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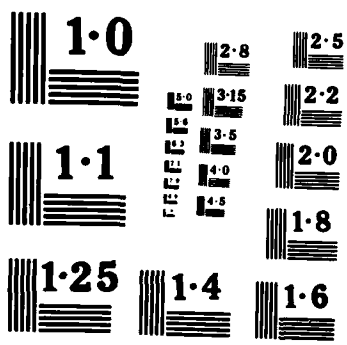
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by

Richard F. Serfozo

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This study shows that when a point process is partitioned into certain uniformly sparce subprocesses, then the subprocesses are asymptotically multivariate Poisson or compound Poisson. Bounds are given for the total-variation distance between the subprocesses and their limits. Several partitioning rules are considered including independent, Markovian, and batch assignments of points.

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PARTITIONS OF POINT PROCESSES: MULTIVARIATE POISSON APPROXIMATIONS

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ABSTRACT

This study shows that when a point process is partitioned into certain uniformly sparse subprocesses, then the subprocesses are asymptotically multivariate Poisson or compound Poisson. Bounds are given for the total-variation distance between the subprocesses and their limits. Several partitioning rules are considered including independent, Markovian, and batch assignments of points. *Additional Keywords: weak convergence; theorems,*

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flows in economic markets. In some instances the partitioning rule is implicit: if N is a point process in which each point has one of several attributes, then the numbers of points with these attributes form a partition of N.

In this paper, we present multivariate Poisson and compound Poisson limit theorems for several partitions. These are weak convergence results for point processes in the setting described in Kallenberg (1975). We also give bounds on the total- variation distance between the partitions and their limits. Related Poisson approximations were developed by Hodges and Le Cam (1960), Le Cam (1960), Freedman (1974), Serfling (1975), Brown (1983) and Serfozo (1985). In particular, we discuss partitions with point assignments that are independent (Section 2), Markovian (Section 3) and syncronous (Section 4). Some of the partitions (Section 4) converge to multivariate infinitely divisible processes with independent increments. We conclude by showing (Section 5) that the asymptotic behavior of a partition, under mild conditions, is not affected by time delays in the assignments. This is why time delays do not appear in the previous results.

2. Partitions With Independent Point Assignments

In this section, we study the asymptotic behavior of the following partition. Let $N = \{N(t);\ t \geq 0\}$ be a point process on R_+ with points at the times $T_1 < T_2 < \ldots$ Suppose N is partitioned by the rule that if a point of N appears at time t, then it is assigned instantaneously to subprocess j with probability $p_j(t)$, independently of everything else, where $\sum_{j=1}^{\infty} p_j(t) = 1$, $t \geq 0$. Let X_k denote the subprocess number to which the point at T_k is assigned. Under our assignment rule,

$$P(X_k = j \mid T_k = t;\ X_\ell,\ T_\ell,\ \ell \neq k) = p_j(t), \text{ for each } j, k, t.$$

The resulting partition $(N_1^\dagger, N_2^\dagger, \ldots)$ is given by

$$N_j^\dagger(t) = \sum_{k=1}^{\infty} I(X_k = j) I(T_k \leq t), \quad t \geq 0,$$

where $I(A)$ is the indicator random variable of the event A.

We will consider the behavior of the partition as $p_j(t)$ tends to zero. To this end, we assume that $p_j(t)$ depends on n and denote it by $p_{nj}(t)$. We consider the normalized partition

$$\underset{\sim}{N}_n(t) := (N_{n1}(t), N_{n2}(t), \ldots) := (N_1^\dagger(a_n t), N_2^\dagger(a_n t), \ldots), \quad t \geq 0,$$

which is the original partition with the time scale changed so that the constant a_n is the new time unit. We assume $a_n \to \infty$. Here is a Poisson limit theorem for $\underset{\sim}{N}_n$.

