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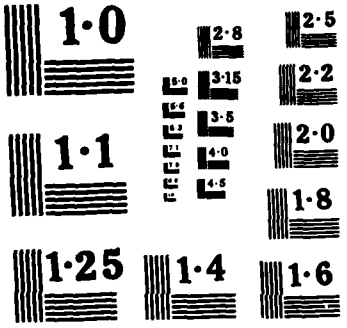

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# COMPUTABLE BOUNDS FOR SOLUTIONS OF INTEGRAL EQUATIONS

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# COMPUTABLE BOUNDS FOR SOLUTIONS OF INTEGRAL EQUATIONS

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## ABSTRACT

Interval integration is used to obtain inclusions of integral operators of the form

$$g(u)(s) = \int g(s,t,u(s),u(t))dt \qquad (1)$$

which can be carried out on a computer. The resulting inclusions, combined with interval iteration, are used to compute guaranteed upper and lower bounds for solutions of integral equations of the form

$$u = g(u) \qquad (2)$$

for $s \in S$. It is also possible to establish existence or nonexistence of solutions of integral equations in given regions on the basis of results of the computation. Examples of applications of this technique to linear and nonlinear integral equations are eigenvalue problems for linear integral operators are given.

AMS (MOS) Subject Classifications: 65G10, 65J10, 65J20, 65R20

Key words: Integral equations, Eigenvalue problems, Error bounds, Interval integration, Interval iteration

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## SIGNIFICANCE AND EXPLANATION

This report is the text of an invited address to be given at the 11th IMACS World Congress, to be held in Oslo, Norway, on August 5-9, 1985. The theories of interval integration, interval iteration and their applications to integral equations have been developed in previous papers. This report surveys these results, but concentrates on computer implementation by discussing various ways in which inclusions of integral transforms can actually be represented on a computer. The computations based on these inclusions are self-validating, in that existence of solutions and upper and lower bounds for the solutions of linear and nonlinear integral equations and eigenvalue problems can be guaranteed on the basis of computed results alone. It is assumed, of course, that the computing environment supports interval computation. Since support for interval arithmetic is now required by the IEEE standard for floating-point arithmetic, and provided by Pascal-SC for microcomputers and ACRITH for IBM 370 mainframe computers, the need to make special provision for interval arithmetic is only necessary on outdated systems.

Some simple examples illustrating the techniques presented in the paper are cited.



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# COMPUTABLE BOUNDS FOR SOLUTIONS OF INTEGRAL EQUATIONS

L. B. Rall


## Summary

Interval integration is used to obtain inclusions of integral operators of the form

$$g(u)(s) = \int_T g(s,t,u(s),u(t))dt \qquad (1)$$

which can be carried out on a computer. The resulting inclusions, combined with interval iteration, are used to compute guaranteed upper and lower bounds for solutions of integral equations of the form

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for $s \in S$. It is also possible to establish existence or nonexistence of solutions of integral equations in given regions on the basis of results of the computation. Examples of applications of this technique to linear and nonlinear integral equations and eigenvalue problems for linear operators are given.


## Integral Equations

The integral equation expressed by (2) includes many of the varieties of integral equations of importance in applications. For example, for

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$$g(s,t,u(s),u(t)) = f(x) + \lambda K(s,t)u(t), \tag{3}$$

the result is the linear integral equation of Fredholm type and second kind,

$$u(s) = f(x) + \lambda \int_T K(s,t)u(t)dt, \tag{4}$$

while the choice

$$g(s,t,u(s),u(t)) = f(s,t,u(t)), \tag{5}$$

gives the nonlinear integral equation of Urysohn type

$$u(s) = \int_T f(s,t,u(t))dt, \tag{6}$$

and so on [12]. In general, S and T are subsets of some possibly high-dimensional real or complex spaces. For definiteness, we will take S = T = [0,1], and consider integral equations of the form

$$u(s) = \int_0^1 g(s,t,u(s),u(t))dt, \tag{7}$$

since the discussion given here will carry over immediately to the more general case. It is also worth noting that (7) includes Volterra integral equations as the special case $g(s,t,v,w) \equiv 0$ for $t > s$.

## Interval Analysis

Our purpose is to find functions $\underline{u}(t)$, $\bar{u}(t)$ such that

$$\underline{u}(s) \leq u(s) \leq \bar{u}(s) \tag{8}$$

$0 \leq s \leq 1$, in other words, an interval-valued function

$$U(s) = [\underline{u}(s), \bar{u}(s)] \tag{9}$$

such that $u(s) \in U(s)$ for $s \in [0,1]$ [1]. The process described in [12] for this purpose is a combination of interval integration [1] and interval iteration [9].

