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A Family of Curves for the Rough Surface Reflection Coefficient

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This report describes a family of theoretical curves for the rough surface reflection coefficient and an envelope in which these curves lie. One of these curves fits the experimental data well; other experimental data points lie within the envelope.

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CONTENTS

A FAMILY OF CURVES FOR THE ROUGH SURFACE REFLECTION COEFFICIENT

INTRODUCTION

Recently, Miller, Brown, and Vegh [1] gave a short derivation that showed the rough surface reflection coefficient R, or roughness factor, was given by

$$R \equiv |\,\bar{E}/E_0\,\Gamma_0\,| = \exp\,(-2(2\pi g)^2\,)I_0(\,2(2\pi g)^2\,) \tag{1}$$

where

$$g \equiv (\sigma\,\sin\,\psi)/\lambda$$

Here $\bar{E}$ is the average electric field, E_0 the field due to the direct wave, Γ_0 the smooth sea reflection coefficient, σ the standard deviation of the sea surface elevation, ψ the grazing angle for specular reflection, λ the electromagnetic wavelength, $I_0(z)$ the modified Bessel function $J_0(iz)$, and $\bar{E}/E_0\,\Gamma_0$ the normalized coherent reflected field; g is a measure of the effective surface roughness or simply surface roughness [2, p. 10].

Equation (1) was first obtained by Brown and Miller [3] in 1974. This result agrees well with the experimental results obtained by C.I. Beard [4] in the range $0 \leqslant g \leqslant 0.3$ rad. An earlier theoretical result of W.S. Ament [5] had agreed with Beard's experimental curve only in the range $0 \leqslant g \leqslant 0.1$ rad. Ament's result is simply Eq. (1) without the I_0 factor (Fig. 1).
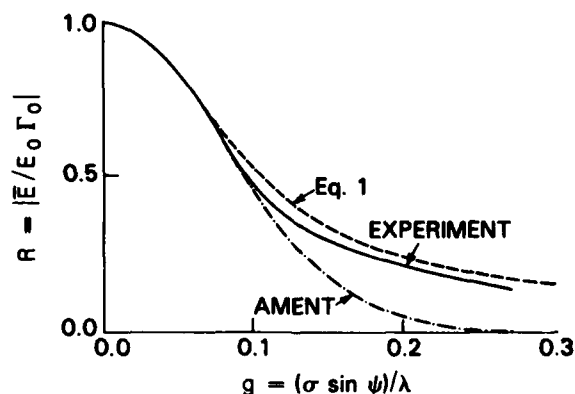

Fig. 1—Comparison of theoretical and experimental results
for the reflection coefficient

Equation (1) was derived in Ref. 1 by taking the Fourier transform

$$\bar{E}/E_0\,\Gamma_0 = \int_{-\infty}^{\infty} \exp\left[4\,\pi i\,\frac{y}{\lambda}\,\sin\,\psi\right]D(y)\,dy \tag{2}$$

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of the probability density function $D(y)$ for sea wave elevation y given by Eq. (3). Equation (2) was obtained by Ament [5] and may be understood by observir.g that $D(y)dy$ is the frequency of occurrence of a plane wave that is reflected from the sea surface with a grazing angle ψ and at a point between y and $y + dy$. The expression $(4 \pi y \sin \psi)/\lambda$ represents the phase of the electromagnetic wave after reflection.

The statistical model for sea wave elevation y used in obtaining Eq. (1) is

$$y = H \sin \frac{2\pi}{\Lambda} x \tag{3}$$

where sea wave crest height H is distributed normally and x is distributed uniformly in the interval $[-\Lambda/4, \Lambda/4]$, Λ, denoting water wavelength. This model implies that the sea surface is divided into a large number of domains and that within each domain the elevation variation is a single sinusoid with random Guassian amplitude. Other statistical models of the ocean surface may be found in Refs. 6 and 7.

In this report, we use a theoretical density for H derived by Rice [8] and by Cartwright and Longuet-Higgins [9], together with the model of the sea surface elevation given by Eq. (3) to obtain a family of curves for the rough surface reflection coefficient. We obtain an envelope wherin these curves lie. Additionally, we obtain the distribution function for sea wave elevation under the stated assumptions.

PRELIMINARY RESULTS

Rice and Cartwright and Longuet-Higgins have derived an expression for the probability density of H which may be expressed in the form

$$K(H,\epsilon) = \frac{\epsilon}{\sigma_H \sqrt{2\pi}} \exp\left[\frac{-H^2}{2\epsilon^2 \sigma_H^2}\right] + \frac{\sqrt{1-\epsilon^2}}{2\sigma_H^2} H \exp\left[\frac{-H^2}{2\sigma_H^2}\right] \left[1 + \text{erf}\left[\frac{\sqrt{2}}{2} \frac{H}{\sigma_H} \frac{\sqrt{1-\epsilon^2}}{\epsilon}\right]\right] \tag{4}$$

Here erf (z) is the error function defined by

$$\text{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)\, dt$$

σ_H is the standard deviation of H, and $0 < \epsilon < 1$. The parameter ϵ, known as the spectral width parameter, is described fully in Refs. 9 and 10, p. 515. It may easily be seen that

$$\lim_{\epsilon \to 1^-} K(H,\epsilon) = \frac{1}{\sigma_H \sqrt{2\pi}} \exp\left[-\frac{1}{2} \frac{H^2}{\sigma_H^2}\right] \qquad |H| < \infty \tag{5}$$

and

$$\left.\begin{array}{ll} \lim_{\epsilon \to 0^+} K(H,\epsilon) = \dfrac{1}{\sigma_H^2} H \exp\left[-\dfrac{1}{2} \dfrac{H^2}{\sigma_H^2}\right] & H \geqslant 0 \\[2ex] = 0 & H < 0 \end{array}\right\} \tag{6}$$

