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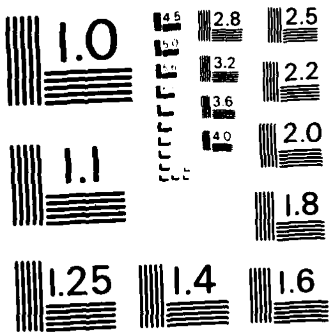

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Final Technical Report

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Print Author(s) Name(s) Here

J.B. McLeod

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This report describes the work that has been done in Oxford during the past three years under support from the U.S. Army.
The work is described under the two main headings of Physical Sciences and Biological Sciences, with these main headings subdivided. There then follows lists of visitors, research workers and publications.

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Final Report

This report describes the work that has been done in Oxford during the past three years under support from the U.S. Army.

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A. PHYSICAL SCIENCES

1. Modelling and Applications

Theoretical mechanics is a fruitful source of interesting nonlinear partial differential equations and our work on nonlinear wave equations and variational inequalities is directly motivated by fundamental problems in fluid and solid mechanics.

Industrial applications also give rise to many interesting and novel models and the Oxford Study Groups with Industry provide a continuing source of such applications. These include a variety of free and moving boundary problems, in particular phase change problems of various kinds. Parabolic equations occur more frequently than any others in industrial applications, and we have lately come across several "forward-backward" equations in which the time-like direction changes sign. Counter-current heat or mass transfer devices clearly lead to such equations. Other parabolic equations have arisen in the modelling of combustion in porous media, which also leads to a free boundary problem, and heat flow in turbine blades.

A particular nonlinear diffusion equation, the so-called "porous medium equation", has received much attention in Oxford and elsewhere in the past few years. Amongst many other applications, we have investigated it as a model for the growth of volcanoes.

Other equations investigated under this heading have been

(i) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\left|\frac{\partial u}{\partial t}\right| ,$$

which arises as the limit of a hyperbolic equation describing underwater cable oscillations, and

$$\text{(ii)}\quad \frac{\partial}{\partial x}\left\{h^3(1+cp)\,\frac{\partial p}{\partial x}\right\} = \frac{\partial}{\partial t}\{h(1+cp)\},\quad p(\pm\tfrac{1}{2},t) = 0,$$

where $h(x,t) = w(t) + c_0 x\phi(t)$,

$$\phi - \phi_0 = c_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} xp\,dx,$$

and

$$c_2 \frac{d^2w}{dt^2} + (w - 1) = F\sin t + c_3 \int_{-\frac{1}{2}}^{\frac{1}{2}} p\,dx,$$

with $p = w - 1 = \dot{w} = 0$ at $t = 0$,

which is a model for a tilting gas bearing.

2. Nonlinear Wave and Diffusion Equations Soluble by Inverse Scattering or Bäcklund Transformations

(i) Inverse Scattering

A number of nonlinear wave equations of importance in application, such as the Korteweg-de Vries equation, the sine-Gordon equation and the nonlinear Schrödinger equation, have the property that they can be solved by the method of inverse scattering, which in effect reduces their solution to that of a linear integral equation, the so-called Gel'fand-Levitan equation. This leads to a rich field of investigation into the properties of similarity solutions of the equations. Similarity solutions are frequently important for studying asymptotic properties of general solutions, and we have been able to show that the similarity solutions of these equations are solutions of the class of ordinary differential equations (known as the Painlevé transcendents) which have the property that singularities of solutions (other than poles) depend only on the equation and not on the particular solution. We have explored this connection between the partial differential equations and the Painlevé transcendents and have obtained important implications for both. In particular, we have been able to give a global asymptotic behaviour for solutions of the Painlevé transcendents which is comparable with the sort of results that are normally expected only for linear equations.