Theorem 2.1. Suppose $N(t)/t \xrightarrow{\mathcal{D}} \lambda$, a positive constant, and, for each j, there is a measurable function $r_j: R_+ \to R_+$ such that

$$\lim_{n\to\infty} a_n p_{nj}(a_n t) = r_j(t), \text{ uniformly on finite intervals.}$$

Then $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$, where $\underset{\sim}{N} = (N_1, N_2, \ldots)$ is a vector of independent Poisson processes with respective intensities $\lambda r_1, \lambda r_2, \ldots$

Comments 2.2. Although $N_{n1}, N_{n2}, \ldots$ are generally dependent, their limits $N_1, N_2, \ldots$ are independent. We have assumed, for simplicity, that the original process N does not depend on n. Theorem 2.1 also applies, however, when N is a function $N^{(n)}$ of n such that

$$N^{(n)}(t_n)/t_n \xrightarrow{\mathcal{D}} \lambda \text{ as } n \to \infty, \text{ for each } t_n \to \infty,$$

or, equivalently, that $T_{k_n}^{(n)}/k_n \xrightarrow{\mathcal{D}} \lambda^{-1}$ for each $k_n \to \infty$.

Proof. To prove $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$, it suffices to show that the Laplace functional of $\underset{\sim}{N}_n$ converges to that of $\underset{\sim}{N}$. That is, for each $J \geq 1$ and continuous where the supremum is over all measurable sets. When X and Y are discrete with densities f and g, respectively, then this becomes

$$d(X,Y) = (1/2) \sum_x |f(x) - g(x)|.$$

<u>Corollary</u> 2.3. Suppose $p_{nj}(t) = p_{nj}$, independent of t, and N has stationary increments with finite $\lambda := EN(1)$ and $\sigma^2 := \mathrm{Var}\, N(1)$. Let $q_n = \sum_{j=1}^J p_{nj}$, and $\underset{\sim}{N}_n^J = (N_{n1},\ldots,N_{nJ})$. If $\underset{\sim}{Z}_n = (Z_{n1},\ldots,Z_{nJ})$ is a vector of independent Poisson random variables with $EZ_{nj} = t\lambda a_n p_{nj}$, then

$$d(\underset{\sim}{N}_n^J(t), \underset{\sim}{Z}_n) \le t\lambda a_n q_n^2, \quad t \ge 0.$$

If $\underset{\sim}{Z} = (Z_1,\ldots,Z_J)$ is a vector of independent Poisson random variables with $EZ_j = t\lambda\alpha$, then

$$d(\underset{\sim}{N}_n^J(t), \underset{\sim}{Z}) \le t\lambda a_n q_n^2 + q_n \sigma\sqrt{ta_n} + t\lambda |a_n q_n - \alpha|, \quad t \ge 0.$$

<u>Proof</u>. This is a special case of Corollary 4.3 below with $\bar{p}_{nk} = q_n$, $d(\underset{\sim}{M}_{nk}^J, \underset{\sim}{M}) = 0$, $EN_n(t) = \lambda a_n t$, and

$$E\left|\sum_{k=1}^{N_n(t)} \bar{p}_{nk} - t\lambda\alpha\right| = E|q_n N(a_n t) - t\lambda\alpha|$$

$$\le q_n\sqrt{\mathrm{Var} N(a_n t)} + t\lambda |a_n q_n - \alpha|.$$

3. <u>Partitions With Markovian Point Assignments</u>

Let $N = \{N(t);\ t \ge 0\}$ be a point process on R_+ with points at $T_1 \le T_2 \le \ldots$ Suppose that N is partitioned by the rule that its point at T_k is assigned instantaneously to subprocess number X_k. The resulting partition is $(N_1^\dagger, N_2^\dagger,\ldots)$ is given by

$$N_j^\dagger(t) = \sum_{k=1}^{\infty} I(X_k = j) I(T_k \le t), \quad t \ge 0,\ j = 1,2,\ldots$$

We assume that the assignment process $X_0, X_1, X_2, \ldots$ is a stationary Markov chain with state space $\{1,2,\ldots\}$, transition probabilities p_{ij}, and distribution $\pi_i = P(X_k = i)$. In this section, we study the asymptotic behavior of this partition as the π_i's tend to zero.