In the first instance we have a Gaussian density as a limiting case; in the second instance we have a Rayleigh density (Fig. 2).
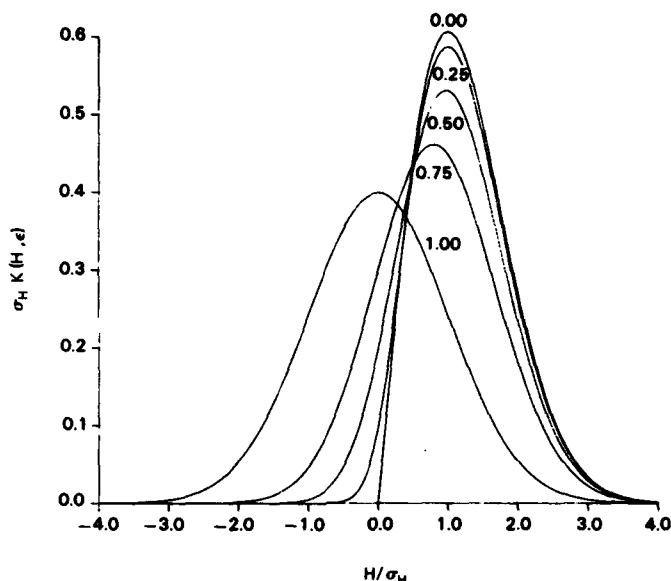
Fig. 2—Density function for the sea wave crest height H for different
values of the spectral width parameter ϵ

It is shown in Appendix A, with the model of sea wave elevation given by Eq. (3) and the density function for wave crest height H given by Eq. (4), that the standard deviation of sea wave elevation σ is given by

$$\sigma = \sigma_H/(\sqrt{2}\,\eta)$$

where η is defined by

$$\eta \equiv \left[1 + \frac{\pi}{2}\,(1 - \epsilon^2) \right]^{-\frac{1}{2}}$$

Then it is shown in Appendix B that the probability density function for y is given by

$$D(y,\epsilon) = \frac{\epsilon}{2\pi^{3/2}\,\eta\sigma}\,\exp\left[\frac{-y^2}{8\epsilon^2\eta^2\sigma^2} \right] K_0\left[\frac{y^2}{8\epsilon^2\eta^2\sigma^2} \right]$$

$$+ \frac{\sqrt{1-\epsilon^2}}{\pi\eta\sigma}\,\exp\left[\frac{-y^2}{4\eta^2\sigma^2} \right] \int_0^\infty \exp\,(-s^2)\,\mathrm{erf}\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon}\left(s^2 + \frac{y^2}{4\eta^2\sigma^2} \right)^{1/2} \right] ds \qquad (7)$$

where K_0 is a Bessel function of imaginary argument defined by Eq. (D5) in Appendix D. Graphs for the cumulative distribution function for sea wave elevation

$$P(y,\epsilon) \equiv \int_{-\infty}^y D(t,\epsilon)\,dt \qquad (8)$$

are given in Fig. 3. An alternate form for Eq. (7) is developed in Appendix D. When substituted into Eq. (8), a formula for $P(y,\epsilon)$ is obtained that is suitable for numerical quadrature. This formula for $P(y,\epsilon)$ is also given in Appendix D. Yet another form for Eq. (7) is given in Appendix E, together with a formula for the special case $D(y,\sqrt{2}/2)$ which is of interest and will be discussed.
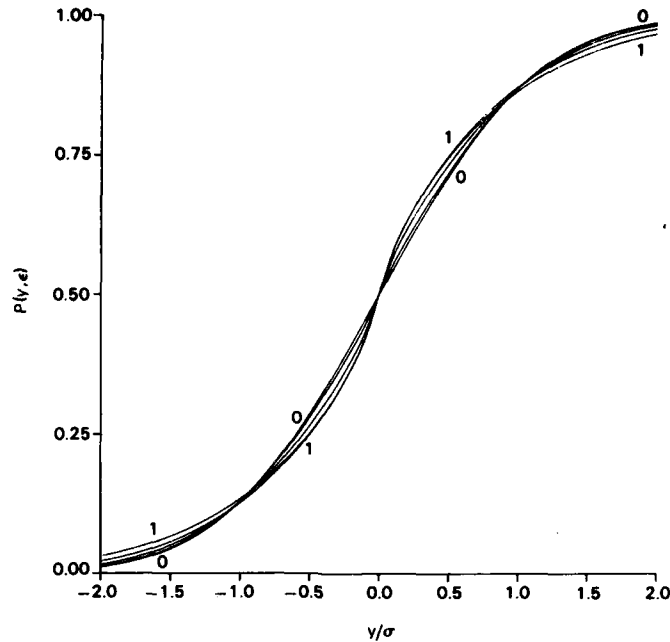
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Fig. 3—Cumulative distribution function for sea wave elevation y
for different values of the spectral width parameter ϵ

THE REFLECTION COEFFICIENT

Substituting the expression for $D(y,\epsilon)$ given by Eq. (7) into Eq. (2), we obtain the following expression for the rough surface reflection coefficient R:

$$R(g,\epsilon) = \epsilon^2 \exp\left(-2\epsilon^2\eta^2 (2\pi g)^2\right) I_0\left(2\epsilon^2\eta^2 (2\pi g)^2\right) + \sqrt{1-\epsilon^2} \exp\left(-4\eta^2 (2\pi g)^2\right)$$

$$-\frac{1}{2}\epsilon^2 (1-\epsilon^2)\, \Phi_1\left[\frac{3}{2}, 1; 2; \epsilon^2, -4\epsilon^2\eta^2 (2\pi g)^2\right] \tag{9}$$

Here $\Phi_1(\alpha, \beta; \gamma; x, y)$ is a degenerate hypergeometric function in two variables x and y, defined for $|x| < 1$, $|y| < \infty$ by

$$\Phi_1(\alpha, \beta; \gamma; x, y) \equiv \sum_{n,m=0}^{\infty} \frac{(\alpha, n+m)\,(\beta, n)}{(\gamma, n+m)\, n!m!}\, x^n y^m \tag{10}$$

where the Pochhammer symbol (a,n) is defined by a ratio of gamma functions:

$$(a,n) \equiv \Gamma(a+n)/\Gamma(a) = a(a+1) \ldots (a+n-1)$$

This result for the reflection coefficient is obtained in Appendix C.