(ii) The Porous Medium Equation

The simplest version of the porous medium equation is $\frac{\partial}{\partial x}\left(u^{m-1}\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t}$. Like the K d V equation, this equation also possesses Bäcklund-type transformations in the case $m = -1$, in which case it can be reduced to the linear heat equation. More generally a method exists for relating solutions for a value of $m = m_0$ to those for $m = m_0 - 2$. In the so-called "fast diffusion" case, $m < 1$, certain estimates on the Cauchy data have been derived for the solution to exist for all $t \geq 0$, and we can heuristically relate the phenomenon of finite-time extinction in the case $m \leq -1$ to that of finite-time blow-up in the "slow-diffusion" case, $m \geq 1$.

3. Analysis of Diffusion Equations

(i) Blow-up of Solutions

We have investigated the blow-up of positive solutions of nonlinear diffusion equations of the form

$$u_t = \Delta u + f(u),$$

holding in some domain Ω in R^n with $u = 0$ on $\partial\Omega$. The nonlinearity $f(u)$ is typically of the form $u^p (p > 1)$ or e^u. The questions of interest are whether the solution blows up and, if it does, how it blows up and in particular whether the set at which it first becomes infinite consists of just one point or can be a region of non-zero measure. We have proved a number of results for the problem, under very general conditions, including:

(a) the solution does not blow up near the boundary,

(b) if $n = 1$, and the initial data $u(x,0)$ has just one "hump", without necessarily being symmetric about the hump, then blow-up, if it occurs, occurs at just one point,

(c) the result (b) can be extended to spherically symmetric solutions in any number of dimensions,

(d) estimates can be obtained for the rate of blow-up.

(ii) Combustion

We have investigated several problems on combustion theory, including the existence and stability of diffusion flames, and the existence and uniqueness of the so-called Liñán-type problems (ordinary differential equations which govern the structure of the combustion in the reaction-diffusion zone).

(iii) Critical Sobolev Exponents

We have been interested in the equation

$$-\Delta u = u^p + \lambda u,$$

holding in some domain Ω in R^n, with $u = 0$ on $\partial\Omega$. Here p is the so-called critical Sobolev exponent, i.e. $p = (n + 2)/(n - 2)$, where n is the dimension of the space. The significance of this value of p is that, if p is less than the critical value, then existence of the solutions of the problem can be obtained by variational techniques, and this has been known and studied for some considerable time. If p is the critical value, then the variational techniques fail, and indeed, if $\lambda = 0$ and Ω is, say, convex, then it is known that there is no solution. The introduction of the term λu can however change this, and Brezis and Nirenberg have recently proved that, if $n = 3$ and Ω is the unit sphere, then there exists a positive solution if and only if $\frac{1}{4}\pi^2 < \lambda < \pi^2$.

This result is not only intriguing as a problem in analysis. It also has significance for other fields, such as isoperimetric inequalities, differential geometry, and solutions of the Yang-Mills equation. It does seem that the case of critical Sobolev exponents is unexpectedly prevalent, and that the more we understand it, the better.

To this end we have studied two aspects of the problem. First, when $n = 3$ and Ω is a sphere, a celebrated theorem by Gidas, Ni and Nirenberg says that the solution must be radial, so that the problem is in fact and O.D.E. But the Brezis-Nirenberg proof is a P.D.E. proof, and we have succeeded in rectifying this by giving a simple proof using O.D.E. methods.

More important is the extension of the result to the situation where Ω is not a sphere but a general domain in R_3. While work on this is not yet complete, the answer seems to be as follows.

Let $G(x,y,\lambda)$ be the Green's function for the operator $-\Delta + \lambda$ in Ω, with Dirichlet boundary conditions. As is well known,

$$G(x,y,\lambda) = \frac{1}{4\pi|x-y|} - h(x,y,\lambda),$$

where $h(x,y,\lambda)$ is a smooth function in Ω, with the properties that

$$h(x,y,0) > 0, \tag{1}$$

$$\frac{\partial}{\partial\lambda} h(x,y,\lambda) < 0.$$

If we consider $h(x,x,\lambda)$, this is then a decreasing function of λ, and so there is, in view of (1), a first value of $\lambda > 0$, say λ_0, such that $h(x,x,\lambda)$ has 0 as its minimum, at x_0, say. Then the range of λ for which a solution exists is $\lambda_0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω, and further, as $\lambda \downarrow \lambda_0$, the solution tends to a spike-type solution with spike at x_0.