Assume that the partition depends on n and consider the finite segment

$$\underset{\sim}{N}_n(t) := (N_{n1}(t),\ldots,N_{nJ}(t)) := (N_1^{\dagger}(a_n t),\ldots,N_J^{\dagger}(a_n t)),\ t > 0,$$

with time unit $a_n := [\sum_{j=1}^{J}\pi_j(1 - p_{jj})]^{-1}$ and $J < \infty$ fixed. The form of a_n and the need for finite J emanates from our analysis.

We will assume that p_{ij} depends on n such that

$$\text{(3.1)} \qquad \sup_i p_{ij} \to 0 \quad \text{as } n \to \infty \quad \text{for each } j = 1,\ldots,J.$$

This implies that $\pi_j \to 0$ as $n \to \infty$ for each $j = 1,\ldots,J$, and hence $a_n \to \infty$. Another consequence is that

$$q_i := \sum_{j=1}^{J} p_{ij} - p_{ii} \to 0 \text{ as } n \to \infty.$$

This q_i is the probability that the point assignment changes from subprocess i to any other subprocess $j \neq i$ in $\{1,\ldots,J\}$. Keep in mind that p_{ij}, π_i and q_i depend on n, but we are not appending an n to them.

Next, we assume that there are probabilities $r_1,\ldots,r_J$ summing to one such that

$$\text{(3.2)} \qquad \sup_i (p_{ij} - q_i r_j) I(q_i > 0) \to 0 \quad \text{for each } j = 1,\ldots,J,$$

and that

$$\text{(3.3)} \qquad \sup_i \pi_i(1 - r_i) I(q_i = 0) / \sum_k \pi_k q_k \to 0 \quad \text{as } n \to \infty.$$

We also assume, for simplicity, that N has stationary increments with finite $\lambda := EN(1)$ and $\sigma^2 := VarN(1)$; that $p_{jj}, r_j, \lambda, \sigma$ are independent of n; and that $0 < p_{jj} < 1$, $r_j > 0$, $\pi_i > 0$ for each i and $j = 1,\ldots,J$. Let

$$\delta_n := 1/2\Big[\sum_i \pi_i \sum_{\substack{j=1\\ j\neq i}}^{J} |p_{ij} I(q_i > 0)/q_i - r_j| + \sum_i \pi_i (1 - r_i) I(q_i = 0)\Big].$$

And let $\underset{\sim}{N}(t) := (N_1(t),\ldots,N_J(t))$ be a vector of independent compound Poisson processes such that the atoms of N_j appear at the rate λr_j and their size has the geometric density $g_j(\ell) := p_{jj}^{\ell-1}(1 - p_{jj})$, $\ell \geq 1$. Note that when $p_{jj} = 0$, then N_j is a Poisson process.

Theorem 3.1. If $\{X_k\}$ is independent of N, then

$$d(\underset{\sim}{N}_n(t), \underset{\sim}{N}(t)) \leq \sum_{j=1}^{J} \pi_j (1 - p_{jj}) + t\lambda a_n(\delta_n + \sum_i \pi_i q_i^2) + [t a_n^{-1}(\sigma - \lambda) + t\lambda a_n \sum_i \pi_i q_i^2]^{1/2}, \quad t \geq 0. \tag{3.4}$$

If (3.1) - (3.3) hold, then the right-hand side of (3.4) converges to zero as $n \to \infty$. If $N(t)/t \xrightarrow{\mathcal{D}} \lambda$ as $t \to \infty$ and, for each J, (3.1) - (3.3) hold and

$$\lim_{n\to\infty} P(X_{[ma_n]} = j | X_0 = i)/\pi_j = 1, \quad \text{for each } m \geq 1,\ j = 1,\ldots,J, \tag{3.5}$$

then $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$.

Comments Assumptions (3.1) - (3.3) are not used for the first assertion. The inequality (3.4) implies that the subprocesses $N_{n1},\ldots,N_{nJ}$ are approximately independent Poisson or compound Poisson processes when the right-hand side of (3.4) is near zero. Note that the independence of $\{X_k\}$ and N is invoked for (3.4) but not for the other assertions; this independence can be relaxed as in Theorem 4.4. Theorem 3.1 also applies when N is dependent on n; the assumption $N(t)/t \to \lambda$ would have to be modified as in Comments 2.2.