Graphs for the reflection coefficient $R(g,\epsilon)$ as a function of surface roughness g are given in Fig. 4. The envelope of these curves are effectively $R(g,1)$ which forms the upper envelope, and $R(g,0)$ which forms the lower envelope. The upper envelope corresponds to the Guassian density given by Eq. (5) and for which the reflection coefficient is given by Eq. (1)

$$R(g,1) = \exp\left(-2(2\pi g)^2\right) I_0\left(2(2\pi g)^2\right)$$
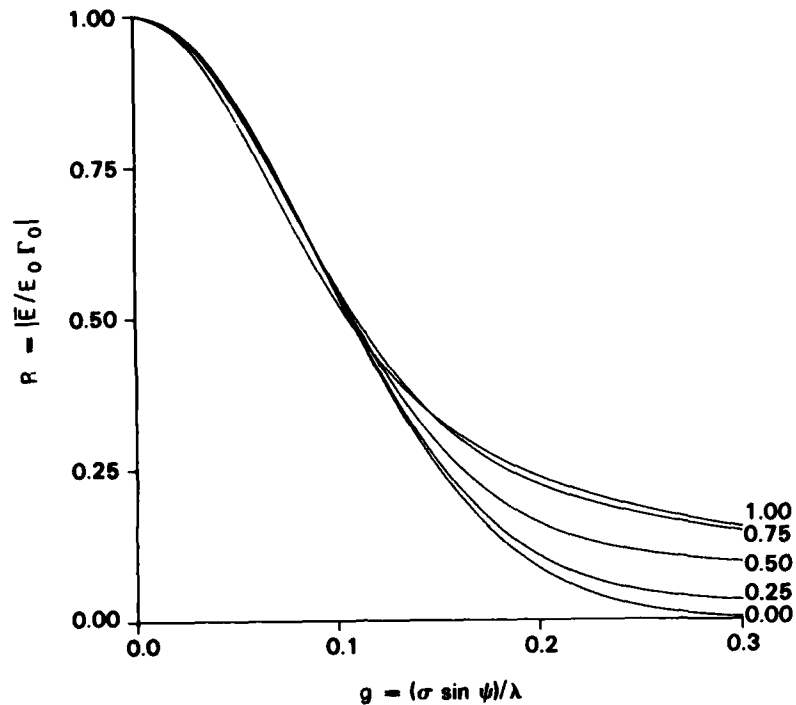
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Fig. 4—Theoretical curves for the reflection coefficient for different values
of the spectral width parameter ϵ

The lower envelope corresponds to the Rayleigh density given by Eq. (6), and for which the reflection coefficient is given by

$$R(g,0) = \exp\left[\frac{-8}{2+\pi}(2\pi g)^2\right]$$

Both these limiting envelopes may be incorporated into the general equation, provided we define the third term in Eq. (9) to be zero when $\epsilon = 1$.

SOME OBSERVATIONS

Note that for surface roughness $0 \leqslant g \leqslant 0.1$ rad, $R(g,\epsilon)$ for all ϵ and Ament's result $R(g) = \exp[-2(2\pi g)^2]$ are similar and agree with Beard's experimental curve. This is not surprising since all theoretical results must be consistent with the Rayleigh criterion [2]. On the interval $0.1 \leqslant g \leqslant 0.3$ rad, numerical calculations reveal that the experimental curve of Beard corresponds almost exactly to $R(g,0.7)$. Noting that $0.7 \approx \sqrt{2}/2$, a closed form for $D(y,\sqrt{2}/2)$ is derived in Appendix E, since this case is of special interest (Fig. 5). Other data points in the domain $0.1 \leqslant g \leqslant 0.3$ rad given by Beard [4, Fig. 2] lie within the two envelopes $R(g,1)$ and $R(g,0)$ (Fig. 6).

CONCLUSION

Statistical models for the sea surface have been put forward and used to derive a family of curves for the rough surface reflection coefficient. One of these curves fits the experimental data curve. Other experimental data points lie within the envelope of the derived curves. There is, therefore, a theoretical basis for the experimental data when the surface roughness exceeds 0.1 rad.
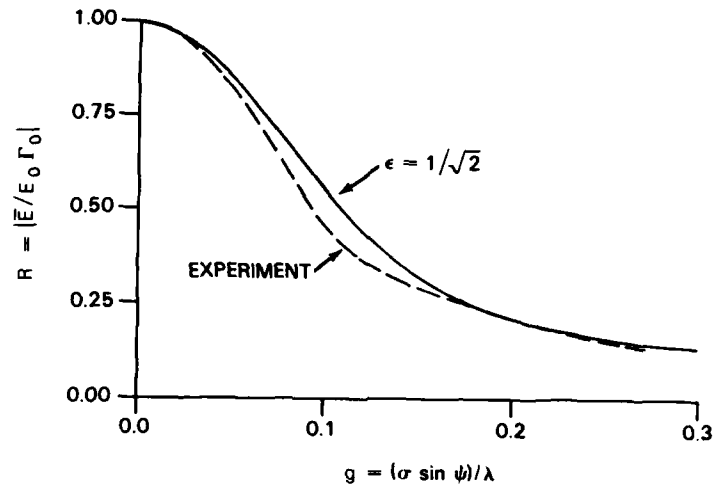
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Fig. 5—Comparison of Beard's experimental curve for the reflection
coefficient and the theoretical curve for $\epsilon = 1/\sqrt{2}$
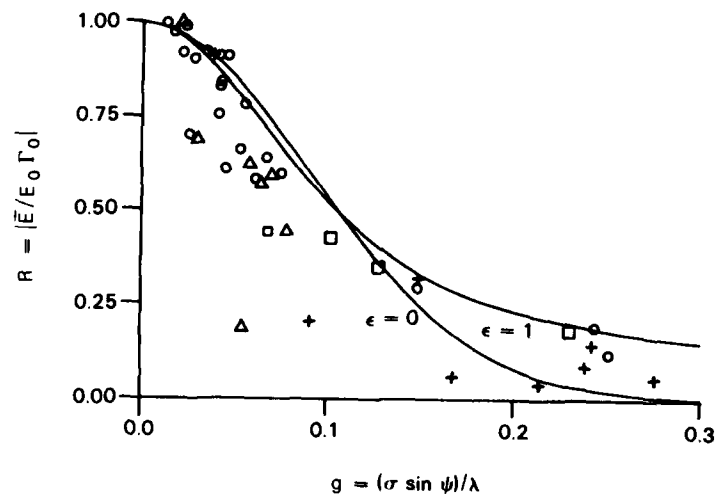

