 5.3. The last case is one that occurs very frequently in practice representing a right angle cut-out (L-shaped domain). The results are listed

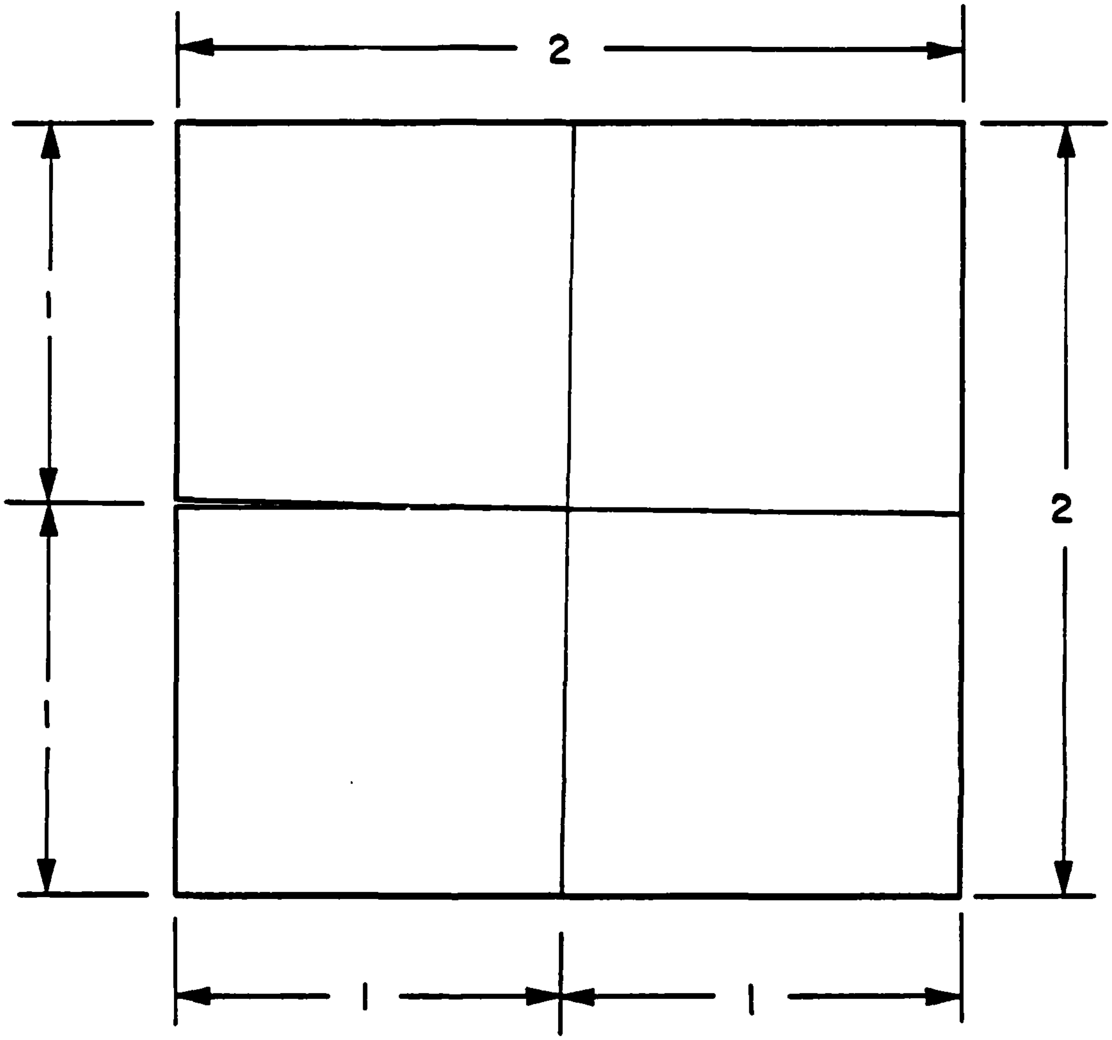


**Figure 5.1**

Element mesh for model problem with a solid angle of 360°

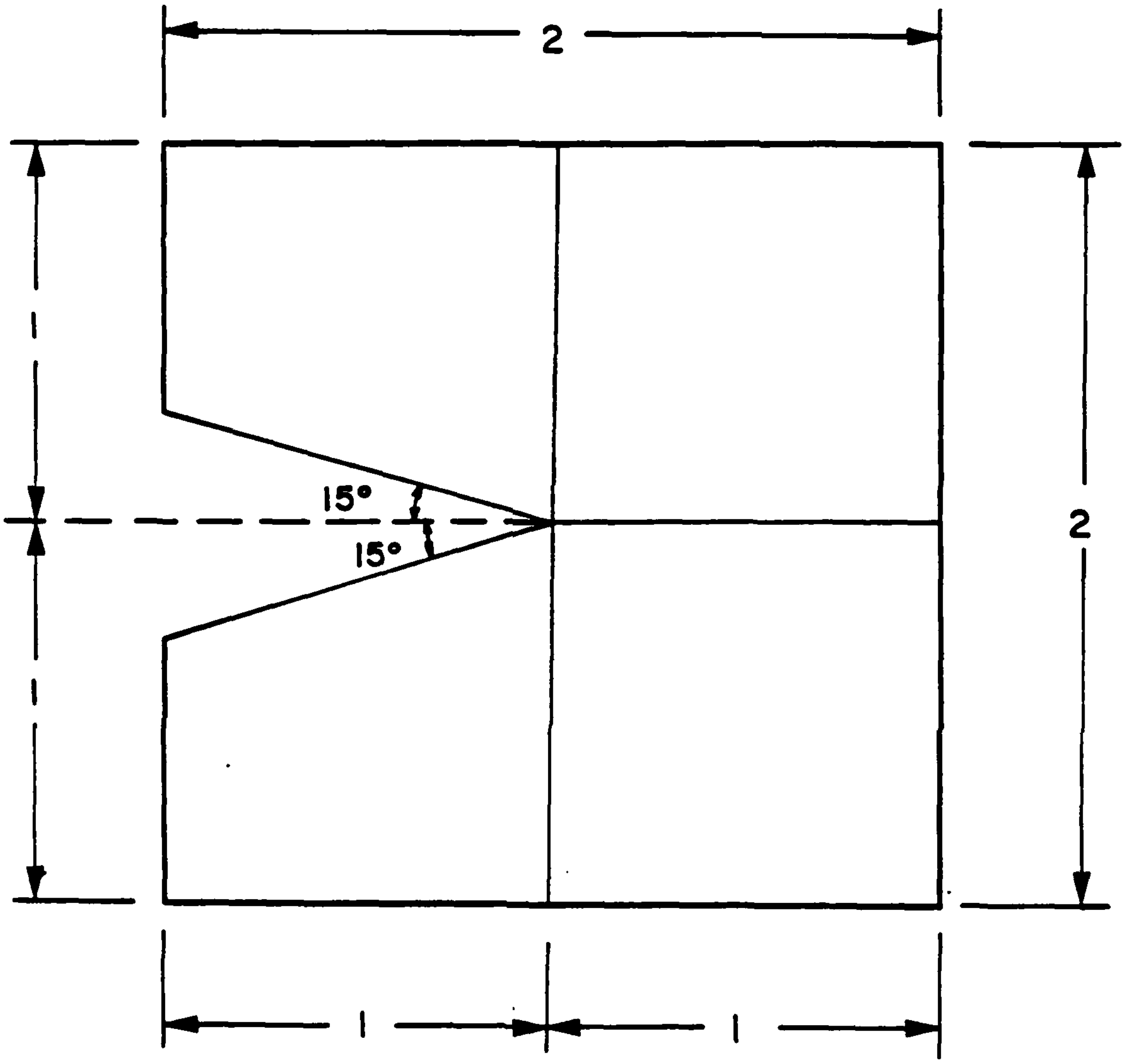


Figure 5.2

Element mesh for model problem with a solid angle of 330°

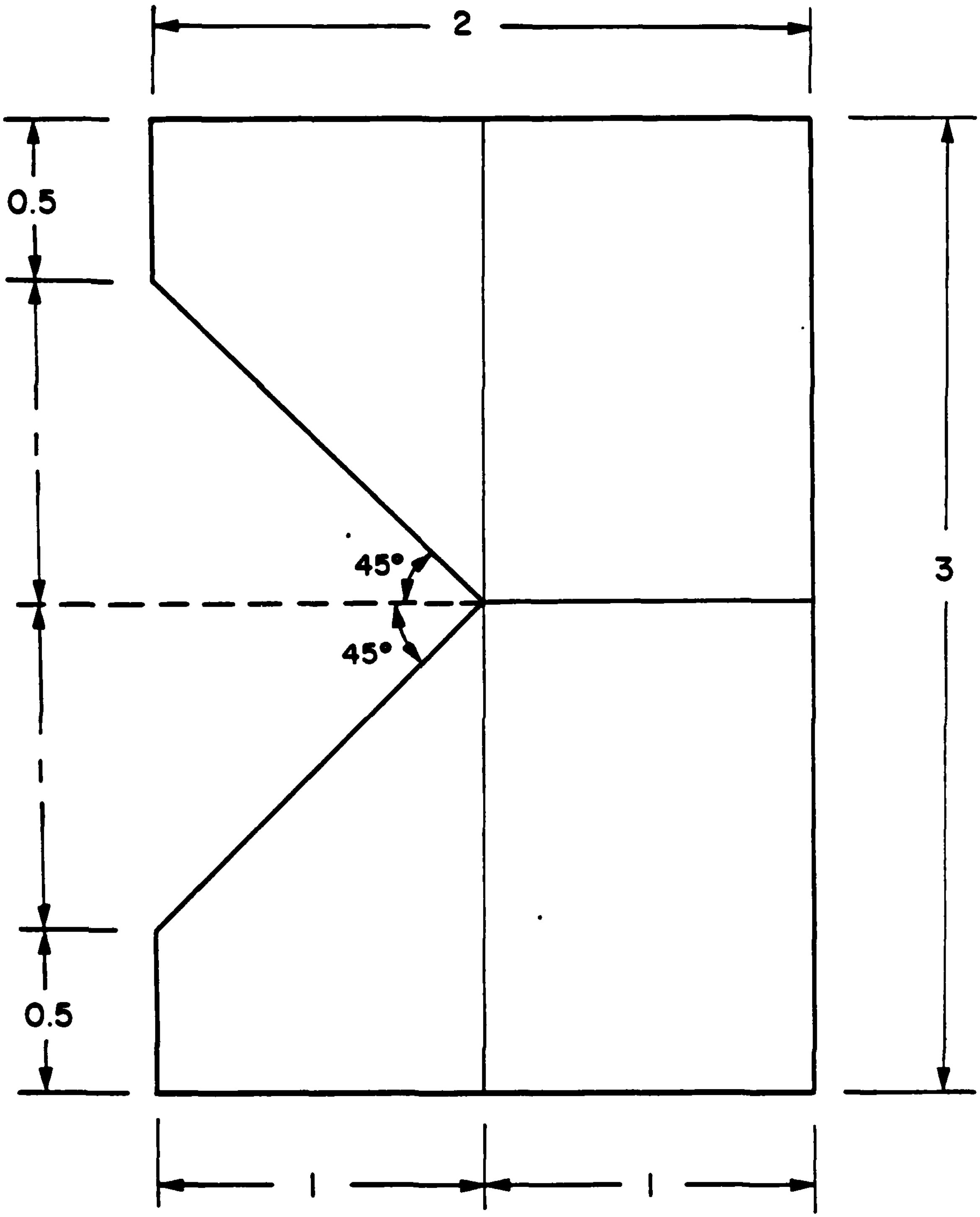


Figure 5.3

Element mesh for model problem with a solid angle of 270°

in tables 5.1, 5.2, 5.3 and are shown graphically in figures 5.4 through 5.12. The theoretically predicted rates of convergence are also shown in the figures. The rate of convergence in energy is equal to twice the smallest eigenvalue, which is the first mode I eigenvalue. For the stress intensity factors the theoretical rate of convergence is equal to twice the corresponding eigenvalue, since the tractions corresponding to mode II have no influence on the mode I stress intensity factor and vice versa. It can be seen from the figures that the theoretical rates of convergence are verified numerically and even for the coarse mesh chosen a relatively small number of degrees of freedom is required to give answers within required engineering accuracy (usually 1 to 5 percent).

Table 5.1

Convergence results for the model problem of figure 5.1 (solid angle = 360°).

Exact $K_I$ = 1.0 Exact $K_{II}$ = 1.0 Exact energy (from extrapolation) ≈ 1.676686.

| polynomial order p | Degrees of freedom | $K_I$ | relative error in $K_I$ | $K_{II}$ | relative error in $K_{II}$ | Energy | relative error in energy |
|---|---|---|---|---|---|---|---|