Consider the last term in (3.6). Let g be the probability density on $\{0,1,...\}^{\infty}$ defined by

$$g(\ell \underset{\sim}{u}_j) = r_j g_j(\ell), \quad \ell = 1,2,...$$

where $\underset{\sim}{u}_j$ is the J-dimensional unit vector with a one in the j-th component and zeros elsewhere. Applying Theorem 1 (expression (1.5)) in Serfozo (1985), we have

$$(3.7) \qquad d(\underset{\sim}{S}_n, \underset{\sim}{N}(t)) \le E \sum_{k=1}^{N(a_n t)} [p_k^2 + d(f_k,g)] + E\left| \sum_{k=1}^{N(a_n t)} p_k - \lambda t\right|,$$

where

$$p_k := P(\underset{\sim}{L}_k \ne \underset{\sim}{0} | X_0,...,X_{k-1}) = \sum_i I(X_{k-1} = i)q_i$$

$$f_k(\ell \underset{\sim}{u}_j) := P(\underset{\sim}{L}_k = \ell \underset{\sim}{u}_j | X_0,...,X_{k-1}, \underset{\sim}{L}_k \ne 0)$$

$$= \sum_{i;i\ne j} I(X_{k-1} = i)(p_{ij}/q_i)I(q_i > 0)g_j(\ell), \quad j = 1,...,J,$$

$$d(f_k,g) := (1/2) \sum_{j=1}^{J} \sum_{\ell=1}^{\infty} |f_k(\ell \underset{\sim}{u}_j) - g(\ell \underset{\sim}{u}_j)|.$$

To evaluate the right-hand side of (3.7), first note that

$$Ep_k = \sum_i \pi_i q_i = a_n^{-1}$$

$$Ep_k^2 = E[\sum_i I(X_{k-1} = i)q_i^2] = \sum_i \pi_i q_i^2$$

$$Ed(f_k,g) = \delta_n.$$

Since N has stationary increments and is independent of the stationary

Markov chain $\{X_k\}$, then

$$(3.8) \qquad E\left[\sum_{k=1}^{N(a_nt)} p_k\right] = EN(a_nt)Ep_1 = \lambda t,$$

$$(3.9) \qquad E\left[\sum_{k=1}^{N(a_nt)} (p_k^2 + d(f_k,g))\right] = \lambda a_n t[\delta_n + \sum_i \pi_i q_i^2],$$

and, by (3.8) and Schwarz's inequality we have

$$(3.10) \qquad E\left|\sum_{k=1}^{N(a_nt)} p_k - \lambda t\right| \leq \left[\operatorname{Var}\sum_{k=1}^{N(a_nt)} p_k\right]^{1/2}$$

$$= [EN(a_nt)\operatorname{Var}p_1 + (Ep_1)^2 \operatorname{Var}N(a_nt)]^{1/2}$$

$$= [ta_n^{-1}(\sigma - \lambda) + t\lambda a_n \sum_i \pi_i q_i^2]^{1/2}.$$

Then using (3.9) and (3.10) in (3.7), combined with (3.6), yields the desired inequality (3.4).

The second assertion of Theorem 3.1 is true since one can show that (3.2) and (3.3) imply $a_n\delta_n \to 0$, and that (3.1) implies

$$a_n \sum_i \pi_i q_i^2 \leq \sum_i \pi_i q_i^2 / \sum_i \pi_i q_i \to 0.$$

To prove the third assertion, consider the multivariate process

$$\underset{\sim}{Y}_n(t) = \sum_{k=1}^{[a_nt]} (I(X_k=1),\ldots,I(X_k=J)), \quad t \geq 0,$$

where $[r]$ denotes the integer part of r. Let $\underset{\sim}{N}^1$ denote the process $\underset{\sim}{N}$ with $\lambda = 1$. The first two assertions apply to $\underset{\sim}{Y}_n$ with $N(t) = [t]$, $\lambda = 1$ and $\sigma = 0$. Thus $\underset{\sim}{Y}_n(t) \xrightarrow{\mathcal{D}} \underset{\sim}{N}^1(t)$ for each t. Since $\{X_k\}$ is stationary and $\underset{\sim}{N}^1$ has stationary increments, then we have $\underset{\sim}{Y}_n(t) - \underset{\sim}{Y}_n(s) \xrightarrow{\mathcal{D}} \underset{\sim}{N}^1(t) - \underset{\sim}{N}^1(s)$