Fig. 6—Comparison of the theoretical limiting cases for the reflection coefficient
and the data given by Beard [4, Fig. 2]

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Appendix A
THE STANDARD DEVIATION OF SEA WAVE ELEVATION

Consider the random variable $y = H \sin \theta$, where H is a random variable with density $K(H, \epsilon)$ given by Eq. (4), and θ is a random variable, independent of H, distributed uniformly on the interval $|\theta| \leqslant \pi/2$. Let $U(\theta)$ be the density of θ, so that

$$\left. \begin{array}{ll} U(\theta) = \pi^{-1} & |\theta| \leqslant \pi/2 \\ \quad\quad = 0 & |\theta| > \pi/2 \end{array} \right\} \tag{A1}$$

The expected value or mean of a random variable x is defined by

$$\mu_x \equiv E[x] \equiv \int_{-\infty}^{\infty} x \, f(x) \, dx$$

where f is the density function for x and the variance is defined by

$$\sigma_x^2 \equiv E[\,(x - E[x])^2\,] = E[\,(x - \mu_x)^2\,] = E[x^2] - \mu_x^2$$

If x and y are independent, then

$$E[xy] = E[x] \, E[y]$$

In our case since H and $\sin \theta$ are also independent,

$$\mu_y = E[H \sin \theta] = E[H] \int_{-\pi/2}^{\pi/2} \sin \theta \, U(\theta) \, d\theta = 0$$

and

$$\sigma_y^2 = E[H^2 \sin^2 \theta] = E[H^2] \int_{-\pi/2}^{\pi/2} \sin^2 \theta \, U(\theta) \, d\theta = \frac{1}{2} E[H^2]$$

Hence

$$2\sigma_y^2 = \sigma_H^2 + \mu_H^2$$

Since

$$\mu_H = \int_{-\infty}^{\infty} H \, K(H, \epsilon) \, dH = \frac{\sqrt{1 - \epsilon^2}}{2\sigma_H^2} \int_{-\infty}^{\infty} H^2 \exp\left[-\frac{1}{2} \frac{H^2}{\sigma_H^2}\right] dH$$

$$= 2\sqrt{2} \, \sigma_H \, \sqrt{1 - \epsilon^2} \int_{0}^{\infty} x^2 \exp(-x^2) \, dx = \sqrt{\pi/2} \, \sigma_H \, \sqrt{1 - \epsilon^2}$$

we have

$$\sigma_H^2 = 2\eta^2\sigma^2 \tag{A2}$$

where

$$\sigma \equiv \sigma_y, \ \eta \equiv \left[1 + \frac{\pi}{2} \left(1 - \epsilon^2 \right) \right]^{-1/2} \tag{A3}$$

Appendix B
THE PROBABILITY DENSITY FUNCTION FOR y

Consider the random variable $y = H \sin \theta$, where H is a random variable with density $K(H,\epsilon)$, $|H| < \infty$ given by Eq. (4), and θ is a random variable, independent of H, with density $U(\theta)$ given by Eq. (A1). Let $D(y,\epsilon)$ be the density function for y. Since θ and H are independent, the joint density $g(\theta,H,\epsilon,)$ of θ and H is given by

$$g(\theta,H,\epsilon) = K(H,\epsilon)\, U(\theta) \qquad |\theta| \leqslant \pi/2,\ |H| < \infty$$
$$0 \qquad\qquad\qquad \text{elsewhere}$$

Set $v = H$ so that $\theta = \sin^{-1}(y/v)$ and let $f(y,v,\epsilon)$ be the joint density of y and v. Then

$$f(y,v,\epsilon) = |J|\, g(\sin^{-1}(y/v),\, v,\epsilon)$$

where the Jacobian determinant

$$|J| = \frac{\partial \theta}{\partial y}\, \frac{\partial H}{\partial v} - \frac{\partial \theta}{\partial v}\, \frac{\partial H}{\partial y}$$

A straightforward computation yields

$$|J| = (v^2 - y^2)^{-1/2}$$

and

$$f(y,H,\epsilon) = \pi^{-1}\, (H^2 - y^2)^{-1/2}\, K(H,\epsilon) \qquad |y| < |H| < \infty$$
$$0 \qquad\qquad\qquad\qquad \text{elsewhere}$$

Now, integrate $f(y,H,\epsilon)$ over H to obtain

$$D(y,\epsilon) = \frac{1}{\pi} \int_{-\infty}^{-|y|} \frac{K(H,\epsilon)\, dH}{\sqrt{H^2 - y^2}} + \frac{1}{\pi} \int_{|y|}^{\infty} \frac{K(H,\epsilon)\, dH}{\sqrt{H^2 - y^2}}$$