| 1 | 17 | 0.71080 | 0.28920 | 0.81864 | 0.18136 | 1.32156 | 0.21180 |
| 2 | 43 | 0.79615 | 0.20355 | 0.89552 | 0.10448 | 1.48569 | 0.11391 |
| 3 | 69 | 0.82619 | 0.17381 | 0.91324 | 0.08676 | 1.51795 | 0.09467 |
| 4 | 103 | 0.85962 | 0.14038 | 0.93325 | 0.06675 | 1.55316 | 0.07346 |
| 5 | 145 | 0.89732 | 0.10268 | 0.95036 | 0.04964 | 1.58509 | 0.05463 |
| 6 | 195 | 0.92187 | 0.07813 | 0.96354 | 0.03646 | 1.60780 | 0.04109 |
| 7 | 253 | 0.93884 | 0.06116 | 0.97169 | 0.02831 | 1.62342 | 0.03177 |
| 8 | 319 | 0.95114 | 0.04886 | 0.97751 | 0.02249 | 1.63444 | 0.02519 |

Table 5.2

Convergence results for the model problem of figure 5.2 (solid angle = 330°).

Exact $K_I = 1.0$ Exact $K_{II} = 1.0$ Exact energy (from extrapolation) $\simeq$ 1.646045.

| polynomial order p | Degrees of freedom | $K_I$ | relative error in $K_I$ | $K_{II}$ | relative error in $K_{II}$ | energy | relative error in energy |
|---|---|---|---|---|---|---|---|
| 1 | 17 | 0.74704 | 0.25296 | 0.85067 | 0.14933 | 1.38485 | 0.15868 |
| 2 | 43 | 0.82378 | 0.17622 | 0.93537 | 0.06463 | 1.51976 | 0.07672 |
| 3 | 69 | 0.85039 | 0.14961 | 0.94749 | 0.05251 | 1.54251 | 0.06290 |
| 4 | 103 | 0.88030 | 0.11970 | 0.96594 | 0.03406 | 1.56997 | 0.04622 |
| 5 | 145 | 0.91434 | 0.08566 | 0.97569 | 0.02431 | 1.59103 | 0.03343 |
| 6 | 195 | 0.93556 | 0.06444 | 0.98388 | 0.01612 | 1.60530 | 0.02475 |
| 7 | 253 | 0.94996 | 0.05004 | 0.98810 | 0.01190 | 1.61476 | 0.01901 |
| 8 | 319 | 0.96028 | 0.03972 | 0.99099 | 0.00901 | 1.62125 | 0.01506 |

Table 5.3

Convergence results for the model problem of figure 5.3 (solid angle = 270°).