Interval Integration

The integral of an interval function U(t) is defined to be

$$\int_0^1 U(s)ds = [\underline{\int_0^1} \underline{u}(s)ds, \overline{\int_0^1} \bar{u}(s)ds], \tag{10}$$

where under and overbars on the integral signs within the brackets denote respectively lower and upper Darboux integrals [1]. Since these Darboux integrals always exist, it follows that all interval (and hence all real) functions are integrable in this sense.

Interval Iteration

Interval iteration for the fixed-point problem (2) works like this [9], [12]: Suppose that one has an interval inclusion G of the operator g, that is, an operator from interval functions to interval functions such that $g(u) \in G(U)$ for each $u \in U$. Then, starting with an intial interval function $U_0$, one computes

$$U_{n+1} = U_n \cap G(U_n). \tag{11}$$

Interval iteration has the following properties [9]:

1. If $u \in U_0$ is a solution of (2), then the interval iteration will converge to

$$U^* = \bigcap_{n=0}^{\infty} U_n, \tag{12}$$

and $u \in U^*$.

2. If for some integer n,

$$U_n \cap G(U_n) = \emptyset, \tag{13}$$

the empty set, then there are no solutions u of (2) in the initial interval $U_0$.

3. If

$$G(U_n) \subset U_n \tag{14}$$

for some integer n, then a solution u of (2) exists in the initial interval $U_0$.

Thus, interval iteration can be used to prove existence of solutions if (14) holds, nonexistence by (13), or to obtain improved bounds for solutions by (12). The problem in actual computation is then to construct an interval inclusion G of the integral operator g in (1). One way to do this is to construct an inclusion G(s,t,U(s),U(t)) of the integrand by using interval arithmetic [6], and then use interval integration to include the operator. Here, by "interval arithmetic", we mean interval inclusions of the ordinary library functions available on a computer in addition to the arithmetic operations themselves. We now present several ways to implement such a construction on an actual computer. It is assumed, of course, that interval computation is properly supported, as in Pascal-SC or ACRITH.

## Inclusion of Integral Operators

### Interval Arithmetic

A straightforward method to construct an inclusion of the integral operator (1) simply uses interval arithmetic. Here, the interval S = T = [0,1] is partitioned into n subintervals

$$S_i = T_i = [s_{i-1}, s_i]$$

with endpoints $s_{i-1} < s_i$, $s_0 = 0$, $s_n = 1$, and widths $w(S_i) = s_i - s_{i-1}$. Further, we define an interval step function U by $U(s) = U_i = [\underline{u}_i, \bar{u}_i]$ for $s \in S_i$. Then, if $u \in U$, it follows that

$$g(s,t,u(s),u(t)) \in G(S_i,T_j,U_i,U_j) \tag{15}$$

for $s \in S_i$, $t \in T_j$. Thus,

$$u(s) \in \sum_{j=1}^{n} G_{ij}(U_i,U_j) \cdot w(T_j), \tag{16}$$

where $G_{ij}(U_i,U_j) = G(S_i,T_j,U_i,U_j)$, because the interval Riemann sum on the right side of (16) contains the interval integral of G(s,t,U(s),U(t)) [1]. In this case, the interval iteration (11) can be carried out in the form

$$U_i^{n+1} = \Big[ \sum_{j=1}^{i-1} G_{ij}(U_i^n, U_j^{n+1}) \cdot w(T_j) + \sum_{j=i}^{n} G_{ij}(U_i^n, U_j^n) \cdot w(T_j) \Big] \cap U_i^n, \tag{17}$$

a "Gauss-Seidel" iteration which makes use of updated information as it is produced, and has the same convergence properties as (11) [13]. This simple approach to solving integral equations has the following advantages:

1. It is easy to implement on a computer.
2. No special properties of g(s,t,v,w) such as positivity or monotonicity are

required.

3. The computation is self-validating, with existence of a solution of the integral equation and lower and upper bounds for it guaranteed if (14) holds.

4. A form of "superconvergence" holds at the nodes $s_i$ for $i = 1,\ldots,n-1$. Since $u(s_i) \in U_i^n$ and $u(s_i) \in U_{i+1}^n$, it follows that

$$u(s_i) \in U_i^n \cap U_{i+1}^n, \tag{18}$$

which results in narrower bounds for $u(s)$ at these points [12].

