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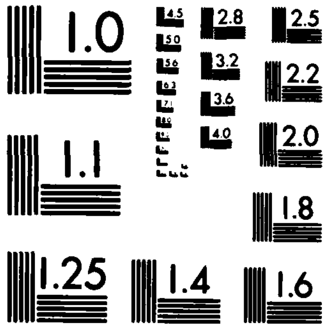

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by

Richard E. Barlow* and Srinivas Iyer**

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COMPUTATIONAL COMPLEXITY OF COHERENT SYSTEMS AND THE RELIABILITY POLYNOMIAL

R. E. Barlow and S. Iyer

Abstract

There are three general methods for system reliability evaluation, namely: 1) Inclusion-Exclusion, 2) Sum of Disjoint Products, and 3) Pivoting. Of these, only pivoting can be applied directly to a logic tree or network graph representation without first finding minimal path (or cut) sets. Domination theory provides the basis for selecting optimal pivoting strategies. Simple proofs of domination theory results for coherent systems are given, based on the reliability polynomial. These results are related to the problem of finding efficient strategies for computing coherent system reliability. The original results for undirected networks are due to Satyanarayana and Chang (1983). Many of the original set theoretic results are due to Huseby (1984). However, he does not use the reliability polynomial to prove his results.

1. INTRODUCTION

Let $C = \{1,2,\ldots,n\}$ be a set of components and $\mathbb{P} = [P_1,\ldots,P_m]$ be a family of min path sets where $P_i \subseteq C$, $P_i \not\subseteq P_j$ for $i \neq j$ and $C = \bigcup_{i=1}^{m} P_i$. Let

$$\phi(S) = \begin{cases} 1 & \text{if } P_i \subseteq S \text{ for some } i \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

$\emptyset$ is a monotonic set function which is one for any min path set and any superset of min path sets.

DEFINITION. $(C,\mathbf{P},\emptyset)$ is a coherent system. (Abbreviated $[C,\mathbf{P}]$)

DEFINITION. The reliability polynomial for $(C,\mathbf{P},\emptyset)$ is

$$h_\emptyset(p) = \sum_{S \subseteq C} \emptyset(S) p^{|S|}(1-p)^{n-|S|}$$

or

$$h_\emptyset(p) = \sum_{i=1}^{n} B_i p^i \qquad (1.2)$$

where $|S|$ is the cardinality of set S.

N.B. This polynomial is relevant to $(C,\mathbf{P},\emptyset)$ regardless of the probability measure assigned to components of C! The coefficients of the reliability polynomial provide useful information about the system. If components operate independently of one another with probability p, then (1.2) is system reliability.

DEFINITION. A formation for C is a set of min paths whose union is C. It is odd (even) if the number of min paths is odd (even).

EXAMPLE. The following simple example of an undirected network will be used to illustrate ideas.

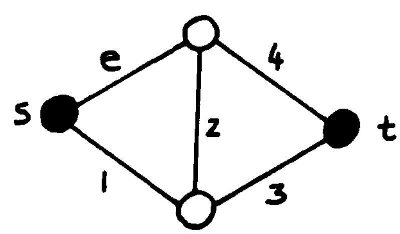

Figure 1

Undirected Two Terminal Network

For this example $C = \{1,2,3,4,e\}$ while $\mathbf{P} = [\{1,3\},\{1,2,4\},\{e,4\},\{e,2,3\}]$. The formations of C are:

$$F_0 = \mathbf{P} = [\{1,3\},\{1,2,4\},\{e,4\},\{e,2,3\}]$$
$$F_1 = [\{1,3\},\{1,2,4\},\{e,2,3\}]$$
$$F_2 = [\{1,3\},\{1,2,4\},\{e,4\}]$$
$$F_3 = [\{1,3\},\{e,4\},\{e,2,3\}]$$
$$F_4 = [\{1,2,4\},\{e,2,3\}]$$
$$F_5 = [\{1,2,4\},\{e,4\},\{e,2,3\}]$$

The reliability polynomial for this example is

$$h_\emptyset(p) = 2p^2 + 2p^3 - 5p^4 + 2p^5 .$$

Notice that the coefficients always sum to 1 since $h_\emptyset(1) = 1$. This is a convenient check on numerical calculations.