Exact $K_I$ = 1.0  Exact $K_{II}$ = 1.0  Exact energy (from extrapolation) $\simeq$ 1.033239.

| polynomial order p | Degrees of freedom | $K_I$ | relative error in $K_I$ | $K_{II}$ | relative error in $K_{II}$ | Energy | relative error in energy |
|---|---|---|---|---|---|---|---|
| 1 | 17 | 0.80954 | 0.19046 | 0.87107 | 0.12893 | 0.90956 | 0.11970 |
| 2 | 43 | 0.87689 | 0.12311 | 0.97308 | 0.02692 | 0.97250 | 0.05878 |
| 3 | 69 | 0.89402 | 0.10598 | 0.98173 | 0.01827 | 0.97901 | 0.05249 |
| 4 | 103 | 0.92086 | 0.07914 | 0.99184 | 0.00816 | 0.99625 | 0.03580 |
| 5 | 145 | 0.94656 | 0.05344 | 0.99636 | 0.00364 | 1.00778 | 0.02464 |
| 6 | 195 | 0.96277 | 0.03723 | 0.99767 | 0.00233 | 1.01483 | 0.01782 |
| 7 | 253 | 0.97218 | 0.02782 | 0.99859 | 0.00141 | 1.01944 | 0.01336 |
| 8 | 319 | 0.97845 | 0.02155 | 0.99914 | 0.00086 | 1.02252 | 0.01038 |

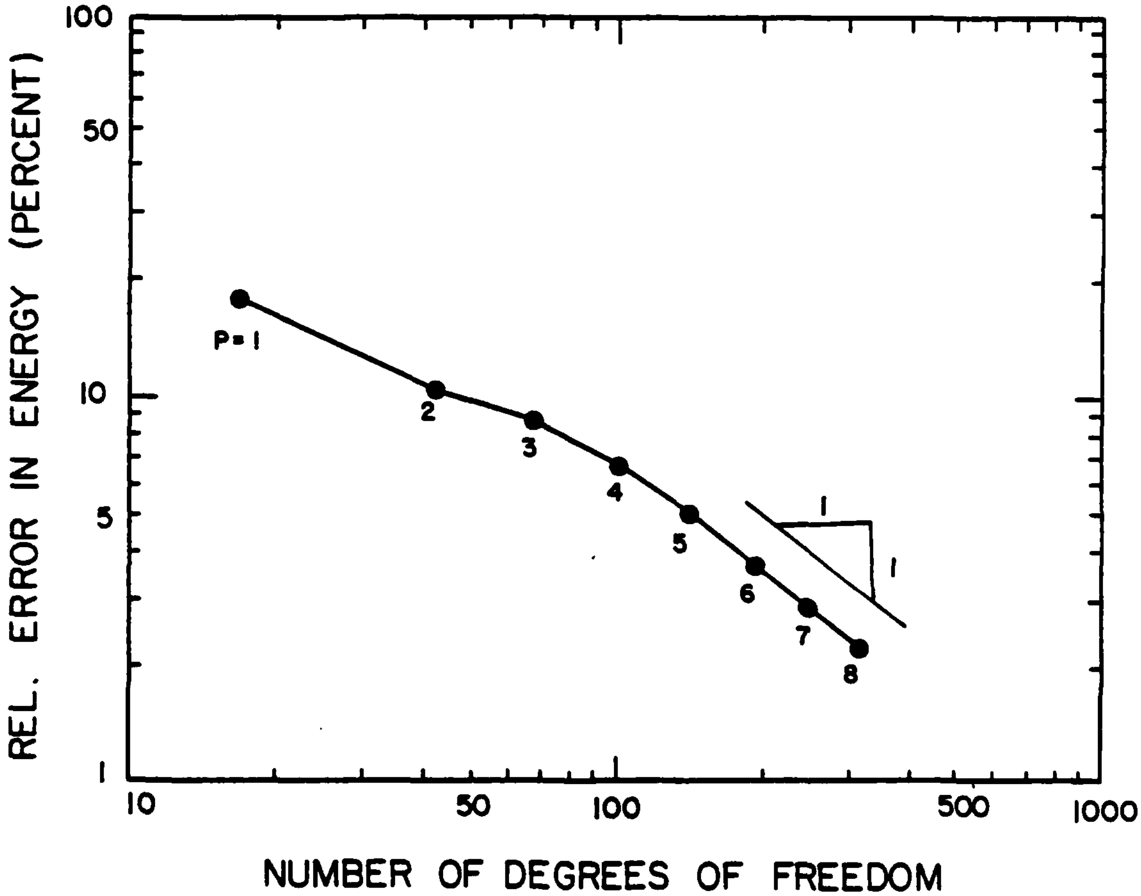


Figure 5.4

Energy convergence for model problem of Figure 5.1

(solid angle = 360°)

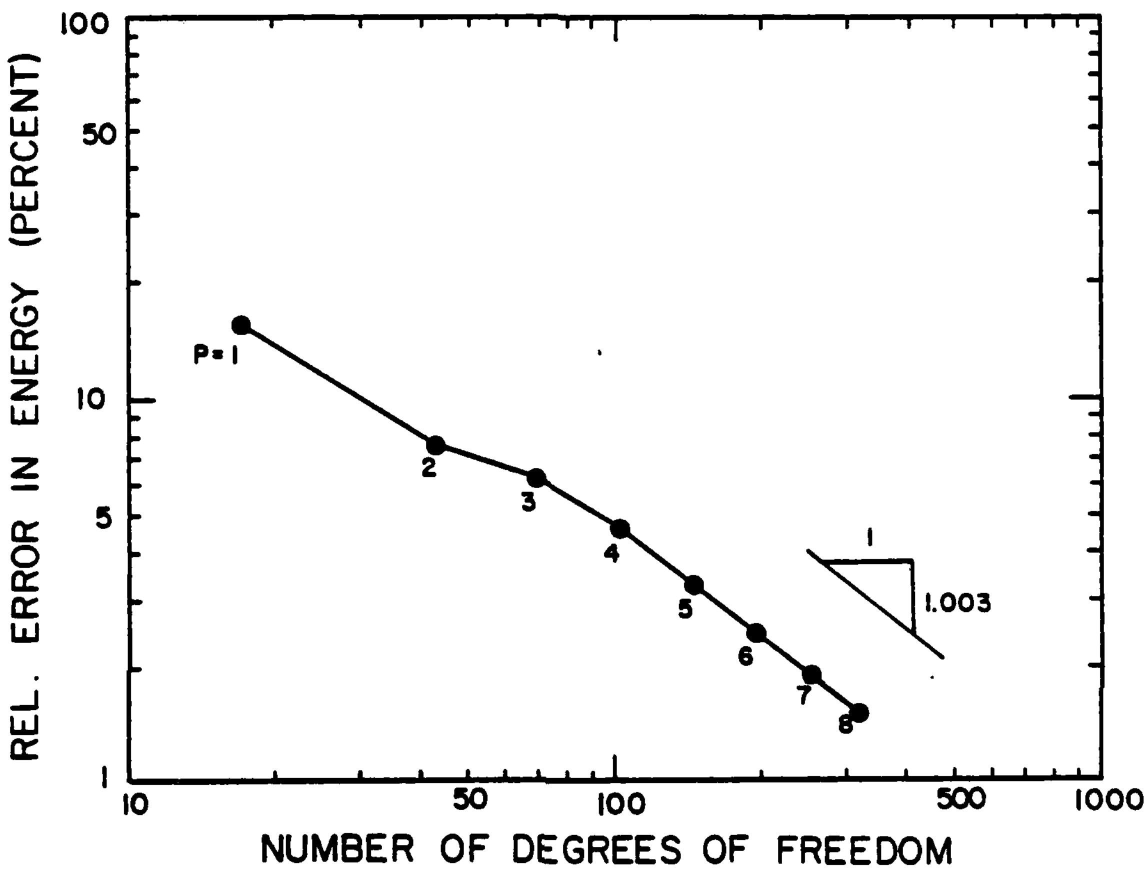


Figure 5.5

Energy convergence for model problem of Figure 5.2

(solid angle = 330°)