4. Other Analytical Investigations

(i) Existence of Steady Vortex Rings

This is a matter of investigating solutions of the equation

$$-\Delta u = f,$$

holding in some domain Ω in R^2, with $u = 0$ on $\partial\Omega$. If the function f is rearranged, (and the application to vortex motion is that f represents the vorticity which does always move in two-dimensional fluid motion so as to remain a rearrangement of itself), then the solution u changes and we are interested in the energy $\int_\Omega |\Delta u|^2$ and in particular in whether its maximum and minimum (as f is rearranged) can actually be attained. We have shown that the maximum is always attained through a specific rearrangement of f, but the minimum is attained if and only if f is of one sign.

(ii) Interior Transition Layers for Elliptic Differential Equations

We have been concerned with equations of the form

$$-\varepsilon^2 \Delta u = f^2(g^2 - u^2)u, \tag{2}$$

holding in some domain Ω in R^1 or R^2, with $u = 0$ on $\partial\Omega$. Here both f and g are smooth functions of x and both are strictly positive. Results for the positive solutions of (2) are well established, but knowledge of other branches of solutions is scarce. In the present work positive and negative solutions are pieced together across lines partitioning the domain and variational arguments are framed in an attempt to locate these nodal lines (where the solution vanishes) so that the composite function is everywhere a solution of (2). Heuristic arguments suggest that there is a close relationship between these nodal lines and lines L which render stationary the functional defined by

$$\int_L fg^3 ds.$$

(iii) Water Waves

We have investigated some properties of solitary waves, particularly those associated with the Froude number F, which is given by $F^2 = c^2/gh$, where c is the velocity of the wave, g the acceleration due to gravity and h the depth of the fluid at infinity. The proof that $F > 1$ for a solitary wave has previously been available only as the result of a long and complicated argument, but we have succeeded in giving a short and simple one.

It had also been conjectured that $F > 1$ implies exponential decay at infinity of the height of the solitary wave, and we have succeeded in proving this conjecture.

(iv) Over-determined Systems

We have investigated a system of partial differential equations involving five equations in two unknown functions. The system arises from attempts to design the most efficient form of flowmeter for measuring fluid flow in pipes (in particular, blood in blood vessels), and was raised at one of the Oxford Study Groups with Industry. We have succeeded in answering the problem completely by giving the most general solution to this over-determined system.

(v) Equations of Catalysis Theory

In the production of chemicals, catalysts are often required to convert gaseous reactants into useful products. Frequently the catalyst is in the form of a porous pellet and the gas must diffuse into the interior of the pellet so that the catalyst there is fully utilised. Depending upon the relative rate of diffusion and reaction, temperature and concentration gradients are set up across the pellet, and their determination is essential for the calculation of the over-all rate of conversion. The modelling of these processes within the pellet leads to a set of parabloic partial differential equations, and a first step in the study of these is to determine whether there exist steadystate solutions, and if so, how many of these there are.

We investigated a particular one-dimensional steady-state equation

$$v'' + \lambda(1 + \beta - v)^p e^{-\gamma/v}$$

with the boundary conditions

$$v'(0) = 0, \quad v(1) = 1,$$

the parameters λ, β, γ being all positive and p a non-negative integer. This seems to be typical of more general situations, and we have proved that if the activation energy γ is sufficiently high, then the number of solutions must be essentially either one or three (depending upon the other parameters in the problem).

(vi) Inequalities

There is a famous inequality due to Hardy, Littlewood and Pólya which states that if h, f, g are positive functions with h spherically symmetric and decreasing, then

$$\iint_{R^n \times R^n} h(|x - y|)f(x)g(y)\,dx\,dy \le \iint_{R^n \times R^n} h(|x-y|)f^*(x)g^*(y)\,dx\,dy,$$

where f^*, g^* are the spherically symmetric decreasing rearrangements of f, g. We have shown that the inequality is in effect strict unless f, g are spherically symmetric and decreasing, and this has applications in various variational situations in proving that the winner in a variational problem has to be spherically symmetric.