By the inclusion-exclusion formula, the coefficient of p^n is the number of odd formations of C minus the number of even formations of C. In our example there are 4 odd formations and 2 even formations so that the coefficient of p^n is 2.

DEFINITION. The Signed Domination, $d(\mathbf{P})$, of $(C,\mathbf{P},\emptyset)$ is the number of odd formations of C minus the number of even formations of C.

For coherent systems the signed domination is the same as the coefficient of p^n. For non-monotonic systems (e.g. logic trees with NOT gates), the coefficient of p^n cannot be interpreted in terms of formations.

DEFINITION. The domination, $D(\mathbf{P})$, of $(C,\mathbf{P},\emptyset)$ is $|d(\mathbf{P})|$.

DEFINITION. The dual system to $(C,\mathbf{P},\emptyset)$ is $(C,\mathbf{K},\emptyset^d)$ where $\mathbf{K} = [K_1,\ldots,K_k]$ are the corresponding min cuts and

$$\emptyset^d(S) = \begin{cases} 1 & \text{if } K_i \subseteq S \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

The reliability polynomial for $(C,\mathbf{K},\emptyset^d)$ is

$$h_\emptyset^d(p) = 1 - h_\emptyset(1-p)$$

so that the signed domination of the dual system satisfies

$$d(\mathbf{K}) = (-1)^{n+1} d(\mathbf{P}) \ . \tag{1.3}$$

It follows that for coherent systems

$$D(\mathbf{P}) = D(\mathbf{K}) \ .$$

EXAMPLE. For a series system $h(p) = p^n$ and the signed domination is 1. For its dual (a parallel system) the signed domination is $(-1)^{n+1}$. However, both have domination 1.

2. PIVOTING

A general method useful for computing system reliability is to pivot on a component, say e. Let $\mathbf{p} = (p_1,\ldots,p_n)$ be the vector of component reliabilities and assume components fail independently of one another. Let $h(\mathbf{p})$ be the coherent system reliability. Then if we pivot on component e, we will have

$$h(\mathbf{p}) = p_e h(1_e,\mathbf{p}) + (1-p_e) h(0_e,\mathbf{p})$$

where $(1_e,\mathbf{p}) = (p_1,\ldots,1_e,\ldots,p_n)$, etc. The systems corresponding to $h(1_e,p)$ and $h(0_e,p)$ are called the minors of $[C,P]$ with respect to e. The minor $[C-e,P_{+e}]$ corresponds to the system with e perfect while the minor $[C-e,P_{-e}]$ corresponds to the system with e failed. P_{-e} are the min paths of P not containing e while P_{+e} is obtained by deleting e from all min paths of P and then discarding any supersets which may now be present. We illustrate the procedure using our previous example.