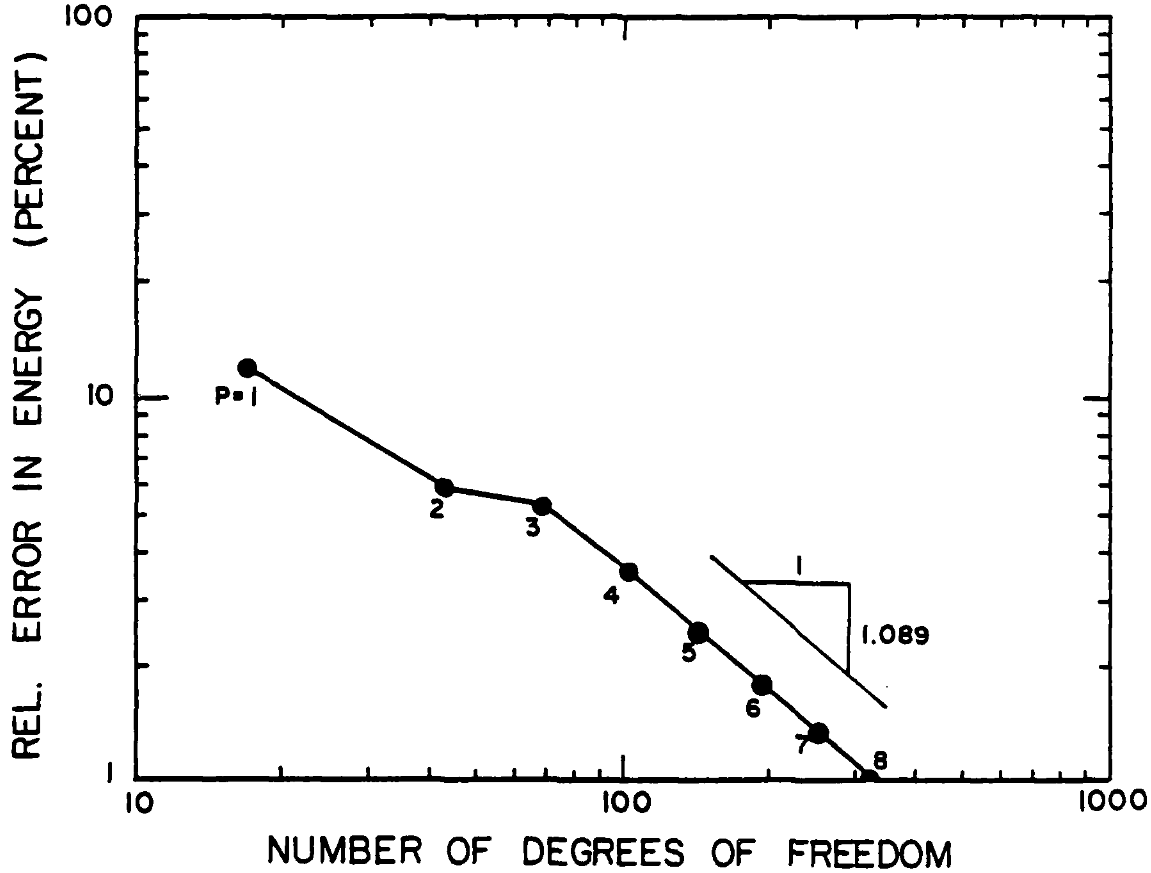


Figure 5.6

Energy convergence for model problem of Figure 5.3

(solid angle = 270°)

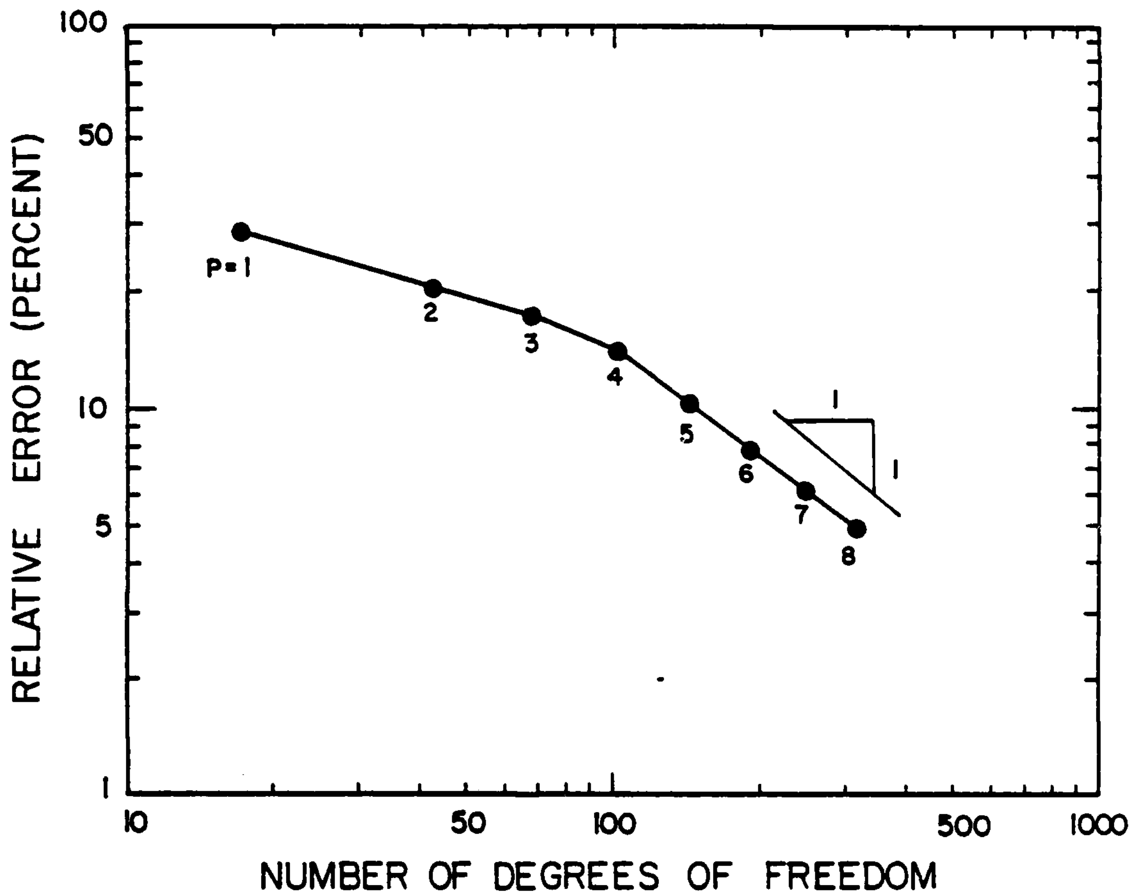


Figure 5.7

Convergence of $K_I$ for model problem of Figure 5.1

(solid angle = 360°)

vicinity of reentrant corners and therefore can be utilized in research in the area of crack initiation. Direct methods for stress computation, whether based on the finite element method of other numerical methods, have the disadvantage that in the vicinity of singular points they incur large errors, therefore they are not well suited for correlation between experimental and analytical data. In order to realize the full benefits of the extraction method, extensive experimentation must be performed with the goal to determine relationships between elastic stress field parameters and failure initiation events. Preliminary experiments with brittle epoxy resin specimens were conducted. Monotonicity was found to exist between the stress intensity factor and failure initiation for a wide range of solid angles. Once theories of failure initiation, based on elastic stress field parameters, are developed and proven the extration method presented herein will provide a basis for the design, analysis and certification of structures with geometric singularities.