Substituting $K(H,\epsilon)$ into this equation and using Eq. (A2) we have

$$D(y,\epsilon) = \frac{\epsilon}{\pi^{3/2}\eta\sigma} \int_{|y|}^{\infty} \frac{\exp\left[\dfrac{-H^2}{4\epsilon^2\eta^2\sigma^2}\right] dH}{\sqrt{H^2 - y^2}} + \frac{\sqrt{1 - \epsilon^2}}{2\pi\eta^2\sigma^2} \int_{|y|}^{\infty} \frac{H \exp\left[\dfrac{-H^2}{4\eta^2\sigma^2}\right] \mathrm{erf}\left[\dfrac{H\sqrt{1 - \epsilon^2}}{2\eta\sigma\epsilon}\right] dH}{\sqrt{H^2 - y^2}}$$

where σ and η are defined by Eq. (A3). A change of variables, namely, $H = |y|\sqrt{x}$ in the former integral and $H^2 - y^2 = s^2$ in the latter integral, gives

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$$D(y,\epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \int_1^\infty \frac{\exp\left[\dfrac{-y^2 x}{4\epsilon^2\eta^2\sigma^2}\right] dx}{\sqrt{x}\ \sqrt{x-1}}$$

$$+ \frac{\sqrt{1-\epsilon^2}}{\pi\eta\sigma} \exp\left[\frac{-y^2}{4\eta^2\sigma^2}\right] \int_0^\infty \exp\left(-s^2\right) \mathrm{erf}\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon}\left(s^2 + \frac{y^2}{4\eta^2\sigma^2}\right)^{1/2}\right] ds$$

Now, using Ref. 11, p. 319, 3.383-3 to evaluate the first integral, Eq. (7) is obtained.

Appendix C
THE REFLECTION COEFFICIENT $R(g,\epsilon)$

Substituting the expression for $D(y,\epsilon)$ given by Eq. (7) into Eq. (2), we obtain

$$R = \frac{\epsilon}{\pi^{3/2}\eta\sigma} \int_0^\infty \cos\left[\frac{4\pi}{\lambda} y \sin\psi\right] \exp\left[\frac{-y^2}{8\epsilon^2\eta^2\sigma^2}\right] K_0\left[\frac{y^2}{8\epsilon^2\eta^2\sigma^2}\right] dy$$

$$+ \frac{2\sqrt{1-\epsilon^2}}{\pi\eta\sigma} \int_0^\infty \int_0^\infty \exp(-s^2) \exp\left[\frac{-y^2}{4\eta^2\sigma^2}\right] \cos\left[\frac{4\pi}{\lambda} y \sin\psi\right] \text{erf}\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon}\left(s^2 + \frac{y^2}{4\eta^2\sigma^2}\right)^{1/2}\right] ds\ dy$$

The first integral is evaluated by making the transformation $x = y^2/8\epsilon^2\eta^2\sigma^2$ and then using Ref. 11, p. 765, 6.755-9. See Ref. 1 for essentially the same computation. We have then

$$R = \epsilon^2 \exp(-2\epsilon^2\eta^2 (2\pi g)^2) I_0(2\epsilon^2\eta^2 (2\pi g)^2) + \frac{2\sqrt{1-\epsilon^2}}{\pi\eta\sigma} \int_0^\infty \int_0^\infty (\cdot)\ ds\ dy$$

where g is defined by $g \equiv (\sigma \sin\psi)/\lambda$ and $(\cdot)$ is the integrand of the double integral. Making the transformation $t = y/2\eta\sigma$ and then changing to polar coordinates gives

$$\frac{2\sqrt{1-\epsilon^2}}{\pi\eta\sigma} \int_0^\infty \int_0^\infty (\cdot)\ ds\ dy = \frac{4\sqrt{1-\epsilon^2}}{\pi} \int_0^\infty \int_0^\infty \cos(8\pi\eta gt) \exp(-t^2-s^2) \text{erf}\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon}\sqrt{t^2+s^2}\right] ds\ dt$$

$$= \frac{4\sqrt{1-\epsilon}}{\pi} \int_0^{\pi/2} \int_0^\infty r \exp(-r^2) \cos(8\pi\eta gr \cos\theta) \text{erf}\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon} r\right] dr\ d\theta$$

Since by Ref. 11, p. 402, 3.715-19,

$$\int_0^{\pi/2} \cos(8\pi\eta gr \cos\theta)\ d\theta = \frac{\pi}{2} J_0(8\pi\eta gr)$$

we have

$$R(g,\epsilon) = \epsilon^2 \exp(-2\epsilon^2\eta^2 (2\pi g)^2) I_0(2\epsilon^2\eta^2 (2\pi g)^2)$$

$$+ 2\sqrt{1-\epsilon^2} \int_0^\infty r \exp(-r^2) J_0(8\pi\eta gr) \text{erf}\left[\frac{\sqrt{1-\epsilon^2}}{\epsilon} r\right] dr \quad . \tag{C1}$$