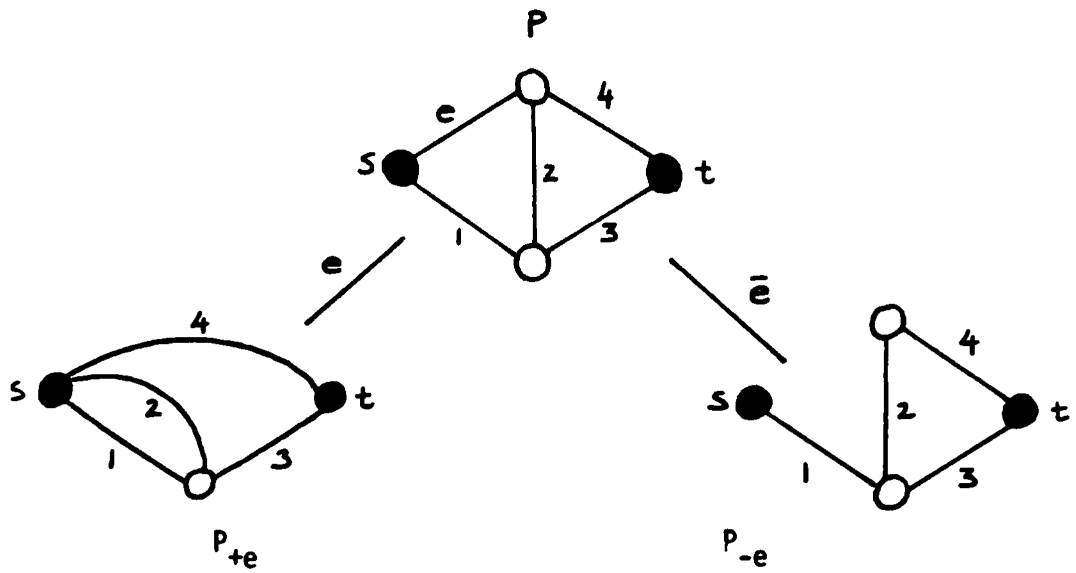

Figure 2

Pivoting on e

Note that we pivoted once and the number of leaves is equal to the domination.

In this example $P_{+e} = [\{1,3\},\{4\},\{2,3\}]$

and $P_{-e} = [\{1,3\},\{1,2,4\}]$.

In P_{+e}, $\{1,2,4\}$ was a superset of $\{4\}$ and was eliminated.

THEOREM 2.1 (The Signed Domination Theorem). For all coherent systems

$$d(\mathbf{P}) = d(\mathbf{P}_{+e}) - d(\mathbf{P}_{-e}) \ . \tag{2.1}$$

Proof. By pivotal decomposition (true for all systems) the reliability polynomial for $(C,\mathbf{P},\emptyset)$ can be written as

$$h_{\emptyset}(p) = ph_{\emptyset}(1_e,p) + (1-p)h_{\emptyset}(0_e,p) \ .$$

Equating coefficients of p^n on both sides of the equation, we have that the coefficient of p^n equals the coefficient of p^{n-1} in $h_{\emptyset}(1_e,p)$ minus the coefficient of p^{n-1} in $h_{\emptyset}(0_e,p)$. The result is not true for non-monotonic systems since the coefficient of p^n does not correspond to the signed domination in this case. Q.E.D.

Using Theorem 2.1 and induction, it is easy to verify the following corollary for undirected networks. An undirected network with $|K|$ distinguished nodes works iff all nodes in K can communicate with each other.

COROLLARY. For a coherent system corresponding to an undirected network with v nodes and $|K| \leqq v$ distinguished nodes

$$d(\mathbf{P}) = (-1)^{n-v+1} D(\mathbf{P}) \ ,$$

where $D(\mathbf{P}) = |d(\mathbf{P})|$ is the domination and n is the number of edges.

DEFINITION. A coherent system $(C,\mathbf{P},\emptyset)$ will be called totally amenable iff for all components e,

$$D(\mathbf{P}) = D(\mathbf{P}_{+e}) + D(\mathbf{P}_{-e}) \tag{2.2}$$

and all minors of minors also satisfy (2.2).

THEOREM 2.2. Undirected networks with $|K|$ distinguished nodes and k-out-of-n systems are totally amenable; i.e. (2.2) holds.

Proof. Use Theorem 2.1 and its Corollary relative to undirected networks with $|K|$ distinguished nodes. Use Theorem 2.1 and induction relative to k-out-of-n systems. Q.E.D.

Huseby (1984) defines a class of coherent systems which he calls regular and which includes undirected networks, k-out-of-n systems and others. Regular coherent systems are totally amenable. Lehman (1964) provides an algorithm based on min paths to determine whether or not a system is regular. It is not known at this time if the class of regular coherent systems is equivalent to the class of totally amenable coherent systems.

3. COMPUTATIONAL COMPLEXITY OF TOTALLY AMENABLE COHERENT SYSTEMS

In considering a strategy for computing system reliability, usually the first idea is to discover modules, compute their reliability and then replace each module in the original system by a super component with that module's reliability. Although clearly prudent, we seek some way of measuring the advantage of this approach in conjunction with pivoting. This can be done for totally amenable systems using domination theory.

MODULES OF COHERENT SYSTEMS. Let $x_i = 1$ if component i works, $x_i = 0$ otherwise and $\mathbf{x} = (x_1,\ldots,x_n)$. Then, with a slight abuse of notation, we define $\phi(\mathbf{x}) = 1$ if the set, S, of indices corresponding to coordinates which are 1 contains a min path set and 0 otherwise.