The main disadvantage of this simple procedure is very slow convergence if iteration is carried out to the point at which $U^n = U^{n+1}$ on the computer [7]. Instead of wasting iterations which result in little improvement in the bounds for the solution, it is better to stop when the decrease in $w(U_i^n \cap U_{i+1}^n)$ falls below a given limit.

Monotonicity Methods

Interval analysis of integral equations is simplified considerably if the function $g(s,t,v,w)$ has known monotonicity properties with respect to $v,w$. The underlying partial ordering for real functions is, of course, componentwise. We say that $v \leq w$ if $v(s) \leq w(s)$ for all $s \in S$. If, for example, $g(s,t,v,w)$ is isotone (monotone increasing) with respect to $v$, and antitone (monotone decreasing) with respect to $w$, then

$$G(s,t,U(s),U(t)) =$$

$$= [g(s,t,\bar{u}(s),\underline{u}(t)),\ g(s,t,\bar{u}(s),\underline{u}(t))] \tag{19}$$

will provide an interval inclusion of $g(s,t,u(s),u(t))$ for $u \in U$. Since interval iteration is inclusion monotone [1], integration of (19) will yield an interval inclusion G of the integral operator g. Methods using monotonicity properties of operators to obtain lower and upper bounds for solutions of functional equations have been investigated extensively

by Collatz [2] and others. In theory, a function of bounded variation can be expressed as the sum of an isotone and an antitone function, so that g(s,t,u(s),u(t)) could be expressed in terms of a finite number of such functions; however, it is not usually possible to perform this construction explicitly [4].

The actual implementation of monotonicity methods is hampered by the fact that is difficult to carry out and represent the transform G(U) on the computer, even for simple interval functions U(s). For example, if $\underline{u}$ and $\overline{u}$ are step functions, then $G([\underline{u},\overline{u}])$ will not in general be a step function, and the representation of more and more complicated functions will be necessary as the iteration proceeds. A solution to this problem is provided by the concept of directed rounding in function space.

## Directed Rounding

For actual computation, we take a set $\Phi$ of functions which can be represented exactly on a computer. For example, $\Phi$ can consist of step functions which have nodes and values which are floating-point numbers. An upward rounding operator $\Delta$ from the set R of real functions into $\Phi$ has the following properties:

$$\text{(i)} \quad \Delta u = u \text{ if } u \in \Phi,$$

$$\text{(ii)} \quad u \leq \Delta u \text{ for } u \in R,$$

$$\text{(iii)} \quad \text{if } u \leq v, \text{ then } \Delta u \leq \Delta v.$$

These conditions imply, among other things, that there is no element v of $\Phi$ such that $u \leq v \leq \Delta v$ [3], hence, this rounding is "optimal" in a certain sense. The downward rounding operator $\nabla$ is defined in a similar way, with the inequality signs reversed. Thus, if one has the same monotonicity properties for the function g(s,t,u(s),u(t)) as in (19), an interval inclusion of g can be computed by

$$G(U) = [\nabla\int_0^1 G(s,t,\underline{u}(s),\bar{u}(t))dt,$$

$$\Delta\int_0^1 G(s,t,\bar{u}(s),\underline{u}(t))dt], \tag{20}$$

An an example of this type of computation, consider the integrand

$$g(s,t,u(s),u(t)) = 1 + \frac{\lambda u(s)u(t)}{s + t}, \tag{21}$$

which arises in radiative transfer theory, where $0 \leq \lambda \leq 0.5$ [12]. Here, $\Phi$ will be the set of step functions, and it follows from (21) that the integrand is isotone for positive functions. Let $f \in \Phi$ have value $f_i$ in the subinterval $T_i$ of $[0,1]$. Then, for $s \in S_i$,

$$g(f)(s) = 1 + \lambda s f_i \cdot \sum_{j=1}^{n} f_j \cdot \ln\{\frac{s + s_j}{s + s_{j-1}}\}, \tag{22}$$

a monotone function which can be rounded downward and upward to bound the integral operator [10]. Numerical computations using this method for bounding the integral operator show both faster convergence and applicability to an extended range of $\lambda$ as compared to the simple approximation to the integral operator using interval arithmetic [10]. For example, (14) holds when the operator is approximated by interval Riemann sums only for $0 \leq \lambda \leq$ 0.365, while the use of (22) extends this range to $0 \leq \lambda \leq 0.451$ [10]. The number of iterations is also drastically reduced over those reported in [7].