# 7. SUMMARY AND CONCLUSIONS

The generalized influence function method for the extraction of stress intensity factors in plane elasticity was evaluated from the point of view of its potential utility in engineering applications. The method was implemented and applied in conjunction with the p-version of the finite element method. The method yields the amplitudes of all terms in the asymptotic expansion of the linear elastic field in the neighborhood of sharp corners and/or sudden changes in boundary conditions. In this investigation only reentrant corners with free-free boundary conditions were considered.

A model problem for which the exact solution is known was used to test the implementation and for evaluating the method. Convergence studies were performed and the theoretically predicted rates of convergence were verified numerically. The method was found to be accurate and reliable. Even with coarse meshes and relatively small number of degrees of freedom, it was possible to compute the stress intensity factors to within a few percent relative error.

The engineering importance of the extraction method presented herein is that it provides good qualitative and quantitative understanding of the stress field in the

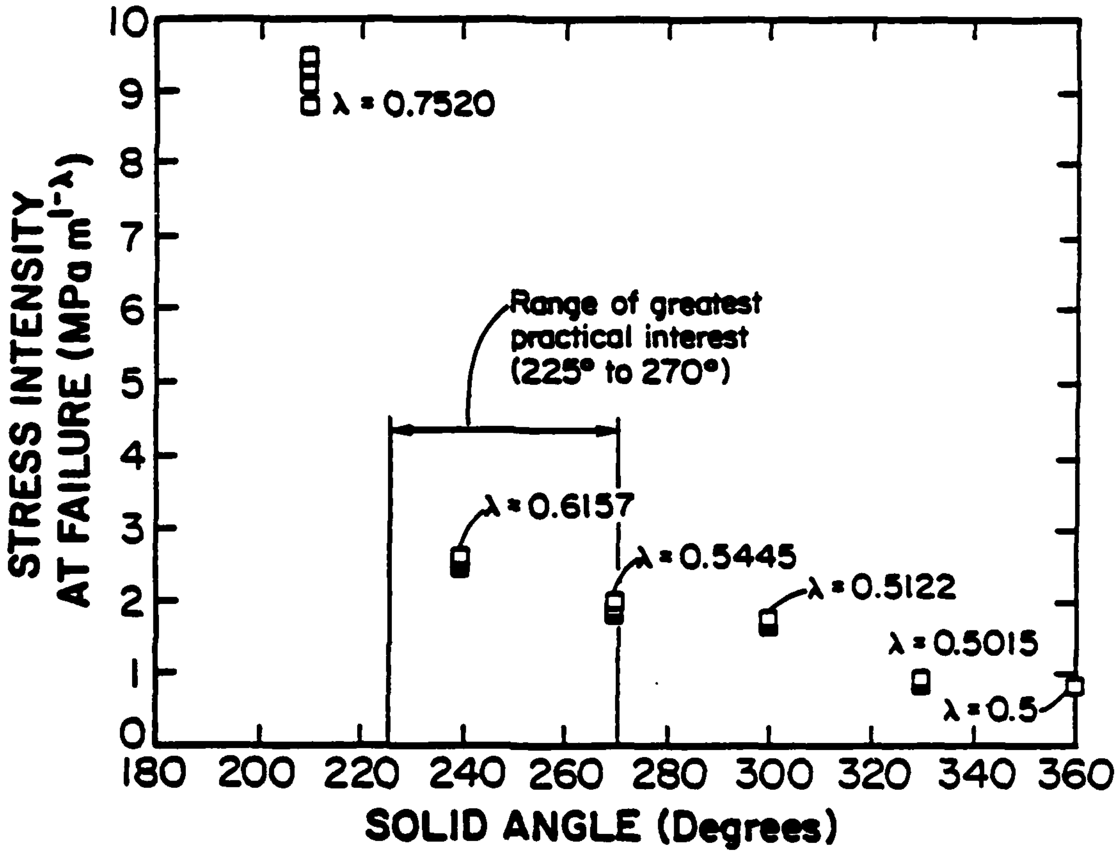


Figure 6.3

Stress intensity ($K_I^*$) at failure vs. solid angle

Table 6.1

Results of experiments

| Solid Angle | First Eigenvalue (λ) | Specimen | Specimen dimensions (mm) Width | Ligament | Thickness | Failure Load (N) | Computed $K_I^*$ ($MPa\ m^{1-\lambda}$) |
|---|---|---|---|---|---|---|---|
| 360° | 0.5 | 2.5 | 38.1 | 28.4 | 3.2 | 657 | 0.805 |
| 330° | 0.5015 | 1.5 | 38.1 | 28.6 | 3.3 | 804 | 0.849 |
| | | 1.6 | 38.1 | 28.6 | 3.3 | 824 | 0.882 |
| | | 2.1 | 38.1 | 28.6 | 3.3 | 785 | 0.839 |
| | | 4+ | 31.3 | 22.2 | 3.2 | 647 | 0.863 |
| 300° | 0.5122 | 3.4 | 38.1 | 28.7 | 3.3 | 1491 | 1.645 |
| | | 3.5 | 38.1 | 28.7 | 3.3 | 1549 | 1.724 |
| | | 4.1 | 38.1 | 28.7 | 3.3 | 1461 | 1.614 |
| 270° | 0.5445 | 3.1 | 38.1 | 26.7 | 3.3 | 1451 | 1.955 |
| | | 3.2 | 38.1 | 26.7 | 3.3 | 1383 | 1.841 |
| | | 3.3 | 38.1 | 26.7 | 3.3 | 1461 | 1.968 |
| 240° | 0.6157 | 4.2 | 38.1 | 29.0 | 3.3 | 1648 | 2.602 |
| | | 4.3 | 38.1 | 29.0 | 3.3 | 1608 | 2.530 |
| | | 4.4 | 38.1 | 29.0 | 3.3 | 1589 | 2.509 |
| 210° | 0.7520 | 1.1 | 38.1 | 28.2 | 3.3 | 3452 | 9.073 |
| | | 1.2 | 38.1 | 28.2 | 3.3 | 3364 | 8.800 |
| | | 1.3 | 38.1 | 28.2 | 3.3 | 3609 | 9.463 |
| | | 4.5 | 38.1 | 28.2 | 3.2 | 3383 | 9.315 |