To obtain a closed form for R, we show the following: for any real numbers α and $\beta \neq 0$,

$$\int_0^\infty r \exp(-r^2) J_0(\alpha r) \operatorname{erf}(\beta r)\, dr$$

$$= \frac{1}{2} \exp\left[\frac{-\alpha^2}{4}\right] - \frac{1}{4}\beta(1+\beta^2)^{-3/2} \Phi_1\left[\frac{3}{2}, 1; 2; \frac{1}{1+\beta^2}, \frac{-\alpha^2}{4(1+\beta^2)}\right] \qquad (C2)$$

where Φ_1 is a degenerate hypergeometric function of two variables defined by Eq. (10). Substituting Ref. 11, p. 931, 8.252-4

$$\operatorname{erf}(\beta r) = 1 - \frac{2\beta}{\pi} \exp(-\beta^2 r^2) \int_0^\infty \frac{\exp(-r^2 t^2)\, dt}{t^2 + \beta^2} \qquad\qquad r > 0$$

in the left side of Eq. (C2) and noting that Ref. 11, p. 717, 6.631-4 for $p > 0$ and any real q

$$\int_0^\infty r \exp(-pr^2) J_0(qr)\, dr = \frac{1}{2p} \exp\left[\frac{-q^2}{4p}\right] \qquad (C3)$$

we obtain

$$\frac{1}{2} \exp\left[\frac{-\alpha^2}{4}\right] - \frac{2\beta}{\pi} \int_0^\infty \left\{ \int_0^\infty r \exp[-(1+\beta^2+t^2)r^2] J_0(\alpha r)\, dr \right\} \frac{dt}{\beta^2 + t^2}$$

Now applying Eq. (C3) again to the integral in braces we obtain

$$\frac{1}{2} \exp\left[\frac{-\alpha^2}{4}\right] - \frac{\beta}{\pi} \int_0^\infty \frac{\exp\left[\dfrac{-\alpha^2}{4(1+\beta^2+t^2)}\right]}{(1+\beta^2+t^2)(\beta^2+t^2)}\, dt$$

and on making the transformation $t = \sqrt{1+\beta^2}\, \sqrt{x^{-1} - 1}$ we have

$$\frac{1}{2} \exp\left[\frac{-\alpha^2}{4}\right] - \frac{\beta}{2\pi(1+\beta^2)^{3/2}} \int_0^1 x^{1/2}(1-x)^{-1/2}\left[1 - \frac{x}{1+\beta^2}\right]^{-1} \exp\left[\frac{-\alpha^2 x}{4(1+\beta^2)}\right]\, dx$$

Finally, using Ref. 11, p. 321, 3.385 we obtain Eq. (C2).

Letting $\alpha = 8\pi\eta g$, $\beta = \sqrt{1-\epsilon^2}/\epsilon$ in Eq. (C2) and substituting the result into Eq. (C1) gives Eq. (9). We may also write

$$R(g,\epsilon) = \epsilon^2 \exp(-2\epsilon^2\eta^2(2\pi g)^2) I_0(2\epsilon^2\eta^2(2\pi g)^2) + \sqrt{1-\epsilon^2} \exp(-4\eta^2(2\pi g)^2)$$

$$- \pi^{-1}\epsilon^2(1-\epsilon^2) \int_0^1 x^{1/2}(1-x)^{-1/2}(1-\epsilon^2 x)^{-1} \exp(-4\epsilon^2\eta^2(2\pi g)^2 x)\, dx \qquad (C4)$$

This expression for R is used in Appendix D.

Appendix D
THE CUMULATIVE DISTRIBUTION $P(y,\epsilon)$

Defining $\omega \equiv (4\pi/\lambda) \sin \psi$, we obtain from Eq. (C4)

$$R(\omega,\epsilon) = \epsilon^2 \exp\left(-\frac{1}{2}\epsilon^2\eta^2\sigma^2\omega^2\right)I_0\left(\frac{1}{2}\epsilon^2\eta^2\sigma^2\omega^2\right) + \sqrt{1-\epsilon^2}\exp\left(-\eta^2\sigma^2\omega^2\right)$$

$$-\frac{1}{\pi}\epsilon^2(1-\epsilon^2)\int_0^1 x^{1/2}(1-x)^{-1/2}(1-\epsilon^2x)^{-1}\exp\left(-\epsilon^2\eta^2\sigma^2\omega^2x\right)dx$$

Since we may write Eq. (2)

$$R(\omega,\epsilon) = \int_{-\infty}^{\infty} e^{i\omega y} D(y,\epsilon)\, dy$$

we must have

$$D(y,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega y} R(\omega,\epsilon)\, d\omega$$

Substituting the expression for $R(\omega,\epsilon)$ into this equation gives

$$D(y,\epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma}\exp\left(-y^2/8\epsilon^2\eta^2\sigma^2\right)K_0\left(y^2/8\epsilon^2\eta^2\sigma^2\right) + \frac{1}{\pi}\sqrt{1-\epsilon^2}\int_0^{\infty}\cos(\omega y)\exp\left(-\eta^2\sigma^2\omega^2\right)d\omega$$

$$-\frac{\epsilon^2(1-\epsilon^2)}{\pi^2}\int_0^{\infty}\cos(\omega y)\left\{\int_0^1 x^{1/2}(1-x)^{-1/2}(1-\epsilon^2x)^{-1}\exp\left(-\epsilon^2\eta^2\sigma^2\omega^2x\right)dx\right\}d\omega$$

Noting that Ref. 11, p. 496, 3.952-9

$$\int_0^{\infty}\exp\left(-\beta^2x^2\right)\cos(\alpha x)\, dx = \frac{\sqrt{\pi}}{2\beta}\exp\left(-\alpha^2/4\beta^2\right)$$

we may integrate the single and double integrals with respect to ω to obtain

$$\frac{\sqrt{1-\epsilon^2}}{2\pi^{1/2}\eta\sigma}\exp\left(\frac{-y^2}{4\eta^2\sigma^2}\right) - \frac{\epsilon(1-\epsilon^2)}{2\pi^{3/2}\eta\sigma}\int_0^1 \frac{\exp\left(-y^2/4\epsilon^2\eta^2\sigma^2x\right)}{(1-\epsilon^2x)\sqrt{1-x}}\, dx$$

for the latter two terms in the equation for $D(y,\epsilon)$. On making the transformation $x(t+1) = 1$, we have another form for $D(y,\epsilon)$, viz.