## Iterative Residual Correction

The method of interval iteration can be combined in a simple way with other approximate methods for the solution of integral equations. The result gives essentially a method for choosing the initial interval $U_0$ in which the solution u of (2) is sought. Suppose, for example, that a numerical solution of the integral equation yields the

approximation $u_0 = u_0(s)$ to a solution. If this solution is believed to be correct within $\varepsilon$, then it is natural to start the interval iteration with $U_0(s) = u_0(s) + \varepsilon \cdot [-1,1]$, for example. If it turns out that (14) is satisfied for $n = 0$, then the interval computation immediately validates the existence of the solution $u$ in the interval $U_0$, and also the error bound $\varepsilon$ [3]. The interval iteration can then be continued if it is desired to obtain a better error bound for the approximate solution than $\varepsilon$. The same applies to $U_n$ if (14) is satisfied for some value of $n > 0$.

The advantages to this approach are that it may be much quicker to obtain a good approximation to the solution of the integral equation using floating-point arithmetic instead of interval arithmetic, and the width of the initial interval $U_0$ is usually very small. This is important because interval inclusions of functions on intervals usually have excess width over the range of the function on that interval, that is, $w(F(X))$ can be much larger than $w(f(X))$ for a given real function $f$ and real interval $X$. However, it is known that $w(F(X)) \to w(f(X))$ as $w(X) \to 0$ if $f$ is continuous [5]. This convergence result implies that interval calculations conducted with intervals of small width will estimate the values of the included real function more accurately, and the iteration will hence also converge more rapidly to whatever tolerance limit is set by the solver of the equation or the computing equipment used.

In terms of the residual

$$r_0 = g(u_0) - u_0,$$

one can write

$$u - u_0 = r_0 + g(u) - g(u_0),$$

and if $G'$ denotes an interval extension of the Fréchet derivative $g'$ of $g$, then the interval mean value theorem [11] gives

$$u - u_0 \in r_0 + G'(U)(U - u_0),$$

or, for $U_0 = u_0 + \varepsilon \cdot [-1,1]$,

$$u - u_0 \in r_0 + \varepsilon \cdot G'(U_0) \cdot [-1,1].$$

## Eigenvalue Problems

A simple application of the interval method for solution of integral equations is to the eigenvalue problem

$$u(s) = \lambda \int_a^b K(s,t)u(t)dt \tag{31}$$

for Fredholm integral operators K with kernel $K(s,t)$. It follows from the above that if $U(s)$ is an interval function such that $0 \notin (U,U)$ and U contains an eigenfunction u of K, then the eigenvalue $\lambda$ belonging to the eigenfunction u will be contained in the interval Rayleigh quotient

$$\Lambda = (KU,U)/(U,U), \tag{32}$$

where the scalar products are formed by interval integration, and the division is done in interval arithmetic [8]. This process forms the basis of an interval iteration method. Starting with $U_0$, one can compute $\Lambda_0$ by (32). Having $U_n$ and $\Lambda_n$, the next step is to iterate

$$V_{n+1} = \Lambda_n K V_n \cap V_n \tag{29}$$

to convergence, starting with $V_0 = U_n$. If V* denotes the converged value of V, then one takes

$$U_{n+1} = V^* \tag{30}$$

and then

$$\Lambda_{n+1} = \Lambda_n \cap \frac{(KU_{n+1}, U_{n+1})}{(U_{n+1}, U_{n+1})} . \tag{31}$$

The interval Rayleigh quotient (32) has the advantage that it gives lower and upper bounds for all eigenvalues of K, including the smallest. Most methods for approximating the smallest eigenvalue give only upper bounds [8]. For example, let K(s,t) be the Green's function for the differential operator $-y''$ on $[0,\pi]$ with boundary conditions $y(0) = y(\pi) = 0$. One can start with $U_0(s)$ defined by

$$\bar{u}(s) = 1, \quad 0 \leq s \leq \pi,$$

$$\underline{u}(s) = (2/\pi)s, \quad 0 \leq s \leq \pi/2, \tag{32}$$

$$\underline{u}(s) = (2/\pi)(\pi - s), \quad \pi/2 \leq s \leq \pi,$$

and obtain $\Lambda_0 = [0.4052, 2.5033]$, while a second iteration gives $\Lambda_1 = [0.6792, 1.9454]$, which provides better guaranteed lower and upper bounds for the smallest eigenvalue $\lambda = 1$ of the problem.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT - cont'd.

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