In the preceding sections, the partitions converge to vectors of independent processes. But here, the syncronous point assignments lead to the convergence of N_n to vectors of dependent processes. These limiting processes are as follows. Suppose $\underset{\sim}{N} = (N_1, N_2, \ldots)$ is a multivariate infinitely divisible point process with independent increments and Laplace functional

$$E\exp\{-\sum_{j=1}^{J} \int_{R_+} f_j(t) N_j(dt)\} = \exp\{-\int_{R_+} \sum_{\underset{\sim}{m}} [1 - \exp(-\sum_{j=1}^{J} m_j f_j(t))] \mu(\underset{\sim}{m} \times dt)\},$$

where μ is the canonical measure on $\{0,1,\ldots\}^{\infty} \times R_+$ that satisfies

$$(4.1) \qquad \sum_{\underset{\sim}{m}} [1 - \exp(-\sum_{j=1}^{J} m_j)] \mu(\underset{\sim}{m} \times [0,t]) < \infty, \quad t > 0.$$

This is a multivariate analogoue of the point processes in Chapter 7 of Kallenberg (1975), or in Kerstan, Matthes and Mecke (1978). When $\mu(\underset{\sim}{m} \times dt) = f(\underset{\sim}{m}) \lambda(dt)$, then $\underset{\sim}{N}$ is a compound Poisson process whose atom locations in R_+ are Poisson with intensity measure λ and its vector-valued atom sizes have the density f; we simply say that $\underset{\sim}{N}$ is multivariate compound Poisson(λ, f). In case f is concentrated on $\{0,1\}^{\infty}$, then $\underset{\sim}{N}$ is multivariate Poisson with intensity λ. In either case, the $N_1, N_2, \ldots$ are independent when $\mu(\underset{\sim}{m} \times dt) = \sum_j I(\underset{\sim}{m} = m_j \underset{\sim}{u}_j) \mu_j(m_j \times dt)$, where $\mu_1, \mu_2, \ldots$ are measures on $\{0,1,\ldots\} \times R_+$.

For the following results, we assume that the parent process N and the partition depend on n, and we let $T_k^{(n)}$ denote T_k and $\underset{\sim}{M}_k^{(n)} :=$ $(M_{k1}, M_{k2}, \ldots)$.

<u>Theorem</u> 4.1. Suppose that

$$(4.2) \qquad T_{k_n}^{(n)}/k_n \xrightarrow{\mathcal{D}} 1, \text{ for each } k_n \to \infty,$$

and that $\underset{\sim}{M}_1^{(n)}, \underset{\sim}{M}_2^{(n)}, \ldots$ are independent and satisfy

$$(4.3) \qquad \lim_{n\to\infty} \max_{k\le ma_n} P((M_{k1}^{(n)},\ldots,M_{kJ}^{(n)}) \ne \underset{\sim}{0}) = 0 \quad \text{for each } J,m.$$

Then $\underset{\sim}{N}_n$ converges in distribution to some $\underset{\sim}{N}$ if and only if there is a measure μ, as above, that satisfies (4.1) and is such that

$$(4.4) \qquad \lim_{n\to\infty} \sum_{k=[a_n s]}^{[a_n t]} P(\underset{\sim}{M}_k^{(n)} = \underset{\sim}{m}) = \mu(\underset{\sim}{m} \times (s,t])$$

for each $s < t$ with $\mu(\underset{\sim}{m} \times \{s\}) = \mu(\underset{\sim}{m} \times \{t\}) = 0$. In this case, the multivariate process $\underset{\sim}{N}$ is infinitely divisible with independent increments and canonical measure μ.