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$$D(y,\epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \exp\left(\frac{-y^2}{8\epsilon^2\eta^2\sigma^2}\right) K_0\left(\frac{y^2}{8\epsilon^2\eta^2\sigma^2}\right) + \frac{\sqrt{1-\epsilon^2}}{2\pi^{1/2}\eta\sigma} \exp\left(\frac{-y^2}{4\eta^2\sigma^2}\right)$$

$$- \frac{\epsilon(1-\epsilon^2)}{2\pi^{3/2}\eta\sigma} \exp\left(\frac{-y^2}{4\epsilon^2\eta^2\sigma^2}\right) \int_0^\infty \frac{\exp\left(\frac{-y^2}{4\epsilon^2\eta^2\sigma^2}\,x\right)}{\sqrt{x}\,\sqrt{1+x}\,(1-\epsilon^2+x)}\,dx$$

Substituting $D(t,\epsilon)$ into Eq. (8), noting that

$$\int_{-\infty}^u \exp(-t^2)\,dt = \frac{\sqrt{\pi}}{2}(1 + \mathrm{erf}\,u) \tag{D1}$$

and integrating with respect to t gives

$$P(y,\epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \int_{-\infty}^y \exp\left(\frac{-t^2}{8\epsilon^2\eta^2\sigma^2}\right) K_0\left(\frac{t^2}{8\epsilon^2\eta^2\sigma^2}\right) dt + \frac{\sqrt{1-\epsilon^2}}{2}\left[1 + \mathrm{erf}\left(\frac{y}{2\eta\sigma}\right)\right]$$

$$- \frac{\epsilon^2(1-\epsilon^2)}{2\pi}\left\{\int_0^\infty \frac{dx}{x^{1/2}(1+x)(1-\epsilon^2+x)} + \int_0^\infty \frac{\mathrm{erf}\left(\frac{y\sqrt{1+x}}{2\epsilon\eta\sigma}\right) dx}{x^{1/2}(1+x)(1-\epsilon^2+x)}\right\} \tag{D2}$$

Using Ref. 11, p. 289, 3.223-1, we find

$$\int_0^\infty \frac{dx}{x^{1/2}(1+x)(1-\epsilon^2+x)} = \frac{\pi}{\epsilon^2}\left(\frac{1}{\sqrt{1-\epsilon^2}} - 1\right) \tag{D3}$$

Making the transformation $x = \sinh^2 t$ in the last term of Eq. (D2), we obtain

$$\int_0^\infty \frac{\mathrm{erf}\left(\frac{y\sqrt{1+x}}{2\epsilon\eta\sigma}\right) dx}{x^{1/2}(1+x)(1-\epsilon^2+x)} = 2\int_0^\infty \frac{\mathrm{erf}\left(\frac{y\cosh t}{2\epsilon\eta\sigma}\right) dt}{\cosh t\,(\cosh^2 t - \epsilon^2)} \tag{D4}$$

Evaluate $K_0(t^2/8\epsilon^2\eta^2\sigma^2)$ using Ref. 12, p. 85

$$K_0(s) \equiv \int_0^\infty \exp(-s\cosh x)\,dx \qquad s > 0, \tag{D5}$$

substitute the result into the first term of Eq. (D2), integrate with respect to t, and use Eq. (D1) to obtain

$$\frac{\epsilon}{2\pi^{3/2}\eta\sigma} \int_{-\infty}^y \exp\left(\frac{-t^2}{8\epsilon^2\eta^2\sigma^2}\right) K_0\left(\frac{t^2}{8\epsilon^2\eta^2\sigma^2}\right) dt$$

$$= \frac{\sqrt{2}\epsilon^2}{2\pi}\left\{\int_0^\infty \frac{dx}{\sqrt{1+\cosh x}} + \int_0^\infty \frac{\mathrm{erf}\left(\frac{y\sqrt{1+\cosh x}}{2\sqrt{2}\epsilon\eta\sigma}\right) dx}{\sqrt{1+\cosh x}}\right\} \tag{D6}$$

Now, making the transformation $\cosh x = 2t + 1$ and using Ref. 11, p. 289, 3.222-2, we have

$$\int_0^\infty \frac{dx}{\sqrt{1 + \cosh x}} = \frac{1}{\sqrt{2}} \int_0^\infty \frac{dt}{\sqrt{t} \ (t + 1)} = \frac{\pi}{\sqrt{2}} \tag{D7}$$

Finally, combining equations (D2), (D3), (D4), (D6), and (D7), we obtain

$$P(y,\epsilon) = \frac{1}{2} + \frac{\sqrt{1 - \epsilon^2}}{2} \, \text{erf}\left(\frac{y}{2\eta\sigma} \right) - \frac{\epsilon^2(1 - \epsilon^2)}{\pi} \int_0^\infty \frac{\text{erf}\left(\dfrac{y \cosh s}{2\epsilon\eta\sigma} \right) ds}{\cosh s \ (\cosh^2 s - \epsilon^2)}$$

$$+ \frac{\sqrt{2}\epsilon^2}{2\pi} \int_0^\infty \frac{\text{erf}\left(\dfrac{y\sqrt{1 + \cosh s}}{2\sqrt{2}\epsilon\eta\sigma} \right) ds}{\sqrt{1 + \cosh s}}$$

This expression for $P(y,\epsilon)$, which is suited for numerical evaluation, was used to obtain the graphs in Fig. 3.