5. Free and Moving Boundary Problems

This is an area which has been studied in Oxford for many years and several different aspects are bing considered at the moment. A summary of some of the work done up to 1981 is given in Weak and variational methods for moving

C. VISITOR PROGRAMME

The following have visited Oxford during the period of the grant, with their stays supported either wholly or in part from the grant.

G. Avila	(Brasilia)
C. Bennewitz	(Uppsala)
G. Caginaffo	(Carnegie-Mellon)
Y. Chin	(Peking)
E. Cox	(Dublin)
C. Elliott	(London)
P.C. Fife	(Arizona)
A. Friedman	(Northwestern)
S.P. Hastings	(Buffalo)
M.A. Herrero	(Madrid)
H. Lange	(Cologne)
B. Nicolaenko	(Los Alamos)
J.B. Rosser	(Madison)
P. Sachdev	(Bangalore)
L.A. Segel	(Weizmann Institute)
C.A. Stuart	(Lausanne)
K. Tomoeda	(Osaka)
J.L. Vasquez	(Madrid)
E.M. Veling	(Amsterdam)
R. Villella-Bressan	(Padua)
A.T. Winfree	(Purdue)

D. RESEARCH ASSISTANTS

The following have been supported as research assistants either wholly or in part from the grant.

P.A. Arcuri	P.C. Meek
S.D. Howison	M. Shillor
A.A. Lacey	A. Wheeler
D.A. Lawson	K. Wilmott

E. PUBLICATIONS

1. T.B. Benjamin and A.D. Cocker, Liquid drops suspended by soap films, Pt. I. General formulation and the case of axial symmetry, Proc. Roy. Soc. Lond. A 394 (1984), 19-32.

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of two. A study in one space dimension of several popular methods shows this problem is common. The implicit midpoint rule did the best out of those tested. In general, efficiency suffers markedly when accuracy is required. Our model has been submitted for publication.

3. Numerical Methods

For eqns. (1) in two space dimensions on arbitrary domains we discretise in space with quadratic isoparametric finite elements, giving near second order convergence and piecewise quadratic approximation of the domain boundary. This transforms the PDEs into a large set of sparsely coupled nonlinear ODEs. For the Thomas system the ODEs have a stiffness ratio in excess of 10^6. Hence we use a variable-order, variable-step Gear solver to step through time. Our program, although quite efficient, takes 10 - 200 minutes CPU time per pattern.

For the rabies model, eqns (3), in two space dimensions we have used a variable-order, variable-step Adams method after discretising in space with finite elements. To study the computed wavespeed in one space dimension we use the Euler, High Order Taylor, Trapezium rule and Implicit Midpoint rule methods. It is common for these methods to produce waveforms travelling at speeds in error by up to a factor of two.

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1. Källen, A., Arcuri, P., Murray, J.D., (1983) to appear.
2. Kaplan, C., (ed) (1979) Rabies: The facts, Oxford University Press.
3. Kauffman, S., Shymko, R., Trabert, K. (1978) [illegible], 259.

of the disease, μ is the death rate, and D is related to the size of the forays into other fox territories made by the disoriented infectives. We assume that the normally territorial fox doesn't "diffuse", ignoring the expeditions made by adolescent foxes striking out on their own. We also assume logistic growth for normal foxes, and that the incubation period is unimportant to the overall dynamics. The equations may be nondimensionalised to give

$$\frac{\partial u}{\partial t} = ru(1 - \frac{u}{k}) - uv$$

$$\frac{\partial v}{\partial t} = u(v - d) + \nabla^2 u \quad . \qquad (3^*)$$

We are interested in the speed of propagation of the epizootic, and in ecological strategies for controlling the spread of the disease.

A standard phase plane analysis follows from a first integral of the system and gives conditions for the existence of travelling waves as well as a lower bound on the wavespeed. Further results have been found using simple energy methods which are useful numerically.