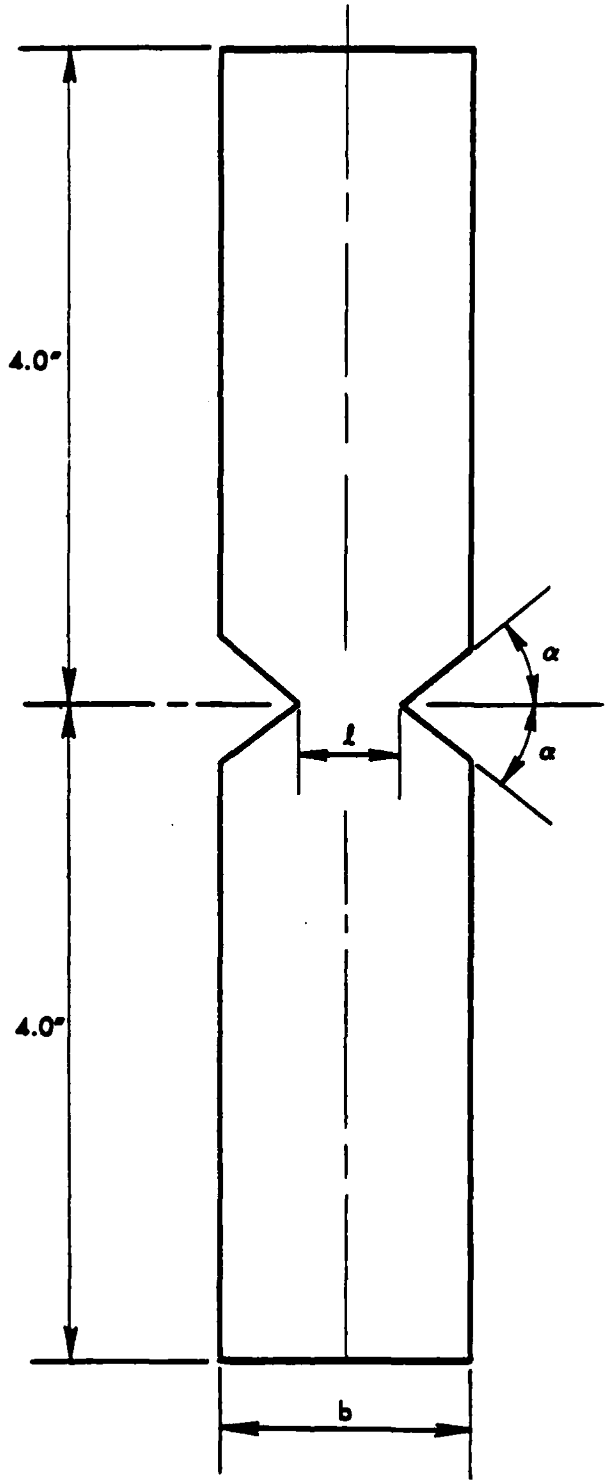


Figure 6.2

Typical test specimen

Eighteen speciments were tested, a typical specimen is shown in figure 6.2. The dimensions of the specimens, the failure load and the stress intensity factors at failure ($K_I^*$) are given in Table 6.1. The dimension of $K_I^*$ depends on the strength of the singularity, which is the corresponding eigenvalue. Due to the brittleness of the material, the failure initiation and failure events occured at virtually the same load level. The experimental results are plotted in figure 6.3. Monotonicity of the critical value of the generalized stress intensity factor $K_I^*$ is observed. It is hoped that future experimental work will establish precise failure initiation criteria in terms of generalized stress intensity factors and clarify their range of applicability.

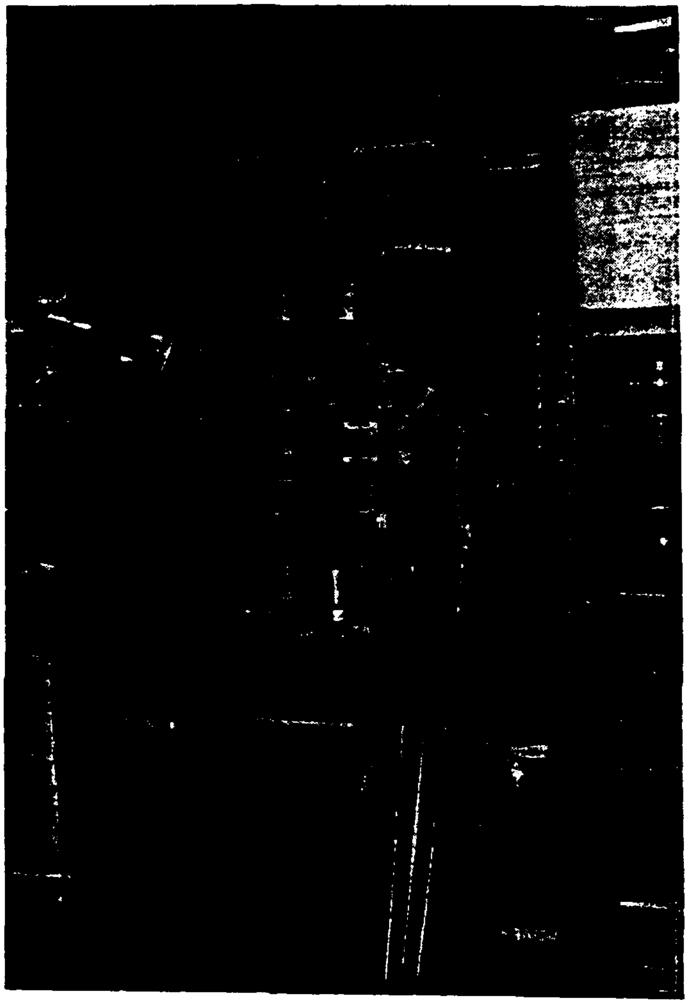

Figure 6.1

Experimental arrangement

the range of solid angles within which the hypothesis is valid. In the context "valid" is meant in the sense: "sufficiently accurate for the purpose of predicting crack initiation in practical applications".

This is one of several possible hypotheses. An extensive amount of experimental work is required in order to identify the minimum number (and kind) of material parameters that need to be measured in order to be able to predict crack initiation events with about the same accuracy and reliability that crack extension is predictable today. The ability to compute with high accuracy the elastic stress field parameters for any sharp notch configuration allows alternate hypotheses to be made and makes this a very promising research area.

The material used for the experiments was a two component epoxy system: 83 percent Shell Epon 828 resin, 17 percent Shell Epon curing agent Z. Modulus of elasticity of this material was 4.3 GPa and Poisson's ratio 0.38-0.39 (based on ultrasonic measurements). The resin was cured 24 hours at room temperature and 12 hours at 65 degrees Celsius. The equipment was an INSTRON testing machine with a 500 kgf load cell. The test arrangement is shown in figure 6.1.

## 6. PRELIMINARY EXPERIMENTS

A number of preliminary experiments were performed in order to obtain baseline data for the hypothesis that, in the case of sharp reentrant corners, crack initiation is controlled by the intensity of the stress singular term.

This hypothesis implies that crack extension and crack initiation are not fundamentally different phenomena; both events can be correlated with the intensity of the stress singular term at the reentrant corner, provided that the solid angle is not much smaller than 360 degrees. Of course, it is very likely that, in the case of smaller solid angles, the intensities of the higher order terms may also have to be considered.

Assuming this hypothesis to be valid, one would expect the intensity of the stress singular term at failure initiation to be some monotonic function of the solid angle. Of course, monotonicity itself is not sufficient evidence that the hypothesis is valid. A number of carefully designed experiments involving various stress fields must be performed and the outcome (i.e. the crack initiation event) successfully predicted from baseline data, representing critical material stress intensities, in order to establish the validity of the hypothesis and