<u>Proof</u>. Define

$$\gamma_n(t) := a_n^{-1} \sum_k I(T_k^{(n)} \le a_n t), \qquad \underset{\sim}{\xi}_k^{(n)}(t) := \underset{\sim}{M}_k^{(n)} I(k \le a_n t)$$

$$\underset{\sim}{M}_n(t) := (M_{n1}(t), M_{n2}(t),\ldots) := \sum_k \underset{\sim}{\xi}_k^{(n)}(t), \quad t \ge 0.$$

Then we can write $N_{nj}(t) = M_{nj}(\gamma_n(t))$, $t \ge 0$. Assumption (4.2) implies that $\gamma_n \overset{\mathcal{D}}{\to} \gamma$ where $\gamma(t) = t$, $t \ge 0$. Thus, by Lemma 3.2, the statement $\underset{\sim}{N}_n \overset{\mathcal{D}}{\to} \underset{\sim}{N}$ is equivalent to $\underset{\sim}{M}_n \overset{\mathcal{D}}{\to} \underset{\sim}{N}$. In addition, note that $\underset{\sim}{M}_n = \sum \underset{\sim}{\xi}_k^{(n)}$, where $\underset{\sim}{\xi}_1^{(n)}, \underset{\sim}{\xi}_2^{(n)},\ldots$ are independent and satisfy, by (4.3),

$$\max_k P((\xi_{k1}^{(n)}(t),\ldots,\xi_{kJ}^{(n)}(t)) \ne 0)$$

$$= \max_{k\le ta_n} P((M_{k1}^{(n)},\ldots,M_{kJ}^{(n)}) \ne 0) \to 0 \quad \text{as } n \to \infty.$$

Now, if (4.4) holds, then by a multivariate point process version of Theorem 7.2 of Kallenberg (1975), we know that $\underset{\sim}{M}_n \overset{\mathcal{D}}{\to} \underset{\sim}{N}$, where $\underset{\sim}{N}$ is as described in the last assertion of Theorem 4.1. Thus $\underset{\sim}{N}_n \overset{\mathcal{D}}{\to} N$.

Conversely, suppose $\underset{\sim}{N}_n \overset{\mathcal{D}}{\to}$ some $\underset{\sim}{N}$. Then $\underset{\sim}{M}_n \overset{\mathcal{D}}{\to} \underset{\sim}{N}$, and by a multivariate version of Theorem 6.1 of Kallenberg (1975), the limit $\underset{\sim}{N}$ must be infinitely divisible. Furthermore, $\underset{\sim}{N}$ has independent increments

a_n-additively null. And the right-hand side of (4.6) converges to zero when there is a $\lambda > 0$ such that

$$\sup_n E|N_n(t) - \lambda a_n t| < \infty, \quad \lim_{n\to\infty} E\Big|\sum_{k=1}^{a_n t} p_{nk} - \lambda^{-1} ct\Big| = 0,$$

and Ed_{nk} and $\bar{p}_{nk}$ are a_n-additively null.

Proof. First note that we can write $\underset{\sim}{N}_n(t) = \sum_{k=1}^{N_n(t)} \underset{\sim}{M}_k^{(n)}$ and that $N_n(t)$ is an $F_k^{(n)}$- stopping time. Thus, parts (a) and (b) follow from Theorem 1 of Serfozo (1985). To prove part (c), first note that for any real numbers r_{nk},

$$\sum_{k=1}^{N_n(t)} r_{nk} \leq a_n^{-1} N_n(t) \sup_m \sum_{k=ma_n}^{(m+1)a_n} r_{nk}.$$

Using this with the property $E\sup_n U_n = \sup_n EU_n$ and $E|p_{nk} - \bar{p}_{nk}| \leq \sigma_{nk}$, one can see that the right-hand side of (4.5) is bounded by

$$a_n^{-1} EN_n(t) \sup_m \sum_{k=ma_n}^{(m+1)a_n} (Ed_{nk} + \bar{p}_{nk}^{2} + \sigma_{nk}),$$

and this converges to zero under the hypotheses of (c). A similar argument shows that the right-hand side of (4.6) converges to zero; here one uses

$$\begin{aligned} E\Big|\sum_{k=1}^{N_n(t)} p_{nk} - ct\Big| &\leq E\Big|\sum_{k=1}^{N_n(t)} p_{nk} - \sum_{k=1}^{\lambda a_n t} p_{nk}\Big| \\ &\quad + E\Big|\sum_{k=1}^{\lambda a_n t} p_{nk} - ct\Big| \\ &\leq E|N_n(t) - \lambda a_n t| \sup_m \sum_{k=ma_n}^{(m+1)a_n} \bar{p}_{nk} + o(1). \end{aligned}$$