After estimating the parameters in (3) we turned to numerical simulations. Rabies will inevitably return to Britain, so we considered a domain with the shape of Britain and introduced a single rabid fox on the coastline. Travelling waves were produced, and this simple model suggests optimal strategies for control in the form of intensive culling in a small buffer zone to contain the outbreak. The strategy is supported by field data (see Källen et al. 1983, or Kaplan 1977).

An interesting numerical problem arose from the desire to compute accurate wavespeeds. Using standard methods in two space dimensions the speed was in error by up to stability of the developing patterns is greatly increased by nonhomogeneous boundary conditions. Also once a pattern is established it is very persistent with respect to quite large changes in the parameters. This robustness is a new feature in reaction-diffusion patterns and has far-reaching implications as regards the applications to biology. For example, we have constructed prepattern models for chick limb chondrogenesis and for Lepidopteran wing ocelli formation which may explain a number of experimental results, and which suggest new experiments. (We are in fact currently interacting with experimentalists on the chick limb model.)

Regarding the simulations, we have constructed parameter space "maps" which show the effect on patterns of varying various parameters. The equations are intrinsicially difficult to solve (they are <u>very</u> stiff), and each map requires at least 20 computer runs which take 10 to 200 minutes CPU time each.

2. <u>Travelling Wave Models</u>

Another large class of phenomena in biology are those which exhibit travelling waves, epidemiology providing many examples. We have developed a simple model for the spread of a rabies epizootic among foxes. The model equations are

$$\frac{\partial S}{\partial t} = rS(1 - \frac{S}{K}) - \sigma IS$$
$$\frac{\partial I}{\partial t} = \sigma IS - \mu I + D\nabla^2 I \qquad (3)$$

where S and I are the population densities of susceptible and infected foxes, r is the birth rate, K is the carrying capacity, σ is a measure of the infectiousness

We have considered one and two space dimensions in simulations, but our analysis holds for higher dimensions. For simplicity we restrict the parameters in (1) so that there is only one steady state, $\tilde{u}$, $\tilde{v}$, which we use as initial condition for perturbation analysis.

We are interested in creating a specific pattern or sequence of patterns by a continuous change of the parameters; moreover, we demand robustness of these patterns with respect to perturbations in any parameter. For homogeneous boundary conditions, i.e. $u - \tilde{u} = 0$ or $\frac{\partial u}{\partial n} = 0$ or $\frac{\partial u}{\partial n} + k(u - \tilde{u}) = 0$, the standard linear analysis of (1a) gives conditions on γ, β, f and g for diffusive instability (see e.g. Murray 1981). Extending this analysis and with the aid of simulations of (1), we have shown why homogeneous boundary conditions produce a model system sensitive to perturbations; furthermore, the importance of the number of space dimensions was demonstrated. We have also shown how to adjust the parameters to excite selectively single modes, i.e. eigenfunctions u_k of the Laplacian operator:

$$\nabla^2 u_k + k^2 u_k = 0 \tag{2}$$

on Ω with homogeneous boundary conditions on $\partial\Omega$.

We have also considered nonhomogeneous boundary conditions, i.e. $u = u_b \neq \tilde{u}$, so that the boundary acts as a source or sink of the morphogens. At the same time biological evidence was reported in the literature suggesting these boundary conditions are plausible (see Meinhardt [illegible] for discussion). Results obtained indicate these boundary conditions should produce much more robust systems, and our numerical simulations confirm these predictions. The

B. BIOLOGICAL SCIENCES

The following is a summary of the work that has been done. Work on all the problems is continuing.

1. Morphogenesis Prepattern Models

The formation of biological patterns from homogeneous tissues is of paramount importance in understanding morphogenesis. Reaction-diffusion models were first applied by Turing (1952), and have recently found many applications, e.g. Murray (1981), Meinhardt (1978, 1983), Kauffman et al. (1978), etc. However, except for simple gradients, the patterns produced by these models are very sensitive to perturbations in domain shape, size, boundary and initial conditions, and other parameters.