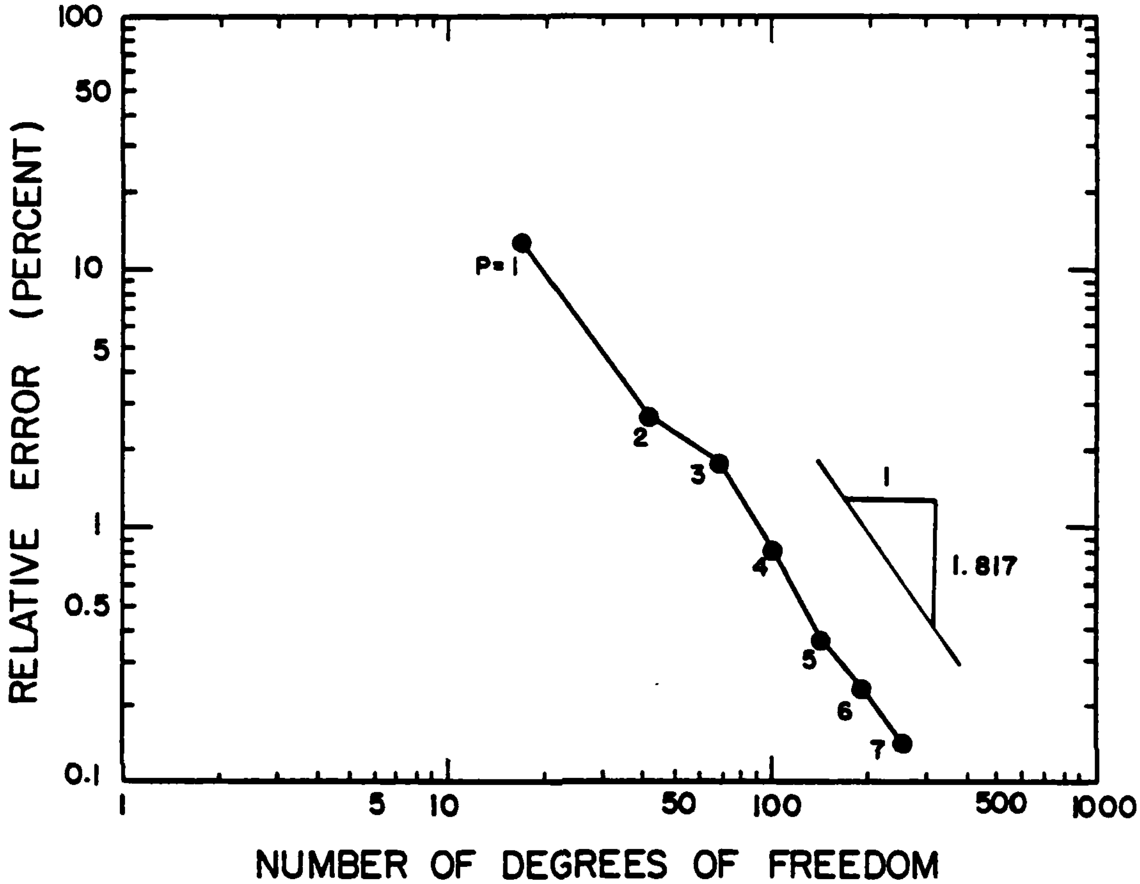


Figure 5.12

Convergence of $K_{II}$ for model problem of Figure 5.3

(solid angle = 270°)

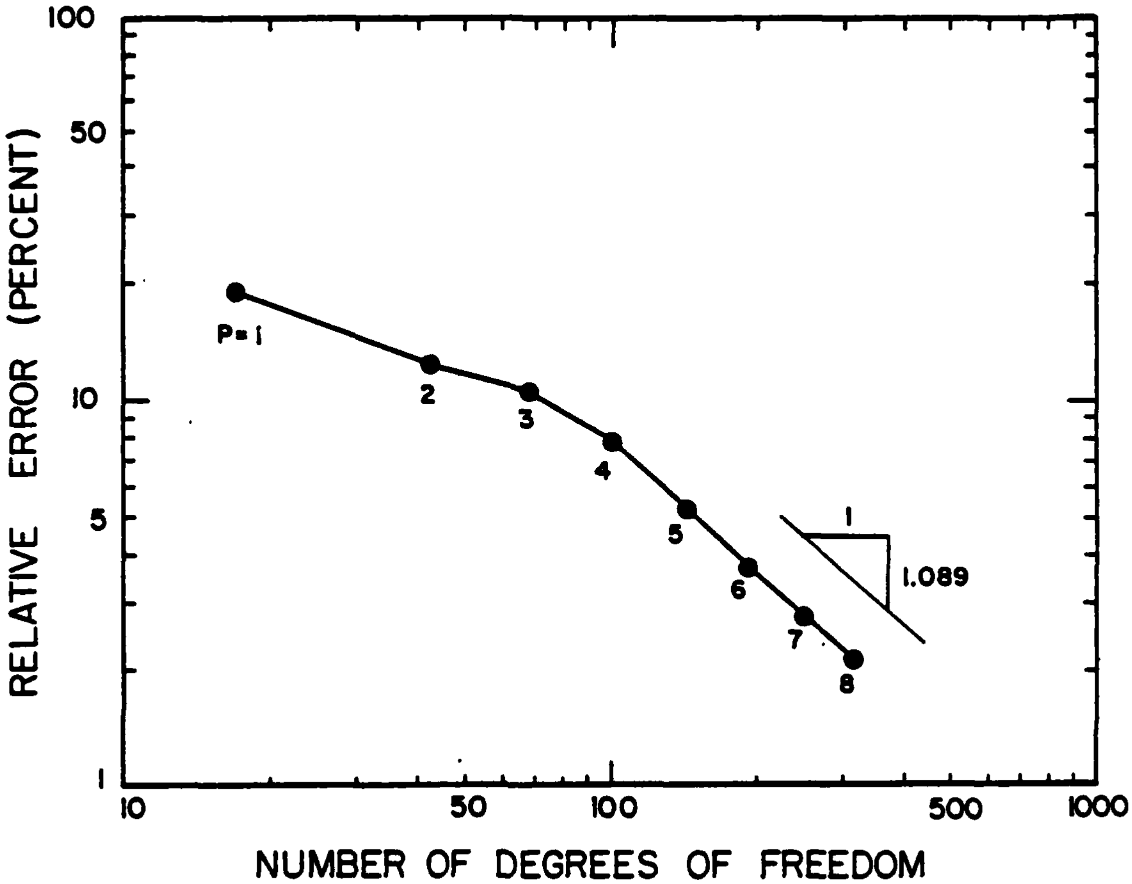


Figure 5.11

Convergence of $K_I$ for model problem of Figure 5.3

(solid angle = 270°)

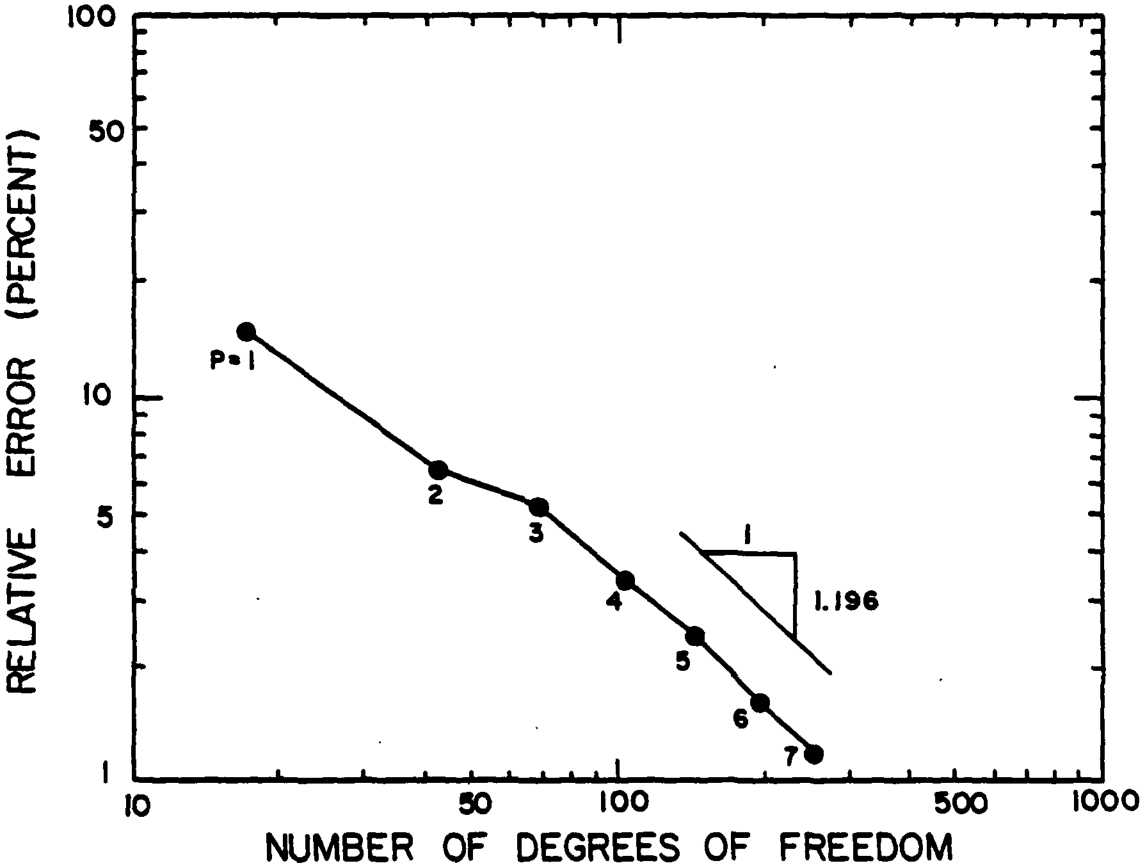


Figure 5.10

Convergence of $K_{II}$ for model problem of Figure 5.2

(solid angle = 330°)