<u>Theorem</u> 5.1. Suppose

$$T^{(n)}_{k_n}/k_n \xrightarrow{\mathcal{D}} 1 \text{ for each } k_n \to \infty, \tag{5.1}$$

and

$$\varepsilon_n := a_n^{-1} \sup_k \max\{D^{(n)}_{k\ell} : \ell \le M^{(n)}_k\} \xrightarrow{\mathcal{D}} 0 \quad \text{as } n \to \infty. \tag{5.2}$$

Then $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$ if and only if $\underset{\sim}{N}^*_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$.

<u>Proof</u>. First, suppose $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$. Clearly, for any j and s < t,

$$\begin{aligned} N^*_{nj}(s,t] &= \sum_k \sum_{\ell=1}^{M^{(n)}_k} I(X^{(n)}_{k\ell} = j) I(a_n s < T^{(n)}_k \le a_n t) \\ &\le \sum_k \sum_{\ell=1}^{M^{(n)}_k} I(X^{(n)}_{k\ell} = j) I(a_n s < T^{(n)}_k + D^{(n)}_{k\ell} \le a_n(t + \varepsilon_n)) \\ &= N_{nj}(s, t + \varepsilon_n], \end{aligned} \tag{5.3}$$

and, similarly,

$$\begin{aligned} N^*_{nj}(s,t] &\ge \sum_k \sum_{\ell=1}^{M^{(n)}_k} I(X^{(n)}_{k\ell} = j) I(a_n(s + \varepsilon_n) < T^{(n)}_k + D^{(n)}_{k\ell} \le a_n t) \\ &= N_{nj}(s + \varepsilon_n, t] \quad \text{when } s + \varepsilon_n < t. \end{aligned} \tag{5.4}$$

One can show that $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$ implies that $(N_{nj}(s,t + \varepsilon_n], N_{nj}(s + \varepsilon_n,t]) \xrightarrow{\mathcal{D}}$ $(N_j(s,t], N_j(s,t])$ for any s < t with $N_j\{s\} = N_j\{t\} = 0$ a.s. This and (5.3), (5.4) imply $N^*_{nj}(s,t] \xrightarrow{\mathcal{D}} N_j(s,t]$. This reasoning readily generalizes to yield $\underset{\sim}{N}^*_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$.

Conversely, suppose $\underset{\sim}{N}^*_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$. Note that we can write $N^*_{nj}(t) =$ $M_{nj}(\gamma_n(t))$, where $\gamma_n(t) := a_n^{-1} \sum_k I(T^{(n)}_k \le a_n t)$ and

$$M_{nj}(t) := \sum_{k=1}^{\infty} \sum_{\ell=1}^{M^{(n)}_k} I(X^{(n)}_{k\ell} = j) I(k \le a_n t), \quad t \ge 0,\ j = 1,2,\ldots$$

Since (5.1) implies that $\gamma_n \xrightarrow{\mathcal{D}} \gamma$, where $\gamma(t) = t$, it follows by Lemma 3.2 that $\underset{\sim}{N}_n^* \xrightarrow{\mathcal{D}} \underset{\sim}{N}$ implies $\underset{\sim}{M}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$. Now, similarly, to (5.3) and (5.4),

$$M_{nj}(s,t-\varepsilon_n] \le N_{nj}(s,t] \le M_{nj}(s-\varepsilon_n,t], \quad s < t-\varepsilon_n.$$

Using this and $\underset{\sim}{M}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$ in an argument analogous to the preceding one yields $\underset{\sim}{N}_n \xrightarrow{\mathcal{D}} \underset{\sim}{N}$.

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