Let χ_j, $j = 1,\dots,r$ be corresponding indicators for modules of $(C,\mathbf{P},\emptyset)$. Let Φ be the module organizing structure function such that

$$\emptyset(\mathbf{x}) = \Phi[\chi_1(\mathbf{x}),\dots,\chi_r(\mathbf{x})] .$$

Modules are coherent sub-systems. Their components do not overlap.

THEOREM 3.1. The domination of $(C,\mathbf{P},\emptyset)$ corresponding to the modular decomposition

$$\emptyset(\mathbf{x}) = \Phi[\chi_1(\mathbf{x}),\dots,\chi_r(\mathbf{x})]$$

satisfies

$$D(\emptyset) = D(\Phi) \prod_{j=1}^{r} D(\chi_j) \tag{3.1}$$

where $D(\emptyset) = D(\mathbf{P})$, again by an abuse of notation.

<u>Proof</u>. The reliability polynomial of $\emptyset$ can be written in terms of that of Φ and of the χ_j, $j = 1,\dots,r$ as follows

$$h_\emptyset(p) = h_\Phi[h_{\chi_1}(p),\dots,h_{\chi_r}(p)] .$$

The coefficient of p^n on the left hand side is equal to the coefficient of p^r corresponding to h_Φ, since Φ has r supercomponents, times the product of the coefficients of p^{n_j} corresponding to each of the modules where module j has n_j components. Q.E.D.

Clearly if each module is either series or parallel and the organizing structure function is either series or parallel then the domination of the system is 1.

DEFINITION. For a coherent system, components i and j are in series (parallel) if whenever i is in a min path (min cut) so is j and vice versa.

DEFINITION. A series (parallel) replacement consists of replacing a series (parallel) system by a "supercomponent" with the same reliability.

COROLLARY 3.2. The domination of a coherent system is invariant under series and parallel replacements.

Proof. If $\chi_1, \chi_2, \ldots, \chi_r$ correspond to series and/or parallel modules, then by (3.1)

$$D(\emptyset) = D(\Phi)$$

where Φ is the organizing structure function. Hence the domination of the modified system is the same as that of the original system. Q.E.D.

DEFINITION. A coherent system is series-parallel iff it can be reduced to a single component by series and parallel replacements alone.

DEFINITION. A coherent system is series-parallel complex (s-p complex) iff it has no components in series or in parallel.

THEOREM 3.3. A totally amenable coherent system is series-parallel iff its domination is 1.

Proof. Suppose $(C,\mathbf{P},\emptyset)$ is totally amenable and $D(\mathbf{P}) = 1$. Then by (2.2), for any $e \in C$

$$D(\mathbf{P}) = D(\mathbf{P}_{+e}) + D(\mathbf{P}_{-e}) = 1$$

which implies that either $D(\mathbf{P}_{+e}) = 1$ and $D(\mathbf{P}_{-e}) = 0$ or vice versa. Suppose $D(\mathbf{P}_{-e}) = 0$, then $\bigcup_{P \in \mathbf{P}_{-e}} P = C - e$ meaning that the min paths in $\mathbf{P}_{-e}$ do not contain e and at least one other component, say j. Hence e and j were in series in the original system and a series replacement is possible for the original system. By a similar argument, if $D(\mathbf{P}_{+e}) = 0$, then e is in parallel with another component in the original system and a parallel replacement is possible. Continuing in this way we see that $(C,\mathbf{P},\emptyset)$ can be reduced to a single component by series and parallel replacements. Q.E.D.

Note that the simple example on page 2 is totally amenable, s-p complex and its domination is 2.

Huseby (1984) proves that for every coherent system which is s-p complex, there exists a component e such that <u>both</u> minors $[C,\mathbf{P}_{+e}]$ and $[C,\mathbf{P}_{-e}]$ are coherent. The only way they could fail to be coherent would be if one or the other contained an irrelevant component. If a system contains an irrelevant component, its domination is zero since it has no formations in terms of minimal paths. Of course, coherent systems have no irrelevant components. However, a coherent system can still have domination zero. For example, the domination is 0 for directed cyclic networks.

THEOREM 3.4. If a s-p complex, coherent system $(C,\mathbf{P},\emptyset)$ is totally amenable, then $D(\mathbf{P}) > 1$.