We have studied the Thomas reaction-diffusion equations, Thomas (1975), as a general model for morphogenetic prepatterns with chondrogenesis in chick limb bud (see Wolpert 1977 for biological background) and ocelli formation in Lepidopteran wings (see Nijhout 1980) as particular cases. The model equations are

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma f(u,v)$$

$$\frac{\partial v}{\partial t} = \beta\nabla^2 v + \gamma g(u,v) \qquad (1a)$$

with

$$f(u,v) = u_0 - u - \sigma \frac{uv}{1+u+ku^2}$$

$$g(u,v) = \alpha(v_0 - v) - \sigma \frac{uv}{1+u+ku^2} \qquad (1b)$$

on some domain Ω with appropriate boundary conditions.

$$V(y)\,\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2}$$

in $0 < x < 1$, $0 < y < 1$, where $V(y)$ can change sign. Eigenfunction expansions for such problems lead to interesting questions of completeness which we have investigated. The implications of such analysis for finding an efficient numerical scheme are currently being studied.

certain behaviour and confirm the accuracy of numerical programs. Numerical solutions extend the range of understanding of heat transfer into the required areas of practical concern.

We have developed methods for the accurate and efficient solution of nonlinear heat equations. These methods can find the heat flux as accurately as the temperature itself. Moving boundaries for two-phase materials can also be handled.

For accurate modelling the diffusivity of the material of interest needs to be experimentally determined, and fairly accurately. The glycerin/air test has been used to determine the diffusivity of a layered material. We have numerically simulated the response of a two-layer medium in this test, and outlined a procedure for evaluating the diffusivities of each of the two layers from the experimental output.

(ii) Soil Deformation

A new project has arisen from modelling soil behaviour. The work involves developing a coupled set of hyperbolic/elliptic differential equations, mildly nonlinear in the elastic zone, and very nonlinear in the plastic zone, of soil deformation. A major difficulty lies in understanding the boundary conditions and the ensuing singularities in the solution. A numerical program has been developed and will be compared with experiments being run in the Engineering Department.

(iii) Counter-Current Problems

As a result of studies of counter-current heat exchanges and other industrial devices, we have become interested in problems typified by between the continuum of steady states which are possible under critical conditions.

(v) The porous medium equation mentioned in 2(ii) also has free boundaries at which $u = 0$ when $m > 1$. One feature of such an equation which has both mathematical and physical interest is the existence of a "waiting time" before the free boundary (i.e. the boundary of the region where $u(x,t) > 0$) begins to move. By constructing specific similarity solutions and using comparison theorems, it has been possible to obtain results about the existence and length of the waiting time for general initial data, and further conjectures have been made which still await proof.

(vi) As a result of the research into free and moving boundary problems carried out here since 1972, researchers from Oxford have been involved in the organisation of a series of interdisciplinary international conferences in the subject, involving both theory and applications. The first of these was in Oxford in 1974 and this was followed by Gatlinburg 1977, Durham (U.K.) 1978, Montecatini (Italy) 1981, and now Bordeaux 1984. The U.S. Army has generously supported these conferences and participated in many of the discussions.

4. Numerical Methods

(i) Heat Transfer in Turbine Blades

Heat transfer preformance in turbine blades is of vital concern to aircraft engine designers, and an understanding of this requires an ability to solve the nonlinear diffusion equation (with time-dependent coefficients) in at least two space dimensions. Approximate analytical solutions [illegible] occur and to understand the physical significance of the weak solution of the mathematical model. Some other research on weak solutions of Stefan problems in thin layers (motivated by oil extraction problems) has also been carried out.