Appendix E
A CLOSED FORM FOR THE DENSITY FUNCTION $D(y,\sqrt{2}/2)$

To give a closed form for the probability density function $D(y,\sqrt{2}/2)$, we first show that for any real k and μ

$$\pi^{1/2} \int_0^\infty \exp(-s^2)\, \text{erf}\,(k\sqrt{\mu^2+s^2})\, ds$$

$$= \tan^{-1} k + \frac{k}{1+k^2} \int_0^{\frac{1}{2}\mu^2(1+k^2)} \exp\left[\frac{1-k^2}{1+k^2}\,s\right] K_0(s)\, ds \qquad (E1)$$

To show this, note that

$$\frac{d}{d\mu}\,\text{erf}\,(k\sqrt{\mu^2+s^2}) = \frac{2k}{\sqrt{\pi}}\,\frac{\mu}{\sqrt{\mu^2+s^2}}\,\exp(-k^2\mu^2)\,\exp(-k^2s^2)$$

and hence

$$\text{erf}\,(k\sqrt{\mu^2+s^2}) = \frac{2k}{\sqrt{\pi}}\,\exp(-k^2s^2)\int_0^{|\mu|}\frac{t\,\exp(-k^2t^2)}{\sqrt{s^2+t^2}}\,dt + \text{erf}\,(k|s|)$$

Making the transformation $t = |\mu|\sqrt{y}$ gives

$$\text{erf}\,(k\sqrt{\mu^2+s^2}) = \frac{k\mu^2}{\sqrt{\pi}}\,\exp(-k^2s^2)\int_0^1\frac{\exp(-k^2\mu^2 y)}{\sqrt{s^2+\mu^2 y}}\,dy + \text{erf}\,(k|s|)$$

Multiplying this by $\exp(-s^2)$ and integrating, we have

$$\int_0^\infty \exp(-s^2)\,\text{erf}\,(k\sqrt{\mu^2+s^2})\, ds = \frac{k\mu^2}{\sqrt{\pi}}\int_0^1 \exp(-k^2\mu^2 y)\left\{\int_0^\infty \frac{\exp[-(1+k^2)s^2]}{\sqrt{s^2+\mu^2 y}}\,ds\right\}dy$$

$$+ \int_0^\infty \exp(-s^2)\,\text{erf}\,(ks)\, ds$$

Using Ref. 11, p. 649, 6.285-1, we have

$$\int_0^\infty \exp(-s^2)\,\text{erf}\,(ks)\, ds = \frac{\tan^{-1}k}{\sqrt{\pi}}$$

The integral in braces is evaluated by making the transformation $s^2 = x$ and then using Ref. 11, p. 316, 3.364-3 to obtain

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$$\int_0^\infty \exp\,(-s^2)\,\operatorname{erf}\,(k\,\sqrt{\mu^2 + s^2})\,ds$$

$$= \frac{\tan^{-1}k}{\sqrt{\pi}} + \frac{k\mu^2}{2\sqrt{\pi}}\int_0^1 \exp\left[\frac{1}{2}\mu^2\,(1-k^2)\,y\right]K_0\left[\frac{1}{2}\mu^2\,(1+k^2)\,y\right]dy$$

Finally, making the transformation $\mu^2(1 + k^2)\,y = 2s$, we have Eq. (E1).

Using $k = \sqrt{1 - \epsilon^2}/\epsilon$, $\mu = y/2\eta\sigma$ in Eq. (E1) and substituting the result into Eq. (7) gives

$$D(y,\epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma}\exp\left[\frac{-y^2}{8\epsilon^2\eta^2\sigma^2}\right]K_0\left[\frac{y^2}{8\epsilon^2\eta^2\sigma^2}\right]$$

$$+ \frac{\sqrt{1-\epsilon^2}}{\pi^{3/2}\eta\sigma}\exp\left[\frac{-y^2}{4\eta^2\sigma^2}\right]\left\{\cos^{-1}\epsilon + \epsilon\sqrt{1-\epsilon^2}\int_0^{y^2/8\epsilon^2\eta^2\sigma^2}\exp\left[\,(2\epsilon^2 - 1)s\,\right]K_0(s)\,ds\right\}\quad\text{(E2)}$$

It may be shown by using Ref. 12, p. 87 that

$$\int_0^x K_0(s)\,ds = \frac{\pi}{2}x[\,K_0(x)\,L_{-1}(x) + K_1(x)\,L_0(x)\,]$$

where $K_\nu(x)$ are modified Bessel functions and $L_\nu(x)$ are modified Struve functions. Defining

$$a \equiv 4/\sqrt{4 + \pi},$$

we obtain

$$D(y,\sqrt{2}/2) = \frac{\exp\,(-y^2/a^2\sigma^2)}{2(2\pi)^{1/2}\,a\sigma}\left\{1 + \left[\frac{2}{\pi} + \frac{y^2}{a^2\sigma^2}\,L_{-1}\left[\frac{y^2}{a^2\sigma^2}\right]\right]K_0\left[\frac{y^2}{a^2\sigma^2}\right]\right.$$

$$\left.+ \frac{y^2}{a^2\sigma^2}\,K_1\left[\frac{y^2}{a^2\sigma^2}\right]L_0\left[\frac{y^2}{a^2\sigma^2}\right]\right\}\quad\text{(E3)}$$

which is a closed form for $D(y,\sqrt{2}/2)$.

The functions $\int_0^x K_0\,(t)\,dt$, $K_\nu(x)$, $L_\nu(x)$ may be computed using tables found in Ref. 13 so that either Eq. (E2) or Eq. (E3) may be used to compute $D(y,\sqrt{2}/2)$. We remark that it may be shown for $0 < |\alpha| < 1$, $0 < x$

$$\int_0^x \exp\,(\alpha t)K_0(t)\,dt = \operatorname{sgn}\alpha\left\{\exp\,(\alpha x)\left[\int_0^\infty \frac{\cos\,(\alpha x t)\,dt}{(1 + t^2)\,\sqrt{1 + \alpha^2 t^2}}\right.\right.$$

$$\left.\left.+ \int_0^\infty \frac{t\,\sin\,(\alpha x t)\,dt}{(1 + t^2)\,\sqrt{1 + \alpha^2 t^2}}\,\right] - \frac{\cos^{-1}(|\alpha|)}{\sqrt{1 - \alpha^2}}\right\}$$

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