<u>Proof</u>. The proof is by induction on the number, n, of components. It is easy to verify that all coherent systems of orders 2 and 3 are totally amenable and their dominations are positive. Suppose all

coherent, totally amenable systems of orders less than or equal to $n-1$ have positive domination. Let $[C,\mathbf{P}]$ be a s-p complex, coherent system which is totally amenable. Huseby (1984) proved that there exists e such that both $[C-e,\mathbf{P}_{+e}]$ and $[C-e,\mathbf{P}_{-e}]$ are coherent. Since $[C,\mathbf{P}]$ is totally amenable, so are the minors and

$$D(\mathbf{P}) = D(\mathbf{P}_{+e}) + D(\mathbf{P}_{+e})$$

by (2.2). By the induction assumption $D(\mathbf{P}_{+e}) > 0$ and $D(\mathbf{P}_{-e}) > 0$ imply $D(\mathbf{P}) > 1$. Q.E.D.

THE FACTORING ALGORITHM FOR COMPUTING SYSTEM RELIABILITY.

1. Perform all possible series-parallel replacements so that the system is s-p complex;

2. Choose a component e such that both $[C,\mathbf{P}_{+e}]$ and $[C,\mathbf{P}_{-e}]$ are coherent (this can always be done for coherent s-p complex systems);

3. Again perform all possible series-parallel replacements;

4. Repeat this procedure until the system is reduced to a single component.

THEOREM 3.5. If a coherent system is totally amenable, then the Factoring Algorithm requires exactly $D(\mathbf{P}) - 1$ pivots. Any other strategy for choosing components to pivot on would require at least as many pivots.

Proof [See Satyarayana and Chang (1983)]. For totally amenable coherent systems

$$D(\mathbf{P}) = D(\mathbf{P}_{+e}) + D(\mathbf{P}_{-e})$$

by (2.2). Since we can always choose e so that

$$D(\mathbf{P}_{+e}) > 0 \quad \text{and} \quad D(\mathbf{P}_{-e}) > 0$$

after all possible series-parallel replacements have been performed, the number of leaves in the corresponding binary computational tree is $D(\mathbf{P})$. The number of nodes in the binary computational tree is $2D(\mathbf{P}) - 1$ and the total number of pivots is $D(\mathbf{P}) - 1$ since the $D(\mathbf{P})$ leaves of the binary computational tree are series-parallel systems with domination one. Q.E.D.

If, after all possible series-parallel replacements, a totally amenable coherent system has a modular representation

$$\phi(\mathbf{x}) = \Phi[\chi_1(\mathbf{x}), \ldots, \chi_r(\mathbf{x})]$$

then by (3.1) the factoring algorithm will require

$$D(\phi) - 1 = D(\Phi) \prod_{j=1}^{r} D(\chi_j) - 1 \tag{3.2}$$

pivots. However, if we first use the factoring algorithm to compute the modular reliabilities and then the system reliability, the minimum total number of pivots required, namely

$$[D(\Phi)-1] + \sum_{j=1}^{r} [D(\chi_j)-1]$$

will be much less, in general, than (3.2).

Notice that were χ_j series-parallel then $[D(\chi_j)-1] = 0$ and pivoting on components in a series-parallel system would be wasteful in time and effort.

k-OUT-OF-n SYSTEMS. Although k-out-of-n systems are totally amenable, the factoring algorithm is a very poor algorithm for this system. Suppose components fail independently of one another and component i has reliability p_i, $i = 1,\ldots,n$. In this case a well known algorithm based on generating functions is available. The computational running time of program 2 in Barlow and Heidtmann (1985) is $k(n-k+1) \leqq n^2/4$. The signed domination for k-out-of-n systems is

$$d(\mathbf{P}) = (-1)^{(k+n)}\binom{n-1}{k-1}$$

and

$$D(\mathbf{P}) = \frac{(n-1)\cdots(n-k+1)}{(k-1)!}$$

so that the factoring algorithm can be very bad in this case. Program 1 in Barlow and Heidtmann (1985) computes exact k-out-of-n reliability. The running time in this case is of the order $n^2/2$.

FINAL REMARKS. For logic trees without NOT gates an algorithm based on pivoting and modular reduction can be devised. For totally amenable systems (with the exception of k-out-of-n systems) it should be far superior to current methodology.

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