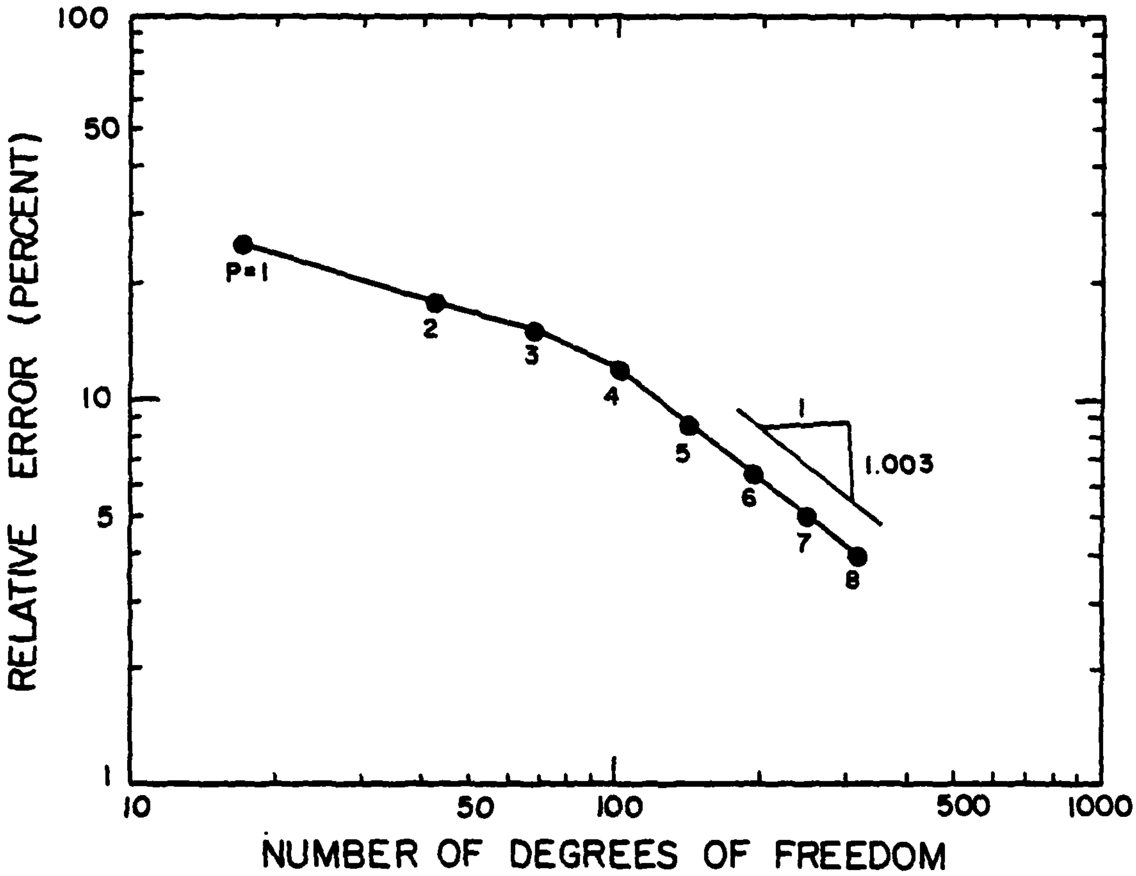


Figure 5.9

Convergence of $K_I$ for model problem of Figure 5.2

(solid angle 330°)

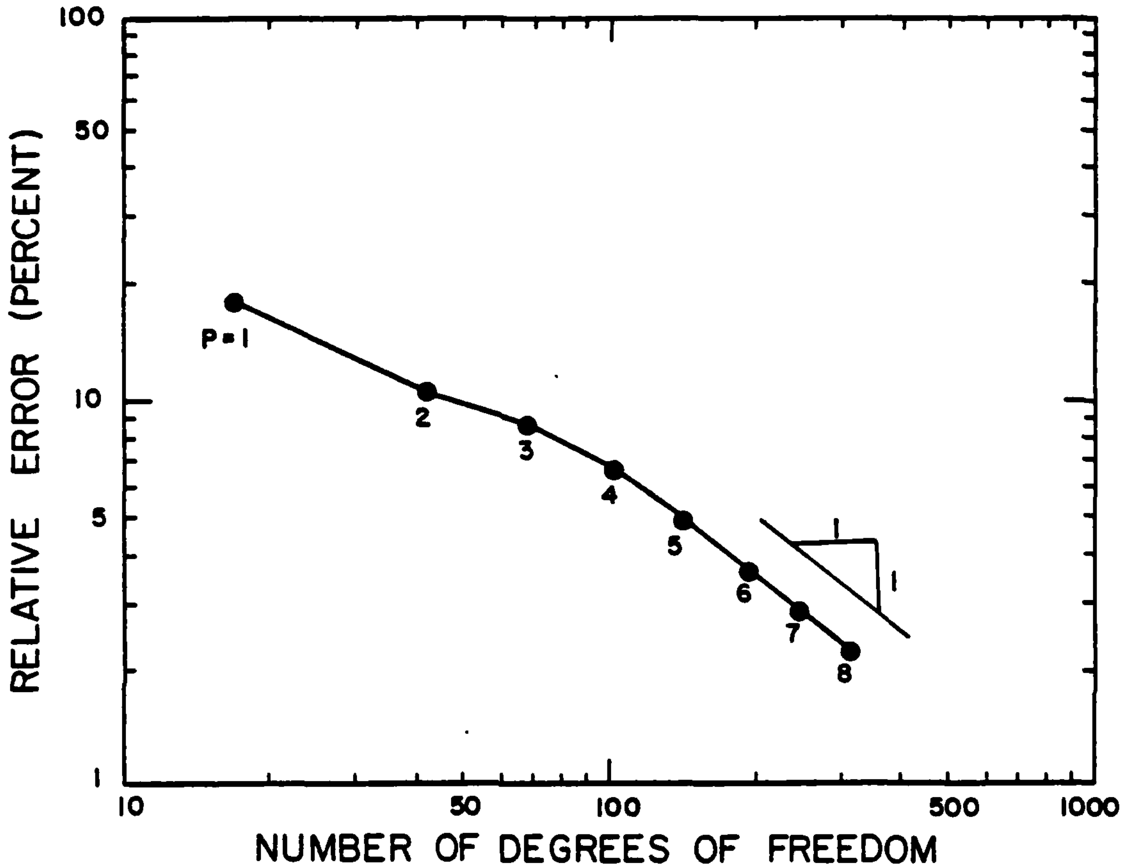


Figure 5.8

Convergence of $K_{II}$ for model problem of Figure 5.1

(solid angle = 360°)

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