(iv) Thermal Runaway with Phase Changes

Thermal runaway is here defined as heating of a host medium by a foreign body as a result of conversion of mechanical to thermal energy in a way which does not decrease with time. It is important in a number of practical applications including friction welding, stick-slip motion in earthquakes and meltdown of nuclear reactors; it may also be a mechanism for the motion of magna or ore bodies under the action of gravity. In several of these situations, the host medium should really be modelled as a fluid whose temperature decreases exponentially with temperature, the temperature rise resulting from friction caused by the applied force. These models are unwieldy to analyse even in the asymptotic limit of rapidly varying viscosity. The current study concerns the simulation of such flows by Stefan problems in which the host medium melts to a constant viscosity liquid at a prescribed temperature. In appropriate parameter regimes, the motion of the molten layer can be analysed using lubrication theory and the resulting nonlinear diffusion equations enable a new kind of criticality to be defined in which steady motion can only occur when a certain prescribed force acts on the foreign body. Larger or smaller forces result in indefinite heating or cooling respectively and a stability analysis is still needed to distinguish for combustion waves, which travel through the porous medium in response to air being blown in at velocity v. It has been discovered recently that these combustion waves cease to exist when $v = v_{crit}$, and what happens when $v > v_{crit}$ is being looked at currently.

(iii) Stefan Problems

The problem of alloy solidification has been studied intensively during the past decade but remains largely unsolved. The principal difficulties are the analysis of the field equations, which are a pair of diffusion equations coupled through the phase boundary conditions, and the occurrence of singularities in the phase boundary even when very simple Stefan type models are adopted. No-one has yet developed an acceptable regularisation procedure for these simple models, but we have recently carried out an analysis of singularity development in Stefan problems in two or more space dimensions. This work has been done in the hope that it will indicate which kinds of regularisation are likely to be successful. Our results indicate that blow-up can occur more readily in two space dimensions than in one, and that cusp development is generic.

Some work on the regularisation problem has also been carried out under the U.S. Army Grant using the so-called "phase field" model which smoothes the interface and tends to a surface tension model as a certain parameter tends to zero.

The existence of mushy or "two-phase" regions in classical Stefan problems has also been of long-standing interest in Oxford and further work has recently been carried out to establish rigorous conditions for mushy regions to which turns on the reaction above a critical temperature; the second stage involves the diffusion of oxygen from the mainstream gas to the site of the reaction in the fibres, and it is this stage which controls the strength of the reaction. This second stage does not appear in conventional flame theory, but is a feature of the present work.

(b) The simplest physical processes involve heat storage in the solid, radiation and conduction of heat through the porous material, heat transfer between the solid and the gas, and convection of mass and heat by the gas movement. Highly nonlinear radiation of heat through the combustion zone, together with heat storage in the solid, help give porous medium combustion an entirely different behaviour to that of conventional combustion theory.

(c) At its simplest the above considerations imply that a highly nonlinear coupled set of partial differential equations governs the behaviour of the combustion. These equations comprise two first-order hyperbolic equations and one second-order parabolic. An efficient and robust numerical method for the solution of these equations has been found, which can be readily extended to more complicated practial situations. The new numerical method takes only about four times the effort required to solve a linear heat equation; it copes with the strong nonlinearities by a Newton iteration.

(d) The success of the analytical and numerical work is shown by the good agreement with experiment of the numerical results boundary problems by C.M. Elliott and J.R. Ockendon (Pitman Research Notes in Mathematics, No. 59).

(i) Elliptic Variational Inequalities

We have been interested in the existence and qualitative behaviour of solutions of a variational inequality which relates to the suspension of a liquid drop on a soap film. The modelling of the physical problem was done by Professor T.B. Benjamin and a research student A.D. Cocker, and the analysis of the problem by A.D. Cocker, Professor A. Friedman and Dr. J.B. McLeod. We have also analysed the related problem of a rigid body resting on a membrane.

(ii) Combustion in a Porous Medium

Combustion in a porous medium involves the study of problems such as fire or flame fronts burning through loosely packed coal beds or tobacco fibre. We have successfully modelled this process, in that theoretical and numerical results agree to about 10% with the corresponding experimental results. The key experimental variables are temperature in the burning material, gas velocity through the porous material, and products of the combustion process.

The simplest model involves coupled parabolic/hyperbolic partial differential equations, an area in which little previous research has been done. Some results of the work are as follows.

(a) The simplest model (to give realistic results) of the combustion process is a two-stage reaction: the first stage is of the standard Arrhenius type, and acts as a switch

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